

# Attractiveness and stability for Riemann–Liouville fractional systems

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**Abstract.** We propose a novel approach to study the asymptotic behavior of solutions to Riemann–Liouville (RL) fractional equations. It is shown that the standard Lyapunov approach is not suited and an extension employing two (pseudo) state spaces is needed. Theorems of Lyapunov and LaSalle type for general multi-order (commensurate or non-commensurate) nonlinear RL systems are stated. It is shown that stability and passivity concepts are thus well defined and can be employed in  $\mathcal{L}^2$ -control. Main applications provide convergence conditions for linear time-varying and nonlinear RL systems having the latter a linear part plus a Lipschitz term. Finally, computational realizations of RL systems, as well as relationships with Caputo fractional systems, are proposed.

**Keywords:** fractional differential equations, Riemann–Liouville derivative, stability, attractiveness, multi-order, nonlinear systems.

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## 1 Introduction

The study of asymptotic properties of integer order systems have been greatly developed through the use of the Lyapunov method and the corresponding stability notions [14, Chapter 3]. The asymptotic stability of equilibrium points, in the Lyapunov sense, has become the main goal in control applications. Seemingly different techniques as passivity [14, Chapter 10] and input-output stability [14, Chapter 6], have deep connections with the Lyapunov functions.

Fractional systems have attracted the attention of researchers due to their non-local properties, which allow to model complex phenomena [12, 13]. In particular, RL systems present interesting mathematical challenges due to their singularities at the initial time (see [3] for the study of positive solutions). However, the Lyapunov theory has been mainly developed for Caputo fractional systems [6, 10], since the formulation for RL fractional order systems present, as we will see in Section 2, an obstacle related to the initial conditions, that has remained unnoticed in the main papers on the subject [17, 19]. Alternative methods like the Gronwall

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inequality and the Barbalat lemma have limited applicability in fractional systems [7, 11]. Integral equation approaches posit restrictive conditions for asymptotic stability (e.g. complete positive kernels [4]).

We aim to develop a Lyapunov approach to study asymptotic properties of RL systems. It involves the definition of suited stability and passivity concepts, the introduction of Lyapunov-like theorems and the illustration of their applicability. Our underlying objective is to show that RL systems have as good properties as the Caputo systems – which have become pervasive in the literature – even in simulation aspects. Moreover, we show that multi-order systems (where each equation can have a different derivation order) can be easier to deal with RL than Caputo derivative.

In Section 3, we establish asymptotic results for RL systems, taking as a model the theory of Lyapunov functions. Specifically, similar results to Lyapunov, LaSalle and passivity theorems for mixed or multi-order nonlinear RL systems (also called non-commensurate) are obtained, which are contributions to the revised literature (cf. the recent work [21] where the systems are commensurate and restricted to a specific class of nonlinearity). Though the Lyapunov theory is powerful in many ways, it has the weakness that the Lyapunov function is not explicit from the equations and has to be constructed. In most of the cases, however, a quadratic function is enough. It turns out that quadratic functions are also effective for RL systems due to a quadratic inequality recently stated [2, 17].

In Section 4, we provide main applications of our approach finding conditions for convergence of fractional time-varying linear and nonlinear systems. Finally, in Section 5, we propose a realization of RL systems based on standard software and relationships among Caputo and RL systems.

## 2 Preliminaries

In this section, we examine stability notions and relevant features of RL systems. The central concept in fractional calculus is the Riemann–Liouville fractional integral. For a function  $f : [0, T] \rightarrow \mathbb{C}$ , it is given by [15, eq. 2.1.1],

$$I^\alpha f(t) := [I^\alpha f(\cdot)](t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad (2.1)$$

where  $\alpha \in \mathbb{R}_{>0}$  and, without loss of generality, the initial time of the fractional integral is fixed at  $t = 0$ . It is well defined for locally integrable functions and directly generalizes the Cauchy formula for repeated integration [7, Lemma 1.1].

The Riemann–Liouville fractional derivative of order  $\alpha$  is defined by [15, eq. 2.1.5]

$${}^R D^\alpha f := D^m I^{m-\alpha} f, \quad (2.2)$$

where  $m = \lceil \alpha \rceil$  and the Caputo derivative is given by  ${}^C D^\alpha f := I^{m-\alpha} D^m f$  (see [7, §2, 3] for a formal definition).

The Riemann–Liouville systems for  $0 < \alpha \leq 1$  are defined by the following initial value problem

$$\begin{cases} {}^R D^\alpha x := f(x, t) \\ \lim_{t \rightarrow 0^+} I^{1-\alpha} x(t) = b, \end{cases} \quad (2.3)$$

where  $x(t), f(x(t), t) \in \mathbb{R}^n$  for all  $t > 0$  and  $b \in \mathbb{R}^n$ . The initial condition  $\lim_{t \rightarrow 0^+} I^{1-\alpha} x(t) = b$  implies that, whenever  $\alpha < 1$  and  $b \neq 0$ ,  $x \notin \mathcal{C}[0, T]$  for any  $T > 0$ , i.e. it is not a continuous

function on  $[0, T]$ . Indeed, if  $x \in \mathcal{C}[0, T]$ , then  $x$  is bounded on  $[0, T]$  but a bounded function yields  $\lim_{t \rightarrow 0^+} I^{1-\alpha} x(t) = 0$ , since  $|I^{1-\alpha} x(t)| \leq [I^{1-\alpha} C](t) = O(t^{1-\alpha})$  where  $C$  is a constant bound on  $x$ . The behavior of  $x$  around  $t = 0$  is given by  $t^{\alpha-1}$  [7], whereby the solution necessarily satisfies  $\lim_{t \rightarrow 0^+} x(t) = \pm\infty$ .

The stability concept is central in system theory and is usually meaning the Lyapunov notion. It characterizes the dynamic behavior in a neighborhood of a given set, for a system of trajectories  $x(\cdot)$ . In its simpler version [14, Definition 3.2], it states that an equilibrium point  $x_e$  is Lyapunov stable, if for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that, if  $\|x(0) - x_e\| < \delta$ , then  $\|x(t) - x_e\| < \epsilon$  for all  $t \geq 0$ , where  $\|\cdot\|$  denotes the Euclidean norm.

Not only  $x(0) = \pm\infty$  makes the above stability notion unsuited but also the very concept of an equilibrium point – a point  $x_e \in \mathbb{R}^n$  such that if a system starts at  $x_e$ ,  $x(t) \equiv x_e$  for all  $t \geq 0$  – is meaningless, since on the one hand  $\lim_{t \rightarrow 0^+} x(t) = \pm\infty$  and on the other, the initial condition is not specified in terms of  $x$ . In the same way, global boundedness of solutions cannot be stated, unless that it is defined after some time  $T > 0$ .

Like the Caputo derivative [1], RL derivative has the following property when  $0 < \alpha \leq 1$

$${}^R D^\alpha x^2(t) \leq 2x(t) {}^R D^\alpha x(t) \quad \forall t \geq 0, \quad (2.4)$$

whenever  $x \in \mathcal{C}[0, T]$  or  $x \in \mathcal{C}^\beta[0, T] := \{u : [0, T] \rightarrow \mathbb{R} : |u(t) - u(t-h)| = O(h^\beta) \text{ uniformly for } 0 \leq t-h < t \leq T\}$  for any  $1 > \beta > \alpha/2$  [2]. In [17], inequality (2.4) was proved under the stronger requirement of continuous differentiability (since it is used integration by parts).

Both references for (2.4) posit unsuited conditions when  $x$  is a nontrivial solution of an RL system, since it belongs at best to  $\mathcal{C}(0, T)$  [7]. However, from the proof in [2], we observe that (2.4) holds for every  $t > 0$  whenever  $x \in \mathcal{C}(0, T)$ . If  $f = f(\cdot, \cdot)$  is continuous and Lipschitz continuous in its first argument, then uniqueness  $\mathcal{C}(0, T)$  solutions are obtained [9]. Note that in this step, the condition  $f(0, t) \equiv 0$  becomes necessary for attractiveness of  $x = 0$ . Hereafter, we assume that these condition are satisfied for system (2.3).

Inequality (2.4) can be extended in at least two relevant ways. By an induction argument (see details in [2]), it can be proved that

$${}^R D^\alpha x^n(t) \leq n x^{n-1} {}^R D^\alpha x(t) \quad \forall t > 0, \quad (2.5)$$

for any even natural number  $n$  and if  $x \geq 0$ , for any odd natural number. Second, for the vector case and using the same (algebraic) reasoning as in [8], where it was deduced for the Caputo operator, one obtains for any  $P \in \mathbb{R}^{n \times n}, P > 0$ ,

$${}^R D^\alpha [x^T P x](t) \leq 2x^T P {}^R D^\alpha x(t) \quad \forall t > 0, \quad (2.6)$$

where  $x(t) \in \mathbb{R}^n$  for all  $t > 0$  and  $x^T$  is the transpose vector.

To end this section, we recall some results which will be referenced along the paper. The Mittag-Leffler function is defined as  $E_{\alpha, \beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$  [7], where  $\alpha, \beta > 0$  and  $z \in \mathbb{C}$ . The Lebesgue spaces are defined as  $\mathcal{L}^p := \{f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n \mid (\int_{\mathbb{R}_{\geq 0}} \|f\|^p dx)^{\frac{1}{p}} < \infty\}$  for  $p \geq 1$ , where  $\mathbb{R}_{\geq 0}$  denotes the set of nonnegative real numbers. The Barbalat lemma [14] is given as follows.

**Lemma 2.1.** *If  $f(t)$  is a uniformly continuous function, such that  $\lim_{t \rightarrow \infty} \int_0^t f(\tau) d\tau$  exists and is finite, then  $\lim_{t \rightarrow \infty} f(t) = 0$ . In particular, if  $f(t)$  is uniformly continuous function and  $f \in \mathcal{L}^1$  then  $\lim_{t \rightarrow \infty} f(t) = 0$ .*

Finally, LaSalle's theorem [14] can be expressed as follows.

**Theorem 2.2.** *Let  $f(x)$  be a locally Lipschitz function defined over a domain  $D \subset \mathbb{R}^n$  and  $\Omega \subset D$  be a compact set such that solutions of  $\dot{x} = f(x)$  starting in  $\Omega$  remain in  $\Omega$ . Let  $V(x)$  be a continuously differentiable function defined over  $D$  such that  $\dot{V}(x) \leq 0$  in  $\Omega$ . Let  $E$  be the set of all points in  $\Omega$  where  $\dot{V}(x) = 0$ . Then every solution starting in  $\Omega$  approaches to  $E$  as  $t \rightarrow \infty$ .*

### 3 Main results

In this section, we propose two complementary but clearly differentiated approaches to determine asymptotic properties of RL systems.

#### 3.1 Attractiveness approach

Instead of asymptotic stability, this approach provides conditions for the attractiveness of  $x = 0$  and boundedness of the solutions after any finite time. The strength of this approach relies on the next theorem, which is an analog of the Lyapunov direct theorem [14, Theorem 3.1] or the fractional Lyapunov theorem for Caputo systems [10, 16]. For it, we need some definitions. A function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is a positive definite function if it is nonnegative,  $V(x) \neq 0$  for all  $x \neq 0$  and  $V(0) = 0$ . More generally, function  $V = V(t, x)$  is positive definite if there exist positive definite functions  $W_1, W_2 : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  such that  $W_2(x) \geq V(x, t) \geq W_1(x)$  for every  $t \geq 0$  and  $x \in \mathbb{R}^n$ .

**Theorem 3.1.** *Suppose that for system (2.3) there exists  $k > 0$  and a positive definite function  $V(x, t)$  such that for any initial condition*

$${}^R D^\alpha V(x(t), t) \leq -kV(x(t), t) \quad \forall t > 0, \forall x \in \mathbb{R}^n. \quad (3.1)$$

*Then,  $\lim_{t \rightarrow \infty} x(t) = 0$ , i.e.  $x = 0$  is globally attractive and  $x$  remains bounded on  $[T, \infty)$  for any  $T > 0$ . If (3.1) holds for some initial conditions  $b \in \Omega \subset \mathbb{R}^n$ , the precedent statement is local.*

*Proof.* Since the comparison principle holds for RL equations (see for instance [16, Remark 6.2]) and  $V \geq 0$ , it is enough for the first claim to study (3.1) in the equality.

The equation  ${}^R D^\alpha y = -ky$  has  $y(t) = bt^{\alpha-1} E_{\alpha, \alpha}(-kt^\alpha)$  as the unique solution ([15, eq. 4.1.10], see [7] for uniqueness), where  $\lim_{t \rightarrow 0^+} I^{1-\alpha} y(t) = b$ . From [6, Proposition 4],  $y \rightarrow 0$  as  $t \rightarrow \infty$ . By comparison,  $0 \leq V(x(t), t) \leq y(t)$ . Then,  $V(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Hence,  $W_1(x(t)) \rightarrow 0$  which implies  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

Since  $x \in \mathcal{C}(0, T]$  for any  $T > 0$  and  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ , we conclude that  $x$  remains bounded on  $[T, \infty)$ .  $\square$

#### Remark 3.2.

- (i) When  $\alpha < 1$ , and using (2.2),  ${}^R D^\alpha V \leq 0$  is equivalent to  $\frac{d}{dt} I^{1-\alpha} V \leq 0$ . Hence,  $I^{1-\alpha} V$  is nonnegative, decreasing function and then, it converges. If  $V(t) = V(x(t), t)$  is uniformly continuous and differentiable,  $\lim_{t \rightarrow \infty} V(t) = 0$  [11]. Thus, in contrast to  $\alpha = 1$  (see Theorem 2.2), condition  ${}^R D^\alpha V \leq 0$  could be enough to guarantee attractiveness instead of (3.1).
- (ii) Unlike Lyapunov functions, it is allowed  $\lim_{t \rightarrow 0^+} V(x(t)) = +\infty$ , for radially unbounded function  $V$  such as  $V(x) = x^T x$ .

With positive definite functions based on terms of type  $x^{2n}$  and inequality (2.6), Theorem 3.1 can be very useful to get asymptotic properties of RL systems, as illustrated in the following example.

**Example 3.3.** Consider the system

$$\begin{cases} {}^R D^\alpha x = -x + xy^T y \\ {}^R D^\alpha y = -x^T x y - y, \end{cases} \quad (3.2)$$

where  $x(t) \in \mathbb{R}^n$ ,  $y(t) \in \mathbb{R}^m$  for every  $t \geq 0$ . The polynomial character of the right-hand side of (3.2) assures the continuity required to apply inequality (2.6). Then,

$$\begin{cases} {}^R D^\alpha x^T x \leq -2x^T x + 2x^T x y^T y \\ {}^R D^\alpha y^T y \leq -2x^T x y^T y - 2y^T y. \end{cases}$$

By defining  $2V(x, y) = x^T x + y^T y$ , we have

$${}^R D^\alpha V \leq -x^T x - y^T y = -2V.$$

Therefore, by applying Theorem 3.1,  $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} y(t) = 0$ .

Example 3.3 also shows a drawback of this approach, shared by the similar approach for Caputo systems [10]. If the first equation of (3.2) has derivation order  $\alpha$  and the second  $\beta \neq \alpha$ , there is not a clear positive definite function candidate, even for such a simple system. The next approach is intended to deal with these more general systems, by finding a common order which, as suggested by the defining equation (2.2), must be an integer number.

### 3.2 Lyapunov mixed order approach

This approach will be shown in the proof of the next theorem. For it, we need some definitions. Consider the following system

$${}^R D^{\alpha_i} x_i(t) = f_i(x_1, \dots, x_n, t) \quad (3.3)$$

where  $0 < \alpha_i \leq 1$ ,  $x_i(t) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ ,  $f_i : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  and the initial conditions are given by  $\lim_{t \rightarrow 0^+} I^{1-\alpha_i} x_i(t) = b_i \in \mathbb{R}$  for  $i \in \{1, \dots, n\}$ . Let  $x := (x_1, \dots, x_n)^T$  and  $f := (f_1, \dots, f_n)^T$ . We refer the space defined by  $\zeta(t) := ([I^{\alpha_1} x_1](t), \dots, [I^{\alpha_n} x_n](t))$  as the *initial condition space*. Conditions for  $(0, \infty)$ -continuity of solutions to system (3.3) were studied at [9]. As before, it is required continuity of functions  $f_i$  in  $t$  and Lipschitz continuity in the  $x$  uniformly on  $t$ . Then, to assert the attractiveness of  $x = 0$ , we also require  $f(0, \dots, 0, t) \equiv 0$ . Finally, a class  $\mathcal{K}$  function is a strictly increasing function  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  such that  $f(0) = 0$  [14, Definition 4.2, p. 144].

**Theorem 3.4.** Consider system (3.3) and the function  $w(x, t) := x^T(t)f(x, t)$ .

- (i) If  $w(x(t), t) \leq 0$  for all  $t \geq 0$ , then  $w \in \mathcal{L}^1$  and stability of system (3.3) holds in the space defined by its initial conditions.
- (ii) If  $w$  is negative definite and there exists  $T_0$  such that  $w(t) = (x(t), t)$  is uniformly continuous for  $t > T_0$ , then  $\lim_{t \rightarrow \infty} x(t) = 0$ .



When  $0 < \alpha_i = \alpha < 2$  for  $i = 1, \dots, n$ , known result states that if the spectrum of the matrix  $A := (a_{ij})$  holds that

$$\sigma(A) \subset \left\{ \lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| > \frac{\alpha\pi}{2} \right\}, \quad (3.5)$$

then asymptotic convergence to zero when  $u = 0$  and bounded solutions for bounded  $u$  is achieved. Consider  $0 < \alpha_i \leq 1$  and  $u \equiv 0$ . Assuming that all the eigenvalues  $\lambda_i$  of  $A$  belong to  $\mathbb{R}_{<0}$ , we have

$$x_1 D^{\alpha_1} x_1 + \dots + x_n D^{\alpha_n} x_n = x^T A x.$$

Using (2.6), we get

$$D^{\alpha_1} x_1^2 + \dots + D^{\alpha_n} x_n^2 \leq 2 \sum_{i=1}^n \lambda_i x_i^2$$

and thus,

$$\frac{d}{dt} (\xi_1 + \dots + \xi_n) \leq 2 \sum_{i=1}^n \lambda_i x_i^2 \leq 0,$$

where  $\xi_i := I^{1-\alpha_i} x_i^2$ . From Theorem 3.4 (i),  $\int_0^t \sum_{i=1}^n |\lambda_i| x_i^2 dt$  is bounded. From Theorem 3.4 (ii) and the type of solutions of linear systems,  $x_i \rightarrow 0$  as  $t \rightarrow \infty$ . If  $u \neq 0$ , then

$$\frac{d}{dt} (\xi_1 + \dots + \xi_n) \leq \lambda x^T x + x^T u(t),$$

where  $\lambda$  is the maximum eigenvalue of  $A$ . If  $x$  is unbounded as  $t \rightarrow \infty$ , then there exists a time  $T$  such that for all  $t > T$ ,  $\frac{d}{dt} (\xi_1 + \dots + \xi_n) \leq 0$ , implying that  $I^{1-\alpha_i} x_i^2(t) < \infty$  for all  $t > T$ . Since  $x$  is unbounded, there exists  $i_0$  such that  $|x_{i_0}| > C > 0$  for all  $t > T$ , then  $I^{1-\alpha_{i_0}} x_{i_0}^2(t)$  diverges, which is a contradiction. Then  $x$  is bounded for  $t > a > 0$ .

Theorem 3.4 was obtained by employing inequality (2.6) with  $P$  the identity matrix. We can obtain more general variables  $\xi$  like  $I^{1-\alpha} x^T P x$ , more general function  $V(\xi)$  and a more general result following the same line of reasoning. Considering this, we present a generalization of the second direct Lyapunov and LaSalle theorems (Theorem 2.2), in the sense that the Lyapunov function is defined on variables which are obtained by a dynamical transformation of the system's original variables.

**Theorem 3.7.** Consider that for system (3.3) there exist a function  $\xi$  and a positive definite function  $V = V(\xi, t)$  such that

$$\frac{d}{dt} V(\xi, t) \leq w(x, t), \quad (3.6)$$

where  $w(x(t), t) \in \mathbb{R}$  for all  $t > 0$  and  $V(\xi(0), 0) < \infty$ .

- (i) If  $w(x(t), t) \leq 0$  for all  $t \geq 0$ , then  $w \in \mathcal{L}^1$  and system (3.3) is stable in the space of initial conditions.
- (ii) If  $w(x(t), t) \leq -\gamma(\|x(t)\|)$  for all  $t > 0$ , where  $\gamma$  is a class  $\mathcal{K}$  function, and there exists a time  $T_0 > 0$  such that  $w(t) := w(x(t), t) : [T_0, \infty) \rightarrow \mathbb{R}$  is uniformly continuous, then  $\lim_{t \rightarrow \infty} x(t) = 0$ .
- (iii) Suppose that  $w \leq 0$  is not explicitly depending on time i.e.  $w = w(x(t))$ , and that is radially unbounded, continuous on  $x$  and uniformly continuous as a time function on  $[T_0, \infty)$  for some  $T_0 > 0$ . Let  $E := \{x \in \mathbb{R}^n | w(x) = 0\}$ . Then,  $x$  converges to the set  $E$ .

*Proof.* (i) and (ii) are proved along the same lines than the proof of Theorem 3.4.

(iii)  $V$  is bounded, since it is positive definite and non-increasing. Integrating equation (3.6), we conclude that  $\int_0^\infty |w(x(t))| dt < \infty$ , that is  $w \in \mathcal{L}^1$ . From Lemma 2.1,  $w(t) \rightarrow 0$  as  $t \rightarrow \infty$ . In particular, there exists  $T'$  such that  $w$  is bounded for all  $t > T_0$ . Since  $w$  is radially unbounded, then  $x$  is also bounded for all  $t > T_0$ .

If  $x$  does not converge to  $E$  as  $t \rightarrow \infty$ , there exists  $\epsilon > 0$  and a sequence  $(t_n)_{n \in \mathbb{N}}$  with  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$ , such that  $d(x(t_n), E) := \inf_{e \in E} \|x(t_n) - e\| > \epsilon$ . From boundedness of the solutions after  $T_0 > 0$ , we can assume that there exist  $0 < \epsilon_1 < \epsilon_2$  and a sequence  $(\tilde{t}_n)_{n \in \mathbb{N}}$  with  $\tilde{t}_n \rightarrow \infty$  as  $n \rightarrow \infty$ , such that  $\epsilon_1 \leq d(x(\tilde{t}_n), E) \leq \epsilon_2$ . From continuity of  $w = w(x)$ , there exists  $\delta > 0$  such that  $-w(x) > \delta$  for any  $x \in \{x \in \mathbb{R}^n : \epsilon_1 \leq d(x, E) \leq \epsilon_2\}$ . The latter is due to the fact that this set is compact and, thus,  $-w$  has a minimum on it, which cannot be 0 since  $x \notin E$  if  $d(x, E) = \epsilon \neq 0$ . Therefore,  $-w(\tilde{t}_n) = -w(x(\tilde{t}_n)) > \delta > 0$  for  $(\tilde{t}_n)_{n \in \mathbb{N}}$  with  $\tilde{t}_n \rightarrow \infty$  as  $n \rightarrow \infty$ . But this contradicts the fact that  $w$  converges to zero. Hence,  $x$  converges to  $E$  as  $t \rightarrow \infty$ .  $\square$

**Remark 3.8.** Asymptotic stability on the initial condition space follows from convergence of  $V \rightarrow 0$  as  $t \rightarrow \infty$ . In the construction of Theorem 3.4, this implies  $I^{1-\alpha} x^T x \rightarrow 0$ . From the fractional Barbalat lemma in [11],  $I^{1-\alpha} x^T x \rightarrow 0$  implies  $x \rightarrow 0$  as  $t \rightarrow \infty$  when  $x$  is uniformly continuous. Thus, asymptotic stability of  $\xi = 0$  implies attractiveness of  $x = 0$ .

This approach also provides a base to deal with fractional order input-output systems, as shown in the following example.

**Example 3.9.** Consider the system

$$\begin{cases} {}^R D^\alpha x = Ax + Bu \\ y = B^T P x \\ \lim_{t \rightarrow 0^+} I^{1-\alpha} x(t) = b, \end{cases} \quad (3.7)$$

where  $x(t) \in \mathbb{R}^n$ ,  $y(t), u(t) \in \mathbb{R}^m$  for all  $t > 0$ ,  $0 < \alpha \leq 1$ ,  $b \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ . Let  $P \in \mathbb{R}^{n \times n}$  be a positive definite matrix such that  $A^T P + PA = Q$  for  $Q$  a negative definite matrix. If  $A$  is Hurwitz (i.e. if every eigenvalue of  $A$  has strictly negative real part), such a  $P$  always exists for any  $Q < 0$  [14]. Defining  $V(\xi(t)) = \xi(t) = [I^{1-\alpha} x^T P x](t)$  and applying inequality (2.6), we have

$$\frac{d}{dt} V(\xi) \leq x^T Q x + x^T P B u \leq y^T u,$$

and by integration, we get the passivity inequality (see e.g. [14, §10.3])

$$\int_0^t [y^T u] d\tau \geq V(t) - V(0) \geq -V(0) \quad \forall t > 0.$$

Therefore, system (3.7) is passive and  $V = V(\xi)$  is an storage function. The interesting fact is that passivity is an additive property, in the sense that if passive subsystems are neutrally interconnected, the overall system is also passive. In particular, systems of different differentiation order can be analyzed independently of their orders from the passive view.

Based on the fact that the passivity inequality can be verified for RL systems as shown in Example 3.9, we present the following  $\mathcal{L}^2$ -control of nonlinear fractional systems.

**Theorem 3.10.** Consider the input-output system

$$\begin{cases} [{}^R D^{\alpha_i} x_i](t) = f_i(x, u, t) & i = 1, \dots, n \\ y = h(x) \\ \lim_{t \rightarrow 0^+} I^{1-\alpha} x(t) = b, \end{cases} \quad (3.8)$$

where  $0 < \alpha_i \leq 1$ ,  $b \in \mathbb{R}^n$ ,  $x(t) := (x_1(t), \dots, x_n(t))^T \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$ ,  $y(t) \in \mathbb{R}^m$ ,  $f_i(x(t), u(t), t) \in \mathbb{R}$  for each  $i = 1, \dots, n$  and  $t \geq 0$ .  $f_i, h$  are continuous functions and Lipschitz continuous in the first argument with  $h(0) = 0$  and  $f(0, u, t) \equiv 0$ . Let  $\xi$  be a function of time such that there exists a positive definite function  $V = V(\xi, t)$  with  $V(\xi(0), 0) < \infty$  and

$$\frac{d}{dt} V(\xi, t) \leq [y^T u](t), \quad \forall t > 0. \quad (3.9)$$

Let  $u = -ky$  for any  $k > 0$  a real number. Then, the origin  $\xi = 0$  is stable and  $y \in \mathcal{L}^2$ . Moreover, if  $y$  is uniformly continuous, then  $\lim_{t \rightarrow \infty} y(t) = 0$  and if the system is detectable, then  $\lim_{t \rightarrow \infty} x(t) = 0$ .

*Proof.* From the hypothesis, we have

$$\frac{d}{dt} V(\xi, t) \leq -ky^T y \leq 0.$$

Hence, as in the proof of Theorem 3.7, the origin  $\xi = 0$  is stable. Since  $V \geq 0$  and non-increasing,  $V$  converges to some value and, in particular, is bounded. Thus,

$$\int_0^\infty [y^T y](\tau) d\tau \leq V(0) < \infty.$$

Then,  $y \in \mathcal{L}^2$ . If  $y$  is uniformly continuous, we conclude  $\lim_{t \rightarrow \infty} y(t) = 0$  from Lemma 2.1. If, in addition, the system is detectable, by definition of a detectable system [20],  $\lim_{t \rightarrow \infty} x(t) = 0$ .  $\square$

## 4 Main applications

In this section, we present applications of the two approaches developed in Section 3, showing their usefulness to prove asymptotic properties. The first deals with time-varying linear systems. We recall the definition of persistently exciting functions [18, p. 102]

$$PE(n) := \left\{ f : [0, \infty) \rightarrow \mathbb{R}^n \mid (\exists \epsilon, T_0 > 0) : \int_t^{t+T_0} f(\tau) f^T(\tau) d\tau \geq \epsilon I_n, \forall t > 0 \right\}, \quad (4.1)$$

where  $I_n$  is the identity matrix of order  $n$ . Any (quasi) periodic function belongs to this set (e.g.  $f(t) = (\sin(wt), \cos(wt)) \in PE(2)$ ).

**Theorem 4.1.** Consider the following linear time-varying system

$${}^R D^\alpha x(t) = -f(t) f^T(t) x(t), \quad (4.2)$$

where  $0 < \alpha \leq 1$  and  $x(t) \in \mathbb{R}^n$  for all  $t \geq 0$ . Let  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$  be a bounded continuous function. Then,  $x^T f \in \mathcal{L}^2$  and if  $f \in PE(n)$ ,  $\lim_{t \rightarrow \infty} x(t) = 0$ .

*Proof.* Since  $f$  is a bounded function,  $ff^T x$  is Lipschitz in  $x$ . Hence,  $x$  is continuous for  $t > 0$ . Defining  $w(x, t) = -x^T(t)f(t)f^T(t)x(t) = -(x^T(t)f(t))^2$ , we use Theorem 3.4(i) to conclude that  $x^T f \in \mathcal{L}^2$ .

Equation (4.2) and inequality (2.6) imply

$$\frac{d}{dt}\xi(t) := \frac{d}{dt}[I^{1-\alpha}x^T x](t) \leq -x^T(t)f(t)f^T(t)x(t) \leq 0. \quad (4.3)$$

Hence  $\xi$ , a nonnegative and non-increasing function, converges to some limit  $L < \infty$ . By integrating equation (4.3) on the interval  $[t, t + T_0]$  for any  $t > 0$ , we get

$$\xi(t + T_0) - \xi(t) \leq -\int_t^{t+T_0} x^T(\tau)f(\tau)f^T(\tau)x(\tau)d\tau = -\int_t^{t+T_0} [x^T(\tau)f(\tau)]^2 d\tau.$$

Using the Cauchy–Schwartz inequality for the  $\mathcal{L}^2[t, t + T_0]$  internal product of functions 1 and  $x^T f$ , we have

$$\xi(t + T_0) - \xi(t) \leq -\frac{1}{T_0} \left( \int_t^{t+T_0} |x^T(\tau)f(\tau)| d\tau \right)^2$$

or equivalently,

$$\sqrt{T_0}(\xi(t) - \xi(t + T_0))^{1/2} \geq \int_t^{t+T_0} |x^T(\tau)f(\tau)| d\tau. \quad (4.4)$$

We will manipulate the right-hand side of (4.4). By adding zero, we get

$$\int_t^{t+T_0} |x^T(\tau)f(\tau)| d\tau \geq \int_t^{t+T_0} |x^T(t)f(\tau)| d\tau - \int_t^{t+T_0} |[x^T(t) - x^T(\tau)]f(\tau)| d\tau. \quad (4.5)$$

Since  $f \in PE(n)$ , we can use [18, Theorem 2.16(c)] with  $u = \frac{x(t)}{\|x(t)\|}$ , to show that there exist  $\epsilon$  and  $T_0$  such that the following bound holds for any  $t > 0$

$$\int_t^{t+T_0} |x^T(t)f(\tau)| d\tau \geq \|x(t)\|T_0\epsilon,$$

where we have used that for  $x(t) = 0$  this inequality trivially holds. On the other hand, since  $f$  is bounded, i.e.  $\|f\|_\infty = f_{\max} < \infty$ , we obtain the following bound

$$\int_t^{t+T_0} |[x^T(t) - x^T(\tau)]f(\tau)| d\tau \leq f_{\max}T_0 \sup_{\tau \in [t, t+T_0]} \|x(t) - x(\tau)\|.$$

Since  $x$  is continuous on  $[t, t + T_0]$ , it reaches its maximum for some  $\tau_{\max} \in [t, t + T_0]$  and we have

$$\int_t^{t+T_0} |[x^T(t) - x^T(\tau)]f(\tau)| d\tau \leq f_{\max}T_0 [\|x(t)\| + \|x(\tau_{\max})\|].$$

By replacing the above bounds in (4.5) and (4.4), we obtain

$$\sqrt{T}(\xi(t) - \xi(t + T_0))^{1/2} \geq \|x(t)\|T_0\epsilon + f_{\max}T_0 [\|x(t)\| + \|x(\tau_{\max})\|] \geq 0.$$

Since  $\xi$  converges to some limit,  $\xi(t) - \xi(t + T_0) \rightarrow 0$  as  $t \rightarrow \infty$ . Therefore,  $x \rightarrow 0$ , as  $t \rightarrow \infty$ .  $\square$

From Theorem 4.1, we obtain the following corollary.

**Corollary 4.2.** Consider the following linear time-varying system

$${}^R D^\alpha x(t) = -A(t)x(t), \quad (4.6)$$

where  $0 < \alpha \leq 1$ ,  $x(t) \in \mathbb{R}^n$ ,  $A(t) \in \mathbb{R}^{n \times n}$  for all  $t \geq 0$ . If  $A$  is a bounded matrix function such that  $A(t) \geq g(t)I_n$  for all  $t \geq 0$ , where  $g \geq 0$  is a bounded function such there exist  $\epsilon, \delta : \int_0^t g(\tau)d\tau \geq \epsilon t + \delta$  for all  $t \geq 0$ , then  $\lim_{t \rightarrow \infty} x(t) = 0$ .

*Proof.* By defining  $2V = x^T x$ , using the hypothesis and inequality (2.6) (the solution is continuous because  $A(t)x$  is Lipschitz and  $A$  is bounded), we get

$${}^R D^\alpha V(t) = -x^T(t)A(t)x(t) \leq -2g(t)V.$$

By the comparison lemma [16, Remark 6.2], using that the hypothesis implies that  $\sqrt{g} \in PE(1)$  and Theorem 4.1, we conclude that  $\lim_{t \rightarrow \infty} V(t) = 0$  and, therefore,  $\lim_{t \rightarrow \infty} x(t) = 0$ .  $\square$

From Theorem 4.1 we also obtain the following improvement to Theorem 3.1.

**Corollary 4.3.** Suppose that for system (2.3), there exist a positive definite function  $V(x, t)$ , a real number  $\alpha \in (0, 1]$ , a bounded scalar time function  $k \geq 0$ , and  $\epsilon, \delta > 0$  with  $\int_0^t k(\tau)d\tau \geq \epsilon t + \delta$  for all  $t \geq 0$ , such that

$${}^R D^\alpha V(t) \leq -k(t)V(t), \quad (4.7)$$

for all  $t > 0$ . Then,  $\lim_{t \rightarrow \infty} x(t) = 0$ .

*Proof.* By the comparison lemma [16, Remark 6.2], using that the hypothesis on the function  $k$  implies that  $k^{1/2} \in PE(1)$  and Corollary 4.2, we conclude that  $\lim_{t \rightarrow \infty} V(t) = 0$  and, therefore,  $\lim_{t \rightarrow \infty} x(t) = 0$ .  $\square$

The next result deals with linear systems with a Lipschitz term added. Since nonlinear systems can be approximated around equilibrium points in that way by Taylor's theorem, the following results can be applied to nonlinear systems to determine the local attractiveness of the equilibrium points. This kind of systems has been studied for the Caputo derivative (see relevant contributions in [5]). For RL derivative we improve the result in [17], by proving convergence with a consistent procedure and weakening the condition on the added term.

**Theorem 4.4.** Consider the following system

$${}^R D^\alpha x(t) = Ax(t) + f(x(t), t), \quad (4.8)$$

where  $0 < \alpha \leq 1$ ,  $A \in \mathbb{R}^{n \times n}$ ,  $x(t) \in \mathbb{R}^n$ ,  $f : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ ,  $f(0, t) = 0$  for every  $t \geq 0$ . Let  $f$  to be a Lipschitz continuous function in its first argument, uniformly on  $t$ , of Lipschitz constant  $L$ . If there exists a constant matrix  $P > 0$  such that  $A^T P + PA + \eta I_n = Q < 0$ , where  $\eta := L^2 + \|P\|^2$ , then  $\lim_{t \rightarrow \infty} x(t) = 0$ .

*Proof.* From the Lipschitz assumption,  $x \in \mathcal{C}(0, T]$ . Using (2.6), we obtain

$$\begin{aligned} {}^R D^\alpha x^T P x &\leq 2x^T P A x + x^T P f \\ &= x^T (PA + A^T P)x + 2x^T P f. \end{aligned} \quad (4.9)$$

Since  $f(0, t) = 0$  and the Lipschitz assumption, we have that  $\|f(x(t), t)\| \leq L\|x(t)\|$  for every  $t \geq 0$ . Using that  $2ab \leq a^2 + b^2$  for any  $a, b \in \mathbb{R}$ , we obtain the bound

$$\begin{aligned} 2x^T P f &\leq x^T P^2 x + f^T f \\ &\leq (\|P\|^2 + L^2)x^T x. \end{aligned}$$

Replacing this bound in (4.9) and using  $A^T P + PA + \eta I_n = Q$ , we obtain

$$\begin{aligned} {}^R D^\alpha x^T P x(t) &\leq x^T Q x \\ &\leq \frac{\lambda_{Q,M}}{\lambda_{P,m}} x^T P x, \end{aligned}$$

where  $\lambda_{Q,M}$ ,  $\lambda_{P,m}$  are the maximum and the minimum eigenvalues of  $Q$  and  $P$ , respectively. Calling  $V = x^T P x$ , we obtain  ${}^R D^\alpha V(t) \leq \frac{\lambda_{Q,M}}{\lambda_{P,m}} V$  and, from Theorem 3.1,  $\lim_{t \rightarrow \infty} x(t) = 0$ .  $\square$

**Remark 4.5.** The condition on  $f$  given by  $\lim_{x \rightarrow 0} \frac{\|f(x, t)\|}{\|x\|} = 0$  uniformly on  $t$ , used in [17], is stronger than the condition  $\|f(x(t), t)\| \leq L\|x(t)\|$  uniformly on  $t$ .

The next is just a comment on an example in [17, §4]. Consider the following fractional system with unbounded delay

$${}^R D^q x(t) = Ax(t) + Bx(t - \tau(t)) + F_1(x(t)) + F_2(x(t + \tau(t))), \quad (4.10)$$

where  $\frac{d}{dt}\tau \leq d < 1$ ,  $0 < q \leq 1$ ,  $x(t) \in \mathbb{R}^n$  for any  $t > 0$ ,  $A, B$  are constant real matrices of suited dimensions and  $F_1, F_2$  are vector functions. For certain constant matrices  $P, Q > 0$ , define

$$V(t) := I^{1-q}[x^T P x](t) + \int_{t-\tau(t)}^t x^T Q x d\tau.$$

This time function is not positive definite in  $x$ , since  $x(t_0) = 0$  for some  $t_0$  does not implies that  $V(t_0) = 0$ . It could be positive definite but in a function space of elements of type  $x_{[0,t]}$  (i.e. the function  $x$  on the interval  $[0, t]$ ). Whether or not a Lyapunov theory exists for this case it is not specified in [17].

To compute  $\dot{V}$  with the aid of the inequality  $D^\alpha(x^T P x) \leq x^T P D^\alpha x$ , the hypothesis of continuous differentiability (or simply continuity in our case) is not verified. Assuming that it can be done, there exist constant matrices  $N_1, N_2 < 0$  such that

$$\frac{d}{dt}V(t) \leq x^T(t)N_1x(t) + x^T(t - \tau(t))N_2x(t - \tau(t)),$$

(see [17]). Using Theorem 3.4,  $x \in \mathcal{L}^2$  and, under uniform continuity,  $\lim_{t \rightarrow \infty} x(t) = 0$ .

## 5 Relationship to Caputo fractional systems

In this section, we compare RL and Caputo systems in terms of their computational and mathematical realizations.

## 5.1 Implementation

The main advantage of Caputo systems, which explains their dominant presence in the literature, is that they can be easily implemented in computational software (such as MATLAB/Simulink), since their initial condition terms are, as in integer order systems, the value of their variables at the initial time; meanwhile, in RL systems, they are the limit of their fractional integrals. We will see a way to overcome this computational obstacle.

Consider the following system

$$\begin{cases} {}^R D^\alpha x := f(x, t) \\ \lim_{t \rightarrow 0^+} I^{1-\alpha} x(t) = b, \end{cases}$$

where  $0 < \alpha < 1$ ,  $b \in \mathbb{R}^n$  and  $f(x, t), x(t) \in \mathbb{R}^n$  for all  $t \geq 0$ . From (2.2), this system can be rewritten as

$$\begin{cases} \frac{d}{dt} I^{1-\alpha} x(t) = f(x, t) \\ \lim_{t \rightarrow 0^+} I^{1-\alpha} x(t) = b, \end{cases}$$

and, by defining  $\zeta = I^{1-\alpha} x$ , and using that  ${}^R D^{1-\alpha} I^{1-\alpha} x = x$  for any  $x$  locally integrable function [15, Lemma 2.4], we can write

$$\begin{cases} \frac{d}{dt} \zeta(t) = f({}^R D^{1-\alpha} \zeta, t) \\ \lim_{t \rightarrow 0^+} \zeta(t) = b. \end{cases}$$

The term  ${}^R D^{1-\alpha} \zeta(t) =: \eta(t)$  can be seen, for simulation purposes, as a function evaluated obtained from the operator  ${}^R D^{1-\alpha} = \frac{d}{dt} I^\alpha$  applied to the past values of function  $\zeta$ . Thus, the only operator required is the fractional order integral. From locally integrable assumption on  $x$ ,  $\zeta$  is continuous at zero, whereby we obtain the following implementable system

$$\begin{cases} \frac{d}{dt} \zeta(t) = f(x, t) \\ x(t) = {}^R D^{1-\alpha} \zeta(t) \\ \zeta(0) = b. \end{cases}$$

The procedure can be directly extended to systems of the form (3.3). We will show a simple example.

**Example 5.1.** The scalar system

$${}^R D^\alpha x = -x$$

can be expressed as  $\frac{d}{dt} \zeta(t) = -{}^R D^{1-\alpha} \zeta$ , or by integration,

$$\zeta(t) = \zeta(0) - \int \frac{d}{dt} I^{1-\alpha} \zeta.$$

Using the NID-block of Matlab/Simulink, where the fractional integral is approximated in the Laplace domain by integer filters, this system is simulated in Fig. 5.1, with  $\alpha = 0.9$  and  $\zeta(0) = 1$ . It is seen that  $x(0)$  takes high values near zero (in theory,  $\lim_{t \rightarrow 0^+} x(t) = +\infty$  and  $\lim_{t \rightarrow \infty} x(t) = 0$ ) and  $\zeta$  is decreasing, since its integer derivative is  $-x$ .

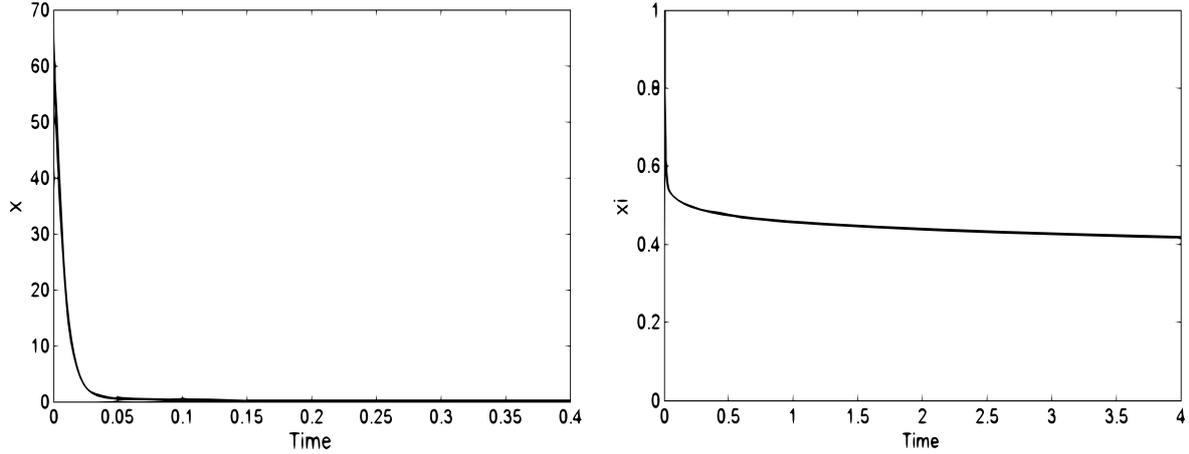


Figure 5.1: Time evolution of functions  $x$  (on the left) and  $\xi$  (on the right).

## 5.2 Caputo systems

To enlarge the application of our results, we show a way to associate a RL system to Caputo one. Consider the Caputo system

$$\begin{cases} {}^C D^\alpha x = f(x, t) \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \quad (5.1)$$

where  $x(t), f(x, t) \in \mathbb{R}^n$  for all  $t > 0$  and  $0 < \alpha \leq 1$ . The solutions to this system are  $\mathcal{C}[0, T]$  provided that, for instance,  $f$  is Lipschitz continuous in its first argument. Using [15, eq. 2.4.8],

$${}^C D^\alpha x = {}^R D^\alpha x - x(0)t^{-\alpha},$$

we obtain the following RL associated system

$$\begin{cases} {}^R D^\alpha x = f(x, t) + x(0)t^{-\alpha} =: \tilde{f}(x, t) \\ x(0) = x_0 \in \mathbb{R}^n. \end{cases}$$

However, its initial condition is finite  $\|x(0)\| < \infty$  and the solution is continuous at  $[0, T]$  for every  $T > 0$ . Hence, the RL system must have null initial condition, that is,  $\lim_{t \rightarrow 0} I^{1-\alpha} x(t) = 0$ , and then the forcing function  $x(0)t^{-\alpha}$  yields nontrivial solutions. Consider as an example the scalar case  $f(x) = ax$ . Then, the associated RL system is given by

$$\begin{cases} {}^R D^\alpha x = ax + \frac{x(0)}{\Gamma(1-\alpha)} t^{-\alpha} \\ \lim_{t \rightarrow 0} I^{1-\alpha} x(t) = 0. \end{cases}$$

The solution to this equation would be given by [15, eq. 4.1.14]

$$x(t) = \frac{x(0)}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha, \alpha}(a(t-\tau)^\alpha) \tau^{-\alpha} d\tau,$$

but note that the forcing function is not continuous at zero. Equating to the known solution of the Caputo initial problem, we must have

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha, \alpha}(a(t-\tau)^\alpha) \tau^{-\alpha} d\tau \equiv E_\alpha(at^\alpha),$$

which is an equality in Laplace domain, since  $\mathcal{L}[t^{\alpha-1}E_{\alpha,\alpha}(at^\alpha)](s) = \frac{1}{s^\alpha+a}$ ,  $\mathcal{L}\left[\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right](s) = s^{\alpha-1}$  and  $\mathcal{L}[E_\alpha(at^\alpha)](s) = \frac{s^{\alpha-1}}{s^\alpha+a}$ .

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