Optimal decay estimates for solutions to damped second order ODE’s

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Abstract. In this paper we derive optimal decay estimates for solutions to second order ordinary differential equations with weak damping. The main assumptions are Kurdyka–Łojasiewicz gradient inequality and its inverse.

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1 Introduction

In this paper we study long-time behavior for solutions of damped second order ordinary differential equations

\[ \ddot{u} + g(\dot{u}) + \nabla E(u) = 0, \] (SOP)

where \( E \in C^2(\Omega) \), \( \Omega \) being an open connected subset of \( \mathbb{R}^n \) and \( g : \mathbb{R}^n \to \mathbb{R}^n \) is a \( C^1 \)-function satisfying \( \langle g(v), v \rangle \geq 0 \) on \( \mathbb{R}^n \). This last condition means that the term \( g(\dot{u}) \) in (SOP) has a damping effect. It is easy to see that energy

\[ \mathcal{E}(u, \dot{u}) = \frac{1}{2} \| \dot{u} \|^2 + E(u) \]

is nonincreasing along solutions. In fact, if \( u \) is a classical solution to (SOP), then

\[ \frac{d}{dt} \mathcal{E}(u(t), \dot{u}(t)) = -\langle g(\dot{v}), v \rangle \leq 0. \]

If \( u : [0, +\infty) \to \Omega \) is a global solution and \( \varphi \) belongs to the \( \omega \)-limit set of \( u \), then \( \mathcal{E}(u(t), \dot{u}(t)) \to \mathcal{E}(\varphi, 0) = E(\varphi) \) as \( t \to +\infty \). In this paper, we derive the exact rate of convergence of \( \mathcal{E}(u(t), \dot{u}(t)) \) to \( E(\varphi) \).

Our main assumption is the Kurdyka–Łojasiewicz gradient inequality (see [10])

\[ \Theta(|E(u) - E(\varphi)|) \leq \| \nabla E(u) \|. \] (KLI)

\[ \Theta(\mathcal{E}(u(t), \dot{u}(t)) - \mathcal{E}(\varphi, 0)) \leq \mathcal{R}(t) \]

where \( \Theta \) is an increasing function satisfying \( \Theta(0) = 0 \) and \( \Theta(x) \to +\infty \) as \( x \to +\infty \).
For linear $g$, the optimal decay estimate was derived in [2]. For nonlinear $g$ (typically satisfying $g'(0) = 0$) some decay estimates were shown in [3,7,8]. Here we derive better decay estimates under additional assumptions on $E$ and we show that these estimates are optimal. We will assume that $E$ satisfies an inverse to (KLI) and some estimates on the second gradient and that $g$ has certain behavior near zero. The present result generalizes the one from [5, Theorem 20] where we worked with the Łojasiewicz gradient inequality, i.e. (KLI) with $\Theta(s) = s^{1-\theta}$ for a constant $\theta \in (0, \frac{1}{2}]$ (see [11]). It also generalizes the result by Haraux (see [9]) and Abdelli, Anguiano, Haraux (see [1]). The present result applies e.g. to functions $E$ and $g$ having the growth near origin as

$$s^a \ln^r (1/s) \ln^r (\ln(1/s)) \ldots \ln^r (\ln \ldots (\ln(1/s)))$$

for some constants $a, r_1, \ldots, r_k$. It also applies to functions $E$ with a non-strict local minimum in $\varphi$.

The paper is organized as follows. In Section 2 we present our notations, basic definitions and the main result. Section 3 contains the proof of the main result.

2 Notations and the main result

By $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ we denote the usual norm and scalar product on $\mathbb{R}^d$. For nonnegative functions $f, g : G \subset \mathbb{R}^d \to \mathbb{R}$ we write $g(x) = O(f(x))$ on $G$ if there exists $C > 0$ such that $g(x) \leq Cf(x)$ for all $x \in G$. We say that $g(x) = O(f(x))$ for $x \to a$ if $g(x) = O(f(x))$ on a neighborhood of $a$. If $f(x) = O(g(x))$ and $g(x) = O(f(x))$, we write $f \sim g$.

We say that a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f(0) = 0$ and $f(s) > 0$ for $s > 0$

- is admissible if $f$ is nondecreasing and there exists $c > 0$ such that $sf'_\pm(s) \leq cf(s)$ for all $s > 0$,
- has property (K) if for every $K > 0$ there exists $C(K) > 0$ such that $f(Ks) \leq C(K)f(s)$ holds for all $s > 0$,
- is $C$-sublinear if there exists $C > 0$ such that $f(t+s) \leq C(f(t)+f(s))$ holds for all $t, s > 0$.

It is easy to see that admissible functions are $C$-sublinear and have property (K) (for proof see Appendix of [4]). Further, for nondecreasing functions property (K) is equivalent to $C$-sublinearity. Moreover, every concave function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is admissible and satisfies $sf'_\pm(s) \leq f(s)$.

Let us introduce the inverse Kurdyka–Łojasiewicz inequality

$$\Theta_1(\|E(u) - E(\varphi)\|) \geq \|\nabla E(u)\|$$

and an inequality for the second gradient

$$\|\nabla^2 E(u)\| \leq \Gamma(\|\nabla E(u)\|).$$

When we say that inequality (KLI) (resp. (IKLI), (2.1)) holds on a set $U$ it means that the inequality holds for all $u \in U$ with a given fixed $\varphi$ and $\Theta$ (resp. $\Theta_1, \Gamma$).
By a solution to (SOP) we always mean a classical solution defined on \([0, +\infty)\). By \(R(u) = \{u(t) : t \geq 0\}\) we denote the range of \(u\). We say that a solution is precompact if \(R(u)\) is precompact in \(\Omega\) (the domain of \(E\)). The \(\omega\)-limit set of \(u\) is 
\[
\omega(u) = \{\varphi \in \Omega : \exists t_n \nearrow +\infty, u(t_n) \to \varphi\}.
\]

By \(c, C, \bar{c}, \bar{C}\) we denote generic constants, their values can change from line to line or from expression to expression.

The main result of the present paper is the following.

**Theorem 2.1.** Let \(u\) be a precompact solution to (SOP) and \(\varphi \in \omega(u)\). Let \(E(\cdot) \geq E(\varphi)\) on \(R(u)\) and let \(E\) satisfy (KLI), (IKLI) and (2.1) on \(R(u)\) with admissible functions \(\Theta, \Theta_1\) and \(\Gamma\), such that \(\Theta(s) \sim \Theta_1(s)\) and \(\Gamma(\Theta(s)) \sim \Theta(s)\Theta'(s)\) for \(s \to 0+\). Let \(g\) satisfies
\[
\langle g(v), v \rangle \geq ch(\|v\|)\|v\|^2, \quad \|g(v)\| \leq Ch(\|v\|)\|v\| 
\]
with an admissible function \(h\) satisfying
\[
\Theta(s) \geq c\sqrt{s} h(\sqrt{s}) \tag{2.3}
\]
for some \(c > 0\) and all \(s \geq 0\). Let us denote
\[
\chi(s) = s h(\sqrt{s}), \quad \Phi_\chi = \int \frac{1}{\chi(s)} ds \tag{2.4}
\]
and assume that \(\psi(s) = s^2h(s)\) is convex. Then
\[
c(-\Phi_\chi)^{-1}(Ct) \leq \mathcal{E}(u(t), \dot{u}(t)) - \mathcal{E}(\varphi, 0) \leq C(-\Phi_\chi)^{-1}(ct)
\]
for some \(c, C > 0\) and all \(t\) large enough.

Let us first mention that if \(E(u) = \|u\|^p, p \geq 2\), then (KLI), (IKLI) hold with \(\Theta(s) \sim \Theta_1(s) = Cs^{1-\theta}, \theta = \frac{p}{2p-2}\) and (2.1) holds with \(\Gamma(s) = Cs^{\frac{1-2\theta}{p-2}}\). If \(h(s) = s^\alpha, \alpha \in (0, 1)\), then condition (2.3) becomes \(\alpha \geq 1 - 2\theta\) and \((-\Phi_\chi)^{-1}(ct) = Ct^{-\frac{1}{2}}\). In this case, we obtain the same result as [5, Theorem 20] and also [9].

**Remark 2.2.**

1. If \((-\Phi_\chi)^{-1}\) has property (K), then the statement of Theorem 2.1 can be written as
\[
\mathcal{E}(u(t), \dot{u}(t)) - E(\varphi) \sim (-\Phi_\chi)^{-1}(t).
\]

2. We can see that the energy decay depends on \(h\) only. In particular, it is independent of \(\Theta\).

3. It is enough to assume that all the assumptions except \(\langle g(v), v \rangle > 0\) for all \(v \neq 0\) hold on a small neighborhood of zero, resp. a small neighborhood of \(\omega(u)\).

4. It follows from (KLI) and [2, Proposition 2.8] that \(\Theta(s) = O(\sqrt{s})\). Hence, by (2.3) function \(h\) must be bounded on a neighborhood of zero and \(\Phi_\chi(t) \to -\infty\) as \(t \to 0+\). So, it is not important which primitive function \(\Phi_\chi\) we take and we have \((-\Phi_\chi)^{-1}(t) \to 0\) as \(t \to +\infty\).

5. Theorem 2.1 does not imply that \(u(t) \to \varphi\) as \(t \to +\infty\). In fact, in [6, Theorem 4] we have shown that \(u(t) \to \varphi\) if \(h\) is large enough, in particular if \(\int_0^t \frac{1}{\Theta(s) h(\Theta(s))} < +\infty\). If this condition is not satisfied, it may happen that \(\omega(u)\) contains more than one point.
6. If \( \varphi \) is an asymptotically stable equilibrium for the gradient system \( \dot{u} + \nabla E(u) = 0 \) (e.g. if \( E \) has a strict local minimum in \( \varphi \) and is convex on a neighborhood of \( \varphi \)) and (KLI), (IKLI) hold on a neighborhood of \( \varphi \), then by [5, Corollary 5] we have \( \|x - \varphi\| \sim \Phi(\Theta(x) - E(\varphi)) \) on a neighborhood of \( \varphi \) where \( \Phi(\Theta(t) = \int_{0}^{1} \Theta. \) In this case, for any solution starting in a neighborhood of \( \varphi \) we have
\[
c(-\Phi^{-1}(Ct)) \leq \|v(t)\|^2 + \Phi^{-1}(\|u(t) - \varphi\|) \leq C(-\Phi^{-1}(ct))
\]
and, especially,
\[
\|u(t) - \varphi\| \leq \Phi(\Theta(\Phi^{-1}(ct)));
\]
so \( u(t) \to \varphi \). We do not have the estimate for \( \|u(t) - \varphi\| \) from below since, at least in one-dimensional case, the solution oscillates and \( u(t_n) = \varphi \) for a sequence \( t_n \nearrow +\infty \) (see [9]).

Example 2.3. Let us consider \( E(u) = F(\|u\|) \) with a real function \( F \) having a strict local minimum \( F(0) = 0 \) and satisfying on a right neighborhood of zero \( CF(s) \geq sF'(s) \geq (1 + \varepsilon)F(s) \) and \( sF''(s) \sim F'(s) \). Moreover, we assume that \( (F')^{-1} \) has property (K). (It is easy to show that any analytic function \( F(s) = \sum_{n=2}^{\infty} a_n s^n \) where \( a_{2n} > 0 \) and any function of the form \( (1.1) \) with \( \alpha > 2, \lambda \in \mathbb{R} \) or \( \alpha = 2, \lambda = \cdots = r_{(-1)} = 0, r_j < 0, r_{j+1}, \ldots, r_k \in \mathbb{R} \) satisfy these assumptions.) Then (KLI), (IKLI) hold with \( \Theta(s) = \frac{\lambda}{\lambda - s} \), since
\[
\Theta(E(u)) = \Theta(F(\|u\|)) = C \frac{F(\|u\|)}{\|u\|} \sim F(\|u\|) = \|\nabla E(u)\|.
\]
Further, (2.1) holds with \( \Gamma(s) = \frac{s}{F} \frac{\Theta}{(F')}^{-1}(s) \) since
\[
\|\nabla^2 E(u)\| \leq CF''(\|u\|) \sim \frac{F''(\|u\|)}{\|u\|} \sim \Gamma(F''(\|u\|)) = \Gamma(\|\nabla E(u)\|),
\]
where the first inequality is due to the fact that the diagonal resp. non-diagonal terms of \( \nabla^2 E(u) \) are
\[
F''(\|u\|) \frac{u_i^2}{\|u\|^2} \quad \text{resp.} \quad \frac{u_i u_j}{\|u\|^2} \left( F''(\|u\|) - \frac{F'(\|u\|)}{\|u\|} \right),
\]
so they are estimated by \( CF''(\|u\|) \). Further, we have
\[
\Theta'(F(s)) = \frac{d}{ds} \Theta(F(s)) = \frac{d}{ds} \frac{F(s)}{F'} = \frac{F'(s) s - F(s)}{s^2 F'(s)} = \frac{1}{s} \left( 1 - \frac{F(s)}{s F'(s)} \right) \sim \frac{1}{s},
\]
so
\[
\Theta(F(s)) \Theta'(F(s)) \sim \frac{1}{s} \Theta(F(s)) \sim \frac{1}{s^2} F(s)
\]
and
\[
\Gamma(\Theta(F(s))) \sim \frac{\Theta(F(s))}{(F')^{-1}(\Theta(F(s)))} \sim \frac{F(s)}{s(F')^{-1}(\frac{F(s)}{s})} \sim \frac{F(s)}{s(F')^{-1}(F'(s))} = \frac{F(s)}{s^2},
\]
hence \( \Gamma(\Theta(s)) \sim \Theta(s) \Theta'(s) \). Then, for any \( g \) satisfying (2.2) with a function \( h \) small enough (such that (2.3) holds) Theorem 2.1 can be applied and we obtain the exact energy decay which depends on \( h \) only and not on \( F \). In particular, if \( h(s) = s^a \) we have \( E(u(t), v(t)) \sim t^{-\frac{a}{2}} \) and if \( h \) is of the form \( (1.1) \), we have by [4, Lemmas 6.5, 6.6]
\[
E(u(t), v(t)) \sim t^{-\frac{a}{2}} \ln^{\frac{a}{2}}(1/t) \ldots \ln^{\frac{a}{2}}(\ldots \ln 1/t).
\]
Let us mention that if \( h \) is equal to (1.1) and such that \( c s \leq h(s) \leq c \) near zero (i.e. \( a \in [0, 1] \) and if \( a \in \{0, 1\} \) we have a sign condition on the first nonzero number \( r_i \)), then \( \psi(s) = s^2 h(s) \) is convex near zero.
3 Proof of Theorem 2.1

Let us write \( v(t) \) instead of \( \dot{u}(t) \) and \( \mathcal{E}(t) \) instead of \( \mathcal{E}(u(t), v(t)) \). We also often write \( u, v \) instead of \( u(t), v(t) \).

First of all, since \( u \) is precompact \( \{E(u(t)) : t \geq 0\} \) is bounded. Therefore, \( \{\mathcal{E}(t) : t \geq 0\} \) is bounded, hence \( v \) is bounded and by (SOP) also \( \ddot{u} = \ddot{v} \) is bounded. Since

\[
\int_0^t \langle g(v), v \rangle = \mathcal{E}(0) - \mathcal{E}(t) \leq K,
\]

we have \( \langle g(v), v \rangle \in L^1((0, +\infty)) \). Then boundedness of \( \dot{v} \) yields convergence of \( \langle g(v(t)), v(t) \rangle \) to 0. Hence \( v(t) \to 0 \) as \( t \to +\infty \) and it follows that \( \mathcal{E}(t) \to \mathcal{E}(\varphi, 0) \). So, we can assume without loss of generality that \( E(\varphi) = 0, \mathcal{E}(\varphi, 0) = 0 \).

In the rest of the proof we will work with

\[
H(t) = \mathcal{E}(t) + \varepsilon B(E(u(t))) \langle \nabla E(u(t)), v \rangle,
\]

where

\[
B(s) = \begin{cases} \frac{1}{\sigma(s)} sh(\sqrt{s}) & s > 0 \\ 0 & s = 0 \end{cases}
\]

and \( \varepsilon > 0 \) is small enough. Let us mention that \( B \) can be unbounded in a neighborhood of zero, but due to (2.3) we have \( \Theta(s)B(s) \leq C\sqrt{s} \), hence \( H \) is continuous even in the points where \( E(u(t)) = 0 \) and in these points we have \( H(t) = \mathcal{E}(t) \). Let us denote \( M := \{t \geq 0 : E(u(t)) > 0\} \) and \( M^c = \{t \geq 0 : E(u(t)) = 0\} \).

We show that \( H(t) \sim \mathcal{E}(t) \). On \( M^c \) it is trivial. On \( M \) we apply (IKLI), Cauchy–Schwarz and Young inequalities and \( \Theta(s)B(s) \leq C\sqrt{s} \) and we obtain

\[
|\varepsilon B(E(u)) \langle \nabla E(u(t)), v \rangle| \leq \varepsilon CB(E(u))\Theta(E(u))\|v\|
\leq \varepsilon CB(E(u))^2\Theta(E(u))^2 + \varepsilon C\|v\|^2
\leq \varepsilon C\mathcal{E}(t),
\]

hence

\[
(1 - \varepsilon C)\mathcal{E}(t) \leq H(t) \leq (1 + \varepsilon C)\mathcal{E}(t)
\]

and taking \( \varepsilon > 0 \) small enough we obtain \( H(t) \sim \mathcal{E}(t) \).

The next step is to show that

\[
0 \leq -H'(t) \sim h(\|v\|)\|v\|^2 + E(u)h\left(\sqrt{E(u)}\right).
\]  

(3.1)

Let us first estimate \( B'(s) \). For any \( s > 0 \) we have

\[
B'(s) = \frac{B(s)}{s} \left(1 + \frac{h'\left(\sqrt{s}\right)\sqrt{s}}{h(\sqrt{s})} - \frac{2s\Theta'(s)}{\Theta(s)}\right) \in \left[\frac{B(s)}{s}(1 - 2C), \frac{B(s)}{s}(1 + C)\right],
\]

where the equality follows by definition of \( B \) and the rest from admissibility of \( h \) and \( \Theta \) (the two fractions in round bracket are nonnegative and bounded above by a constant). Hence, \( |sB'(s)| \leq CB(s) \).
Let $t \in M$. Let us compute $H'(t)$ and use the fact that $u$ solves (SOP) to get
\[
H'(t) = - \langle g(v), v \rangle - \epsilon B(E(u))\|\nabla E(u)\|^2 \\
+ \epsilon B'(E(u))\langle \nabla E(u), v \rangle^2 \\
+ \epsilon B(E(u))\langle \nabla^2 E(u)v, v \rangle \\
+ \epsilon B(E(u))(\langle \nabla E(u), -g(v) \rangle).
\]

Due to (2.2) we have $\langle g(v), v \rangle \sim h(\|v\|)\|v\|^2$ and by definition of $B$, (KLI) and (IKLI) we immediately have $B(E(u))\|\nabla E(u)\|^2 \sim E(u)h(\sqrt{E(u)})$. So,
\[
\langle g(v), v \rangle + \epsilon B(E(u))\|\nabla E(u)\|^2 \sim h(\|v\|)\|v\|^2 + \epsilon CE(u)h\left(\sqrt{E(u)}\right).
\]

We show that the second, third and fourth lines of (3.2) are smaller than this term, then (3.1) is proved.

The second line of (3.2) is less than
\[
\epsilon C \frac{B(E(u))}{E(u)} \Theta(E(u))^2\|v\|^2 \leq \epsilon Ch\left(\sqrt{E(u)}\right)\|v\|^2.
\]

Since $\Gamma$ has property (K) and satisfies $\Gamma(\Theta(s)) \sim \Theta(s)\Theta'(s) \leq C s^{-1} \Theta(s)^2$ and due to (IKLI) and definition of $B$, the third line in (3.2) is less than
\[
\epsilon CB(E(u))\Gamma(\|\nabla E(u)\|)\|v\|^2 \leq \epsilon Ch\left(\sqrt{E(u)}\right)\|v\|^2.
\]

If $E(u) \leq 4C\|v\|^2$, then ($h$ satisfies property (K)) we have $h(\sqrt{E(u)})\|v\|^2 \leq \tilde{C} h(\|v\|)\|v\|^2$ and if $E(u) \geq 4C\|v\|^2$, then $h(\sqrt{E(u)})\|v\|^2 \leq \frac{1}{4C} h(\sqrt{E(u)} E(u))$. So, in either case we have that lines two and three in (3.2) are less than
\[
\epsilon Ch(\|v\|)\|v\|^2 + \frac{1}{4} sh\left(\sqrt{E(u)}\right) E(u),
\]

so they are less than the first line in (3.2) since we can make $\epsilon C$ small by taking $\epsilon$ small enough.

The last line in (3.2) is (by definition of $B$ and (2.3)) less than
\[
\epsilon CB(E(u))\|\nabla E(h(\|v\|))\|v\| \leq \epsilon C \frac{1}{\Theta(E(u))} E(u)h\left(\sqrt{E(u)}\right) h(\|v\|)\|v\| \\
\leq \epsilon C \sqrt{E(u)} h(\|v\|)\|v\|.
\]

Applying the Young inequality $ab \leq \psi(a) + \hat{\psi}(b)$ with $\psi(s) = s^2 h(s)$ and the convex conjugate $\hat{\psi}$ we get
\[
\epsilon C \sqrt{E(u)} h(\|v\|)\|v\| \leq \frac{1}{4} \epsilon \psi\left(\sqrt{E(u)}\right) + \epsilon C \hat{\psi}(\|v\| h(\|v\|)) \\
\leq \frac{1}{4} \epsilon E(u) h\left(\sqrt{E(u)}\right) + \epsilon Ch(\|v\|)\|v\|^2
\]

since $\hat{\psi}(sh(s)) \leq C s^2 h(s)$ due to Lemma 3.1 below. Now, (3.1) is proven on $M$. If $E(u(t)) \to 0$ for $t \to t_0$, we can see that $H'(t) \to -\langle g(v(t_0)), v(t_0) \rangle = E'(t_0)$ (due to the estimates above, all
terms on the right-hand side of (3.2) except the first one tend to zero). By continuity of $H$, we have $H' = E'$ on $M'$, in particular (3.1) holds on $M'$.

We show that $\chi(H(t)) \sim -H'(t)$. In fact,

$$\chi(H(t)) \leq \chi(C(||v||^2 + E(u)))$$

$$\leq C(\chi(||v||^2) + \chi(E(u)))$$

$$= C\left(h(||v||)||v||^2 + E(u)\right)$$

$$\leq -CH'(t),$$

where we applied monotonicity in the first line, C-sublinearity and property (K) in the second line ($\chi$ has these properties by Lemma 3.2 below), definition of $\chi$ in the third line and (3.1) in the last inequality. On the other hand, by Lemma 3.2 also the inverse inequalities in C-sublinearity and property (K) are valid, so we have

$$\chi(H(t)) \geq \chi(c(||v||^2 + E(u)))$$

$$\geq c(\chi(||v||^2) + \chi(E(u)))$$

$$= c\left(h(||v||)||v||^2 + E(u)\right)$$

$$\geq -cH'(t),$$

so $\chi(H(t)) \sim -H'(t)$ is proved.

Let $T = \sup\{ t \geq 0 : H(t) > 0 \}$. For any $t \in (0, T)$ we have proved

$$-\frac{d}{dt}\Phi_\chi(H(t)) = -\frac{H'(t)}{\chi(H(t))} \in [c, C].$$

Integrating this relation from $t_0$ to $t$ we obtain

$$c(t - t_0) - \Phi_\chi(H(t_0)) \leq -\Phi_\chi(H(t)) \leq C(t - t_0) - \Phi_\chi(H(t_0)). \quad (3.3)$$

If $T < +\infty$, then we can see that $-\Phi_\chi(H(t))$ is bounded on $(0, T)$, hence $0 < \lim_{t \to T^-} H(t) = H(T)$, contradiction. Therefore, $T = +\infty$, (3.3) holds for all $t > 0$ and for $t$ large enough we have

$$\check{\epsilon}t \leq c(t - t_0) - \Phi_\chi(H(t_0)) \leq -\Phi_\chi(H(t)) \leq C(t - t_0) - \Phi_\chi(H(t_0)) \leq \check{C}t.$$

Hence

$$c(-\Phi_\chi)^{-1}(\check{\epsilon}t) \leq H(t) \sim E(u(t), v(t)) \leq C(-\Phi_\chi)^{-1}(\check{\epsilon}t),$$

which completes the proof of Theorem 2.1.

**Lemma 3.1.** Let $\psi(s) = s^2h(s)$ and $\check{\psi}(r) = \sup\{rs - \psi(s) : s \geq 0\}$ be the convex conjugate to $\psi$. Then there exists $C > 0$ such that $\check{\psi}(sh(s)) \leq Cs^2h(s)$ for all $s \geq 0$.

**Proof.** Since $\psi$ is convex, the one-sided derivatives $\psi'_\pm(s) = s^2h'_\pm(s) + 2sh(s)$ are nondecreasing functions and the interval $[\psi'_\pm(s_0), \psi'_\pm(s_0)]$ is nonempty. Take $s_0 > 0$ arbitrarily and take $r \in [\psi'_\pm(s_0), \psi'_\pm(s_0)]$. Then the function $s \mapsto rs - \psi(s)$ attains its maximum in $s_0$, hence $\check{\psi}(r) = rs_0 - s_0^2h(s_0)$. Since $r \geq \psi'_\pm(s_0) = s_0^2h'_\pm(s_0) + 2s_0h(s_0) \geq s_0h(s_0)$ and $\psi$ is increasing, we have

$$\check{\psi}(s_0h(s_0)) \leq \check{\psi}(r) = rs_0 - s_0^2h(s_0) \leq \psi'_\pm(s_0)s_0 - s_0^2h(s_0) = s_0^2h'_\pm(s_0) + 2s_0^2h(s_0) - s_0^2h(s_0) \leq (c + 2 - 1)s_0^2h(s_0).$$

$\square$
Lemma 3.2. Function $\chi(s) = sh(\sqrt{s})$ is $C$-sublinear and it has property (K). Moreover, $\chi(s + t) \geq \frac{1}{2}(\chi(s) + \chi(t))$ for all $s, t > 0$ and for every $c > 0$ there exists $\tilde{c} > 0$ such that $\chi(cs) \geq \tilde{c}\chi(s)$.

Proof. Since $h$ has property (K), we have for a fixed $K > 0$

$$\chi(Ks) = Ksh(\sqrt{K}s) \leq KsC(\sqrt{K})h(\sqrt{s}) = KC(\sqrt{K})\chi(s).$$

So, $\chi$ has property (K) and since it is increasing, it is also $C$-sublinear. Since $\chi$ is increasing, we also have $\chi(s + t) \geq \chi(s), \chi(s + t) \geq \chi(t)$ and therefore $\chi(s + t) \geq \frac{1}{2}(\chi(s) + \chi(t))$. From property (K) we have for any fixed $c > 0$

$$\chi(s) = \chi\left(\frac{1}{c}cs\right) \leq C\left(\frac{1}{c}\right)\chi(cs) = \frac{1}{c}\chi(cs)$$

and the last property is proven. \qed

References


