Infinitely many solutions for fractional Kirchhoff–Sobolev–Hardy critical problems

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Abstract. We investigate a class of critical stationary Kirchhoff fractional $p$-Laplacian problems in presence of a Hardy potential. By using a suitable version of the symmetric mountain-pass lemma due to Kajikiya, we obtain the existence of a sequence of infinitely many arbitrarily small solutions converging to zero.

Keywords: fractional $p$-Laplacian, Kirchhoff coefficient, Hardy potentials, critical Sobolev exponent, variational methods.

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1 Introduction

In this paper we consider the following fractional problem

\[
\begin{cases}
    M([u]_{sp}^p)(-\Delta)^s_p u - \gamma \frac{|u|^{p-2}u}{|x|^p} = \lambda w(x)|u|^{q-2}u + \frac{|u|^{p^*_s(\alpha)-2}u}{|x|^\alpha}, & \text{in } \Omega, \\
    u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where $0 < s < 1 < p < \infty$, $0 \leq \alpha < sp < N$, $1 < q < p$, $p^*_s(\alpha) = \frac{p(N-\alpha)}{N-sp} \leq p^*_s(0) = p^*_s$ is the critical Hardy–Sobolev exponent, $\gamma$ and $\lambda$ are real parameters, $w$ is a positive weight whose assumption will be introduced in the sequel and $\Omega \subseteq \mathbb{R}^N$ is a general open set. Naturally, the condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$ disappears when $\Omega = \mathbb{R}^N$.

Here $(-\Delta)^s_p$ denotes the fractional $p$-Laplace operator which, up to normalization factors, may be defined by the Riesz potential as

\[
(-\Delta)^s_p u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} dy, \quad x \in \mathbb{R}^N,
\]

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along any \( u \in C_0^\infty(\mathbb{R}^N) \), where \( B_\varepsilon(x) = \{ y \in \mathbb{R}^N : |x-y| < \varepsilon \} \). See [11, 23] and the references therein for further details on the fractional Sobolev space \( W^{s,p}(\Omega) \) and some recent results on the fractional \( p \)-Laplacian.

Problem (1.1) is fairly delicate due to the intrinsic lack of compactness, which arise from the Hardy term and the nonlinearity with critical exponent \( p_0^*(\alpha) \). For this reason, we strongly need that the Kirchhoff coefficient \( M \) is non-degenerate, namely \( M(t) > 0 \) for any \( t \geq 0 \). Hence, along the paper, we suppose that the Kirchhoff function \( M : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) is continuous and satisfies

\[
(M_1) \quad \inf_{t \in \mathbb{R}_0^+} M(t) = a > 0; \\
(M_2) \quad \text{there exists } \theta \in [1, p_0^*(\alpha)/p), \text{ such that } M(t)t \leq \theta M(t) \text{ for all } t \in \mathbb{R}_0^+, \text{ where } M(t) = \int_0^t M(\tau)d\tau.
\]

Concerning the positive weight \( w \), we assume that

\[
(w) \quad w(x)|x|^{-\frac{4s}{p}\alpha} \in L^p(\mathbb{R}^N), \quad \text{with } r = \frac{p_0^*(\alpha)}{p_0^*(\alpha) - q}.
\]

Condition \((w)\) is necessary, since it guarantees that the embedding \( Z(\Omega) \hookrightarrow L^q(\Omega, w) \) is compact, even when \( \Omega \) is the entire space \( \mathbb{R}^N \). Indeed, the natural solution space for problem (1.1) is the fractional density space \( Z(\Omega) \), that is the closure of \( C_0^\infty(\Omega) \) with respect to the norm \( \langle \cdot \rangle_{s,p} \), given by

\[
[u]_{s,p} = \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\,dxdy \right)^{1/p}.
\]

Thus, by arguing similarly to Lemma 4.1 of [15], we have that the embedding \( Z(\Omega) \hookrightarrow L^q(\Omega, w) \) is compact with

\[
\|u\|_{q,w} \leq C_w[u]_{s,p} \quad \text{for any } u \in Z(\Omega),
\]

where the weighted norm is set by

\[
\|u\|_{q,w} = \left( \int_{\Omega} w(x)|u(x)|^q\,dx \right)^{1/q}
\]

and \( C_w = H^{-1/p}_a \left( \int_{\mathbb{R}^N} w\varphi(x)|x|^{-\frac{4s}{p}\alpha}\,dx \right)^{1/q} \) is a positive constant. Here \( H_a = H(N, p, s, \alpha) \) denotes the best fractional critical Hardy–Sobolev constant, given by

\[
H_a = \inf_{u \in Z(\Omega) \setminus \{0\}} \frac{[u]_{s,p}}{\|u\|_{H_a}^p}, \quad \|u\|_{H_a}^p = \int_{\Omega} |u(x)|^{p_0^*(\alpha)}\,dx / |x|^{\alpha}, \quad \text{for any } u \in Z(\Omega) \setminus \{0\}.
\]

Of course number \( H_a \) is well-defined and strictly positive for any \( \alpha \in [0, ps] \), since Lemma 2.1 of [15]. We observe that when \( \alpha = 0 \) then \( H_0 \) coincides with the critical Sobolev constant, while when \( \alpha = sp \) then \( H_{sp} \) is the true critical Hardy constant. In order to simplify the notation, throughout the paper we denote the true fractional Hardy constant and the true fractional Hardy norm with \( H = H_{sp} \) and \( \| \cdot \|_H = \| \cdot \|_{H_{sp}} \) in (1.3) when \( \alpha = sp \).

When \( s = 1 \) and \( p = 2 \), our problem (1.1) is related to the celebrated Kirchhoff equation

\[
\rho u_{tt} - \left( \frac{P_0}{H} + \frac{E}{2L} \int_0^L |u_x|^2\,dx \right) u_{xx} = 0,
\]
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proposed by Kirchhoff [21] in 1883 as a nonlinear generalization of D’Alembert’s wave equation for free vibrations of elastic strings. This model describes a vibrating string, taking into account the changes in the length of the string during vibrations. In the equation (1.4) \( u = u(x,t) \) is the transverse string displacement at the space coordinate \( x \) and time \( t \), \( L \) is the length of the string, \( h \) is the area of the cross section, \( E \) is Young’s modulus of the material, \( \rho \) is the mass density, and \( P_0 \) is the initial tension. The early studies devoted to the Kirchhoff model were given by Bernstein [6], Lions [22] and Pohozaev [26].

In the nonlocal setting, Fiscella and Valdinoci [17] proposed a stationary Kirchhoff variational model in smooth bounded domains of \( \mathbb{R}^N \), which takes into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string, given by Caffarelli et al. in [8]. In other words, the problem studied in [17] is the fractional version of the Kirchhoff equation (1.4). Starting from [17], a great attention has been devoted to the study of fractional Kirchhoff problems; see for example [1–3, 9, 13–16, 24, 27].

The true local version of problem (1.1), namely when \( M \equiv 1 \) and \( s = 1 \), given by

\[
\begin{cases}
-\Delta_p u - \frac{|u|^{p-2}u}{|x|^\theta} = \lambda w(x)|u|^{p-2}u + \frac{|u|^{p^*(a)-2}u}{|x|^\alpha}, & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\] (1.5)

has been widely studied in [10,12,18,19]. In these works, the authors proved the existence of infinitely many solutions of (1.5), when the parameter \( \lambda \) is controlled by a suitable threshold depending on the following Sobolev–Hardy–critical constant

\[
S_\gamma = \inf_{W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega \left( |\nabla u(x)|^p - \gamma \frac{|u(x)|^p}{|x|^\theta} \right) dx}{\left( \frac{\int_\Omega |u(x)|^{p^*(a)} dx}{|x|^\alpha} \right)}^{\frac{p}{p^*(a)}}.
\]

In order to overcome the lack of compactness, due to the presence of two Hardy potentials in (1.5), they exploit a concentration compactness principle, applied to the combined norm \( \int_\Omega \left( |\nabla u|^p - \gamma \frac{|u|^p}{|x|^\theta} \right) dx \) and to the critical norm \( \int_\Omega \frac{|u|^{p^*(a)}}{|x|^\alpha} dx \). Because of the bi–nonlocal nature of the problem (1.1), the same approach of [10,12,18,19] can not work in our case. Indeed, due to the presence of a Kirchhoff coefficient \( M \), for which the equation in (1.1) is no longer a pointwise identity, we have difficulties in considering a combined norm. Since \( \Omega \) could be unbounded, we can not apply a concentration compactness argument because of the nonlocal nature of \( (-\Delta)_\alpha^\mu \), as well explained in Section 2.3 of [25]. For these reasons, we use a tricky analysis of the energy functional which allows us to handle the two Hardy potentials in (1.1); see Sections 2 and 3.

Thus, we get the next multiplicity result for (1.1), which involves the main geometrical parameter \( \kappa_\sigma = \kappa(\sigma) \) defined by

\[
\kappa_\sigma = \frac{a(\sigma - \theta p)}{\theta(\sigma - p)},
\] (1.6)

for any \( \sigma \in (p\theta, p^*_\alpha(a)) \). A parameter similar to (1.6) already appeared in [9]. Clearly \( \kappa_\sigma \leq a \), since \( \theta \geq 1 \) and \( p\theta \leq \sigma \). When \( \theta = 1 \) in \( (M_2) \), we observe that parameter \( \kappa_\sigma = a \) does not depend by the choice of \( \sigma \). As shown in Section 2 of [9], the situation \( \theta = 1 \) holds true in other cases, besides the obvious one \( M \equiv a \).

Now, we are ready to state the main result of the present paper.
Theorem 1.1. Let $N > ps > \alpha \geq 0$, $q \in (1, p)$, with $s \in (0, 1)$ and $p \in (1, \infty)$. Assume that $M$ and $w$ satisfy assumptions $(M_1)$–$(M_2)$ and $(w)$.

Then, for any $\sigma \in (p\theta, p_s^*(\alpha))$ and for any $\gamma \in (-\infty, \kappa_c H)$, there exists $\lambda = \lambda(\sigma, \gamma) > 0$ such that for any $\lambda \in (0, \bar{\lambda})$ problem (1.1) admits a sequence of solutions $\{u_n\}_n$ in $Z(\Omega)$ with the energy functional $J_{\gamma, \lambda}(u_n) < 0$, $J_{\gamma, \lambda}(u_n) \to 0$ and $\{u_n\}_n$ converges to zero as $n \to \infty$.

The proof of Theorem 1.1 is obtained by applying suitable variational methods and consists of several steps. In Section 2 we study the compactness property of the Euler-Lagrange functional associated with (1.1). After that, in Section 3, we introduce a truncated functional which allows us to apply the symmetric mountain pass lemma in [20]. Finally, we prove that the critical points of the truncated functional are indeed solutions of the original problem (1.1).

2 The Palais–Smale condition

Throughout the paper we assume that $N > ps > \alpha \geq 0$, $s \in (0, 1)$, $p \in (1, \infty)$, $q \in (1, p)$, $(M_1)$–$(M_2)$ and $(w)$, without further mentioning.

According to the variational nature, (weak) solutions of (1.1) correspond to critical points of the truncated functional $J_{\gamma, \lambda} : Z(\Omega) \to \mathbb{R}$, defined by

$$J_{\gamma, \lambda}(u) = \frac{1}{p} \mathcal{M}(\|u\|_{s,p}^p) - \frac{\gamma}{p} \|u\|_p^p - \frac{\lambda}{q} \|u\|_q^q - \frac{1}{p^*_s(\alpha)} \|u\|_{H_s}^{p^*_s(a)}.$$

Note that $J_{\gamma, \lambda}$ is a $C^1(Z(\Omega))$ functional and for any $u, \varphi \in Z(\Omega)$

$$\langle J'_{\gamma, \lambda}(u), \varphi \rangle = M(\|u\|_{s,p}^p)\langle u, \varphi \rangle_{s,p} - \gamma \langle u, \varphi \rangle_H - \lambda \langle u, \varphi \rangle_{q,w} - \langle u, \varphi \rangle_{H_s},$$

where

$$\langle u, \varphi \rangle_{s,p} = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}[u(x) - u(y)] \cdot [\varphi(x) - \varphi(y)]}{|x - y|^{N+sp}} dx dy,$$

$$\langle u, \varphi \rangle_{q,w} = \int_{\Omega} \frac{w(x)|u(x)|^{q-2}u(x)\varphi(x)}{|x|^{N+sp}} dx,$$

$$\langle u, \varphi \rangle_H = \int_{\Omega} \frac{|u(x)|^{p-2}u(x)\varphi(x)}{|x|^{N+sp}} dx, \quad \langle u, \varphi \rangle_{H_s} = \int_{\Omega} |u(x)|^{1/2} u(x) \varphi(x) \frac{dx}{|x|^a}.$$

Now, we discuss the compactness property for the functional $J_{\gamma, \lambda}$, given by the Palais–Smale condition. We recall that $\{u_n\}_n \subset Z(\Omega)$ is a Palais–Smale sequence for $J_{\gamma, \lambda}$ at level $c \in \mathbb{R}$ if

$$J_{\gamma, \lambda}(u_n) \to c \quad \text{and} \quad J'_{\gamma, \lambda}(u_n) \to 0 \quad \text{in } (Z(\Omega))' \quad \text{as } n \to \infty. \quad (2.2)$$

We say that $J_{\gamma, \lambda}$ satisfies the Palais–Smale condition at level $c$ if any Palais–Smale sequence $\{u_n\}_n$ at level $c$ admits a convergent subsequence in $Z(\Omega)$.

Lemma 2.1. Let $c < 0$.

Then, for any $\sigma \in (p\theta, p_s^*(\alpha))$ and any $\gamma \in (-\infty, \kappa_c H)$ there exists $\lambda_0 = \lambda_0(\sigma, \gamma) > 0$ such that for any $\lambda \in (0, \lambda_0)$, the functional $J_{\gamma, \lambda}$ satisfies the Palais–Smale condition at level $c$.

Proof. Fix $\sigma \in (p\theta, p_s^*(\alpha))$ and $\gamma \in (-\infty, \kappa_c H)$. Since $\gamma < \kappa_c H \leq aH$, there exists a number $\tilde{c} \in [0, 1)$ such that $\gamma^+ = \tilde{c} a H$. Thus, let us consider $\lambda_0 = \lambda_0(\sigma, \gamma) > 0$ sufficiently small such that

$$\left(1 - \frac{1}{p_s^*(\alpha)}\right)^{-\frac{p^*_s(a)}{p^*_s(\alpha)-q}} \left[ \lambda_0 \left( \frac{1}{\sigma} - \frac{1}{p^*_s(\alpha)} \right) \|w\|_{s,p} \right]^{-\frac{p^*_s(a)}{p^*_s(\alpha)-q}} < \left(1 - \tilde{c} a H\right)^{-\frac{p^*_s(a)}{p^*_s(\alpha)-q}} \quad (2.3)$$
where \( q < p < p_s^*(\alpha) \), \( a \) is set in \((M_1)\), while \( H_a \) is given in \((1.3)\).

Fix \( \lambda \in (0, \lambda_0) \). Let \( \{u_n\}_n \) be a \((PS)_c\) sequence in \( Z(\Omega) \). We first show that \( \{u_n\}_n \) is bounded. By using the assumptions \((M_1)\) and \((M_2)\), and the inequalities \((1.2)\) and \((1.3)\), we get

\[
\mathcal{J}_{\gamma, \lambda}(u_n) - \frac{1}{\sigma} \langle \mathcal{J}_{\gamma, \lambda}'(u_n), u_n \rangle \geq \left( \frac{1}{p_{\theta}} - \frac{1}{\sigma} \right) M([u_n]_{s,p})^p [u_n]_{s,p} - \frac{\gamma^+}{H} \left( \frac{1}{p} - \frac{1}{\sigma} \right) [u_n]_{s,p}^p \\
- \lambda \left( \frac{1}{q} - \frac{1}{\sigma} \right) C_\theta [u_n]_{s,p}^q - \left( \frac{1}{p_{s}^*(\alpha)} - \frac{1}{\sigma} \right) \|u_n\|_{H_a}^{p_{s}^*(\alpha)} \\
\geq v[u_n]_{s,p}^p - \lambda \left( \frac{1}{q} - \frac{1}{\sigma} \right) C_\theta [u_n]_{s,p}^q \\
- \left( \frac{1}{p_{s}^*(\alpha)} - \frac{1}{\sigma} \right) \|u_n\|_{H_a}^{p_{s}^*(\alpha)},
\]

(2.4)

where

\[
v = \left( \frac{1}{p_{\theta}} - \frac{1}{\sigma} \right) a - \frac{\gamma^+}{H} \left( \frac{1}{p} - \frac{1}{\sigma} \right) > 0
\]

(2.5)
in view of \((1.6)\) and the fact that \( \sigma > p_{\theta} \geq p \) and \( \gamma \in (-\infty, \kappa_c H) \). Thus, by \((2.2)\) there exists \( \beta > 0 \) such that as \( n \to \infty \)

\[
c + \beta [u_n]_{s,p}^q + o(1) \geq v[u_n]_{s,p}^p,
\]

which implies at once that \( \{u_n\}_n \) is bounded in \( Z(\Omega) \), being \( q < p \).

Therefore, using arguments similar to Lemma 4.1 of [15], there exists a subsequence, still denoted by \( \{u_n\}_n \), and a function \( u \in Z(\Omega) \) such that

\[
\begin{align*}
    u_n \to u & \quad \text{in } Z(\Omega), \\
    [u_n]_{s,p} & \to d, \\
    u_n \to u & \quad \text{in } L^p(\Omega, |x|^{-sp}), \quad \|u_n - u\|_{H} \to 1, \\
    u_n \to u & \quad \text{in } L^{p_{s}^*(\alpha)}(\Omega, |x|^{-a}), \quad \|u_n - u\|_{H_{a}} \to \ell, \\
    u_n \to u & \quad \text{in } L^q(\Omega, w), \quad u_n \to u \text{ a.e. in } \Omega
\end{align*}
\]

(2.6)
as \( n \to \infty \).

Furthermore, as shown in the proof of Lemma 2.4 of [9], by \((2.6)\) the sequence \( \{U_n\}_n \), defined in \( \mathbb{R}^{2N} \setminus \text{Diag } \mathbb{R}^{2N} \) by

\[
(x, y) \mapsto U_n(x, y) = \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x - y|^{N+sp}},
\]
is bounded in \( L^p(\mathbb{R}^{2N}) \) as well as \( U_n \to U \text{ a.e. in } \mathbb{R}^{2N} \), where

\[
U(x, y) = \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}}.
\]

Thus, up to a subsequence, we get \( U_n \to U \) in \( L^p(\mathbb{R}^{2N}) \), and so as \( n \to \infty \)

\[
\langle u_n, \varphi \rangle_{s,p} \to \langle u, \varphi \rangle_{s,p}
\]

(2.7)

for any \( \varphi \in Z(\Omega) \), since \( |\varphi(x) - \varphi(y)| \cdot |x - y|^{-N+sp} \in L^p(\mathbb{R}^{2N}) \). Similarly, \((2.6)\) and Proposition A.8 of [4] imply that \( |u_n|^{p-2}u_n \to |u|^{p-2}u \text{ in } L^p(\Omega, |x|^{-sp}) \) and \( |u_n|^{p_{s}^*(\alpha)-2}u_n \to |u|^{p_{s}^*(\alpha)-2}u \text{ in } L^{p_{s}^*(\alpha)}(\Omega, |x|^{-a}) \), from which as \( n \to \infty \)

\[
\langle u_n, \varphi \rangle_H \to \langle u, \varphi \rangle_H, \quad \langle u_n, \varphi \rangle_{H_{a}} \to \langle u, \varphi \rangle_{H_{a}},
\]

(2.8)
for any \( \varphi \in Z(\Omega) \).

Thanks to (2.6), by using Hölder inequality it results
\[
\lim_{n \to \infty} \int_{\Omega} w(x) |u_n(x)|^{q-2} u_n(x)(u_n(x) - u(x)) \, dx = 0. \tag{2.9}
\]
Consequently, from (2.2), (2.6)–(2.9) we deduce that, as \( n \to \infty \)
\[
o(1) = \langle J_{\gamma,\lambda}'(u_n), u_n - u \rangle = M([u_n]_{s,p})^{p} \|u_n\|_{s,p}^{p} - M([u_n]_{s,p}) \langle u_n, u \rangle_{s,p}
- \gamma \int_{\Omega} |u_n(x)|^{p-2} u_n(x)(u_n(x) - u(x)) \frac{dx}{|x|^a}
- \lambda \int_{\Omega} w(x) |u_n(x)|^{q-2} (u_n(x) - u(x)) \, dx
- \int_{\Omega} |u_n(x)|^{p^*_a(a)-2} u_n(x)(u_n(x) - u(x)) \, dx \frac{dx}{|x|^a}
= M([u_n]_{s,p}) ([|u_n|_{s,p}^{p} - |u|_{s,p}^{p}] - \gamma (\|u_n\|_{H}^{p} - \|u\|_{H}^{p} - \|u\|_{H}^{p} + o(1).
- \|u_n\|_{H}^{p} + \|u\|_{H}^{p} + o(1). \tag{2.10}
\]
Furthermore, by using (2.6) and the celebrated Brézis and Lieb Lemma in [7], we have
\[
\|u_n\|_{H}^{p} = \|u_n - u\|_{H}^{p} + \|u\|_{H}^{p} + o(1),
\|u_n\|_{H}^{p} = \|u_n - u\|_{H}^{p} + \|u\|_{H}^{p} + o(1), \tag{2.11}
\]
as \( n \to \infty \). By applying again the Brézis and Lieb Lemma [7] to
\[
\frac{(u_n - u)(x) - (u_n - u)(y)}{|x-y|^p} \in L^p(\mathbb{R}^2_N)
\]
we can see that
\[
[u_n]_{s,p}^{p} = [u_n - u]_{s,p}^{p} + [u]_{s,p}^{p} + o(1) \quad \text{as } n \to \infty. \tag{2.12}
\]
Therefore, combining (2.6), the continuity of \( M \) and relations (2.10)–(2.12), we have proved the crucial formula
\[
M(d^p) \lim_{n \to \infty} [u_n - u]_{s,p}^{p} = \gamma \lim_{n \to \infty} \|u_n - u\|_{H}^{p} + \lim_{n \to \infty} \|u_n - u\|_{H}^{p} = \gamma \ell^p + \ell^p(a). \tag{2.13}
\]
Now, let us rewrite the formula (2.13) as
\[
(1 - \tilde{c})M(d^p) \lim_{n \to \infty} [u_n - u]_{s,p}^{p} + \tilde{c}M(d^p) \lim_{n \to \infty} [u_n - u]_{s,p}^{p} = \gamma \ell^p + \ell^p(a),
\]
with \( \tilde{c} \in [0,1] \) fixed at the beginning of the proof. By (M1) and (1.3), we have
\[
(1 - \tilde{c})a H \ell \leq (1 - \tilde{c})M(d^p) \lim_{n \to \infty} [u_n - u]_{s,p}^{p} + \tilde{c}M(d^p) \lim_{n \to \infty} [u_n - u]_{s,p}^{p} \leq \gamma \ell^p + \ell^p(a).
\]
Therefore, since \( \gamma^+ = \tilde{c} a H \), we obtain
\[
\ell^p(a) \geq (1 - \tilde{c})a H \ell^p,
\]
from which, assuming by contradiction that \( \ell > 0 \), we get
\[
\ell^p(a) \geq [(1 - \tilde{c})a H \ell^p]^{\frac{1}{p}}. \tag{2.14}
\]
Exploiting (2.4) and (2.5), taking the limit as $n \to \infty$, and by using (2.2), (2.6), (2.10), assumption (w), Hölder inequality and Young inequality, we can infer
\[
c \geq \left( \frac{1}{\sigma} - \frac{1}{p^*_s(\alpha)} \right) \left( \ell_p^{p^*_s(\alpha)}(A) + \|u\|^p_{H^s} \right) - \lambda \left( \frac{1}{q} - \frac{1}{\sigma} \right) \|w\| \|u\|^q_{H^s}.
\]

Finally, by (2.14) we get
\[
0 > c \geq \left( \frac{1}{\sigma} - \frac{1}{p^*_s(\alpha)} \right) \left[ (1 - \widetilde{c}) a H_s \right]^{\frac{p^*_s(\alpha)}{p^*_s(\alpha) - q}} \\
- \left( \frac{1}{\sigma} - \frac{1}{p^*_s(\alpha)} \right)^{-\frac{q}{p^*_s(\alpha) - q}} \left[ \lambda \left( \frac{1}{q} - \frac{1}{\sigma} \right) \|w\| \right]^{\frac{p^*_s(\alpha)}{p^*_s(\alpha) - q}} > 0,
\]
where the last inequality follows from (2.3). This is impossible, so $\ell = 0$.

Now, let us assume by contradiction that $\iota > 0$. Then, from $(M_1)$, (1.3) and (2.13) we have
\[
M(d^p) \lim_{n \to \infty} \|u_n - u\|_{s,p}^p = \gamma \lim_{n \to \infty} \|u_n - u\|_{H^s}^p \\
< a H \lim_{n \to \infty} \|u_n - u\|_{H^s}^p \leq M(d^p) \lim_{n \to \infty} \|u_n - u\|_{s,p}^p,
\]
which gives a contradiction. Therefore, $\iota = 0$ and by using again $(M_1)$ and (2.13) it follows that $u_n \to u$ in $Z(\Omega)$ as $n \to \infty$, as claimed.

\section{The truncated functional}

In this section we prove that problem (1.1) admits a sequence of solutions which goes to zero. Firstly, we recall the definition of genus and some its fundamental properties; see [29] for more details.

Let $E$ be a Banach space and $A$ a subset of $E$. We say that $A$ is symmetric if $u \in A$ implies that $-u \in A$. For a closed symmetric set $A$ which does not contain the origin, we define the genus $\mu(A)$ of $A$ as the smallest integer $k$ such that there exists an odd continuous mapping from $A$ to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such a $k$, we put $\mu(A) = \infty$. Moreover, we set $\mu(\emptyset) = 0$.

Let us denote by $\Sigma_k$ the family of closed symmetric subsets $A$ of $E$ such that $0 \notin A$ and $\mu(A) \geq k$. Then we have the following result.

\textbf{Proposition 3.1.} Let $A$ and $B$ be closed symmetric subsets of $E$ which do not contain the origin. Then we have

(i) If there exists an odd continuous mapping from $A$ to $B$, then $\mu(A) \leq \mu(B)$.

(ii) If there is an odd homeomorphism from $A$ onto $B$, then $\mu(A) = \mu(B)$.

(iii) If $\mu(B) < \infty$, then $\mu(A \setminus B) \geq \mu(A) - \mu(B)$. 
(iv) The n-dimensional sphere $S^n$ has a genus of $n + 1$ by the Borsuk–Ulam Theorem.

(v) If $A$ is compact, then $\mu(A) < \infty$ and there exist $\delta > 0$ and a closed and symmetric neighborhood $N_\delta(A) = \{x \in E : \|x - A\| \leq \delta\}$ of $A$ such that $\mu(N_\delta(A)) = \mu(A)$.

Now, we state the following variant of symmetric mountain pass lemma due to Kajikija [20].

**Lemma 3.2.** Let $E$ be an infinite-dimensional Banach space and let $I \in C^1(E, \mathbb{R})$ be a functional satisfying the conditions below:

1. $I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the local Palais–Smale condition; that is, for some $c^* > 0$, in the case when every sequence $\{u_n\}_n$ in $E$ satisfying $I(u_n) \to c < c^*$ and $I'(u_n) \to 0$ in $E^*$ has a convergent subsequence;

2. For each $n \in \mathbb{N}$, there exists an $A_n \in \Sigma_n$ such that $\sup_{u \in A_n} I(u) < 0$.

Then either (i) or (ii) below holds.

(i) There exists a sequence $\{u_n\}_n$ such that $I'(u_n) = 0$, $I(u_n) < 0$ and $\{u_n\}_n$ converges to zero.

(ii) There exist two sequences $\{u_n\}_n$ and $\{v_n\}_n$ such that $I'(u_n) = 0$, $I(u_n) = 0$, $u_n \neq 0$, $\lim_{n \to \infty} u_n = 0$, $I'(v_n) = 0$, $I(v_n) < 0$, $\lim_{n \to \infty} I(v_n) = 0$ and $\{v_n\}_n$ converges to a non-zero limit.

**Remark 3.3.** It is worth to point out that in [20] the functional $I$ verifies the Palais–Smale condition in global. Anyway, a careful analysis of the proof of Theorem 1 in [20], allows us to deduce that the result in [20] holds again if $I$ satisfies the local Palais–Smale condition with the critical levels below zero.

Let us note that the functional $J_{\gamma, \lambda}$ is not bounded from below in $Z(\Omega)$. Indeed, assumption $(M_1)$ implies that $M(t) > 0$ for any $t \in \mathbb{R}^+_0$ and consequently by $(M_2)$ we have

$$\frac{M(t)}{\mathcal{M}(t)} \leq \frac{\theta}{\overline{T}}.$$ 

Thus, integrating on $[1, t]$, with $t > 1$, we get

$$\mathcal{M}(t) \leq \mathcal{M}(1)t^\theta \quad \text{for any } t \geq 1.$$

From this, by using (1.2) and (1.3), for any $u \in Z(\Omega)$ we have

$$J_{\gamma, \lambda}(tu) \leq t^{p^\theta} \mathcal{M}(1) \left[|u|_{\mathcal{P}}^{p^\theta} - t^p \|u\|_H^p - t^{\frac{q}{q}} \|u\|_{q,w}^q \right]$$

$$- \frac{t^p}{p^\gamma(\alpha)} \frac{1}{p_s^*(\alpha)} \|u\|_{p_s^*(\alpha)}^q \to -\infty \quad \text{as } t \to \infty.$$

Now, fix $\gamma \in (-\infty, aH)$ and $\lambda > 0$ and let us consider the function

$$Q_{\gamma, \lambda}(t) = \frac{1}{p} \left( a - \frac{\gamma^+}{H} \right) t^p - \frac{\lambda C_w}{q} t^q - \frac{1}{p_s^*(\alpha) H_s} t^{p_s^*(\alpha)}.$$

Choose $R_1 > 0$ such that

$$\frac{1}{p} \left( a - \frac{\gamma^+}{H} \right) R_1^p = \frac{1}{p_s^*(\alpha) H_s} R_1^{p_s^*(\alpha)}$$

and define

$$\lambda^* = \frac{C_w}{2q R_1^q} \left[ \left( a - \frac{\gamma^+}{H} \right) R_1^p - \frac{1}{p_s^*(\alpha) H_s} R_1^{p_s^*(\alpha)} \right].$$

In particular, if

$$\lambda \leq \lambda^*$$

(by the Borsuk–Ulam Theorem).
such that $Q_{\gamma, \lambda^*}(R_1) > 0$. Let us set

$$R_0 = \max\{t \in (0, R_1) : Q_{\gamma, \lambda^*}(t) \leq 0\}.$$  \hfill (3.3)

Taking in mind the fact that $Q_{\gamma, \lambda}(t) \leq 0$ for $t$ near zero, since $q < p < p_s^*(\alpha)$, and $Q_{\gamma, \lambda^*}(R_1) > 0$, we can infer that $Q_{\gamma, \lambda^*}(R_0) = 0$.

Choose $\phi \in C^\infty([0, \infty))$ such that $0 \leq \phi(t) \leq 1$, $\phi(t) = 1$ for $t \in [0, R_0]$ and $\phi(t) = 0$ for $t \in [R_1, \infty)$. Thus, we consider the truncated functional

$$\tilde{J}_{\gamma, \lambda}(u) = \frac{1}{p} \|u\|^p_{s,p} - \frac{\gamma}{p} \|u\|^p_{H^s} - \frac{\lambda}{q} \|u\|^q_{V^q} - \frac{\phi([u]_{s,p})}{p_s^*(\alpha)} \|u\|^{p_s^*(\alpha)}.$$

It immediately follows that $\tilde{J}_{\gamma, \lambda}(u) \to \infty$ as $[u]_{s,p} \to \infty$, by $(M_1)$, since $\gamma \in (-\infty, aH)$ and $q < p$. Hence, $\tilde{J}_{\gamma, \lambda}$ is coercive and bounded from below. Now, we prove a local Palais–Smale result for the truncated functional $\tilde{J}_{\gamma, \lambda}$.

**Lemma 3.4.** For any $\gamma \in (-\infty, aH)$, there exists $\bar{\lambda} > 0$ such that, for any $\lambda \in (0, \bar{\lambda})$

(i) if $\tilde{J}_{\gamma, \lambda}(u) \leq 0$ then $[u]_{s,p} \leq R_0$, and for any $v$ in a small neighborhood of $u$ we have $J_{\gamma, \lambda}(v) = \tilde{J}_{\gamma, \lambda}(v)$;

(ii) $\tilde{J}_{\gamma, \lambda}$ satisfies a local Palais–Smale condition for $c < 0$.

**Proof.** Let us choose $\bar{\lambda}$ sufficiently small such that $\bar{\lambda} \leq \min\{\lambda_0, \lambda^*\}$, where $\lambda_0$ is defined in Lemma 2.1 and $\lambda^*$ in (3.2). Fix $\lambda < \bar{\lambda}$.

(i) Let us assume that $\tilde{J}_{\gamma, \lambda}(u) \leq 0$.

If $[u]_{s,p} \geq R_1$, then by using $(M_1)$, (1.2), (1.3), the definition of $\phi(t)$ and the fact that $\lambda < \lambda^*$, we obtain

$$\tilde{J}_{\gamma, \lambda}(u) \geq \frac{1}{p} \left( \frac{\lambda^* C_{\Omega}}{q} \right) \|u\|^p_{s,p} - \lambda^* \|u\|^q_{V^q} > 0,$$

where the last inequality follows from $q < p$ and $Q_{\gamma, \lambda^*}(R_1) > 0$. Thus we get a contradiction because of $0 \geq \tilde{J}_{\gamma, \lambda}(u) > 0$.

When $[u]_{s,p} < R_1$, by using $(M_1)$, (1.2), (1.3), $\lambda < \lambda^*$, the definition of $\phi(t)$, we can infer

$$0 \geq \tilde{J}_{\gamma, \lambda}(u) \geq Q_{\gamma, \lambda}([u]_{s,p}) \geq Q_{\gamma, \lambda^*}([u]_{s,p}).$$

From the definition of $R_0$ we deduce that $[u]_{s,p} \leq R_0$. Moreover, for any $u \in B_{\frac{R_0}{2}}(0)$ we have that $J_{\gamma, \lambda}(u) = \tilde{J}_{\gamma, \lambda}(u)$.

(ii) Being $\tilde{J}_{\gamma, \lambda}$ a coercive functional, every Palais–Smale sequence for $\tilde{J}_{\gamma, \lambda}$ is bounded. Thus, since $\lambda < \lambda_0$, by Lemma 2.1 we deduce a local Palais–Smale condition for $\tilde{J}_{\gamma, \lambda} \equiv \tilde{J}_{\gamma, \lambda}$ at any level $c < 0$. \hfill \Box

Taking into account that $Z(\Omega)$ is reflexive and separable (see Appendix A in [28]), we can find a sequence $\{\varphi_n\}_n \subset Z(\Omega)$ such that $Z(\Omega) = \text{span}\{\varphi_n : n \in \mathbb{N}\}$. For any $n \in \mathbb{N}$ we can set $X_n = \text{span}\{\varphi_n\}$ and $Y_n = \oplus_{i=1}^n X_i$.

**Lemma 3.5.** For any $\gamma \in (-\infty, aH)$, $\lambda > 0$ and $k \in \mathbb{N}$, there exists $\varepsilon = \varepsilon(\gamma, \lambda, k) > 0$ such that

$$\mu(\tilde{J}_{\gamma, \lambda}^{-\varepsilon}) \geq k,$$

where $\tilde{J}_{\gamma, \lambda}^{-\varepsilon} = \{u \in Z(\Omega) : \tilde{J}_{\gamma, \lambda}(u) \leq -\varepsilon\}$. 
Proof. Fix $\gamma \in (-\infty, aH)$, $\lambda > 0$ and $k \in \mathbb{N}$. Since $Y_k$ is finite dimensional, there exist two positive constants $c_1(k)$ and $c_2(k)$ such that for any $u \in Y_k$

$$c_1(k)|u|_{s,p}^p \leq \|u\|_{H}^p \quad \text{and} \quad c_2(k)|u|_{q,w}^q \leq \|u\|_{q,w}^q.$$  

(3.4)

By using (3.4), for any $u \in Y_k$ such that $|u|_{s,p} \leq R_0$, we can infer

$$\tilde{\mathcal{J}}_{\gamma,\lambda}(u) = \mathcal{J}_{\gamma,\lambda}(u) \leq \frac{M^*}{p} |u|_{s,p}^p + \frac{\gamma - c_1(k)}{p} |u|_{s,p}^p - \frac{\lambda}{q} c_2(k)|u|_{q,w}^q,$$

(3.5)

with $M^* = \max_{\tau \in [0,R_0]} M(\tau) < \infty$, by continuity of $M$. Now, let $\varrho$ be a positive constant such that

$$\varrho < \min \left\{ R_0, \left[ \frac{\lambda c_2(k)p}{q(M^* + \gamma - c_1(k))} \right]^{\frac{1}{p-q}} \right\}.$$  

(3.6)

Then, for any $u \in Y_k$ such that $|u|_{s,p} = \varrho$, by (3.5) we get

$$\tilde{\mathcal{J}}_{\gamma,\lambda}(u) \leq \varrho^p \left[ \frac{M^* + \gamma - c_1(k)}{p} \varrho^{p-q} - \frac{\lambda c_2(k)}{q} \right] < 0,$$

(3.7)

where the last inequality follows from (3.6). Hence we can find a constant $\varepsilon = \varepsilon(\gamma, \lambda, k) > 0$ such that $\tilde{\mathcal{J}}_{\gamma,\lambda}(u) \leq -\varepsilon$ for any $u \in Y_k$ such that $|u|_{s,p} = \varrho$. As a consequence

$$\{ u \in Y_k : |u|_{s,p} = \varrho \} \subset \{ u \in \mathcal{Z}(\Omega) : \tilde{\mathcal{J}}_{\gamma,\lambda}(u) \leq -\varepsilon \} \setminus \{0\}.$$  

By using (ii) and (iv) of Proposition 3.1 we have the thesis. \qed

For any $c \in \mathbb{R}$ and any $k \in \mathbb{N}$, let us define the set

$$K_c = \{ u \in \mathcal{Z}(\Omega) : \tilde{\mathcal{J}}_{\gamma,\lambda}'(u) = 0 \text{ and } \tilde{\mathcal{J}}_{\gamma,\lambda}(u) = c \}$$

and the number

$$c_k = \inf_{\mathcal{A} \in \Sigma_k} \sup_{u \in A} \tilde{\mathcal{J}}_{\gamma,\lambda}(u).$$  

(3.8)

Lemma 3.6. For any $\gamma \in (-\infty, aH)$, $\lambda > 0$ and $k \in \mathbb{N}$, we have that $c_k < 0$.

Proof. Fix $\gamma \in (-\infty, aH)$, $\lambda > 0$ and $k \in \mathbb{N}$. Then, by using Lemma 3.5 we can find a positive constant $\varepsilon$ such that $\mu(\tilde{\mathcal{J}}_{\gamma,\lambda}^{-\varepsilon}) \geq k$. Moreover, $\tilde{\mathcal{J}}_{\gamma,\lambda}^{-\varepsilon} \in \Sigma_k$ since $\tilde{\mathcal{J}}_{\gamma,\lambda}$ is a continuous and even functional. Taking into account that $\tilde{\mathcal{J}}_{\gamma,\lambda}(0) = 0$, we have $0 \notin \tilde{\mathcal{J}}_{\gamma,\lambda}^{-\varepsilon}$ and $\sup_{u \in \tilde{\mathcal{J}}_{\gamma,\lambda}^{-\varepsilon}} \tilde{\mathcal{J}}_{\gamma,\lambda}(u) \leq -\varepsilon$.

Therefore, recalling that $\tilde{\mathcal{J}}_{\gamma,\lambda}$ is bounded from below, we get

$$-\infty < c_k = \inf_{\mathcal{A} \in \Sigma_k} \sup_{u \in A} \tilde{\mathcal{J}}_{\gamma,\lambda}(u) \leq \sup_{u \in \tilde{\mathcal{J}}_{\gamma,\lambda}^{-\varepsilon}} \tilde{\mathcal{J}}_{\gamma,\lambda}(u) \leq -\varepsilon < 0.$$ \qed

Lemma 3.7. Let $\gamma \in (-\infty, aH)$ and $\lambda \in (0, \bar{\lambda})$, where $\bar{\lambda}$ is given by Lemma 3.4. Then all $c_k$ are critical values for $\tilde{\mathcal{J}}_{\gamma,\lambda}$ and $c_k \to 0$ as $k \to \infty$.

Proof. Fix $\gamma \in (-\infty, aH)$ and $\lambda > 0$. It is easy to see that $c_k \leq c_{k+1}$ for all $k \in \mathbb{N}$. By Lemma 3.6 it follows that $c_k < 0$, so we can assume that $c_k \to \varepsilon \leq 0$. Since $\tilde{\mathcal{J}}_{\gamma,\lambda}$ satisfies the Palais–Smale condition at level $c_k$ by Lemma 3.4, we can argue as in [29] to see that all $c_k$ are critical value of $\tilde{\mathcal{J}}_{\gamma,\lambda}$.\qed
Now, we prove that $\bar{c} = 0$. We argue by contradiction, and we suppose that $\bar{c} < 0$. In view of Lemma 3.4, we know that $K_\bar{c}$ is compact, so, by applying part (v) of Proposition 3.1, we can deduce that $\mu(K_\bar{c}) = k_0 < \infty$ and there exists $\delta > 0$ such that $\mu(K_\delta) = \mu(N_\delta(K_\bar{c})) = k_0$. By Theorem 3.4 of [5], there exists $\varepsilon \in (0, \bar{c})$ and an odd homeomorphism $\eta : Z(\Omega) \to Z(\Omega)$ such that

$$\eta(\tilde{J}_{\gamma,\lambda}^{\bar{c}+\varepsilon} \setminus N_\delta(K_\varepsilon)) \subset \tilde{J}_{\gamma,\lambda}^{\bar{c}-\varepsilon}.$$  

Now, taking into account that $c_k$ is increasing and $c_k \to \bar{c}$, we can find $k \in \mathbb{N}$ such that $c_k > \bar{c} - \varepsilon$ and $c_k + k_0 \leq \bar{c}$. Take $A \in \Sigma_{k+k_0}$ such that $\sup_{u \in A} \tilde{J}_{\gamma,\lambda}(u) < \bar{c} + \varepsilon$. By using part (iii) of Proposition 3.1, we obtain

$$\mu(A \setminus N_\delta(K_\varepsilon)) \geq \mu(A) - \mu(N_\delta(K_\varepsilon)) \quad \text{and} \quad \mu(\eta(A \setminus N_\delta(K_\varepsilon))) \geq k,$$

from which $\eta(A \setminus N_\delta(K_\varepsilon)) \in \Sigma_k$. Thus

$$\sup_{u \in \eta(A \setminus N_\delta(K_\varepsilon))} \tilde{J}_{\gamma,\lambda}(u) \geq c_k > \bar{c} - \varepsilon.$$  

(3.10)

However, in view of (3.7) and (3.9) we can see that

$$\eta(A \setminus N_\delta(K_\varepsilon)) \subset \eta(\tilde{J}_{\gamma,\lambda}^{\bar{c}+\varepsilon} \setminus N_\delta(K_\varepsilon)) \subset \tilde{J}_{\gamma,\lambda}^{\bar{c}}.$$

which gives a contradiction in virtue of (3.10). Therefore, $\bar{c} = 0$ and $c_k \to 0$. \hfill \Box

Proof of Theorem 1.1. Let $\sigma \in (pt_\gamma p_\gamma^*(a))$, $\gamma \in (-\infty, \kappa_\sigma H)$ and $\lambda \in (0, \bar{\lambda})$. Since $\kappa_\sigma \leq a$, putting together Lemma 3.4, Lemma 3.5, Lemma 3.6 and Lemma 3.7, we can see that $\tilde{J}_{\gamma,\lambda}$ verifies all the assumptions of Lemma 3.2. Therefore, the thesis follows by point (i) of Lemma 3.4. \hfill \Box

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