Location of solutions for quasi-linear elliptic equations with general gradient dependence

Dumitru Motreanu\textsuperscript{1} and Elisabetta Tornatore\textsuperscript{2}

\textsuperscript{1}University of Perpignan, Department of Mathematics, Perpignan, 66860, France
\textsuperscript{2}Università degli Studi di Palermo, Dipartimento di Matematica e Informatica, Palermo, 90123, Italy

Received 25 September 2017, appeared 10 December 2017
Communicated by Gabriele Bonanno

Abstract. Existence and location of solutions to a Dirichlet problem driven by \((p,q)\)-Laplacian and containing a (convection) term fully depending on the solution and its gradient are established through the method of subsolution-supersolution. Here we substantially improve the growth condition used in preceding works. The abstract theorem is applied to get a new result for existence of positive solutions with a priori estimates.

Keywords: quasi-linear elliptic equations, gradient dependence, \((p,q)\)-Laplacian, subsolution-supersolution, positive solution.

2010 Mathematics Subject Classification: 35J92, 35J25.

1 Introduction

The aim of this paper is to study the following nonlinear elliptic boundary value problem

\[
\begin{aligned}
-\Delta_p u - \mu \Delta_q u &= f(x, u, \nabla u) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
\]

by means of the method of subsolution-supersolution on a bounded domain \(\Omega \subset \mathbb{R}^N\). For regularity reasons we assume that the boundary \(\partial\Omega\) is of class \(C^2\). In order to simplify the presentation we suppose that \(N \geq 3\). The lower dimensional cases \(N = 1, 2\) are simpler and can be treated by slightly modified arguments.

In the statement of problem \((P_{\mu})\), there are given real numbers \(\mu \geq 0\) and \(1 < q < p\). The leading differential operator in \((P_{\mu})\) is described by the \(p\)-Laplacian and \(q\)-Laplacian, namely

\[
\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) \quad \text{and} \quad \Delta_q u = \text{div}(|\nabla u|^{q-2}\nabla u).
\]

Hence if \(\mu = 0\), problem \((P_{\mu})\) is governed by the \(p\)-Laplacian \(\Delta_p\), whereas if \(\mu = 1\), it is driven by the \((p,q)\)-Laplacian \(\Delta_p + \Delta_q\), which is an essentially different type of nonlinear operator.

The right-hand side of the elliptic equation in \((P_{\mu})\) is expressed through a Carathéodory function \(f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}\), i.e., \(f(\cdot, s, \xi)\) is measurable for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^N\) and \(f(x, \cdot, \cdot)\)

\textsuperscript{52}Corresponding author.
Emails: motreanu@univ-perp.fr (D. Motreanu), elisa.tornatore@unipa.it (E. Tornatore).
is continuous for a.e. \( x \in \Omega \). We emphasize that the term \( f(x, u, \nabla u) \) (often called convection term) depends not only on the solution \( u \), but also on its gradient \( \nabla u \). This fact produces serious difficulties of treatment mainly because the convection term generally prevents to have a variational structure for problem \( (P_\mu) \), so the variational methods are not applicable.

Existence results for problem \( (P_\mu) \) or for systems of equations of this form have been obtained in [1,4–7,10–12]. Location of solutions through the method of subsolution-supersolution in the case of systems involving \( p \)-Laplacian operators has been investigated in [3]. Here, in the case of an equation possibly involving the \((p,q)\)-Laplacian, we focus on the location of solutions within ordered intervals determined by pairs of subsolution-supersolution of problem \( (P_\mu) \) under a much more general growth condition on the right-hand side \( f(x, u, \nabla u) \) (see hypothesis \((H)\) below). We also provide a new result guaranteeing the existence of positive solutions to \( (P_\mu) \).

The functional space associated to problem \( (P_\mu) \) is the Sobolev space \( W_0^{1,p}(\Omega) \) endowed with the norm

\[ ||u|| = \left( \int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}}. \]

Its dual space is \( W^{-1,p'}(\Omega) \), with \( p' = p/(p - 1) \), and the corresponding duality pairing is denoted \( \langle \cdot, \cdot \rangle \).

A solution of problem \( (P_\mu) \) is understood in the weak sense, that is any function \( u \in W_0^{1,p}(\Omega) \) such that

\[ \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx + \mu \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx = \int_\Omega f(x, u, \nabla u) v \, dx \]

for all \( v \in W_0^{1,p}(\Omega) \).

Our study of problem \( (P_\mu) \) is based on the method of subsolution-supersolution. We refer to [2, 9] for details related to this method. We recall that a function \( \bar{\pi} \in W^{1,p}(\Omega) \) is a supersolution for problem \( (P_\mu) \) if \( \bar{\pi} \geq 0 \) on \( \partial \Omega \) and

\[ \int_\Omega \left( |\nabla \bar{\pi}|^{p-2} \nabla \bar{\pi} + \mu |\nabla \bar{\pi}|^{p-2} \nabla \bar{\pi} \right) \nabla v \, dx \geq \int_\Omega f(x, \bar{\pi}, \nabla \bar{\pi}) v \, dx \]

for all \( v \in W_0^{1,p}(\Omega), \, v \geq 0 \) a.e. in \( \Omega \). A function \( \underline{u} \in W^{1,p}(\Omega) \) is a subsolution for problem \( (P_\mu) \) if \( \underline{u} \leq 0 \) on \( \partial \Omega \) and

\[ \int_\Omega \left( |\nabla \underline{u}|^{p-2} \nabla \underline{u} + \mu |\nabla \underline{u}|^{p-2} \nabla \underline{u} \right) \nabla v \, dx \leq \int_\Omega f(x, \underline{u}, \nabla \underline{u}) v \, dx \]

for all \( v \in W_0^{1,p}(\Omega), \, v \geq 0 \) a.e. in \( \Omega \).

In the sequel we suppose that \( N > p \) (if \( N \leq p \) the treatment is easier). Then the critical Sobolev exponent is \( p^* = \frac{Np}{N-p} \).

Given a subsolution \( \underline{u} \in W^{1,p}(\Omega) \) and a supersolution \( \bar{\pi} \in W^{1,p}(\Omega) \) for problem \( (P_\mu) \) with \( \underline{u} \leq \bar{\pi} \) a.e. in \( \Omega \), we assume that \( f : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) satisfies the growth condition:

\((H)\) There exist a function \( \sigma \in L^\gamma'(\Omega) \) for \( \gamma' = \frac{\gamma}{\gamma-1} \) with \( \gamma \in (1, p^*) \) and constants \( a > 0 \) and \( \beta \in [0, \frac{p}{(p^*)'}) \) such that

\[ |f(x, s, \zeta)| \leq \sigma(x) + a|\zeta|^\beta \]

for a.e. \( x \in \Omega \), all \( s \in [\underline{u}(x), \bar{\pi}(x)] \), \( \zeta \in \mathbb{R}^N \).
Notice that, under assumption \((H)\), the integrals in the definitions of the subsolution \(u\) and the supersolution \(\overline{u}\) exist.

Our main goal is to obtain a solution \(u \in W^{1,p}_0(\Omega)\) of problem \((P_\mu)\) with the location property \(\underline{u} \leq u \leq \overline{u}\) a.e. in \(\Omega\). This is done through an auxiliary truncated problem termed \((T_{\lambda, \mu})\) depending on a positive parameter \(\lambda\) (for any fixed \(\mu \geq 0\)). It is shown in Theorem 2.1 that whenever \(\lambda > 0\) is sufficiently large, problem \((T_{\lambda, \mu})\) is solvable. The next principal step is performed in Theorem 3.1, where it is proven by adequate comparison that every solution \(R\) with some notation. The Euclidean norm on \(\lambda\) on \(\Omega\) is denoted by \(|·|\).

This section is devoted to the study of an auxiliary problem related to problem \((P_\mu)\), where it is proven by adequate comparison that every solution \(u \in W^{1,p}_0(\Omega)\) of problem \((T_{\lambda, \mu})\) is within the ordered interval \([\underline{u}, \overline{u}]\) determined by the subsolution-supersolution, that is \(\underline{u} \leq u \leq \overline{u}\) a.e. in \(\Omega\). Then the expression of the equation in \((T_{\lambda, \mu})\) enables us to conclude that \(u\) is actually a solution of the original problem \((P_\mu)\) verifying the location property \(\underline{u} \leq u \leq \overline{u}\) a.e. in \(\Omega\). We emphasize that Theorem 2.1 improves all the growth conditions for the convection term \(f(x, u, \nabla u)\) considered in the preceding works.

Finally, in Theorem 4.1, the procedure to construct solutions located in ordered intervals \([\underline{u}, \overline{u}]\) is conducted to guarantee the existence of a positive solution to problem \((P_\mu)\). It is also worth mentioning that this result provides a priori estimates for the obtained solution.

## 2 Auxiliary truncated problem

This section is devoted to the study of an auxiliary problem related to problem \((P_\mu)\). We start with some notation. The Euclidean norm on \(\mathbb{R}^N\) is denoted by \(|·|\) and the Lebesgue measure on \(\mathbb{R}^N\) by \(|·|_N\). For every \(r \in \mathbb{R}\), we set \(r^+ = \max\{r, 0\}\), \(r^- = \max\{-r, 0\}\), and if \(r > 1\), \(r' = \frac{1}{r-1}\).

Let \(\underline{u}\) and \(\overline{u}\) be a subsolution and a supersolution for problem \((P_\mu)\), respectively, with \(\underline{u} \leq \overline{u}\) a.e. in \(\Omega\) such that hypothesis \((H)\) is satisfied. We consider the truncation operator \(T : W^{1,p}_0(\Omega) \to W^{1,p}_0(\Omega)\) defined by

\[
Tu(x) = \begin{cases} 
\overline{u}(x), & u(x) > \overline{u}(x), \\
\underline{u}(x), & \underline{u}(x) \leq u(x) \leq \overline{u}(x), \\
u(x), & u(x) < \underline{u}(x),
\end{cases}
\] (2.1)

which is known to be continuous and bounded.

By means of the constant \(\beta\) in hypothesis \((H)\) we introduce the cut-off function \(\pi : \Omega \times \mathbb{R} \to \mathbb{R}\) defined by

\[
\pi(x, s) = \begin{cases} 
(s - \overline{u}(x))^{\frac{\beta}{p'}} & s > \overline{u}(x), \\
0, & \underline{u}(x) \leq s \leq \overline{u}(x), \\
-(u(x) - s)^{\frac{\beta}{p'}} & s < \underline{u}(x).
\end{cases}
\] (2.2)

We observe that \(\pi\) satisfies the growth condition

\[
|\pi(x, s)| \leq c|s|^{\frac{\beta}{p'}} + \varrho(x) \quad \text{for a.e. } x \in \Omega, \text{ all } s \in \mathbb{R},
\] (2.3)

with a constant \(c > 0\) and a function \(\varrho \in L^{\frac{p'}{p}}(\Omega)\). Here it is used that \(\underline{u}, \overline{u} \in W^{1,p}_0(\Omega) \subseteq L^p(\Omega)\) and \(\beta < \frac{p}{(p')}\). By (2.3), the fact that \(\beta < \frac{p}{(p')}\) and Rellich–Kondrachov compactness embedding theorem, it follows that the Nemitskij operator \(\Pi : W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega)\) given
by \( \Pi(u) = \pi(\cdot, u(\cdot)) \) is completely continuous. Moreover, (2.2) leads to

\[
\int_{\Omega} \pi(x, u(x))u(x) \, dx \geq r_1 \| u \|_{L^{\frac{p}{p-1}}(\Omega)}^{\frac{p}{p-1}} - r_2 \quad \text{for all } u \in W_0^{1,p}(\Omega),
\]

with positive constants \( r_1 \) and \( r_2 \).

Next we consider the Nemytskij operator \( N : [u, \pi] \to W^{-1,p'}(\Omega) \) determined by the function \( f \) in \( (P_\rho) \), that is

\[
N(u)(x) = f(x, u(x), \nabla u(x)),
\]

which is well defined by virtue of hypothesis \((H)\).

With the data above, for any \( \lambda > 0 \) let the auxiliary truncated problem associated to \((P_\rho)\) be formulated as follows

\[
-\Delta_p u - \mu \Delta_q u + \lambda \Pi(u) = N(Tu).
\]

For problem \((T_{\lambda, \mu})\) we have the following result.

**Theorem 2.1.** Let \( u \) and \( \overline{u} \) be a sub solution and a supersolution of problem \((P_\rho)\), respectively, with \( u \leq \overline{u} \) a.e. in \( \Omega \) such that hypothesis \((H)\) is fulfilled. Then there exists \( \lambda_0 > 0 \) such that whenever \( \lambda \geq \lambda_0 \) there is a solution \( u \in W_0^{1,p}(\Omega) \) of the auxiliary problem \((T_{\lambda, \mu})\).

**Proof.** For every \( \lambda > 0 \) we introduce the nonlinear operator \( A_\lambda : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) \) defined by

\[
A_\lambda u = -\Delta_p u - \mu \Delta_q u + \lambda \Pi(u) - N(Tu).
\]

Due to (2.3) and \((H)\), the operator \( A_\lambda \) is bounded.

We claim that \( A_\lambda \) in (2.5) is a pseudomonotone operator. In order to show this, let a sequence \( \{u_n\} \subset W_0^{1,p}(\Omega) \) satisfy

\[
u_n \to u \quad \text{in } W_0^{1,p}(\Omega)
\]

and

\[
\limsup_{n \to \infty} \langle A_\lambda u_n, u_n - u \rangle \leq 0.
\]

Recalling from \((H)\) that \( \sigma \in L^{\gamma'}(\Omega) \) with \( \gamma < p^* \), by Hölder’s inequality, (2.6) and the Rellich–Kondrachov compact embedding theorem we get

\[
\int_{\Omega} |\nabla (Tu_n)|^p \, dx \to 0 \quad \text{as } n \to +\infty.
\]

Let us show that

\[
\int_{\Omega} |\nabla u_n|^p \, dx \to 0 \quad \text{as } n \to +\infty.
\]

The definition of the truncation operator \( T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega) \) in (2.1) yields

\[
\int_{\Omega} |\nabla (Tu_n)|^p \, dx = \int_{\{u_n < \underline{u}\}} |\nabla \underline{u}|^p |u_n - u| \, dx + \int_{\{u_n \leq \underline{u} \leq \overline{u}\}} |\nabla u_n|^p |u_n - u| \, dx + \int_{\{u_n > \overline{u}\}} |\nabla \overline{u}|^p |u_n - u| \, dx.
\]

Using Hölder’s inequality, (2.6) and the Rellich–Kondrachov compact embedding theorem, as well as the inequality \( \frac{p}{p-1} < p^* \), enables us to find that

\[
\int_{\{u_n < \underline{u}\}} |\nabla \underline{u}|^p |u_n - u| \, dx \leq \| \nabla \underline{u} \|_{L^p(\Omega)}^p \| u_n - u \|_{L^{\frac{p}{p-1}}(\Omega)} \to 0,
\]

\[
\int_{\{u_n \leq \underline{u} \leq \overline{u}\}} |\nabla u_n|^p |u_n - u| \, dx \leq \| \nabla u_n \|_{L^p(\Omega)}^p \| u_n - u \|_{L^{\frac{p}{p-1}}(\Omega)} \to 0,
\]

\[
\int_{\{u_n > \overline{u}\}} |\nabla \overline{u}|^p |u_n - u| \, dx \leq \| \nabla \overline{u} \|_{L^p(\Omega)}^p \| u_n - u \|_{L^{\frac{p}{p-1}}(\Omega)} \to 0.
\]
\[
\int_{\{u \leq u_{n} \leq \bar{u}\}} |\nabla u_{n}|^{\beta}|u_{n} - u| \, dx \leq \|\nabla u_{n}\|_{L^{p}(\Omega)}^{\beta}\|u_{n} - u\|_{L^{\frac{p}{\beta}}(\Omega)} \to 0,
\]
\[
\int_{\{u > \bar{u}\}} |\nabla u_{n}|^{\beta}|u_{n} - u| \, dx \leq \|\nabla u_{n}\|_{L^{p}(\Omega)}^{\beta}\|u_{n} - u\|_{L^{\frac{p}{\beta}}(\Omega)} \to 0.
\]

Therefore (2.9) holds true.

Taking into account (2.8), (2.9), hypothesis (H) and the fact that \( u \leq Tu_{n} \leq \bar{u} \) a.e. in \( \Omega \) for every \( n \), it turns out that

\[
\lim_{n \to \infty} \int_{\Omega} f(x, Tu_{n}, \nabla (Tu_{n}))(u_{n} - u) \, dx = 0. \tag{2.10}
\]

Using (2.3), (2.6) and the inequality \( \frac{p}{p - \beta} < p^{*} \), the same type of arguments yields

\[
\lim_{n \to \infty} \int_{\Omega} \pi(x, u_{n})(u_{n} - u) \, dx = 0. \tag{2.11}
\]

Due to (2.10) and (2.11), inequality (2.7) becomes

\[
\limsup_{n \to \infty} (-\Delta_{p} u_{n} - \mu\Delta_{q} u_{n}, u_{n} - u) \leq 0.
\]

Through the \((S)_{+}\) property of the operator \(-\Delta_{p} - \mu\Delta_{q}\) (see [9, pp. 39–40]) and (2.6), we obtain the strong convergence \( u_{n} \to u \) in \( W^{1,p}_{0}(\Omega) \), thus

\[
-\Delta_{p} u_{n} - \mu\Delta_{q} u_{n} \to -\Delta_{p} u - \mu\Delta_{q} u. \tag{2.12}
\]

Taking into account (2.12) and that \( u_{n} \to u \) in \( W^{1,p}_{0}(\Omega) \), we get

\[
A_{\lambda} u_{n} \to A_{\lambda} u, \quad \langle A_{\lambda} u_{n}, u_{n} \rangle \to \langle A_{\lambda} u, u \rangle,
\]

which ensures that the operator \( A_{\lambda} \) is pseudomonotone.

Now we prove that the operator \( A_{\lambda} : W^{1,p}_{0}(\Omega) \to W^{-1,p'}(\Omega) \) is coercive meaning that

\[
\lim \frac{\langle A_{\lambda} u, u \rangle}{\|u\|} = +\infty.
\]

The expression of \( A_{\lambda} \) in (2.5) allows us to find

\[
\langle A_{\lambda} u, u \rangle \geq \|\nabla u\|_{L^{p}(\Omega)}^{p} + \lambda \int_{\Omega} \pi(x, u) u \, dx - \int_{\Omega} f(x, Tu, \nabla (Tu)) u \, dx. \tag{2.13}
\]

Notice that by virtue of (2.1), it holds \( u \leq Tu \leq \bar{u} \) a.e. in \( \Omega \) for every \( u \in W^{1,p}_{0}(\Omega) \), so we can use hypothesis (H) with \( s = (Tu)(x) \) for a.e. \( x \in \Omega \). Then, combining with Young’s inequality and Sobolev embedding theorem, we infer for each \( \varepsilon > 0 \) that

\[
\left| \int_{\Omega} f(x, Tu, \nabla (Tu)) u \, dx \right| \leq \int_{\Omega} \left( \sigma|u| + a|\nabla (Tu)|^{\beta}|u| \right) \, dx \leq \|\sigma\|_{L^{q'}(\Omega)}\|u\|_{L^{q}(\Omega)} + \varepsilon\|\nabla u\|_{L^{p}(\Omega)}^{p} + c_{1}(\varepsilon)\|u\|_{L^{p}(\Omega)}^{\frac{p}{p'}} + c_{2}\|u\|_{L^{\frac{p}{p'}}(\Omega)} \leq \varepsilon\|u\|^{p} + c_{1}(\varepsilon)\|u\|_{L^{p}(\Omega)}^{\frac{p}{p'}} + d\|u\|, \tag{2.14}
\]

\[
|\int_{\Omega} \left( \sigma|u| + a|\nabla (Tu)|^{\beta}|u| \right) \, dx| \leq \int_{\Omega} \left( \sigma|u| + a|\nabla (Tu)|^{\beta}|u| \right) \, dx \leq \|\sigma\|_{L^{q'}(\Omega)}\|u\|_{L^{q}(\Omega)} + \varepsilon\|\nabla u\|_{L^{p}(\Omega)}^{p} + c_{1}(\varepsilon)\|u\|_{L^{p}(\Omega)}^{\frac{p}{p'}} + c_{2}\|u\|_{L^{\frac{p}{p'}}(\Omega)} \leq \varepsilon\|u\|^{p} + c_{1}(\varepsilon)\|u\|_{L^{p}(\Omega)}^{\frac{p}{p'}} + d\|u\|,
\]
with positive constants \(c_1(\varepsilon)\) (depending on \(\varepsilon\)), \(c_2\), \(d\).

Inserting (2.4) and (2.14) in (2.13), it turns out that

\[
\langle A_\lambda u, u \rangle \geq (1 - \varepsilon)\|u\|^p + (\lambda r_1 - c_1(\varepsilon))\|u\|_{L^p(\Omega)}^p - d\|u\| - \lambda r_2. \tag{2.15}
\]

Choose \(\varepsilon \in (0, 1)\) and \(\lambda > \frac{c_1(\varepsilon)}{r_1}\). Then (2.15) implies that the operator \(A_\lambda\) is coercive.

Since the operator \(A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)\) is bounded, pseudomonotone and coercive, it is surjective (see [2, p. 40]). Therefore we can find \(u \in W_0^{1,p}(\Omega)\) that solves equation \((T_{\lambda,}\mu)\), which completes the proof. \(\square\)

3 Main result

We state our main abstract result on problem \((P_p)\).

**Theorem 3.1.** Let \(\underline{u}\) and \(\overline{u}\) be a subsolution and a supersolution of problem \((P_p)\), respectively, with \(\underline{u} \leq \overline{u}\) a.e. in \(\Omega\) such that hypothesis \((H)\) is fulfilled. Then problem \((P_p)\) possesses a solution \(u \in W_0^{1,p}(\Omega)\) satisfying the location property \(\underline{u} \leq u \leq \overline{u}\) a.e. in \(\Omega\).

**Proof.** Theorem 2.1 guarantees the existence of a solution of the truncated auxiliary problem \((T_{\lambda,}\mu)\) provided \(\lambda > 0\) is sufficiently large. Fix such a constant \(\lambda\) and let \(u \in W_0^{1,p}(\Omega)\) be a solution of \((T_{\lambda,}\mu)\).

We prove that \(u \leq \overline{u}\) a.e. in \(\Omega\). Acting with \((u - \overline{u})^+ \in W_0^{1,p}(\Omega)\) as a test function in the definition of the supersolution \(\overline{u}\) of \((P_p)\) and in the definition of the solution \(u\) for the auxiliary truncated problem \((T_{\lambda,}\mu)\) results in

\[
\langle -\Delta_p \overline{u} - \mu \Delta_q \overline{u}, (u - \overline{u})^+ \rangle \geq \int_{\Omega} f(x, \overline{u}, \nabla \overline{u}) (u - \overline{u})^+ \, dx \tag{3.1}
\]

and

\[
\langle -\Delta_p u - \mu \Delta_q u, (u - \overline{u})^+ \rangle + \lambda \int_{\Omega} \Pi(u)(u - \overline{u})^+ \, dx = \int_{\Omega} f(x, Tu, \nabla (Tu))(u - \overline{u})^+ \, dx. \tag{3.2}
\]

From (3.1), (3.2) and (2.1) we derive

\[
\int_{\Omega} (|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla \overline{u}) \nabla (u - \overline{u})^+ \, dx \\
+ \mu \int_{\Omega} (|\nabla u|^{q-2}\nabla u - |\nabla \overline{u}|^{q-2}\nabla \overline{u}) \nabla (u - \overline{u})^+ \, dx + \lambda \int_{\Omega} \pi(x, u)(u - \overline{u})^+ \, dx \\
\leq \int_{\Omega} (f(x, Tu, \nabla (Tu)) - f(x, \overline{u}, \nabla \overline{u})) (u - \overline{u})^+ \, dx \\
= \int_{\{u > \overline{u}\}} (f(x, Tu, \nabla (Tu)) - f(x, \overline{u}, \nabla \overline{u})) (u - \overline{u}) \, dx = 0.
\tag{3.3}
\]

Since

\[
\int_{\Omega} (|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla \overline{u}) \nabla (u - \overline{u})^+ \, dx \\
= \int_{\{u > \overline{u}\}} (|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla \overline{u}) \nabla (u - \overline{u}) \, dx \geq 0
\]
and
\[
\int_{\Omega} (|\nabla u|^{q-2} \nabla u - |\nabla \bar{u}|^{q-2} \nabla \bar{u}) \nabla (u - \bar{u})^+ \, dx
= \int_{\{u > \bar{u}\}} (|\nabla u|^{q-2} \nabla u - |\nabla \bar{u}|^{q-2} \nabla \bar{u}) (\nabla u - \nabla \bar{u}) \, dx \geq 0,
\]
we are able to derive from (2.2) and (3.3) that
\[
\int_{\{u > \bar{u}\}} (u - \bar{u})^{\frac{p}{2}} \, dx = \int_{\Omega} \pi(x, u)(u - \bar{u})^+ \, dx \leq 0.
\]
It follows that \( u \leq \bar{u} \) a.e. in \( \Omega \).

In an analogous way, by suitable comparison we can show that \( u \leq u \) a.e. in \( \Omega \). Consequently, the solution \( u \) of the auxiliary truncated problem \((T_{\lambda, \mu})\) satisfies \( Tu = u \) and \( \Pi(u) = 0 \) (see (2.1) and (2.2)), so it becomes a solution of the original problem \((P_{\mu})\), which completes the proof. \( \square \)

## 4 Existence of positive solutions

In this section we focus on the existence of positive solutions to problem \((P_{\mu})\). The idea is to construct a subsolution \( \underline{u} \in W^{1,p}(\Omega) \) and a supersolution \( \bar{u} \in W^{1,p}(\Omega) \) with \( 0 < \underline{u} \leq \bar{u} \) a.e. in \( \Omega \) for which Theorem 3.1 can be applied. In this respect, inspired by [6, 8], we suppose the following assumptions on the right-hand side \( f \) of \((P_{\mu})\):

\( (H1) \) There exist constants \( a_0 > 0, b > 0, \delta > 0 \) and \( r > 0 \), with \( r < p - 1 \) if \( \mu = 0 \) and \( r < q - 1 \) if \( \mu > 0 \), such that
\[
\left( \frac{a_0}{b} \right)^{\frac{1}{r-1}} < \delta
\]
and
\[
f(x, s, \xi) \geq a_0 s^r - b s^{p-1} \quad \text{for a.e. } x \in \Omega, \text{ all } 0 < s < \delta, \xi \in \mathbb{R}^N.
\]

\( (H2) \) There exists a constant \( s_0 > \delta \), with \( \delta > 0 \) in \((H1)\), such that
\[
f(x, s_0, 0) \leq 0 \quad \text{for a.e. } x \in \Omega.
\]

Our result on the existence of positive solutions for problem \((P_{\mu})\) is as follows.

**Theorem 4.1.** Assume \((H1)\), \((H2)\) and that
\[
|f(x, s, \xi)| \leq \sigma(x) + a|\xi|^\beta \quad \text{for a.e. } x \in \Omega, \text{ all } s \in [0, s_0], \xi \in \mathbb{R}^N,
\]
with a function \( \sigma \in L^\gamma(\Omega) \) for \( \gamma \in [1, p^*] \) and constants \( a > 0 \) and \( \beta \in \left(0, \frac{p}{(p^*)^*}\right) \). Then, for every \( \mu \geq 0 \), problem \((P_{\mu})\) possesses a positive smooth solution \( u \in C_0^1(\mathbb{T}) \) satisfying the a priori estimate \( u(x) \leq s_0 \) for all \( x \in \Omega \) (\( s_0 \) is the constant in \((H2)\)).

**Proof.** With the notation in hypothesis \((H1)\), consider the following auxiliary problem
\[
\begin{cases}
-\Delta_p u - \mu \Delta_q u + b |u|^{p-2} u = a_0(u^+)^r & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(4.4)
We are going to show that there exists a solution \( u \in C^1_0(\Omega) \) of problem (4.4) satisfying \( u > 0 \) in \( \Omega \) and
\[
b\|u\|_{L^{p-r-1}(\Omega)}^r \leq a_0. \tag{4.5}
\]
To this end, we consider the Euler functional associated to (4.5), that is the \( C^1 \)-function \( I : W^{1,p}_0(\Omega) \to \mathbb{R} \) defined by
\[
I(u) = \frac{1}{p} \int_\Omega (|\nabla u|^p + b|u|^p) \, dx + \frac{H}{q} \int_\Omega |\nabla u|^q \, dx - \frac{a_0}{r+1} \int_\Omega (u^+)^{r+1} \, dx
\]
whenever \( u \in W^{1,p}_0(\Omega) \). From the assumption on \( r \) in hypothesis (H1) and Sobolev embedding theorem, it is easy to prove that \( I \) is coercive. Since \( I \) is also sequentially weakly lower semicontinuous, there exists \( \bar{u} \in W^{1,p}_0(\Omega) \) such that
\[
I(\bar{u}) = \inf_{u \in W^{1,p}_0(\Omega)} I(u).
\]
On the basis of the conditions \( r < p-1 \) if \( \mu = 0 \) and \( r < q-1 \) if \( \mu > 0 \) (see hypothesis (H1)), it is seen that for any positive function \( v \in W^{1,p}_0(\Omega) \) and with a sufficiently small \( t > 0 \), there holds \( I(tv) < 0 \), so \( \inf_{u \in W^{1,p}_0(\Omega)} I(u) < 0 \). This enables us to deduce that \( \bar{u} \) is a nontrivial solution of (4.4). Testing equation (4.4) with \( -u^- \) yields \( \bar{u} > 0 \). By the nonlinear regularity theory and strong maximum principle we obtain that \( \bar{u} \in C^1_0(\Omega) \) and \( \bar{u} > 0 \) in \( \Omega \).

According to the latter properties, we can utilize \( \bar{u}^{a+1} \), with any \( a > 0 \), as a test function in (4.4). Through Hölder’s inequality and because \( r+1 < p \), this leads to
\[
b\|u\|_{L^{p+1}(\Omega)}^{p+1} \leq a_0 \int_\Omega |u|^{p+1} \, dx \leq a_0 \|u\|_{L^{q+1}(\Omega)}^{q+1} |\Omega|^{(p-r-1)/(p+1)}.
\]
Letting \( a \to +\infty \) in the inequality
\[
b\|u\|_{L^{q+1}(\Omega)}^{p-r-1} \leq a_0 |\Omega|^{(p-r-1)/(p+1)}
\]
we arrive at (4.5).

We claim that \( \bar{u} \) is a subsolution for problem \( (P_\mu) \). Specifically, due to (4.1) and (4.5), we can insert \( s = \bar{u}(x) \) and \( \zeta = \nabla \bar{u}(x) \) in (4.2), which in conjunction with (4.4) for \( u = \bar{u} \) reads as
\[
\int_\Omega (|\nabla \bar{u}|^{p-2} \nabla \bar{u} + \mu |\nabla \bar{u}|^{q-2} \nabla \bar{u} \nabla v) \nabla v \, dx = \int_\Omega (a_0 \bar{u}' - b \bar{u}^{p-1}) v \, dx 
\leq \int_\Omega f(x, \bar{u}, \nabla \bar{u}) v \, dx
\]
whenever \( v \in W^{1,p}_0(\Omega), v \geq 0 \) a.e. in \( \Omega \). Thereby the claim is proven.

Now we notice that hypothesis (H2) guarantees that \( \bar{u} = s_0 \) is a supersolution of problem \( (P_\mu) \). Indeed, in view of (4.3), we obtain
\[
\int_\Omega (|\nabla \bar{u}|^{p-2} \nabla \bar{u} + \mu |\nabla \bar{u}|^{q-2} \nabla \bar{u}) \nabla v \, dx = \int_\Omega f(x, s_0, 0) v \, dx
\]
for all \( v \in W^{1,p}_0(\Omega), v \geq 0 \) a.e. in \( \Omega \). We point out from assumption (H2) that \( s_0 > \delta \), which in conjunction with (4.1) and (4.5), entails that \( \bar{u} < \bar{u} \) in \( \Omega \).
We also note that hypothesis \((H)\) holds true for the constructed subsolution-supersolution \((u, \varpi)\) of problem \((P_\mu)\). Therefore Theorem 3.1 applies ensuring the existence of a solution \(u \in W^{1,p}_0(\Omega)\) to problem \((P_\mu)\), which satisfies the enclosure property \(u \leq \varpi \text{ a.e. in } \Omega\). Taking into account that \(u > 0\), we conclude that the solution \(u\) is positive. Moreover, the regularity up to the boundary invoked for problem \((P_\mu)\) renders \(u \in C^{1,0}_0(\Omega)\), whereas the inequality \(u \leq \varpi\) implies the estimate \(u(x) \leq s_0\) for all \(x \in \Omega\). This completes the proof. □

**Remark 4.2.** Proceeding symmetrically, a counterpart of Theorem 4.1 for negative solutions can be established.

We illustrate the applicability of Theorem 4.1 by a simple example.

**Example 4.3.** Let \(f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}\) be defined by

\[
f(x, s, \xi) = |s|^r - |s|^{p-1} + \left(2^{\frac{p}{r-1}} - s\right)|\xi|^{\beta}
\]

for all \((x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N\),

where the constants \(r, p, \beta\) are as in conditions \((H)\) and \((H1)\). For simplicity, we have dropped the dependence with respect to \(x \in \Omega\). Hypothesis \((H1)\) is verified by taking for instance \(a_0 = b = 1\) and \(\delta = 2^{\frac{p}{r-1}}\) (see (4.1) and (4.2)). Hypothesis \((H2)\) is fulfilled for every \(s_0 > \delta = 2^{\frac{p}{r-1}}\). It is also clear that the growth condition for \(f\) on \(\Omega \times [0, s_0] \times \mathbb{R}^N\) required in the statement of Theorem 4.1 is satisfied, too. Consequently, Theorem 4.1 applies to problem \((P_\mu)\) with the chosen function \(f(x, s, \xi)\) giving rise to a positive solution belonging to \(C^{1,0}_0(\Omega)\).

**Acknowledgements**

The authors were partially supported by INdAM - GNAMPA Project 2015.

**References**


