



Radial solutions to semilinear elliptic equations via linearized operators

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Received 27 March 2017, appeared 27 April 2017

Communicated by Patrizia Pucci

Abstract. Let u be a classical solution of semilinear elliptic equations in a ball or an annulus in \mathbb{R}^N with zero Dirichlet boundary condition where the nonlinearity has a convex first derivative. In this note, we prove that if the N -th eigenvalue of the linearized operator at u is positive, then u must be radially symmetric.

Keywords: semilinear elliptic equation, nonconvex domain, radial solution, symmetry.

2010 Mathematics Subject Classification: 35B06, 35J05.

1 Introduction

Let $N \geq 2$ and Ω be a ball or an annulus centered at zero in \mathbb{R}^N . We study symmetry properties of classical solutions to the following semilinear elliptic equation

$$\begin{cases} -\Delta u = f(|x|, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function of class C^1 with respect to the second variable.

A classical tool to study this problem is the well-known moving plane method which was introduced by Alexandrov and Serrin in [11] and was successfully used by Gidas, Ni and Nirenberg in [5] to prove the radial symmetry of positive solutions to (1.1) when Ω is a ball and f is nonincreasing in the radial variable. However, if u changes sign or Ω is an annulus or f does not have the right monotonicity, then the moving plane method cannot be applied. Indeed, there are counterexamples to the symmetry of solutions if one of these hypotheses fail. For instance, see [4] for the existence of a nonradial solution in an annulus. More recently, it is proved in [6] the bifurcation of nonradial positive solutions from the radial positive solution of equation $-\Delta u = u^p + \lambda u$ in an annulus when the radii of the annulus vary or when the exponent p varies.

Nevertheless, it is natural to expect that the solutions inherit part of the symmetry of the domain at least for some types of nonlinearities or for certain types of solutions, even if u

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changes sign or Ω is an annulus or f does not have the right monotonicity. This topic was first investigated in [9] where Pacella proved that if Ω is a ball or an annulus, f is strictly convex in u , then any solution u to (1.1) with Morse index one is axially symmetric with respect to an axis passing through the origin and nonincreasing in the polar angle from this axis. The conclusion was then expanded to solutions having Morse index less than or equal to N in [10] when Ω is a ball or an annulus and in [7] when Ω is the whole \mathbb{R}^N or the exterior of a ball. Some related examples and counterexamples are given in [1]. Similar results on axial symmetry for minimizers of certain variational problems were obtained in [3] using a completely different approach based on symmetrization techniques.

Instead of axial symmetry, in this paper we are interested in classification of radial solutions of (1.1) in a ball or an annulus, that is, solutions that fully inherits the symmetry of domain Ω . One of the first attempts in this topic is paper [8]. A typical result in [8] is that if Ω is a ball or an annulus, f is convex in its second variable and the second eigenvalue of the linearized operator of (1.1) at u is positive then u must be radially symmetric, regardless of its sign. However, the results of [8] do not apply to sign changing solutions of (1.1) in the case of Lane–Emden–Fowler nonlinearity $f(s) = |s|^{p-1}s$, $p > 1$. Indeed, this nonlinearity f , when considered on the whole real line, is not convex. Utilizing some techniques developed in [10], in this paper we prove general radial symmetry results for solutions to (1.1) in the case where f has its first derivative, with respect to the second variable, convex in the second variable. Our results partially improve results in [8, 10] and can apply to sign changing solutions of (1.1) with a large class of nonlinearities such as $f(|x|, s) = g(|x|)|s|^{p-1}s$, $p \geq 2$ and $f(|x|, s) = g(|x|)e^s$ where g is a continuous function.

2 Preliminaries and main results

In the sequel, we always assume that Ω is a radially symmetric open bounded domain, such as a ball or an annulus centered at zero in \mathbb{R}^N . Let us denote by $\langle v, w \rangle$ the scalar product of v, w in $L^2(\Omega)$, that is $\langle v, w \rangle = \int_{\Omega} v(x)w(x) dx$. For a bounded domain $U \subset \mathbb{R}^N$ and a linear operator $L : H_0^1(U) \rightarrow L^2(U)$, we denote by $\lambda_k(L, U)$ the k -th eigenvalue of L in U with zero Dirichlet boundary conditions.

Let u be a classical solution of (1.1), we recall the linearized operator L_u of (1.1) at u defined by duality as

$$\langle L_u v, w \rangle = \int_{\Omega} \nabla v(x) \nabla w(x) dx - \int_{\Omega} f'_s(|x|, u(x)) v(x) w(x) dx,$$

for any $v, w \in H_0^1(\Omega)$, here we denote f'_s the derivative of f in its second variable. It is well-known that

$$\lambda_1(L_u, \Omega) < \lambda_2(L_u, \Omega) \leq \lambda_3(L_u, \Omega) \leq \dots \leq \lambda_k(L_u, \Omega) \rightarrow \infty.$$

We recall that the Morse index of u is the number of negative eigenvalues of L_u .

We denote the open ball in \mathbb{R}^N of center x and radius $r > 0$ by $B(x, r)$ and the unit sphere in \mathbb{R}^N by S . For a unit vector $e \in S$ we consider the hyperplane $H(e) = \{x \in \mathbb{R}^N : x \cdot e = 0\}$ and write $\sigma_e : \Omega \rightarrow \Omega$ for the reflection with respect to $H(e)$, that is, $\sigma_e(x) = x - 2(x \cdot e)e$ for every $x \in \Omega$. We also denote $\Omega(e) = \{x \in \Omega : x \cdot e > 0\}$.

Our main result is the following theorem.

Theorem 2.1. *Suppose that $f(|x|, \cdot)$ has a convex derivative for every $x \in \Omega$. Then any solution u of (1.1) having $\lambda_N(L_u, \Omega) > 0$ is radially symmetric.*

Remark 2.2. It is proved in [10, Theorem 1.1] that if $f(|x|, \cdot)$ has a convex derivative for every $x \in \Omega$ and u has Morse index less than or equal to N (that is, $\lambda_{N+1}(L_u, \Omega) \geq 0$) then u is axially symmetric with respect to an axis passing through the origin and nonincreasing in the polar angle from this axis. Therefore, Theorem 2.1 gives us a stronger conclusion on the symmetry of u in the case $\lambda_N(L_u, \Omega) > 0$. In other words, with the same assumption on f , if u is a nonradial solution having Morse index less than or equal to N then we can conclude that $\lambda_N(L_u, \Omega) \leq 0 \leq \lambda_{N+1}(L_u, \Omega)$.

Remark 2.3. The assumption $\lambda_N(L_u, \Omega) > 0$ of Theorem 2.1 is strict at least in 2-dimensional case. Indeed, let $N = 2$ and $f(|x|, u) = |u|^{p-1}u + \lambda u$ where $p \geq 2$ and $\lambda < \lambda_1$, here λ_1 denotes the first eigenvalue of the Laplace operator in Ω with zero Dirichlet boundary conditions. In this case, a positive solution u of (1.1) of index 1 can be either found using the famous mountain-pass lemma or by constrained minimization procedure. When Ω is an annulus it can be proved that this positive solution is, in general, not radial (see [4, 6]). This solution is anyway axially symmetric by [10, Theorem 1.1]. Moreover, by Theorem 2.1 and the fact that this non-radial solution has Morse index 1 we obtain $\lambda_2(L_u, \Omega) = 0$. Therefore, this example demonstrates the sharpness of assumption $\lambda_2(L_u, \Omega) > 0$ of Theorem 2.1 in 2-dimensional case.

Remark 2.4. Since $\lambda_N(L_u, \Omega) \geq \lambda_2(L_u, \Omega) > \lambda_1(L_u, \Omega)$, any solution of (1.1) of Morse index zero must be radial by Theorem 2.1.

As an application of Theorem 2.1, we have the following Liouville type theorem for sign changing solutions of (1.1).

Theorem 2.5. *Suppose that $f = f(s)$ does not depend on x and f is convex. Then problem (1.1) has no sign changing solution u such that $\lambda_N(L_u, \Omega) > 0$.*

Remark 2.6. The assumptions of Theorem 2.5 are satisfied for the Lane–Emden–Fowler nonlinearity $f(s) = |s|^{p-1}s$, $p \geq 2$ and the exponential nonlinearity $f(s) = e^s$. Under these assumptions, from Theorem 2.5 it follows that every sign changing solution of (1.1) must satisfy $\lambda_N(L_u, \Omega) \leq 0$.

3 Proofs

We begin with the following elementary lemma.

Lemma 3.1. *Let a unit vector $e \in S$ and $\varepsilon > 0$. Assume that function $u : \Omega \rightarrow \mathbb{R}$ is symmetric with respect to hyperplane $H(d)$ for every $d \in S(e, \varepsilon)$ where $S(e, \varepsilon) = \{d \in S : \arccos(d \cdot e) < \varepsilon\}$. Then u is radially symmetric.*

Proof. We will prove that u is symmetric with respect to hyperplane $H(d)$ for every $d \in S(e, \min\{2\varepsilon, \pi\})$. Indeed, let $d \in S(e, \min\{2\varepsilon, \pi\})$ and put $d_e = \frac{d+e}{|d+e|}$ then $H(e) = \sigma_{d_e}(H(d))$ and

$$\arccos(d_e \cdot e) = \arccos\left(\frac{d+e}{|d+e|} \cdot e\right) = \arccos\left(\frac{\sqrt{d \cdot e + 1}}{2}\right) = \frac{\arccos(d \cdot e)}{2} < \varepsilon.$$

That is, $d_e \in S(e, \varepsilon)$. Now let any $x_0 \in \Omega$ and denote $x_1 = \sigma_d(x_0)$. Since $H(e)$, $\sigma_{d_e}(x_0)$ and $\sigma_{d_e}(x_1)$ are reflection images of $H(d)$, x_0 and x_1 respectively with respect to hyperplane $H(d_e)$,

we have $\sigma_{d_e}(x_1) = \sigma_e(\sigma_{d_e}(x_0))$, which implies $x_1 = \sigma_{d_e}(\sigma_e(\sigma_{d_e}(x_0)))$. Using the fact that u is symmetric with respect to hyperplane $H(d_e)$ and $H(e)$, we obtain

$$u(x_0) = u(\sigma_{d_e}(\sigma_e(\sigma_{d_e}(x_0)))) = u(x_1).$$

Therefore u is symmetric with respect to hyperplane $H(d)$, as desired.

Repeating the previous argument n times, we conclude that u is symmetric with respect to hyperplane $H(d)$ for every $d \in S(e, \min\{2^n \varepsilon, \pi\})$. By choosing n such that $2^n \varepsilon \geq \pi$, we get the axial symmetry of u . \square

We continue with the following lemma.

Lemma 3.2. *Suppose that $f(|x|, \cdot)$ has a convex derivative for every $x \in \Omega$. Then for any solution u of (1.1) having $\lambda_N(L_u, \Omega) > 0$, we may find a unit vector $e \in S$ such that $\lambda_1(L_u^e, \Omega(e)) > 0$, where the linear operator L_u^e is defined as*

$$\langle L_u^e v, w \rangle = \int_{\Omega(e)} \nabla v(x) \nabla w(x) dx - \int_{\Omega(e)} \frac{f'_s(|x|, u(x)) + f'_s(|x|, u(\sigma_e(x)))}{2} v(x) w(x) dx$$

for any $v, w \in H_0^1(\Omega(e))$.

Proof. For any $e \in S$, we denote by $g_e \in H_0^1(\Omega)$ the odd extension in Ω of the positive L^2 -normalized eigenfunction of the operator L_u^e in the half domain $\Omega(e)$ corresponding to the first eigenvalue $\lambda_1(L_u^e, \Omega(e))$. It is clear that g_e depends continuously on e in the L^2 -norm and $g_{-e} = -g_e$ for every $e \in S$. Now we let $\varphi_1, \varphi_2, \dots, \varphi_{N-1} \in H_0^1(\Omega)$ denote L^2 -orthonormal eigenfunctions of L_u corresponding to its eigenvalue $\lambda_1, \lambda_2, \dots, \lambda_{N-1}$. It is well-known that

$$\inf_{\substack{v \in H_0^1(\Omega) \setminus \{0\} \\ \langle v, \varphi_1 \rangle = \dots = \langle v, \varphi_{N-1} \rangle = 0}} \frac{\langle L_u v, v \rangle}{\langle v, v \rangle} = \lambda_N > 0. \quad (3.1)$$

We consider the map $h : S \rightarrow \mathbb{R}^{N-1}$ defined as

$$h(e) = (\langle g_e, \varphi_1 \rangle, \langle g_e, \varphi_2 \rangle, \dots, \langle g_e, \varphi_{N-1} \rangle).$$

Since h is an odd and continuous map defined on the unit sphere $S \subset \mathbb{R}^N$, h must have a zero by the Borsuk–Ulam theorem. This means that there is a direction $e \in S$ such that g_e is L^2 -orthogonal to all eigenfunctions $\varphi_1, \varphi_2, \dots, \varphi_{N-1}$. Therefore $\langle L_u g_e, g_e \rangle > 0$ by (3.1). But since g_e is an odd function,

$$\langle L_u g_e, g_e \rangle = 2 \langle L_u^e g_e, g_e \rangle = 2 \lambda_1(L_u^e, \Omega(e)).$$

which yields that $\lambda_1(L_u^e, \Omega(e)) > 0$. \square

We are now in position to prove our main results.

Proof of Theorem 2.1. Applying Lemma 3.2, we obtain a unit vector $e \in S$ such that $\lambda_1(L_u^e, \Omega(e)) > 0$. By continuity of the first eigenvalue with respect to the potential and the domain (see [2]), we may find $\varepsilon > 0$ such that $\lambda_1(L_u^d, \Omega(d)) > 0$ for all $d \in S(e, \varepsilon)$ where $S(e, \varepsilon)$ is defined as in Lemma 3.1.

We will show that u is symmetric with respect to $H(d)$ for all $d \in S(e, \varepsilon)$ and therefore the radial symmetry of u follows from Lemma 3.1. Indeed, since u and $u \circ \sigma_d$ solve (1.1), we put $w_d(x) = u(x) - u(\sigma_d(x))$ and get

$$\begin{cases} -\Delta w_d - V_d(x)w_d = 0 & \text{in } \Omega(d), \\ w_d = 0 & \text{on } \partial\Omega(d), \end{cases} \quad (3.2)$$

where $V_d(x) = \int_0^1 f'_s(|x|, tu(x) + (1-t)u(\sigma_d(x))) dt$. Using the convexity of f , we have

$$\begin{aligned} V_d(x) &\leq \int_0^1 t f'_s(|x|, u(x)) + (1-t) f'_s(|x|, u(\sigma_d(x))) dt \\ &= \frac{f'_s(|x|, u(x)) + f'_s(|x|, u(\sigma_d(x)))}{2} \end{aligned}$$

for all $x \in \Omega$. Hence, denoting by M_u^d the linearized operator of (3.2)

$$\langle M_u^d v, w \rangle = \int_{\Omega(d)} \nabla v(x) \nabla w(x) dx - \int_{\Omega(d)} V_d(x) v(x) w(x) dx$$

for any $v, w \in H_0^1(\Omega(d))$, we have $\lambda_1(M_u^d, \Omega(d)) \geq \lambda_1(L_u^d, \Omega(d)) > 0$. It follows that $w_d = 0$ because it satisfies (3.2). In other words, u is symmetric with respect to $H(d)$, as desired. \square

Proof of Theorem 2.5. If the sign changing solution u satisfying $\lambda_N(L_u, \Omega) > 0$ exists, then u is radially symmetric by Theorem 2.1. Moreover, this assumption also implies that u has Morse index less than N . Then by [10, Theorem 1.2], u must be nonradial, a contradiction. \square

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