A variational property on the evolutionary bifurcation curves for the one-dimensional perturbed Gelfand problem from combustion theory

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Abstract. We study a variational property on the evolutionary bifurcation curves for the one-dimensional perturbed Gelfand problem from combustion theory

\[
\begin{aligned}
&u''(x) + \lambda \exp\left(\frac{au}{a+u}\right) = 0, \quad -1 < x < 1, \\
&u(-1) = u(1) = 0,
\end{aligned}
\]

where \( \lambda > 0 \) is the Frank–Kamenetskii parameter or ignition parameter, \( a > 0 \) is the activation energy parameter, and \( u \) is the dimensionless temperature.

Keywords: positive solution, exact multiplicity, Turning point, S-shaped bifurcation curve.

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1 Introduction and the main result

In this paper we mainly study a variational property on the evolutionary bifurcation curves of positive solutions for the two-point boundary value problem

\[
\begin{aligned}
&u''(x) + \lambda \exp\left(\frac{au}{a+u}\right) = 0, \quad -1 < x < 1, \\
&u(-1) = u(1) = 0,
\end{aligned}
\]

which is the one-dimensional case of a problem arising in the study of standard models of ignition in a context of thermal combustion, cf. [1,14]. In (1.1), \( \lambda > 0 \) is the Frank–Kamenetskii parameter or ignition parameter, \( a > 0 \) is the activation energy parameter, \( u \) is the dimensionless temperature of the medium, and the reaction term

\[f(u) \equiv \exp\left(\frac{au}{a+u}\right)\]
is the temperature dependence obeying the simple Arrhenius reaction-rate law in irreversible chemical reaction kinetics, see, e.g. Boddington et al. [2]. Notice that, substituting $a = 1/\varepsilon$ ($\varepsilon$ is the reciprocal activation energy parameter) into (1.1), we obviously obtain

$$
\begin{align*}
&u''(x) + \lambda \exp\left(\frac{u}{1+u}\right) = 0, \quad -1 < x < 1, \\
&u(-1) = u(1) = 0.
\end{align*}
$$

(1.2)

This problem (1.2) is the famous one-dimensional perturbed Gelfand problem, cf. [1, 3, 5, 10, 11, 13].

For any $a > 0$, on the $(\lambda, \|u\|_{\infty})$-plane, we study the shape and structure of bifurcation curves $S_a$ of positive solutions of (1.1), defined by

$$
S_a \equiv \{ (\lambda, \|u_\lambda\|_{\infty}) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)} \}.
$$

We say that, on the $(\lambda, \|u\|_{\infty})$-plane, the bifurcation curve $S_a$ is S-shaped if $S_a$ has exactly two turning points at some points $(\lambda^*, \|u_{\lambda^*}\|_{\infty})$ and $(\lambda_*, \|u_{\lambda_*}\|_{\infty})$ where $\lambda_* < \lambda^*$ are two positive numbers such that

(i) $\|u_{\lambda^*}\|_{\infty} < \|u_{\lambda_*}\|_{\infty}$,

(ii) at $(\lambda^*, \|u_{\lambda^*}\|_{\infty})$ the bifurcation curve $S_a$ turns to the left,

(iii) at $(\lambda_*, \|u_{\lambda_*}\|_{\infty})$ the bifurcation curve $S_a$ turns to the right.

See Figure 1.1 (i). In that case for S-shaped bifurcation curve $S_a$ for thermal combustion problem (1.1), the two critical values $\lambda^*$ and $\lambda_*$ correspond to ignition limit and extinction limit respectively. The upper branch of $S_a$ is then known as the explosion branch, and the lower branch the quenching branch. See [9, p. 374].

![Figure 1.1: The global bifurcation of bifurcation curves $S_a$ for $a > 0$.](image)

Huang and Wang [6, Theorem 4] very recently studied global bifurcation of bifurcation curves $S_a$ in the following theorem.

**Theorem 1.1** (See Figure 1.1). Consider (1.1) with varying $a > 0$. Then there exists a critical value $a_0 \approx 4.069$ such that the following assertions (i)–(iii) hold:

(i) (See Figure 1.1 (i).) For $a > a_0$, the bifurcation curve $S_a$ is S-shaped on the $(\lambda, \|u\|_{\infty})$-plane. Let $(\lambda^*, \|u_{\lambda^*}\|_{\infty})$ and $(\lambda_*, \|u_{\lambda_*}\|_{\infty})$ be exactly two turning points of the bifurcation curve $S_a$ satisfying $\lambda_* < \lambda^*$ and $\|u_{\lambda^*}\|_{\infty} < \|u_{\lambda_*}\|_{\infty}$. Then $u_{\lambda_*}$ and $u_{\lambda^*}$ are only two degenerate positive solutions of (1.1).
(ii) (See Figure 1.1 (ii).) For \( a = a_0 \), the bifurcation curve \( S_{a_0} \) is monotone increasing on the \( (\lambda, \|u\|_\infty) \)-plane. Moreover, (1.1) has exactly one (cusp type) degenerate positive solution \( u_{\lambda_0} \).

(iii) (See Figure 1.1 (iii).) For \( 0 < a < a_0 \), the bifurcation curve \( S_a \) is monotone increasing on the \( (\lambda, \|u\|_\infty) \)-plane. Moreover, all positive solutions \( u_\lambda \) of (1.1) are nondegenerate.

Furthermore, Hung and Wang [8] proved that there exists a positive number \( a^* (\approx 4.166) > a_0 \) such that

\[
p_1(a) < \|u_{\lambda^*}\|_\infty < \gamma(a) < p_2(a) < \|u_{\lambda^*}\|_\infty \quad \text{for } a \geq a^*, \tag{1.3}
\]

where

\[
\gamma(a) \equiv \frac{a(a - 2)}{2}, \quad p_1(a) \equiv \frac{a(a - 2) - a \sqrt{a(a - 4)}}{2}, \quad p_2(a) \equiv \frac{a(a - 2) + a \sqrt{a(a - 4)}}{2}. \tag{1.4}
\]

Clearly, \( p_1(a) < \gamma(a) < p_2(a) \) for \( a > 4 \). In addition, for \( a > 4 \), we note that \( (\gamma(a), f(\gamma(a))) \) is the unique inflection point of \( f(u) \) on \( (0, \infty) \), and \( p_1(a) \) and \( p_2(a) \) are two positive zeros of

\[
f(u) - uf'(u) = \frac{\left[u^2 - a(a - 2)u + a^2\right]}{(a + u)^2} \exp\left(-\frac{au}{a + u}\right), \tag{1.5}
\]

which is the \( y \)-intercept of the tangent line to the graph of \( f \) at the point \((u, f(u))\). In this paper, we continue our work [6] and extend the result of (1.3). The following Theorem 1.2 is our main result, in which we show the variation of the values of \( \|u_{\lambda^*}\|_\infty \) and \( \|u_{\lambda^*}\|_\infty \) with varying parameter \( a > a_0 \), where \((\lambda^*, \|u_{\lambda^*}\|_\infty)\) and \((\lambda_0, \|u_{\lambda_0}\|_\infty)\) are defined in Theorem 1.1.

**Theorem 1.2** (See Figures 1.1 (i) and 1.2). Consider (1.1) with varying \( a > a_0 \). Let \((\lambda^*, \|u_{\lambda^*}\|_\infty)\) and \((\lambda_0, \|u_{\lambda_0}\|_\infty)\) be two turning points of the bifurcation curve \( S_a \) satisfying \( \lambda_0 < \lambda^* \) and \( \|u_{\lambda^*}\|_\infty < \|u_{\lambda_0}\|_\infty \). Then there exist two positive numbers \( \hat{\alpha} \approx 4.088, \tilde{\alpha} \approx 4.077 \) satisfying \( a^* > \hat{\alpha} > \tilde{\alpha} > a_0 \) such that:

\[
(1 < \) \quad p_1(a) < \|u_{\lambda^*}\|_\infty < \gamma(a) < p_2(a) < \|u_{\lambda_0}\|_\infty \quad \text{for } a > \hat{\alpha}, \tag{1.6}
\]

\[
\gamma(\hat{\alpha}) = \|u_{\lambda^*}\|_\infty < p_2(\hat{\alpha}) < \|u_{\lambda_0}\|_\infty \quad \text{for } a = \hat{\alpha}, \tag{1.7}
\]

\[
\gamma(\tilde{\alpha}) < \|u_{\lambda^*}\|_\infty < p_2(\tilde{\alpha}) < \|u_{\lambda_0}\|_\infty \quad \text{for } \tilde{\alpha} < a < \hat{\alpha}, \tag{1.8}
\]

\[
\gamma(\hat{\alpha}) < \|u_{\lambda^*}\|_\infty < \|u_{\lambda_0}\|_\infty = p_2(\hat{\alpha}) \quad \text{for } a = \hat{\alpha}, \tag{1.9}
\]

\[
\gamma(a) < \|u_{\lambda^*}\|_\infty < \|u_{\lambda_0}\|_\infty < p_2(a) \quad \text{for } a_0 < a < \tilde{\alpha}, \tag{1.10}
\]

\[
\lim_{\alpha \to \hat{\alpha}} \|u_{\lambda^*}\|_\infty = \lim_{a \to a_0^+} \|u_{\lambda_0}\|_\infty = \|u_{\lambda_0}\|_\infty \approx 4.896. \tag{1.11}
\]

Moreover,

\[
\frac{a \gamma(a)}{p_1(a)} > \frac{\|u_{\lambda^*}\|_\infty}{\|u_{\lambda^*}\|_\infty} > \frac{p_2(a)}{\|u_{\lambda_0}\|_\infty} \quad \text{for } a \geq \hat{\alpha} \quad \text{and} \quad \lim_{a \to a_0^+} \frac{\|u_{\lambda^*}\|_\infty}{\|u_{\lambda^*}\|_\infty} = \infty. \tag{1.12}
\]

The paper is organized as follows: Section 2 contains a few lemmas needed to prove the main result. Finally, Section 3 contains the proof of the main result.
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Figure 1.2: The evolution of bifurcation curves $S_a$ with varying $a \geq a_0 \approx 4.069$. The notations • and ▲ denote the two turning points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda^*, \|u_{\lambda^*}\|_\infty)$, respectively.

2 Lemmas

To prove Theorem 1.2, we develop some new time-map techniques. The time-map formula which we apply to study (1.1) takes the form as follows:

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^\alpha [F(\alpha) - F(u)]^{-1/2} \, du \equiv T_a(\alpha) \quad \text{for } \alpha > 0,$$

(2.1)

where $F(u) \equiv \int_0^u f(t) \, dt$, see Laetsch [12]. (Note that it can be proved that $T_a(\alpha)$ is a twice differentiable function of $\alpha > 0$ for $a > a_0$, and is a differentiable function of $\alpha > 0$. The proofs are easy but tedious and hence we omit them.) So the positive solution $u$ of (1.1) corresponds to

$$\|u\|_\infty = \alpha \quad \text{and} \quad T_a(\alpha) = \sqrt{\lambda}.$$

Thus studying the shape of bifurcation curve $S_a$ on the $(\lambda, \|u\|_\infty)$-plane is equivalent to studying the shape of the time-map $T_a(\alpha)$ on $(0, \infty)$, cf. [6]. By (2.1) and Theorem 1.1, we note that

(i) If $a > a_0$, $T_a(\alpha)$ has exactly two critical points at $\|u_{\lambda^*}\|_\infty < \|u_{\lambda}\|_\infty$ where $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ are exactly two turning points of the S-shaped bifurcation curve $S_a$. See Figure 2.1 (i).

(ii) If $a = a_0$, $T_a(\alpha)$ has exactly one critical point at $\|u_{\lambda_0}\|_\infty$ where $(\lambda_0, \|u_{\lambda_0}\|_\infty)$ is the unique turning point of the monotone bifurcation curve $S_{a_0}$. See Figure 2.1 (ii).

(iii) If $0 < a < a_0$, $T_a(\alpha)$ has no critical points on $(0, \infty)$ and is a strictly increasing function on $(0, \infty)$. See Figure 2.1 (iii).
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Figure 2.1: Graphs of $T_a(\alpha)$ on $(0, \infty)$ for $a > 0$. $\alpha_M = \|u_{\lambda^*}\|_\infty$, $\alpha_m = \|u_{\lambda_1}\|_\infty$ and $\alpha_0 = \|u_{\lambda_0}\|_\infty$.

For $T_a(\alpha)$ in (2.1), we compute that

$$T'_a(\alpha) = \frac{1}{2\sqrt{2\lambda}} \int_0^\lambda \frac{\theta(\alpha) - \theta(u)}{[F(\alpha) - F(u)]^{3/2}} du,$$

(2.2)

where

$$\theta(u) = 2F(u) - uf(u),$$

cf. [8, (3.4) and p. 230]. For the sake of convenience, we let $\gamma = \gamma(a)$, $\gamma' = \gamma'(a)$, $p_1 = p_1(a)$, $p_2 = p_2(a)$ and $p'_2 = p'_2(a)$. First, we need to have the following lemma:

Lemma 2.1. Consider (1.1) with $a > 4$. Then there exists $\hat{a} \in [a_0, a^*]$ such that

$$T'_a(\gamma(a)) \begin{cases} > 0 & \text{for } 4 < a < \hat{a}, \\ = 0 & \text{for } a = \hat{a}, \\ < 0 & \text{for } \hat{a} < a \leq a^* \approx 4.166. \end{cases}$$

(2.3)

Proof of Lemma 2.1. By (2.2), we compute that

$$\frac{\partial}{\partial a} T'_a(\gamma(a)) = \frac{1}{2\sqrt{2\gamma^2(a)}} \int_0^{\gamma(a)} \frac{N(u)}{[F(\gamma(a)) - F(u)]^{3/2}} du,$$

(2.4)

where

$$N(u) \equiv -[F(\gamma) - F(u)] \left\{ \gamma' [\gamma f(\gamma) - uf(u)] + \gamma \int_u^\gamma \frac{s^2}{(a + s)^2} f(s) ds \right\}$$

$$+ \frac{3}{2} [\gamma f(\gamma) - uf(u)] \left\{ \gamma' [\gamma f(\gamma) - uf(u)] + \gamma \int_u^\gamma \frac{s^2}{(a + s)^2} f(s) ds \right\}$$

$$- [F(\gamma) - F(u)] \left\{ \gamma' + \frac{a^2 \gamma' + a \gamma^3}{(a + \gamma)^2} \right\} \gamma f(\gamma)$$

$$+ [F(\gamma) - F(u)] \left\{ \gamma' + \frac{a^2 \gamma' u + a u^2}{(a + u)^2} \right\} uf(u).$$
By [6, Lemma 17], we have that
\[ af(a) - uf(u) \leq \left(1 + \frac{a}{4}\right) [F(a) - F(u)] \quad \text{for } 0 \leq u \leq a \text{ and } a > 4. \] (2.5)
Since we compute and find that, for \(0 \leq u \leq \gamma \) and \(a > 4\),
\[ \frac{a^2 \gamma' \gamma^2 + \gamma^3}{(a + \gamma)^2} = \gamma \quad \text{and} \quad \gamma' [\gamma f(\gamma) - uf(u)] + \gamma \int_u^\gamma \frac{s^2}{(a + s)^2} f(s)ds \geq 0, \]
and by (2.5), we obtain that
\[ N(u) \leq \gamma [F(\gamma) - F(u)] L(u, a) \quad \text{for } 0 \leq u \leq \gamma \text{ and } a > 4, \] (2.6)
where
\[
L(u, a) \equiv \left(\frac{3a}{8} + \frac{1}{2}\right) \int_0^\gamma \frac{s^2}{(a + s)^2} f(s)ds + \left[(a - 1)\left(\frac{3a}{8} - \frac{1}{2}\right) - \gamma\right] f(\gamma) \]
\[
- \left[(a - 1)\left(\frac{3a}{8} - \frac{1}{2}\right) - \frac{a^2(a - 1)u + \gamma u^2}{[a + u]^2}\right] \frac{u}{\gamma} f(u). \] (2.7)
We assert that, for \(4 < a \leq 4.17\),
\[ L(0, a) < 0 \quad \text{and} \quad \frac{\partial}{\partial u} L(u, a) = \begin{cases} < 0 & \text{for } 0 \leq u < v_1, \\ = 0 & \text{for } u = v_1, \quad \text{for some } v_1 \in (0, \gamma). \\ > 0 & \text{for } v_1 < u \leq \gamma \end{cases} \] (2.8)
It is easy to see that \(L(\gamma, a) = 0\) by (2.7). So under (2.8), we observe that \(L(u, a) < 0\) for \(0 \leq u < \gamma\). So by (2.4) and (2.6), we see that \(\frac{\partial}{\partial a} T'_a(\gamma(a)) < 0\) for \(4 < a \leq 4.17\). It follows that \(\frac{\partial}{\partial a} T'_a(\gamma(a)) \leq 0\) for \(a_0 \leq a \leq a^*\) since \(4 < a_0 (\approx 4.069) < a^* (\approx 4.166) < 4.17\). In addition, by Theorem 1.1 (i) and (1.3), we see that
\[ T'_a(\gamma(a)) = \begin{cases} > 0 & \text{for } 4 < a < a_0, \\ < 0 & \text{for } a \geq a^*. \end{cases} \]
Thus there exists \(\hat{a} \in [a_0, a^*)\) such that (2.3) holds. So the proof of Lemma 2.1 is complete.

Next, we divide the proof of assertion (2.8) into next Steps 1–2.

**Step 1.** We prove the first inequality of (2.8). We compute that
\[
\int \frac{s^2}{(a + s)^2} ds = s - \frac{a^2}{a + s} - 2a \ln(a + s). \] (2.9)
Since \(f'(u) > 0\) for \(u \geq 0\), and by (2.7) and (2.9), we compute and obtain that, for \(4 < a \leq 4.17\),
\[
L(0, a) = \left(\frac{3a}{8} + \frac{1}{2}\right) \int_0^\gamma \frac{s^2}{(a + s)^2} f(s)ds + \left[(a - 1)\left(\frac{3a}{8} - \frac{1}{2}\right) - \gamma\right] f(\gamma) \]
\[
= \left(\frac{3a}{8} + \frac{1}{2}\right) \left[\left(\frac{a}{8} - \frac{1}{2}\right) f(\gamma) - \frac{1}{8}(a^2 - a - 4) f(\gamma)\right] \]
\[
= \left(\frac{3a}{8} + \frac{1}{2}\right) \left[\int_0^2 \frac{s^2}{(a + s)^2} f(s)ds + \int_2^\gamma \frac{s^2}{(a + s)^2} f(s)ds\right] \]
\[
\leq \left(\frac{3a}{8} + \frac{1}{2}\right) \left[\int_0^2 \frac{s^2}{(a + s)^2} ds\right] f(2) + \left[\left(\frac{3a}{8} + \frac{1}{2}\right) \int_2^\gamma \frac{s^2}{(a + s)^2} ds - \frac{1}{8}(a^2 - a - 4)\right] f(\gamma) \]
\[
= \frac{1}{16(a + 2)} L_1(a) < 0,
\]
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where

\[ L_1(a) \equiv 4(3a + 4) \left[ 2a + 2 + (a^2 + 2a) \ln \frac{a}{a + 2} \right] \exp \left( \frac{2a}{a + 2} \right) \]

\[ + \left[ 3a^4 + 8a^3 - 30a^2 - 84a - 48 + (12a^3 + 40a^2 + 32a) \ln \left( \frac{2(a + 2)}{a^2} \right) \right] \exp (a - 2) \]

\[ < 0 \quad \text{for} \quad 4 < a \leq 4.17, \]

see Figure 2.2. So the first inequality of (2.8) holds.

\[ \text{Figure 2.2: The graph of } L_1(a) \text{ on } [4, 4.17] \text{ and } L_1(4.17) \approx -69.547. \]

**Step 2.** We prove the second inequality of (2.8). We compute that

\[ \frac{\partial}{\partial u} L(u, a) = \frac{f(u)}{8a(a - 2)(a + u)} L_2(u), \quad (2.10) \]

where

\[ L_2(u) \equiv -(3a - 4) (a^2 - 2) u^4 + (-4a^4 + 10a^3 - 32a) u^3 \]

\[ + (a^5 + 34a^4 - 4a^3 - 48a^2) u^2 - 2a^3 (a - 1) (3a + 4) (a - 4) u \]

\[ - 2a^4 (a - 1) (3a - 4) \]

is a quartic polynomial of \( u \). We compute that, for \( 4 < a \leq 4.17 \),

\[ L_2(0) = -2a^4 (a - 1) (3a - 4) < 0, \quad (2.11) \]

\[ L_2(\gamma) = \frac{a^8}{16} \left\{ (3a - 8) \left[ (4.2 - a) (a + 0.4) + \frac{5a + 8}{25} \right] + 8 \right\} > 0, \quad (2.12) \]

\[ L_2'(0) = -2a^3 (a - 1) (3a + 4) (a - 4) < 0, \quad (2.13) \]

\[ L_2'(\gamma) = \frac{1}{2} a^6 (3a - 4) [(4.2 - a) (a + 0.2) + 0.16] > 0, \quad (2.14) \]

\[ L_2''(0) = \left[ (2a^2 - 8a + (68a^2 - 96)) a^2 \right] > 0, \quad (2.15) \]

\[ L_2''(\gamma) = a^3 \left[ (36 - 9a) a^3 - 10a^2 + (64 - 40a) \right] < 0. \quad (2.16) \]
Since $L''_2(u)$ is a quadratic polynomial with a negative leading coefficient, and by (2.15) and (2.16), there exists $v_2 \in (0, \gamma)$ such that

$$L''_2(u) \begin{cases} > 0 & \text{for } 0 \leq u < v_2, \\ = 0 & \text{for } u = v_2, \\ < 0 & \text{for } v_2 < u \leq \gamma. \end{cases}$$

So by (2.13) and (2.14), there exists $v_3 \in (0, \gamma)$ such that

$$L'_2(u) \begin{cases} < 0 & \text{for } 0 \leq u < v_3, \\ = 0 & \text{for } u = v_3, \\ > 0 & \text{for } v_3 < u \leq \gamma. \end{cases}$$

So by (2.10)–(2.12), there exists $v_1 \in (0, \gamma)$ such that the second inequality of (2.8) holds.

The proof of Lemma 2.1 is complete.

**Lemma 2.2.** Consider (1.1) with $4 < a \leq 4.108$. Then

$$3.6 |F(p_2) - F(u)| \leq A(u) \leq M_a |F(p_2) - F(u)| \quad \text{for } 0 \leq u \leq p_2, \quad (2.17)$$

where

$$A(u) \equiv \frac{p'_2}{p_2} [p_2f(p_2) - uf(u)] + \int_u^{p_2} \frac{s^2}{(a+s)^2} f(s) ds,$$

$$M_a \equiv \frac{p'_2}{p_2} \left( \frac{a}{4} + 1 \right) + \frac{p_2^2}{(a+p_2)^2}.$$

**Proof of Lemma 2.2.** Let

$$U_1(u) \equiv M_a |F(p_2) - F(u)| - A(u) \quad \text{and} \quad U_2(u) \equiv A(u) - 3.6 |F(p_2) - F(u)|.$$

To prove (2.17), it is sufficient to prove that $U_1(u) \geq 0$ and $U_2(u) \geq 0$ for $0 \leq u \leq p_2$.

1) We prove that $U_1(u) \geq 0$ for $0 \leq u \leq p_2$. Clearly, we see that

$$p'_2(a) = \frac{(a-1) \sqrt{a^2 - 4a + a(a-3)}}{\sqrt{a^2 - 4a}} > 0 \quad \text{for } a > 4. \quad (2.18)$$

Since $u^2 / (a + u)^2$ is a strictly increasing function of $u > 0$ for $a > 0$, and by (2.18), we compute and observe that, for $0 \leq u \leq p_2$, 

$$U'_1(u) = \frac{d}{du} \left\{ \int_u^{p_2} \left( M_a - \frac{s^2}{(a+s)^2} \right) f(s) ds - \frac{p'_2}{p_2} [p_2f(p_2) - uf(u)] \right\}$$

$$= \left\{ -M_a + \frac{u^2}{(a+u)^2} + \frac{p'_2}{p_2} \left[ \frac{a^2u}{(a+u)^2} + 1 \right] \right\} f(u)$$

$$= - \left\{ \frac{a(a-1)^2p'_2}{4 (a+u)^2} + \frac{p_2^2}{(a+p_2)^2} - \frac{u^2}{(a+u)^2} \right\} f(u) < 0. \quad (2.19)$$

Since $U_1(p_2) = 0$, and by (2.19), we see that $U_1(u) \geq 0$ for $0 \leq u \leq p_2$. It implies that the second inequality of (2.17) holds.
(II) We prove that $U_2(u) \geq 0$ for $0 \leq u \leq p_2$. We observe that

$$U_2(u) = \frac{p_2^2}{p_2} \left[ p_2 f(p_2) - uf(u) \right] + \int_u^{p_2} \left( \frac{s^2}{(a + s)^2} - 3.6 \right) f(s)ds.$$ 

First, we assert that

$$U_2(0) = p_2' f(p_2) + \int_0^{p_2} \left[ \frac{s^2}{(a + s)^2} - 3.6 \right] f(s)ds > 0 \quad \text{for } 4 < a \leq 4.108. \quad (2.20)$$

Indeed, by (1.4), we observe that

$$\frac{\partial}{\partial a} p_2' f(p_2) = \frac{2af(p_2)}{[a + \sqrt{a(a - 4)}][a(a - 4)]^{3/2}} w_1(a) < 0 \quad \text{for } 4 < a \leq 4.108, \quad (2.21)$$

where

$$w_1(a) \equiv \sqrt{a(a - 4)[a(a - 1)(a - 4) + 1] + a^4 - 7a^3 + 12a^2 - a}.$$ 

See Figure 2.3(i). Clearly,

$$\frac{d}{da} \int_0^{5.7} \left[ \frac{s^2}{(a + s)^2} - 3.6 \right] f(s)ds = -\int_0^{5.7} \frac{s^2 f(s)}{5(a + s)} \left[ 13s^2 + (36a + 10)s + 18a^2 + 10a \right] ds < 0. \quad (2.22)$$

![Figure 2.3: (i) The graph of $w_1(a)$ on $[4, 4.108]$. (ii) The graph of $w_2(a)$ on $[4, 4.108]$.](image)

By (2.18), we compute that

$$p_2(a) \leq p_2(4.108) \approx 5.697 < 5.7 \quad \text{for } 4 < a \leq 4.108. \quad (2.23)$$

So by (2.21)–(2.23), we compute and find that, for $4 < a \leq 4.108$,

$$U_2(0) \geq p_2' f(p_2) + \int_0^{5.7} \left[ \frac{s^2}{(a + s)^2} - 3.6 \right] f(s)ds \geq \left\{ p_2' f(p_2) + \int_0^{5.7} \left[ \frac{s^2}{(a + s)^2} - 3.6 \right] f(s)ds \right\}_{a=4.108} \approx 1.174 > 0.$$
Thus assertion (2.20) holds.

Secondly, we compute and obtain that, for $0 \leq u < p_2$,

$$
\frac{U_2'(u)}{f(u)} = 3.6 - \frac{p_2'}{p_2} \left[ \frac{a^2 u}{(a + u)^2} + 1 \right] - \frac{u^2}{(a + u)^2}, \tag{2.24}
$$

where $w_2(a) = \sqrt{a(a - 4)}(31a - 10) - 5a^2$. See Figure 2.3 (ii). Since $f(u) > 0$ for $u > 0$, and by (2.25) and (2.26), we see that either $U_2'(u) < 0$ for $0 < u < p_2$, or there exists $v_4 \in (0, p_2)$ such that

$$
\begin{cases}
  > 0 & \text{for } 0 \leq u < v_4, \\
  = 0 & \text{for } u = v_4, \\
  < 0 & \text{for } v_4 < u \leq p_2.
\end{cases}
$$

By (2.24), we compute and obtain that

$$
\frac{U_2'(p_2)}{f(p_2)} = -2\frac{p_2'}{p_2} \frac{p_2}{a^2} + 3.6 = \frac{w_2(a)}{10a\sqrt{a^2 - 4a}} < 0 \quad \text{for } 4 < a \leq 4.108, \tag{2.26}
$$

where $w_2(a) = \sqrt{a(a - 4)}(31a - 10) - 5a^2$. Since $f(u) > 0$ for $u > 0$, and by (2.25) and (2.26), we see that either $U_2'(u) < 0$ for $0 < u < p_2$, or there exists $v_4 \in (0, p_2)$ such that

$$
\begin{cases}
  > 0 & \text{for } 0 \leq u < v_4, \\
  = 0 & \text{for } u = v_4, \\
  < 0 & \text{for } v_4 < u \leq p_2.
\end{cases}
$$

Since $U_2(p_2) = 0$, and by (2.20), we further see that $U_2(u) \geq 0$ for $0 \leq u \leq p_2$. It implies that the first inequality of (2.17) holds.

The proof of Lemma 2.2 is complete.

**Lemma 2.3.** Consider (1.1) with $a > 4$. There exists $\tilde{a} \in [a_0, 4.108]$ such that

$$
T_4'(p_2(a)) \begin{cases}
  > 0 & \text{for } 4 < a < \tilde{a}, \\
  = 0 & \text{for } a = \tilde{a}, \\
  < 0 & \text{for } a > \tilde{a}.
\end{cases} \tag{2.27}
$$

**Proof of Lemma 2.3.** We compute that

$$
\frac{\partial}{\partial a} F(p_2) = p_2 f(p_2) + \int_0^{p_2} \frac{t^2}{(a + t)^2} f(t) dt \tag{2.28}
$$

and

$$
\frac{\partial}{\partial a} p_2 f(p_2) = p_2 f(p_2) + \frac{a^2 p_2 p_2' + p_2^3}{(a + p_2)^2} f(p_2). \tag{2.29}
$$
We further compute that, by (2.2), (2.28) and (2.29),
\[
\frac{\partial}{\partial a} T'_d(p_2(a)) = \frac{\partial}{\partial a} \left\{ \frac{1}{2\sqrt{2}} \int_0^1 \frac{\theta(p_2) - \theta(p_2 t)}{(F(p_2) - F(p_2 t))^{3/2}} dt \right\} \quad \text{(let } t = \frac{u}{p_2})
\]
\[
= \frac{1}{2\sqrt{2}} \int_0^1 \left\{ \frac{\partial}{\partial a} \left[ \theta(p_2) - \theta(p_2 t) \right] \right\} \frac{1}{(F(p_2) - F(p_2 t))^{3/2}} dt
\]
\[
- \frac{1}{2\sqrt{2}} \int_0^1 \frac{3}{2} \left[ \theta(p_2) - \theta(p_2 t) \right] \frac{\partial}{\partial a} \left[ F(p_2) - F(p_2 t) \right] dt
\]
\[
= \frac{1}{2\sqrt{2} \bar{p}_2} \int_0^{\bar{p}_2} \frac{F(p_2) - F(u)}{(F(p_2) - F(u))^{3/2}} B(u) - \frac{3}{2} \left[ \theta(p_2) - \theta(u) \right] A(u) du,
\] (2.30)

where \(A(u)\) is defined in Lemma 2.2 and
\[
B(u) \equiv 2A(u) - \left[ p'_2 + p_2 \left( \frac{a^2 p'_2 + p_2^2}{(a + p_2)^2} \right) \right] f(p_2) + \left[ p'_2 \frac{u}{p_2} + \frac{u (a^2 p'_2 u + p_2 u^2)}{p_2^2 (a + u)^2} \right] f(u).
\]

In addition, by [6, Lemma 12], we see that there exists \(\bar{p}_2 \in (0, p_1)\) such that
\[
\theta(p_2) - \theta(u) \begin{cases} > 0 & \text{for } 0 \leq u < \bar{p}_2, \\
= 0 & \text{for } u = \bar{p}_2, \\
< 0 & \text{for } \bar{p}_2 < u < p_2. 
\end{cases}
\] (2.31)

So by Lemma 2.2, we observe that, for \(0 \leq u < \bar{p}_2\),
\[
- \frac{3}{2} \left[ \theta(p_2) - \theta(u) \right] A(u) \leq -5.4 \left[ \theta(p_2) - \theta(u) \right] [F(p_2) - F(u)],
\] (2.32)

and, for \(\bar{p}_2 \leq u \leq p_2\),
\[
- \frac{3}{2} \left[ \theta(p_2) - \theta(u) \right] A(u) \leq -\frac{3}{2} M_a \left[ \theta(p_2) - \theta(u) \right] [F(p_2) - F(u)].
\] (2.33)

By (2.30)–(2.33), we have that
\[
\frac{\partial}{\partial a} T'_d(p_2) \leq \frac{1}{2\sqrt{2} \bar{p}_2} \int_0^{\bar{p}_2} \frac{U_2(u)}{(F(p_2) - F(u))^{3/2}} du - \frac{5.4}{2\sqrt{2} \bar{p}_2} \int_0^{\bar{p}_2} \frac{\theta(p_2) - \theta(u)}{(F(p_2) - F(u))^{3/2}} du
\]
\[
- \frac{3}{4\sqrt{2} \bar{p}_2} \int_0^{\bar{p}_2} \frac{\theta(p_2) - \theta(u)}{(F(p_2) - F(u))^{3/2}} du
\]
\[
= \frac{1}{2\sqrt{2} \bar{p}_2} \int_0^{\bar{p}_2} \frac{B(u)}{(F(p_2) - F(u))^{3/2}} du + \frac{1}{2\sqrt{2} \bar{p}_2} \int_{\bar{p}_2}^{p_2} \frac{C(u)}{(F(p_2) - F(u))^{3/2}} du
\]
\[
- \frac{5.4}{2\sqrt{2} \bar{p}_2} T'_d(p_2),
\] (2.34)

where
\[
C(u) \equiv B(u) - \left( \frac{3}{2} M_a - 5.4 \right) \left[ \theta(p_2) - \theta(u) \right].
\]

We assert that
\[
B(u) < 0 \text{ for } 0 < u < \bar{p}_2 \text{ and } C(u) < 0 \text{ for } \bar{p}_2 \leq u < p_2 \text{ and } 4 < a \leq 4.108.
\] (2.35)
In addition, by [6, Lemma 16], there exists a positive number \( \tilde{a} \) (\( \approx 4.107 \)) such that \( T'_a(p_2(a)) < 0 \) for \( a \geq \tilde{a} \). By Theorem 1.1 (iii), we see that \( T'_a(p_2(a)) > 0 \) for \( 0 < a < a_0 \). It follows that there exists \( \alpha \in [a_0, 4.108) \) such that \( T'_a(p_2(\alpha)) = 0 \). Furthermore, since \( 4 < \tilde{a} < 4.108 \), and by (2.34) and (2.35), we see that
\[
\frac{\partial}{\partial a} T'_a(p_2(a)) \bigg|_{a=\alpha} < 0.
\]
Thus \( \alpha \) is unique and (2.27) holds. We then divide the proof of (2.35) into next Steps 1–3.

**Step 1.** We prove that \( 1 < \bar{p}_2(a) \) for \( 4 < a \leq 4.108 \). Let
\[
\Lambda_a(u) \equiv \theta(u) - \theta(p_2) \quad \text{for} \quad 0 \leq u \leq p_2.
\]
By (2.18), we see that \( 1 < 4 = p_2(4) < p_2(a) \) for \( a > 4 \). So by (2.31), it is sufficient to prove that \( \Lambda_a(1) < 0 \) for \( 4 < a \leq 4.108 \). We compute that
\[
\frac{\partial}{\partial a} \Lambda_a(u) = -2 \int_u^{p_2} \frac{s^2}{(a+s)^2} f(s) ds + \frac{p_2^3 f(p_2)}{(a + p_2)^2} - \frac{u^3 f(u)}{(a+u)^2}.
\]  
(2.36)
Since
\[
u^2 - a^2 u - a^2 < 0 \quad \text{for} \quad 0 \leq u \leq p_2 \quad \text{and} \quad a > 4,
\]
we further compute and obtain that
\[
\frac{\partial}{\partial u} \frac{\partial}{\partial a} \Lambda_a(u) = \frac{u^2 f(u)}{(a+u)^4} (u^2 - a^2 u - a^2) < 0 \quad \text{for} \quad 0 \leq u \leq p_2 \quad \text{and} \quad a > 4.
\]  
(2.37)
So by (2.36) and (2.37), we have that
\[
\frac{\partial}{\partial a} \Lambda_a(u) \bigg|_{u=p_2} = 0 \quad \text{for} \quad 0 \leq u < p_2 \quad \text{and} \quad a > 4.
\]  
(2.38)
By (2.38), we compute and obtain that \( \Lambda_a(1) < \Lambda_{4.108}(1) \approx -0.0356 \) \( < 0 \). Thus \( 1 < \bar{p}_2(a) \) for \( 4 < a \leq 4.108 \).

**Step 2.** We prove that \( B(u) < 0 \) for \( 0 < u < \bar{p}_2 \) and \( 4 < a \leq 4.108 \). Clearly, \( B(p_2) = 0 \). By (2.38), we see that, for \( a > 4 \),
\[
B(0) = 2 \int_0^{p_2} \frac{s^2}{(a+s)^2} f(s) ds - \frac{p_2^3 f(p_2)}{(a + p_2)^2} f(p_2) = \frac{\partial}{\partial a} \theta(p_2) = -\frac{\partial}{\partial a} \Lambda_a(0) < 0.
\]
We assert that there exists \( \mu_1 \in (0, p_2) \) such that
\[
B'(u) \begin{cases} 
< 0 & \text{for} \ 0 \leq u < \mu_1, \\
= 0 & \text{for} \ u = \mu_1, \\
> 0 & \text{for} \ \mu_1 < u < p_2.
\end{cases}
\]  
(2.39)
Thus \( B(u) < 0 \) for \( 0 \leq u < p_2 \). It implies that \( B(u) < 0 \) for \( 0 < u < \bar{p}_2 \).

Next, we prove assertion (2.39). We compute that
\[
B'(u) = \frac{f(u)}{a(a+u)^4 \sqrt{a^2 - 4u}} B(u),
\]  
(2.40)
We compute that 
\[ \bar{B}(u) = a \left[ -u^4 + (-a^2 - 4a)u^3 + (a^4 - 6a^2)u^2 + (a^4 - 4a^3)u - a^4 \right] \]
\[ + \sqrt{a^2 - 4a} (u + a) \left[ (-a - 1)u^3 + (a^3 - 3a)u^2 + (a^3 - 3a^2)u - a^3 \right]. \]

We further compute that 
\[ \bar{B}''(u) = -12 \left[ a + (a + 1) \sqrt{a^2 - 4a} \right] u^2 + \left[ -6a^3 - 24a^2 + 6a (a^2 - a - 4) \sqrt{a^2 - 4a} \right] u \]
\[ + 2 (a^2 - 6) a^3 + 2a^2 (a + 3)(a - 2) \sqrt{a^2 - 4a}. \]

Obviously, the leading coefficient of quadratic polynomial \( \bar{B}''(u) \) is negative and \( \bar{B}''(0) > 0 \). So there exists \( \mu_2 > 0 \) such that
\[ \bar{B}''(u) = \begin{cases} > 0 & \text{for } 0 \leq u < \mu_2, \\ = 0 & \text{for } u = \mu_2, \\ < 0 & \text{for } u > \mu_2. \end{cases} \tag{2.41} \]

We compute that, for \( a > 4 \),
\[ \bar{B}'(0) = a^3 (a - 4)(a + \sqrt{a^2 - 4a}) > 0, \tag{2.42} \]
\[ \bar{B}'(\gamma) = 2a^3 \left[ -2a^2 + 3a + (a - 2) \sqrt{a^2 - 4a} \right] < 2a^3 \left[ -2a^2 + 3a + (a - 2) a \right] \]
\[ = -2a^3 (a - 1) < 0. \tag{2.43} \]

Since \( \gamma(a) < p_2(a) \) for \( a > 4 \), and by (2.41)–(2.43), there exists \( \mu_3 \in (0, p_2) \) such that
\[ \bar{B}'(u) = \begin{cases} > 0 & \text{for } 0 \leq u < \mu_3, \\ = 0 & \text{for } u = \mu_3, \\ < 0 & \text{for } \mu_3 < u < p_2. \end{cases} \tag{2.44} \]

We compute that \( \bar{B}(0) = -a^4 (a + \sqrt{a^2 - 4a}) < 0 \) and \( \bar{B}(p_2) = 0 \) for \( a > 4 \). So by (2.40) and (2.44), assertion (2.39) holds.

**Step 3.** We prove that \( C(u) < 0 \) for \( p_2 \leq u < p_2 \). By Step 1, Lemma 2.2 and (2.38), we observe that, for \( 4 < a \leq 4.108 \),
\[ M_a > 3.6, \quad \theta(p_2) - \theta(1) > 0, \quad \frac{a^2 p_2}{(a + p_2)^2} = 1, \tag{2.45} \]
\[ 2 \int_{1}^{p_2} \frac{s^2}{(a + s)^2} f(s) ds - \frac{p_3^2}{(a + p_2)^2} f(p_2) + \frac{1}{(a + 1)} f(1) = -\frac{\partial}{\partial a} \Lambda_a(1) < 0. \tag{2.46} \]

By (2.18), (2.45) and (2.46), we obtain that, for \( 4 < a \leq 4.108 \),
\[ C(1) = 2 \frac{p_2}{p_2} \left[ p_2 f(p_2) - f(1) \right] + 2 \int_{1}^{p_2} \frac{s^2}{(a + s)^2} f(s) ds - \left[ 2 p_2' + \frac{p_3^2}{(a + p_2)^2} \right] f(p_2) \]
\[ + \left[ p_2' + \frac{p_3 a^2}{p_2 (a + 1)^2} + \frac{1}{(a + 1)^2} \right] f(1) - \left( \frac{3}{2} M_a - 5.4 \right) [\theta(p_2) - \theta(1)] \]
\[ = \frac{p_2'}{p_2} \left[ \frac{a^2}{(a + 1)^2} - 1 \right] f(1) - \frac{3}{2} (M_a - 3.6) [\theta(p_2) - \theta(1)] - \frac{\partial}{\partial a} \Lambda_a(1) < 0. \tag{2.47} \]
We assert that there exists $\mu_4 \in (1, p_2)$ such that
\[
\begin{align*}
\text{either } C'(u) > 0 & \text{ for } 1 < u < p_2, \quad \text{or } C'(u) \begin{cases} < 0 & \text{for } 1 \leq u < \mu_4, \\ = 0 & \text{for } u = \mu_4, \\ > 0 & \text{for } \mu_4 < u \leq p_2. \end{cases} 
\end{align*}
\] (2.48)

By Step 1, we note that $1 < p_2(a)$ for $4 < a \leq 4.108$. Since $C(p_2) = 0$, and by (2.47) and (2.48), we see that $C(u) < 0$ for $p_2 \leq u < p_2$.

Next, we prove assertion (2.48). We compute that
\[
C'(u) = \frac{f(u)}{10a(a-4)\left[a + \sqrt{a(a-4)}\right]^2(a+u)^4} \tilde{C}(u),
\] (2.49)

where
\[
\tilde{C}(u) \equiv a(a-4)\left[(-83a^2 + 141a + 40)u^4 + (83a^4 - 473a^3 + 444a^2 + 160a)u^3 \\
+ (166a^5 - 680a^4 + 566a^3 + 240a^2)u^2 + (63a^6 - 353a^5 + 364a^4 + 160a^3)u \\
- 63a^6 + 101a^5 + 40a^4 \right] \\
+ \sqrt{a(a-4)}\left[(-83a^3 + 307a^2 + 180a)u^4 + (83a^5 - 639a^4 + 968a^3 + 720a^2)u^3 \\
+ (166a^6 - 1012a^5 + 1162a^4 + 1080a^3)u^2 \\
+ (63a^7 - 479a^6 + 728a^5 + 720a^4)u - 63a^7 + 227a^6 + 180a^5 \right].
\]

We further compute that $\tilde{C}''(u) = \psi_2(a)u^2 + \psi_1(a)u + \psi_0(a)$ where
\[
\psi_2(a) \equiv -12a(a-4)\left(83a^3 - 141a - 40\right) - 12a\sqrt{a(a-4)}\left(83a^2 - 307a - 180\right),
\]
\[
\psi_1(a) \equiv 6a^2(a-4)\left(83a^3 - 473a^2 + 444a + 160\right) \\
- 6a^2\sqrt{a(a-4)}\left(-83a^3 + 639a^2 - 968a - 720\right),
\]
\[
\psi_0(a) \equiv 4a^3(a-4)\left(83a^3 - 340a^2 + 283a + 120\right) \\
+ 4a^3\sqrt{a(a-4)}(83a^3 - 506a^2 + 581a + 540).
\]

Since $83a^2 - 307a - 180 < 0$ for $4 < a \leq 4.108$, we observe that, for $4 < a \leq 4.108$,
\[
\psi_2(a) \leq -12a(a-4)\left(83a^2 - 141a - 40\right) - 12a^2(a-4)\left(83a^2 - 307a - 180\right) \\
= -12a(a-4)\left(83a^3 - 224a^2 - 321a - 40\right) < 0.
\]

It implies that the quadratic polynomial $\tilde{C}''(u)$ of $u$ has a negative leading coefficient. Similarly, we observe that $\tilde{C}''(0) = \psi_0(a) > 0$ for $4 < a \leq 4.108$. Then there exists $\mu_5 > 0$ such that
\[
\tilde{C}''(u) \begin{cases} > 0 & \text{for } 0 \leq u < \mu_5, \\ = 0 & \text{for } u = \mu_5, \\ < 0 & \text{for } u > \mu_5. \end{cases}
\] (2.50)
From Figure 2.4, we further see that, for \( 4 < a \leq 4.108 \),
\[
\bar{C}'(1) = a(a-4)(63a^6 - 21a^5 - 747a^4 - 127a^3 + 1480a^2 + 1044a \\
+ 160) + a\sqrt{a(a-4)(63a^6 - 147a^5 - 1047a^4 + 1127a^3} \\
+ 4732a^2 + 3388a + 720) > 0,
\]
(2.51)
\[
\bar{C}'(p_2(a)) = -a^6(a-4)(83a^3 - 473a^2 + 539a + 140) \\
- a^5\sqrt{a(a-4)}(83a^4 - 639a^3 + 1319a^2 - 324a - 80) < 0.
\]
(2.52)
By (2.50)–(2.52), for \( 4 < a \leq 4.108 \), there exists \( \mu_6 \in (1, p_2) \) such that
\[
\bar{C}'(u) \begin{cases} 
> 0 & \text{for } 1 \leq u < \mu_6, \\
= 0 & \text{for } u = \mu_6, \\
< 0 & \text{for } \mu_6 < u \leq p_2.
\end{cases}
\]
(2.53)
We compute that \( \bar{C}(p_2) = 0 \). So by (2.53), we see that either \( \Psi_3(u) > 0 \) for \( 1 < u < p_2 \), or
\[
\bar{C}(u) \begin{cases} 
< 0 & \text{for } 1 \leq u < \eta_6, \\
= 0 & \text{for } u = \eta_6, \\
> 0 & \text{for } \eta_6 < u \leq p_2
\end{cases}
\]
for some \( \eta_6 \in (1, p_2) \).
So by (2.49), (2.48) holds.

The proof of Lemma 2.3 is complete.

By numerical simulations, we compute and find that
(i) \( T'_{4.075}(4.8) \approx -3.461 \times 10^{-4} \) < 0,
(ii) \( T'_{4.075}(p_2(4.075)) \approx 6.596 \times 10^{-5} > 0 \),
(iii) $T'_{4.084}(\gamma(4.084)) \approx 3.351 \times 10^{-4} > 0$.
(iv) $T'_{4.084}(p_2(4.084)) \approx -2.474 \times 10^{-4} < 0$.

In fact, these inequalities can be proved by analytic techniques, see the next lemma. These results as stated in next lemma are needed in the proof of Theorem 1.2.

**Lemma 2.4.** Consider (1.1). The following assertions (i)–(iv) hold.

(i) $T'_{4.075}(4.8) < 0$.
(ii) $T'_{4.075}(p_2(4.075)) = T'_{4.075}\left(\frac{13529}{3200} + \frac{163}{3200}\sqrt{489}\right) > 0$.
(iii) $T'_{4.084}(\gamma(4.084)) = T'_{4.084}\left(\frac{531941}{125000}\right) > 0$.
(iv) $T'_{4.084}(p_2(4.084)) = T'_{4.084}\left(\frac{531941}{125000} + \frac{1021}{125000}\sqrt{21441}\right) < 0$.

**Proof of Lemma 2.4.** The proofs of assertions (i)–(iv) are similar. So we only prove assertion (i) while the proofs of assertions (ii)–(iv) are omitted. Let

$$X_1(u) \equiv -\frac{1}{5} \left( u - \frac{24}{5} \right) (4u + 23), \quad X_2(u) \equiv -\frac{1}{10} \left( u - \frac{24}{5} \right) (9u + 48),$$

$$X_3(u) \equiv -\frac{29}{5000} \left( u - \frac{383}{100} \right)^2 + \frac{10083}{5000}.$$

Assume that $a = 4.075$. By Figure 2.5, we obtain that

$$X_1(u) < F(4.8) - F(u) < X_2(u) \quad \text{for } 0 \leq u \leq 4.8, \quad (2.54)$$

$$X_3(u) [F(4.8) - F(u)] \leq 4.8f(4.8) - uf(u) \quad \text{for } 0 \leq u \leq 4.8. \quad (2.55)$$

The proofs of (2.54) and (2.55) are trivial but rather lengthy, and hence we put them in [7]. Clearly, the quartic polynomials $X_1(u) > 0$ and $X_2(u) > 0$ for $0 \leq u < 4.8$. 

![Figure 2.5](image-url)
We further see that there exists
\[ \varsigma \equiv \frac{383}{100} - \frac{1}{29} \sqrt{2407} \approx 2.138 \]
such that \( 0 < X_3(u) < 2 \) for \( 0 \leq u < \varsigma \), \( X_3(\varsigma) = 2 \) and \( X_3(u) > 2 \) for \( \varsigma < u \leq 4.8 \). So by (2.54) and (2.55), we observe that
\[
T'_{4.075}(4.8) = \frac{5}{48\sqrt{2}} \int_0^{4.8} \frac{2[F(4.8) - F(u)] - 4.8f(4.8) + uf(u)}{[F(4.8) - F(u)]^{3/2}} du \\
\leq \frac{5}{48\sqrt{2}} \int_0^{4.8} \frac{2 - X_3(u)}{\sqrt{F(4.8) - F(u)}} du \\
= \frac{5}{48\sqrt{2}} \left[ \int_\varsigma^0 \frac{2 - X_3(u)}{\sqrt{F(4.8) - F(u)}} du + \int_0^\varsigma \frac{2 - X_3(u)}{\sqrt{F(4.8) - F(u)}} du \right] \\
\leq \frac{5}{48\sqrt{2}} \left[ \int_0^\varsigma \frac{2 - X_3(u)}{\sqrt{X_1(u)}} du + \int_\varsigma^{4.8} \frac{2 - X_3(u)}{\sqrt{X_2(u)}} du \right].
\]

We compute that
\[
\int_0^\varsigma \frac{2 - X_3(u)}{\sqrt{X_1(u)}} du = \left[ \left( -\frac{29u}{40000} + \frac{97121}{8 \times 10^6} \right) \sqrt{-20u^2 - 19u + 552} \right]_0^\varsigma \\
+ \left( \frac{68634343\sqrt{5}}{8 \times 10^8} \arcsin \left( \frac{40}{211}u + \frac{19}{211} \right) \right) \right]_0^\varsigma \\
\approx 0.01391,
\]

\[
\int_\varsigma^{4.8} \frac{2 - X_3(u)}{\sqrt{X_2(u)}} du = \left[ \left( -\frac{29u}{90000} + \frac{11687}{2250000} \right) \sqrt{-90u^2 - 48u + 2304} \right]_\varsigma^{4.8} \\
+ \left( \frac{23277863\sqrt{10}}{45 \times 10^7} \arcsin \left( \frac{15}{76}u + \frac{1}{19} \right) \right) \right]_\varsigma^{4.8} \\
\approx -0.01455.
\]

Thus we obtain that
\[
T'_{4.075}(4.8) \leq \frac{5}{48\sqrt{2}} \left[ \int_0^\varsigma \frac{2 - X_3(u)}{\sqrt{X_1(u)}} du + \int_\varsigma^{4.8} \frac{2 - X_3(u)}{\sqrt{X_2(u)}} du \right] \approx -4.7 \times 10^{-5} < 0.
\]

The proof of Lemma 2.4 is complete. \( \square \)

3 Proof of the main result

Since \( \lim_{a \to \infty} p_1(a) = \lim_{a \to \infty} \frac{a(a-2)-a\sqrt{a(a-4)}}{2} = 1 \) and
\[
p_1'(a) = \frac{(a-1)\sqrt{a^2 - 4a} - a(a-3)}{\sqrt{a^2 - 4a}} < 0 \quad \text{for} \ a > 4,
\]
we obtain that $p_1(a) > 1$ for $a > 4$. Assume that $a > a_0$. By Theorem 1.1, we see that $T_\alpha(u)$ has exactly two critical points, a local maximum at some $\alpha_M(a) = \|u_{\lambda_0}\|_\infty$ and a local minimum at some $\alpha_m(a) = \|u_{\lambda_0}\|_\infty (> \alpha_M(a))$, see Figure 2.1. By [6, Lemma 25], we have that

$$a_M(a) < \lim_{a \to a_0^-} a_M(a) = \lim_{a \to a_0^+} a_m(a) = \|u_{\lambda_0}\|_\infty < a_m(a).$$

(3.1)

Thus (1.11) holds immediately. By [6, Lemma 12], we see that $\theta(p_1) - \theta(u) > 0$ for $0 \leq u < p_2$ and $a > 4$. So by (2.2), we further see that $T_\alpha'(p_1) > 0$ for $a > 4$. Since $a_0 > 4$, we see that $p_1(a) < a_M(a)$ for $a > a_0$. In addition, since $4.8 < p_2(4.075) \approx 5.354$, and by Lemmas 2.1, 2.3 and 2.4, we see that $a_0 < 4.075 < \hat{\alpha} < 4.084 < \hat{\alpha}$.

(3.2)

By Lemma 2.1 and (3.2), it is easy to see that $\gamma(\alpha) = a_m(\alpha)$ or $\gamma(\alpha) = a_M(\alpha)$. Suppose to the contrary that $a_M(\hat{\alpha}) < a_m(\hat{\alpha}) = \gamma(\alpha)$. By [6, Lemma 25(i)], we see that $\gamma(\alpha)$ and $a_M(\alpha)$ are continuous functions of $a > a_0$. So by Lemma 2.1, we observe that $a_M(\alpha) < a_m(\alpha) < \gamma(\alpha)$ for $a_0 < a < \hat{\alpha}$. It implies that $T_\alpha(\alpha)$ has two critical points on $(0, \gamma)$, which is a contradiction by [12, Lemma 3.2]. Thus $\gamma(\alpha) = a_M(\alpha)$. Then since $\gamma'(\alpha) = a - 1 > 0$ for $a > 4$, and by [6, Lemma 25(i)], we see that $\gamma(\alpha)$ and $a_M(\alpha)$ are strictly increasing and strictly decreasing on $(a_0, \infty)$, respectively. So we obtain that

$$\begin{cases}
\gamma(\alpha) = a_M(\alpha) \quad &\text{for } a = \hat{\alpha}, \\
\gamma(\alpha) < a_M(\alpha) \quad &\text{for } a_0 < a < \hat{\alpha}.
\end{cases}$$

(3.4)

By Lemma 2.3, we have that $a_M(\alpha) < p_2(\alpha) < a_m(\alpha)$ for $a > \hat{\alpha}$. So by (3.2) and (3.4),

$$\begin{cases}
\gamma(\alpha) = a_M(\alpha) < p_2(\alpha) < a_M(\alpha) \quad &\text{for } a = \hat{\alpha}, \\
\gamma(\alpha) < a_M(\alpha) < p_2(\alpha) < a_m(\alpha) \quad &\text{for } a_0 < a < \hat{\alpha}.
\end{cases}$$

(3.5)

By Lemma 2.3 and (3.2), it is easy to see that $p_2(\hat{\alpha}) = a_M(\hat{\alpha})$ or $p_2(\hat{\alpha}) = a_m(\hat{\alpha})$. Suppose to the contrary that $p_2(\hat{\alpha}) = a_M(\hat{\alpha}) < a_m(\hat{\alpha})$. Since $p_2(\alpha)$ and $a_M(\alpha)$ are strictly increasing and strictly decreasing on $(a_0, \infty)$ respectively, and by (2.18) and (3.2), we obtain that

$$4.8 < (5.35 \approx) p_2(4.075) < p_2(\hat{\alpha}) = a_M(\hat{\alpha}) < a_M(4.075).$$

It follows that $T_{4.075}'(4.8) > 0$, which is a contradiction by Lemma 2.4(i). Thus $a_M(\hat{\alpha}) < a_m(\hat{\alpha}) = p_2(\hat{\alpha})$. By Lemma 2.3 and continuity of $a_M(\alpha)$ and $p_2(\alpha)$ on $(a_0, \infty)$, we find that $a_M(a) < a_m(a) < p_2(\alpha)$ for $\alpha_0, \hat{\alpha}$. Thus we have that

$$\begin{cases}
\gamma(\alpha) < a_M(\alpha) < a_m(\alpha) = p_2(\alpha) \quad &\text{for } a = \hat{\alpha}, \\
\gamma(\alpha) < a_M(\alpha) < a_m(\alpha) < p_2(\alpha) \quad &\text{for } a_0 < a < \hat{\alpha}.
\end{cases}$$

(3.6)

By (3.3), (3.5) and (3.6), inequalities (1.6)–(1.10) hold.

Finally, we prove (1.12). We compute and observe that

$$\theta'(u) = \frac{t^2 - (a^2 - 2a) t + a^2}{(a + t)^2} f(t) \begin{cases}
> 0 &\text{for } u \in (0, p_1) \cup (p_2, \infty), \\
= 0 &\text{for } u \in \{p_1, p_2\}, \\
< 0 &\text{for } u \in (p_1, p_2)
\end{cases}$$

(3.7)
Since
\[
\frac{a \gamma(a) - p_2(a)}{p_1(a)} = \frac{a(a-1)(a-2) - a\sqrt{a^2 - 4a}}{2} \geq \frac{a(a-1)(a-2) - a^2}{2} \geq 0 \quad \text{for} \quad a \geq \bar{a} > 4,
\]
we see that \(p_1(a) < p_2(a) < a \gamma(a)\) for \(a \geq \bar{a}\). Since \(f'(u) > 0\) for \(u > 0\), and by (3.7), we compute and observe that
\[
\theta(a \gamma) - \theta(p_1) = \int_{p_1}^{a \gamma} \theta'(t) dt = \int_{p_1}^{p_2} \theta'(t) dt + \int_{p_2}^{a \gamma} \theta'(t) dt \\
\geq f(p_2) \left[ \int_{p_1}^{p_2} \frac{t^2 - (a^2 - 2a) t + a^2}{(a^2 + t)^2} dt + \int_{p_2}^{a \gamma} \frac{t^2 - (a^2 - 2a) t + a^2}{(a^2 + t)^2} dt \right] \\
= f(p_2) \left[ \int_{p_1}^{p_2} \frac{t^2 - (a^2 - 2a) t + a^2}{(a^2 + t)^2} dt \right] = f(p_2) \left[ t - \frac{a^3}{a^2 + t^2} - a^2 \ln(a^2 + t) \right]_{p_1}^{a \gamma} \\
= \frac{a}{2 (a^2 - 2a + 2)} \left[ a - \sqrt{a(a - 4)} \right] K(a), \tag{3.8}
\]
where
\[
K(a) \equiv a(a^4 - 6a^3 + 20a^2 - 32a + 20) - \sqrt{a(a - 4)(a^4 - 6a^3 + 12a^2 - 16a + 4)} \\
- 2a(a^2 - 2a + 2) \left[ a - \sqrt{a(a - 4)} \right] \ln \left( \frac{a^2 - 2a + 2}{a - \sqrt{a(a - 4)}} \right).
\]

From Figure 3.1, we observe that \(K(a)\) is a strictly increasing and positive function of \(a \geq 4.06\).

![Graph of K(a) for a ≥ 4.06](image)

Figure 3.1: The graph of \(K(a)\) for \(a \geq 4.06\).

Since \(\bar{a} > a_0 \approx 4.069 > 4.06\), and by (3.8), we have that \(\theta(a \gamma) - \theta(p_1) > 0\) for \(a \geq \bar{a}\). Since \(\theta(0) = 0\), and by (3.7), we observe that \(\theta(a) > \theta(u)\) for \(0 < u < a \gamma(a), a \geq a \gamma(a)\) and \(a \geq \bar{a}\). So by (2.2), we obtain that \(T_u^a(a) > 0\) for \(a \geq a \gamma(a)\) and \(a \geq \bar{a}\). It follows that \(a_m(a) < a \gamma(a)\) for \(a \geq \bar{a}\). So by (1.6)–(1.9) and (3.1), we see that
\[
\frac{a \gamma(a)}{p_1(a)} < \frac{a_m(a)}{\alpha M(a)} = \frac{\|u_{\lambda_n}\|_{\infty}}{\|u_{\lambda_m}\|_{\infty}} > \frac{p_2(a)}{\|u_{\lambda_n}\|_{\infty}} \quad \text{for} \quad a \geq \bar{a},
\]
Thus (1.12) holds.

The proof of Lemma 1.2 is complete.

**Remark 3.1.** By numerical simulations, we find that \( \hat{a} \approx 4.088 \) and \( \tilde{a} \approx 4.077 \).

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**References**


