Fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacians

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Abstract. In this article we extend the Sobolev spaces with variable exponents to include the fractional case, and we prove a compact embedding theorem of these spaces into variable exponent Lebesgue spaces. As an application we prove the existence and uniqueness of a solution for a nonlocal problem involving the fractional $p(x)$-Laplacian.

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1 Introduction

Our main goal in this paper is to extend Sobolev spaces with variable exponents to cover the fractional case.

For a bounded domain with Lipschitz boundary $\Omega \subset \mathbb{R}^n$ we consider two variable exponents, that is, we let $q : \overline{\Omega} \to (1, \infty)$ and $p : \overline{\Omega} \times \overline{\Omega} \to (1, \infty)$ be two continuous functions. We assume that $p$ is symmetric, $p(x,y) = p(y,x)$, and that both $p$ and $q$ are bounded away from 1 and $\infty$, that is, there exist $1 < q_- < q_+ < +\infty$ and $1 < p_- < p_+ < +\infty$ such that $q_- \leq q(x) \leq q_+$ for every $x \in \overline{\Omega}$ and $p_- \leq p(x,y) \leq p_+$ for every $(x,y) \in \overline{\Omega} \times \overline{\Omega}$.

We define the Banach space $L^{q(x)}(\Omega)$ as usual,

$$L^{q(x)}(\Omega) := \left\{ f : \Omega \to \mathbb{R} : \exists \lambda > 0 : \int_{\Omega} \left| \frac{f(x)}{\lambda} \right|^{q(x)} \, dx < \infty \right\},$$

with its natural norm

$$\|f\|_{L^{q(x)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{f(x)}{\lambda} \right|^{q(x)} \, dx < 1 \right\}.$$
Now for $0 < s < 1$ we introduce the variable exponent Sobolev fractional space as follows:

$$W = W^{s,q(x),p(x,y)}(\Omega) := \left\{ f : \Omega \to \mathbb{R} : f \in L^{q(x)}(\Omega) : \right.$$ \[ \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}} < \infty, \text{ for some } \lambda > 0 \right\}, \]

and we set

$$[f]^{s,p(x,y)}(\Omega) := \inf \left\{ \lambda > 0 : \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}} < 1 \right\}$$

as the variable exponent seminorm. It is easy to see that $W$ is a Banach space with the norm

$$\|f\|_W := \|f\|_{L^{q(x)}(\Omega)} + [f]^{s,p(x,y)}(\Omega);$$

in fact, one just has to follow the arguments in [20] for the constant exponent case. For general theory of classical Sobolev spaces we refer the reader to [1, 5] and for the variable exponent case to [8].

Our main result is the following compact embedding theorem into variable exponent Lebesgue spaces. For an analogous theorem for the Sobolev trace embedding we refer to the companion paper [3].

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz bounded domain and $s \in (0, 1)$. Let $q(x)$, $p(x,y)$ be continuous variable exponents with $sp(x,y) < n$ for $(x,y) \in \overline{\Omega} \times \overline{\Omega}$ and $q(x) > p(x,y)$ for $x \in \overline{\Omega}$. Assume that $r : \overline{\Omega} \to (1, \infty)$ is a continuous function such that

$$p^*(x) := \frac{n p(x,y)}{n - sp(x,y)} > r(x) \geq r_- > 1,$$

for $x \in \overline{\Omega}$. Then, there exists a constant $C = C(n, s, p, q, r, \Omega)$ such that for every $f \in W$, it holds that

$$\|f\|_{L^{r(x)}(\Omega)} \leq C \|f\|_W.$$

That is, the space $W^{s,q(x),p(x,y)}(\Omega)$ is continuously embedded in $L^{r(x)}(\Omega)$ for any $r \in (1, p^*)$. Moreover, this embedding is compact.

In addition, when one considers functions $f \in W$ that are compactly supported inside $\Omega$, it holds that

$$\|f\|_{L^{r(x)}(\Omega)} \leq C [f]^{s,p(x,y)}(\Omega).$$

**Remark 1.2.** Observe that if $p$ is a continuous variable exponent in $\overline{\Omega}$ and we extend $p$ to $\overline{\Omega} \times \overline{\Omega}$ as $p(x,y) := \frac{p(x)+p(y)}{2}$, then $p^*(x)$ is the classical Sobolev exponent associated with $p(x)$, see [8].

**Remark 1.3.** When $q(x) \geq r(x)$ for every $x \in \overline{\Omega}$ the main inequality in the previous theorem, $\|f\|_{L^{r(x)}(\Omega)} \leq C \|f\|_W$, trivially holds. Hence our results are meaningful when $q(x) < r(x)$ for some points $x$ inside $\Omega$.

With the above theorem at hand one can readily deduce existence of solutions to some nonlocal problems. Let us consider the operator $\mathcal{L}$ given by

$$\mathcal{L}u(x) := \text{p.v.} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)} - 2(u(x) - u(y))}{|x-y|^{n+sp(x,y)}} \, dy. \quad (1.1)$$
This operator appears naturally associated with the space $W$. In the constant exponent case it is known as the fractional $p$-Laplacian, see [2, 4, 6, 7, 9–11, 13, 14, 17–19] and references therein. On the other hand, we remark that (1.1) is a fractional version of the well-known $p(x)$-Laplacian, given by $\text{div}(\nabla u|^{p(x)-2}\nabla u)$, that is associated with the variable exponent Sobolev space $W^{1, p(x)}(\Omega)$. We refer for instance to [8, 12, 15, 16].

Let $f \in L^{a(x)}(\Omega), a(x) > 1$. We look for solutions to the problem

$$
\begin{cases}
Lu(x) + |u(x)|^{q(x)-2}u(x) = f(x), & x \in \Omega, \\
u(x) = 0, & x \in \partial \Omega.
\end{cases}
$$

(1.2)

Associated with this problem we have the following functional

$$
F(u) := \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{n + sp(x,y)p(x,y)}} \, dy \, dx + \int_\Omega \frac{|u(x)|^{q(x)}}{q(x)} \, dx - \int_\Omega f(x)u(x) \, dx.
$$

(1.3)

To take into account the boundary condition in (1.2) we consider the space $W_0$ that is the closure in $W$ of compactly supported functions in $\Omega$. In order to have a well defined trace on $\partial \Omega$, for simplicity, we just restrict ourselves to $sp_{-} > 1$, since then it is easy to see that $W \subset W^{s, p_{-}}(\Omega) \subset W^{s-1/p_{-}p_{-}}(\partial \Omega)$, with $sp_{-} > 1$, see [1, 20]. Concerning problem (1.2), we shall prove the following existence and uniqueness result.

**Theorem 1.4.** Let $s \in (1/2, 1)$, and let $q(x)$ and $p(x,y)$ be continuous variable exponents as in Theorem 1.1 with $sp_{-} > 1$. Let $f \in L^{a(x)}(\Omega)$, with $1 < a_{-} \leq a(x) \leq a_{+} < +\infty$ for every $x \in \Omega$, such that

$$
\frac{n p(x, x)}{n - sp(x, x)} > \frac{a(x)}{a(x) - 1} > 1.
$$

Then, there exists a unique minimizer of (1.3) in $W_0$ that is the unique weak solution to (1.2).

The rest of the paper is organized as follows: In Section 2 we collect previous results on fractional Sobolev embeddings; in Section 3 we prove our main result, Theorem 1.1, and finally in Section 4 we deal with the elliptic problem (1.2).

## 2 Preliminary results.

In this section we collect some results that will be used along this paper.

**Theorem 2.1** (Hölder’s inequality). Let $p, q, r : \overline{\Omega} \to (1, \infty)$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. If $f \in L^{r(x)}$ and $g \in L^{q(x)}$, then $fg \in L^{p(x)}$ and

$$
\|fg\|_{L^{p(x)}} \leq c \|f\|_{L^{r(x)}} \|g\|_{L^{q(x)}}.
$$

For the constant exponent case we have a fractional Sobolev embedding theorem.

**Theorem 2.2** (Sobolev embedding, [20]). Let $s \in (0, 1)$ and $p \in [1, +\infty)$ such that $sp < n$. Then, there exists a positive constant $C = C(n, p, s)$ such that, for any measurable and compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$, we have

$$
\|f\|_{L^{p, \infty}(\mathbb{R}^n)} \leq C \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n + sp}} \right)^{1/p},
$$
where

\[ p^* = p^*(n, s) = \frac{np}{n - sp} \]

is the so-called “fractional critical exponent”.

Consequently, the space \( W^{s,p}(\mathbb{R}^n) \) is continuously embedded in \( L^q(\mathbb{R}^n) \) for any \( q \in [p, p^*] \).

Using the previous result together with an extension property, we also have an embedding theorem in a domain.

**Theorem 2.3** ([20]). Let \( s \in (0, 1) \) and \( p \in [1, +\infty) \) such that \( sp < n \). Let \( \Omega \subset \mathbb{R}^n \) be an extension domain for \( W^{s,p} \). Then there exists a positive constant \( C = C(n, p, s, \Omega) \) such that, for any \( f \in W^{s,p}(\Omega) \), we have

\[ \|f\|_{L^q(\Omega)} \leq C \|f\|_{W^{s,p}(\Omega)} \]

for any \( q \in [p, p^*] \); i.e., the space \( W^{s,p}(\Omega) \) is continuously embedded in \( L^q(\Omega) \) for any \( q \in [p, p^*] \).

If, in addition, \( \Omega \) is bounded, then the space \( W^{s,p}(\Omega) \) is continuously embedded in \( L^q(\Omega) \) for any \( q \in [1, p^*] \). Moreover, this embedding is compact for \( q \in [1, p^*] \).

### 3 Fractional Sobolev spaces with variable exponents.

**Proof of Theorem 1.1.** Being \( p, q \) and \( r \) continuous, and \( \Omega \) bounded, there exist two positive constants \( k_1 \) and \( k_2 \) such that

\[ q(x) - p(x, x) \geq k_1 > 0 \] (3.1)

and

\[ \frac{np(x, x)}{n - sp(x, x)} - r(x) \geq k_2 > 0, \] (3.2)

for every \( x \in \bar{\Omega} \).

Let \( t \in (0, s) \). Since \( p, q \) and \( r \) are continuous, using (3.1) and (3.2) we can find a constant \( \epsilon = \epsilon(p, r, q, k_2, k_1, t) \) and a finite family of disjoint Lipschitz sets \( B_i \) such that

\[ \Omega = \bigcup_{i=1}^N B_i \quad \text{and} \quad \text{diam}(B_i) < \epsilon, \]

that verify that

\[ \frac{np(z, y)}{n - tp(z, y)} - r(x) \geq \frac{k_2}{2}, \] (3.3)

\[ q(x) \geq p(z, y) + \frac{k_1}{2}, \]

for every \( x \in B_i \) and \( (z, y) \in B_i \times B_i \).

Let

\[ p_i := \inf_{(z,y) \in B_i \times B_i} (p(z, y) - \delta). \]

From (3.3) and the continuity of the involved exponents we can choose \( \delta = \delta(k_2) \), with \( p_1 - 1 > \delta > 0 \), such that

\[ \frac{np_i}{n - tp_i} \geq \frac{k_2}{3} + r(x) \] (3.4)

for each \( x \in B_i \).

It holds that
If we let \( p_i^* = \frac{np_i}{n-tp_i} \), then \( p_i^* \geq \frac{k_2}{r(x)} + r(x) \) for every \( x \in B_i \).

(2) \( q(x) \geq p_i + \frac{k_1}{2} \) for every \( x \in B_i \).

Hence we can apply Theorem 2.3 for constant exponents to obtain the existence of a constant \( C = C(n, p_i, t, \epsilon, B_i) \) such that

\[
\|f\|_{L^{p_i}(B_i)} \leq C \left( \|f\|_{L^{p_i}(B_i)} + [f]_{L^{p_i}(B_i)} \right). \tag{3.5}
\]

Now we want to show that the following three statements hold.

(A) There exists a constant \( c_1 \) such that

\[
\sum_{i=0}^{N} \|f\|_{L_i^{p_i}(B_i)} \geq c_1 \|f\|_{L^{(x)}(\Omega)}.
\]

(B) There exists a constant \( c_2 \) such that

\[
c_2 \|f\|_{L^{(x)}(\Omega)} \geq \sum_{i=0}^{N} \|f\|_{L_i^{p_i}(B_i)}.
\]

(C) There exists a constant \( c_3 \) such that

\[
c_3 [f]_{L^{s,p(x,y)}(\Omega)} \geq \sum_{i=0}^{N} [f]_{L_i^{p_i}(B_i)}.
\]

These three inequalities and (3.5) imply that

\[
\|f\|_{L^{(x)}(\Omega)} \leq C \sum_{i=0}^{N} \|f\|_{L_i^{p_i}(B_i)} \leq C \sum_{i=0}^{N} \left( \|f\|_{L_i^{p_i}(B_i)} + [f]_{L_i^{p_i}(B_i)} \right) \leq C \left( \|f\|_{L^{(x)}(\Omega)} + [f]_{L^{s,p(x,y)}(\Omega)} \right) = C \|f\|_{W},
\]

as we wanted to show.

Let us start with (A). We have

\[
|f(x)| = \sum_{i=0}^{N} |f(x)| \chi_{B_i}.
\]

Hence

\[
\|f\|_{L^{(x)}(\Omega)} \leq \sum_{i=0}^{N} \|f\|_{L_i^{p_i}(B_i)} r\tag{3.6}
\]

and by item (1), for each \( i, p_i^* > r(x) \) if \( x \in B_i \). Then we take \( a_i(x) \) such that

\[
\frac{1}{r(x)} = \frac{1}{p_i^*} + \frac{1}{a(x)}.
\]
Using Theorem 2.1 we obtain
\[
\|f\|_{L^p(B_i)} \leq c \|f\|_{L^p_{\tilde{v}}(B_i)} \|1\|_{L^p(B_i)} = C \|f\|_{L^p_{\tilde{v}}(B_i)}.
\]

Thus, recalling (3.6) we get (A).

To show (B) we argue in a similar way using that \( q(x) > p_i \) for \( x \in B_i \).

In order to prove (C) let us set
\[
F(x, y) := \frac{|f(x) - f(y)|}{|x - y|^s},
\]
and observe that
\[
[f]^{L^p(B_i)} = \left( \int_{B_i} \int_{B_i} \frac{|f(x) - f(y)|^{p_i}}{|x - y|^{n+p_i+sp_i-p_i}} \, dx \, dy \right)^{\frac{1}{p_i}}
= \left( \int_{B_i} \int_{B_i} \left( \frac{|f(x) - f(y)|}{|x - y|^s} \right)^{p_i} \frac{dx \, dy}{|x - y|^{n+(1-s)p_i}} \right)^{\frac{1}{p_i}}
= \|F\|_{L^{p_i}(\mu_{B_i \times B_i})},
\]
(3.7)
where we have used Theorem 2.1 with
\[
\frac{1}{p_i} = \frac{1}{p(x, y)} + \frac{1}{b_i(x, y)},
\]
but considering the measure in \( B_i \times B_i \) given by
\[
d\mu(x, y) = \frac{dxdy}{|x - y|^s}.
\]

Now our aim is to show that
\[
\|F\|_{L^{p(x, y)}(\mu_{B_i \times B_i})} \leq C [f]^{L^{s+p(x, y)}(B_i)}
\]
(3.8)
for every \( i \). If this is true, then we immediately derive (C) from (3.7).

Let \( \lambda > 0 \) be such that
\[
\int_{B_i} \int_{B_i} \frac{|f(x) - f(y)|^{p(x, y)}}{\lambda^{p(x, y)} |x - y|^{n+sp(x, y)}} \, dx \, dy < 1.
\]

Choose
\[
k := \sup \left\{ 1, \sup_{(x, y) \in \Omega} |x - y|^{s-t} \right\} \quad \text{and} \quad \lambda := \lambda k.
\]

Then
\[
\int_{B_i} \int_{B_i} \left( \frac{|f(x) - f(y)|}{\lambda |x - y|^s} \right)^{p(x, y)} \, dx \, dy
= \int_{B_i} \int_{B_i} \frac{|x - y|^{(s-t)p_i}}{k^{p(x, y)}} \frac{|f(x) - f(y)|^{p(x, y)}}{\lambda^{p(x, y)} |x - y|^{n+sp(x, y)}} \, dx \, dy
\leq \int_{B_i} \int_{B_i} \frac{|f(x) - f(y)|^{p(x, y)}}{\lambda^{p(x, y)} |x - y|^{n+sp(x, y)}} \, dx \, dy < 1.
\]
Therefore
\[ \|F\|_{L^{p(x,y)}(\mu, B_i \times B_i)} \leq \lambda k, \]
which implies the inequality (3.8).

On the other hand, when we consider functions that are compactly supported inside \( \Omega \) we can get rid of the term \( \|f\|_{L^{p(x)}(\Omega)} \) and it holds that
\[ \|f\|_{L^{p(x)}(\Omega)} \leq C|f|_{sp(x,y)}(\Omega). \]

Finally, we recall that the previous embedding is compact since in the constant exponent case we have that for subcritical exponents the embedding is compact. Hence, for a bounded sequence in \( W \), \( f_i \), we can mimic the previous proof obtaining that for each \( B_i \) we can extract a convergent subsequence in \( L^{r(x)}(B_i) \).

**Remark 3.1.** Our result is sharp in the following sense: if
\[ p^*(x_0) := \frac{np(x_0,x_0)}{n-sp(x_0,x_0)} < r(x_0) \]
for some \( x_0 \in \Omega \), then the embedding of \( W \) in \( L^{r(x)}(\Omega) \) cannot hold for every \( q(x) \). In fact, from our continuity conditions on \( p \) and \( r \) there is a small ball \( B_\delta(x_0) \) such that
\[ \max_{\Omega_0(x_0) \times B_\delta(x_0)} \frac{np(x,y)}{n-sp(x,y)} < \min_{B_\delta(x_0)} r(x). \]

Now, fix \( q \leq \min_{B_\delta(x_0)} r(x) \) (note that for \( q \geq r(x) \) we trivially have that \( W \) is embedded in \( L^{r(x)}(\Omega) \)). In this situation, with the same arguments that hold for the constant exponent case, one can find a sequence \( f_k \) supported inside \( B_\delta(x_0) \) such that \( \|f_k\|_W \leq C \) and \( \|f_k\|_{L^{q(x)}B_\delta(x_0)} \to +\infty \). In fact, just consider a smooth, compactly supported function \( g \) and take \( f_k(x) = k^ag(kx) \) with \( a \) such that \( ap(x,y) - n + sp(x,y) \leq 0 \) and \( ar(x) - n > 0 \) for \( x, y \in B_\delta(x_0) \).

Finally, we mention that the critical case
\[ p^*(x) := \frac{np(x,x)}{n-sp(x,x)} \geq r(x) \]
with equality for some \( x_0 \in \Omega \) is left open.

## 4 Equations with the fractional \( p(x) \)-Laplacian.

In this section we apply our previous results to solve the following problem. Let us consider the operator \( \mathcal{L} \) given by
\[ \mathcal{L}u(x) := \text{p.v.} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)} - 2(\|u(x) - u(y)\|^{p(x,y)} - 2)}{|x - y|^{n+sp(x,y)}} \, dy. \]

Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^n \) and \( f \in L^{a(x)}(\Omega) \) with \( a_+ > a(x) > a_- > 1 \) for each \( x \in \Omega \). We look for solutions to the problem
\[
\begin{cases}
\mathcal{L}u(x) + |u(x)|^{q(x)-2}u(x) = f(x), & x \in \Omega, \\
u(x) = 0, & x \in \partial\Omega.
\end{cases}
\tag{4.1}
\]
To this end we consider the following functional

\[ \mathcal{F}(u) := \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{n+sp(x,y)p(x,y)}} \, dx \, dy + \int_{\Omega} \frac{|u(x)|^{q(x)}}{q(x)} \, dx - \int_{\Omega} f(x)u(x) \, dx. \numberthis \label{eq:2}
\]

Let us first state the definition of a weak solution to our problem \eqref{eq:1}. Note that here we are using that \( p \) is symmetric, that is, we have \( p(x, y) = p(y, x) \).

**Definition 4.1.** We call \( u \) a weak solution to \eqref{eq:1} if \( u \in W^{s,q(x),p(x,y)}_0(\Omega) \) and

\[
\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)} - 2(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+sp(x,y)}} \, dxdy + \int_{\Omega} |u|^{q(x)} - 2u(x)v(x) \, dx = \int_{\Omega} f(x)v(x) \, dx,
\]

for every \( v \in W^{s,q(x),p(x,y)}_0(\Omega) \).

Now our aim is to show that \( \mathcal{F} \) has a unique minimizer in \( W^{s,q(x),p(x,y)}_0(\Omega) \). This minimizer shall provide the unique weak solution to the problem \eqref{eq:1}.

**Proof of Theorem 1.4.** We just observe that we can apply the direct method of Calculus of Variations. Note that the functional \( \mathcal{F} \) given in \eqref{eq:2} is bounded below and strictly convex (this holds since for any \( x \) and \( y \) the function \( t \mapsto t^{p(x,y)} \) is strictly convex).

From our previous results, \( W^{s,q(x),p(x,y)}_0(\Omega) \) is compactly embedded in \( L^{r(x)}(\Omega) \) for \( r(x) < p^*(x) \), see Theorem 1.1. In particular, we have that \( W^{s,q(x),p(x,y)}_0(\Omega) \) is compactly embedded in \( L^{\frac{n}{n-mp(x,y)}}(\Omega) \).

Let us see that \( \mathcal{F} \) is coercive. We have

\[
\mathcal{F}(u) = \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{n+sp(x,y)p(x,y)}} \, dxdy + \int_{\Omega} \frac{|u(x)|^{q(x)}}{q(x)} \, dx - \int_{\Omega} f(x)u(x) \, dx \\
\geq \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{n+sp(x,y)p(x,y)}} \, dxdy + \int_{\Omega} \frac{|u(x)|^{q(x)}}{q(x)} \, dx - \|f\|_{L^{r(x)}(\Omega)}\|u\|_{L^{\frac{n}{n-mp(x,y)}}(\Omega)} \\
\geq \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{n+sp(x,y)p(x,y)}} \, dxdy + \int_{\Omega} \frac{|u(x)|^{q(x)}}{q(x)} \, dx - C\|u\|_W.
\]

Now, let us assume that \( \|u\|_W > 1 \). Then we have

\[
\frac{\mathcal{F}(u)}{\|u\|_W} \geq \frac{1}{\|u\|_W} \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{n+sp(x,y)p(x,y)}} \, dxdy + \int_{\Omega} \frac{|u(x)|^{q(x)}}{q(x)} \, dx \right) - C \\
\geq \|u\|_W^{\min\{p,q\} - 1} - C.
\]

We next choose a sequence \( u_j \) such that \( \|u_j\|_W \to \infty \) as \( j \to \infty \). Then we have

\[
\mathcal{F}(u_j) \geq \|u_j\|_W^{\min\{p,q\} - 1} - C\|u_j\|_W \to \infty,
\]

and we conclude that \( \mathcal{F} \) is coercive. Therefore, there is a unique minimizer of \( \mathcal{F} \).
Finally, let us check that when $u$ is a minimizer to (4.2) then it is a weak solution to (4.1). Given $v \in W^{s,q(x)}_0(\Omega)$ we compute

$$0 = \frac{d}{dt} \mathcal{F}(u + tv) \bigg|_{t=0} = \int_{\Omega} \int_{\Omega} \frac{d}{dt} \left| u(x) - u(y) + t(v(x) - v(y)) \right|^{p(x,y)}(x) dxdy \bigg|_{t=0}$$

$$+ \int_{\Omega} \frac{d}{dt} \left| u(x) + tv(x) \right|^{q(x)}(x) dx \bigg|_{t=0} - \int_{\Omega} \frac{d}{dt} f(x)(u(x) + tv(x)) dx \bigg|_{t=0}$$

$$= \int_{\Omega} \int_{\Omega} \frac{u(x) - u(y)}{p(x,y)} |x-y|^{q(x)}(x) dxdy$$

$$+ \int_{\Omega} |u(x)|^{q(x)}(x) dx - \int_{\Omega} f(x)v(x),$$

as $u$ is a minimizer of (4.2). Thus, we deduce that $u$ is a weak solution to the problem (4.1).

The proof of the converse (that every weak solution is a minimizer of $\mathcal{F}$) is standard and we leave the details to the reader. \hfill \Box

References


