



Oscillation of second order advanced differential equations

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Abstract. We establish a new technique for deducing oscillation of the second order advanced differential equation

$$(r(t)u'(t))' + p(t)u(\sigma(t)) = 0$$

with help of a suitable equation of the form

$$(r(t)u'(t))' + q(t)u(t) = 0.$$

The comparison principle obtained fills the gap in the theory of oscillation and essentially improves existing criteria. Our technique is based on new monotonic properties of nonoscillatory solutions, and iterated exponentiation is employed. The results are supported with several illustrative examples.

Keywords: second order differential equations, advanced argument, oscillation.

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1 Introduction

We consider the second order functional advanced differential equation

$$(r(t)u'(t))' + p(t)u(\sigma(t)) = 0, \tag{E}$$

where

(H₁) $r(t), p(t) \in C([t_0, \infty))$ are positive;

(H₂) $\sigma(t) \in C^1([t_0, \infty))$, $\sigma'(t) > 0$, $\sigma(t) \geq t$ for $t \geq t_0$.

Throughout the paper we assume that (E) is in a canonical form, that is,

$$(H_3) R(t) = \int_{t_0}^t \frac{1}{r(s)} ds \rightarrow \infty \text{ as } t \rightarrow \infty.$$

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By a solution of (E) we mean a function $u(t)$ with $r(t)u'(t) \in C^1([t_0, \infty))$, which satisfies equation (E) on $[t_0, \infty)$. We consider only those solutions $u(t)$ of (E) which satisfy $\sup\{|u(t)| : t \geq T\} > 0$ for all $T \geq t_0$. A solution of (E) is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is called nonoscillatory. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

There are numerous results dealing with the oscillation of (E) (see [1–10]) for delay case, i.e. $\sigma(t) \leq t$. The authors especially paid attention to a comparison technique which is the most effective tool in the theory of oscillation. Mahfoud in [9] deduced oscillation of the delay equation

$$y''(t) + p(t)u(\sigma(t)) = 0, \quad (E_0)$$

from that of the ordinary equation without deviating argument

$$y''(t) + \frac{p(\sigma^{-1}(t))}{\sigma'(\sigma^{-1}(t))} u(t) = 0.$$

Yang [10] studied oscillation of delay equation (E₀) via oscillation of

$$y''(t) + \frac{\sigma(t)}{t} p(t)u(t) = 0.$$

Many authors used the Riccati transformation to get oscillatory criteria for delay equations.

On the other hand, these techniques established for delay equations fail for advanced differential equations. In particular, there is no corresponding comparison result for advanced equations except that of Kusano [7], where oscillation of functional equation (E) follows from the oscillation of the ordinary equation

$$(r(t)u'(t))' + p(t)u(t) = 0. \quad (E_1)$$

But, here the informations about value of $\sigma(t)$ are lost.

In this paper, we would like to fill this gap in oscillation theory and to establish a new comparison principle for advanced equations.

Remark 1.1. We assume that all functional inequalities hold eventually, i.e., they are satisfied for all t large enough.

2 Preliminary results

Without loss of generality, considering nonoscillatory solutions of (E), we can restrict our attention only to positive ones.

Lemma 2.1. *Assume that $u(t)$ is a positive solution of (E). Then*

$$r(t)u'(t) > 0 \quad \text{and} \quad (r(t)u'(t))' < 0, \quad (2.1)$$

eventually.

Proof. Assume that $u(t)$ is a positive solution of (E). Then $(r(t)u'(t))' < 0$, which implies that $r(t)u'(t)$ is decreasing. If we admit that $r(t)u'(t) < 0$ for $t \geq t_1 \geq t_0$, then there exists a constant $c > 0$ such that $r(t)u'(t) \leq -c$ for $t \geq t_1$. An integration from t_1 to t yields

$$u(t) \leq u(t_1) - c \int_{t_1}^t \frac{1}{r(s)} ds \rightarrow -\infty \quad \text{as } t \rightarrow \infty.$$

This is a contradiction and we conclude that $r(t)u'(t) > 0$. □

We establish the following basic oscillatory criterion that will be improved in successive steps.

Theorem 2.2. *Assume that there exists a constant α such that*

$$R(t) \int_t^\infty p(s) ds \geq \alpha > \frac{1}{4}, \quad (2.2)$$

eventually. Then (E) is oscillatory.

Proof. Condition (2.2) guarantees oscillation of (E_1) , which implies oscillation of (E) (see e.g. [3,7]). \square

The above criterion does not include the advanced argument $\sigma(t)$ and so it is more suitable for (E_1) . But in view of Theorem 2.2, we may assume that the condition (opposite to (2.2)) holds, namely

$$R(t) \int_t^\infty p(s) ds \geq \alpha \quad \text{but} \quad \alpha \leq \frac{1}{4}.$$

In our main results, we adapt criterion (2.2) from Theorem 2.2 to contain also information about the advanced argument.

3 Main results

Our intended comparison technique is based on the new monotonic properties of nonoscillatory solutions.

Theorem 3.1. *Assume that $u(t)$ is a positive solution of (E) and*

$$R(t) \int_t^\infty p(s) ds \geq \alpha > 0, \quad (3.1)$$

eventually. Then there is t_ such that for $t \geq t_*$*

$$\frac{u(t)}{R^\alpha(t)} \uparrow.$$

Proof. Assume that $u(t) > 0$ is a solution of (E). An integration of (E) yields

$$\begin{aligned} r(t)u'(t) &\geq \int_t^\infty p(s)u(\sigma(s)) ds \geq u(\sigma(t)) \int_t^\infty p(s) ds \\ &\geq u(t) \int_t^\infty p(s) ds. \end{aligned}$$

Thus,

$$\begin{aligned} \left(\frac{u(t)}{R^\alpha(t)} \right)' &= \frac{1}{r(t)R^{\alpha+1}(t)} [R(t)r(t)u'(t) - \alpha u(t)] \\ &\geq \frac{u(t)}{r(t)R^{\alpha+1}(t)} \left[R(t) \int_t^\infty p(s) ds - \alpha \right] \geq 0. \end{aligned}$$

The proof is complete. \square

Remark 3.2. Theorem 3.1 provides a new monotonic property of positive solution of (E). In earlier results it is only known that $u(t) \uparrow$ and $u(t)/R(t) \downarrow$. This new information permits us to essentially improve oscillatory criteria for advanced differential equations.

We are now prepared to establish a new comparison result that really contains advanced argument.

Theorem 3.3. *Let (3.1) hold. Assume that differential equation*

$$(r(t)u'(t))' + \left(\frac{R(\sigma(t))}{R(t)}\right)^\alpha p(t)u(t) = 0 \quad (E_2)$$

is oscillatory. Then (E) is oscillatory.

Proof. To the contrary, assume that (E) possesses an eventually positive solution $u(t)$. Since $u(t)/R^\alpha(t)$ is increasing, we conclude that $u(t)$ is a positive solution of the differential inequality

$$(r(t)u'(t))' + \left(\frac{R(\sigma(t))}{R(t)}\right)^\alpha p(t)u(t) \leq 0.$$

By Corollary 2 in [7], the corresponding differential equation (E₂) also has a positive solution. This is a contradiction and the proof is complete. \square

Employing any oscillatory criterion for the oscillation of (E₂), we immediately obtain an oscillation result for (E).

Theorem 3.4. *Let (3.1) hold. Assume that there exists a constant α_1 such that*

$$R(t) \int_t^\infty \left(\frac{R(\sigma(t))}{R(t)}\right)^\alpha p(s)ds \geq \alpha_1 > \frac{1}{4}, \quad (3.2)$$

eventually. Then (E) is oscillatory.

Proof. Condition (3.2) guarantees (see e.g. [3]) oscillation of (E₂), which implies oscillation of (E). \square

We support the results obtained with a series of illustrative examples.

Example 3.5. Consider the second order advanced Euler differential equation

$$y''(t) + \frac{a}{t^2}y(\lambda t) = 0, \quad (E_x)$$

with $a > 0$, $\lambda > 1$. Now $\alpha = a$ and by Theorem 3.4, Eq. (E_x) is oscillatory provided that

$$a\lambda^a > \frac{1}{4}.$$

For e.g. $a = 0.2$ it is required $\lambda \geq 3.051758$ or conversely for given $\lambda = 1.8$ we need $a \geq 0.219712$.

Remark 3.6. It is useful to notice that Koplatadze et al. [6] studied oscillation of advanced differential equations and provided the oscillatory criterion that for (E_x) takes the form $a(2 + \ln \lambda) > 1$. Instead of $\ln \lambda$ our criterion contains the power λ^a .

Remark 3.7. Agarwal et al. [1] studied (E) by comparing it with the first order advanced equation. But this technique again leads to criterion that for (E_x) uses $\ln \lambda$ instead of λ^a . So our criterion essentially improves known results.

If the criterion (3.2) fails ($\alpha_1 \leq 1/4$), then we are able to derive new oscillatory criterion that makes use of the constant α_1 .

Theorem 3.8. *Let (3.1) hold. Assume that $u(t)$ is a positive solution of (E) and*

$$R(t) \int_t^\infty \left(\frac{R(\sigma(s))}{R(s)} \right)^\alpha p(s) ds \geq \alpha_1 > 0, \quad (3.3)$$

eventually. Then there is t_ such that for $t \geq t_*$*

$$\frac{u(t)}{R^{\alpha_1}(t)} \uparrow.$$

Proof. Assume that (E) possesses a positive solution $u(t)$. Since $u(t)/R^{\alpha_1}(t)$ is increasing, we see that

$$\begin{aligned} r(t)u'(t) &\geq \int_t^\infty p(s)u(\sigma(s)) ds \geq \int_t^\infty \left(\frac{R(\sigma(s))}{R(s)} \right)^\alpha u(s)p(s) ds \\ &\geq u(t) \int_t^\infty \left(\frac{R(\sigma(s))}{R(s)} \right)^\alpha p(s) ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \left(\frac{u(t)}{R^{\alpha_1}(t)} \right)' &= \frac{1}{r(t)R^{\alpha_1+1}(t)} [R(t)r(t)u'(t) - \alpha_1 u(t)] \\ &\geq \frac{u(t)}{r(t)R^{\alpha_1+1}(t)} \left[R(t) \int_t^\infty \left(\frac{R(\sigma(s))}{R(s)} \right)^\alpha p(s) ds - \alpha_1 \right] \geq 0. \end{aligned}$$

The proof is complete. □

Theorem 3.9. *Let (3.1) and (3.3) hold. Assume that differential equation*

$$(r(t)u'(t))' + \left(\frac{R(\sigma(t))}{R(t)} \right)^{\alpha_1} p(t)u(t) = 0 \quad (E_3)$$

is oscillatory. Then (E) is oscillatory.

Theorem 3.10. *Let (3.1) and (3.3) hold. Assume that there exists a constant α_2 such that*

$$R(t) \int_t^\infty \left(\frac{R(\sigma(t))}{R(t)} \right)^{\alpha_1} p(s) ds \geq \alpha_2 > \frac{1}{4}, \quad (3.4)$$

eventually. Then (E) is oscillatory.

The proofs of the above theorems are similar to those of Theorem 3.3 and 3.4 and so they are omitted.

Example 3.11. Consider once more the differential equation (E_x). For this equation $\alpha_1 = a\lambda^a$. By Theorem 3.10, Eq. (E_x) is oscillatory provided that

$$a\lambda^{\alpha_1} > \frac{1}{4}.$$

Since $\alpha_1 > a$, Theorem 3.10 improves Theorem 3.4. Now for $a = 0.2$ it is needed only $\lambda \geq 2.5267188$.

We can repeat the above process and improve our oscillatory criteria again and again. To simplify our considerations we use the additional condition that there is a positive constant λ such that

$$\frac{R(\sigma(t))}{R(t)} \geq \lambda > 1, \quad (3.5)$$

eventually. Then, in view of (3.1), conditions (3.3) and (3.4) can be written in simpler forms as

$$\begin{aligned} \alpha_1 &= \lambda^\alpha \alpha > \frac{1}{4}, \\ \alpha_2 &= \lambda^{\alpha_1} \alpha > \frac{1}{4}, \end{aligned}$$

respectively. Repeating the above process, we get the increasing sequence $\{\alpha_n\}_{n=0}^\infty$ defined as follows:

$$\begin{aligned} \alpha_0 &= \alpha, \\ \alpha_{n+1} &= \lambda^{\alpha_n} \alpha. \end{aligned} \quad (3.6)$$

Now, we can generalise the oscillatory criteria presented in Theorems 3.4 and 3.10.

Theorem 3.12. *Let (3.1) and (3.5) hold. Assume that there exists a positive integer n such that $\alpha_j \leq 1/4$ for $j = 0, 1, \dots, n-1$ and*

$$\alpha_n > \frac{1}{4},$$

Then (E) is oscillatory.

Example 3.13. Consider the second order advanced differential equation

$$y''(t) + \frac{0.222}{t^2} y(1.61t) = 0, \quad (E_{x1})$$

For this equation $\alpha_0 = 0.222$ and $\lambda = 1.61$. Simple computations show that

$$\alpha_1 = 0.2467563 \quad \text{and} \quad \alpha_2 = 0.2496828,$$

so Theorems 3.4 and 3.10 fail for (E_{x1}) . But

$$\alpha_3 = 0.2500310 > \frac{1}{4}$$

and Theorem 3.12 guarantees the oscillation of (E_{x1}) .

If we consider the modified equation

$$y''(t) + \frac{0.21894135}{t^2} y(1.7t) = 0, \quad (E_{x2})$$

then some suitable software (e.g. Matlab) is needed to verify that $\alpha_i < 1/4$ for $i = 0, 1, \dots, 7$ and $\alpha_8 = 0.250000006 > 1/4$ and to conclude that (E_{x2}) is oscillatory.

Since the sequence $\{\alpha_n\}_{n=0}^\infty$ is increasing, there exists

$$\rho = \lim_{t \rightarrow \infty} \alpha_n \quad (3.7)$$

and Theorem 3.12 can be reformulated as follows.

Theorem 3.14. *Let (3.1) and (3.5) hold. Assume ρ is defined by (3.7). If*

$$\rho > \frac{1}{4}, \quad (3.8)$$

then (E) is oscillatory.

Since for α and λ given by (3.1) and (3.5) the calculation of ρ is not easy, the above criterion is only theoretical. At the same time it yields an easily verifiable oscillation criterion.

But at first we recall some facts concerning iterated exponentiation. The problem of iterated exponentiation is the evaluation

$$\delta = z^{z^{z^{\dots}}} \quad (3.9)$$

whenever it makes the sense. Euler was the first to prove that this iteration converges for $z \in (e^{-e}, e^{1/e})$. From our point of view it is important for desired δ to find z such that (3.9) holds.

Since (3.9) can be written in the form $\delta = z^\delta$, then necessarily

$$z = \delta^{1/\delta}. \quad (3.10)$$

Employing criteria for convergence of iterative exponentiation, we obtain a sufficient condition for the iterations of z given by (3.10) to converge to σ .

Lemma 3.15. *Assume that $z = \delta^{1/\delta}$, where $\delta < e$ and $z > e^{-e}$. Then (3.9) holds.*

We rewrite the iteration process (3.6) in term of iterated exponentiation to be able to apply Lemma 3.15. It is easy to see that ρ given by (3.7) is the limit of the iterations

$$\rho = \alpha z^{z^{z^{\dots}}}, \quad \text{where } z = \lambda^\alpha.$$

Theorem 3.16. *Let (3.1) and (3.5) hold. If*

$$\alpha > \frac{1}{4e} \quad (3.11)$$

and

$$\alpha \lambda^{1/4} > \frac{1}{4}, \quad (3.12)$$

then (E) is oscillatory.

Proof. By Theorem 3.14, it is sufficient to show that the corresponding $\rho > 1/4$. For that reason, we verify that the condition $\alpha \lambda^{1/4} = 1/4$ yields $\rho = 1/4$.

It is easy to see that

$$\alpha \lambda^{1/4} = 1/4 \Leftrightarrow z = \lambda^\alpha = \left(\frac{1}{4\alpha}\right)^{4\alpha}$$

By Lemma 3.15, for $z = (1/4\alpha)^{4\alpha}$ the following iterated exponentiation is convergent and

$$z^{z^{z^{\dots}}} = \frac{1}{4\alpha}$$

which yields that

$$\rho = \alpha z^{z^{z^{\dots}}} = \frac{1}{4}.$$

The proof is complete. □

Although Theorem 3.16 is based on all previous consecutive improvements of Theorem 3.4, it provides simple easily verifiable oscillatory criterion that do not require any mathematical software for verification.

Example 3.17. Consider again the advanced Euler differential equation (E_x). By Theorem 3.16, Eq. (E_x) is oscillatory provided that

$$a\lambda^{1/4} > \frac{1}{4} \quad \text{and} \quad a > \frac{1}{4e}.$$

For $a = 0.2$ it is sufficient that $\lambda \geq 2.4414063$.

The following example is intended to show that condition (3.11) cannot be relaxed.

Example 3.18. We consider advanced equation

$$y''(t) + \frac{0.043240136481863}{t^2} y(1526t) = 0, \quad (E_{x4})$$

It is easy to verify that (3.12) holds, but (3.11) fails and (E_{x4}) has a positive solution $y(t) = t^{0.1}$.

4 Summary

In the paper, as a consequence of new monotonic properties of nonoscillatory solutions, we introduce a new comparison technique for studying oscillation of second order advanced differential equations. Our results fulfil the gap in oscillation theory.

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