Isospectral Dirac operators

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Abstract. We give the description of self-adjoint regular Dirac operators, on $[0, \pi]$, with the same spectra.

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1 Introduction and statement of result

Let $p$ and $q$ are real-valued, summable on $[0, \pi]$ functions, i.e. $p, q \in L^1_R[0, \pi]$. By $L(p, q, \alpha) = L(\Omega, \alpha)$ we denote the boundary-value problem for canonical Dirac system (see [5,6,9,13,14]):

\[
\ell y \equiv \left\{ B \frac{d}{dx} + \Omega(x) \right\} y = \lambda y, \quad x \in (0, \pi), \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \lambda \in \mathbb{C}, \quad (1.1)
\]
\[
y_1(0) \cos \alpha + y_2(0) \sin \alpha = 0, \quad \alpha \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right), \quad (1.2)
\]
\[
y_1(\pi) = 0, \quad (1.3)
\]

where
\[
B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}.
\]

By the same $L(p, q, \alpha)$ we also denote a self-adjoint operator generated by differential expression $\ell$ in Hilbert space of two component vector-function $L^2([0, \pi]; \mathbb{C}^2)$ on the domain

\[
D = \left\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; \ y_k \in AC[0, \pi], \ (\ell y)_k \in L^2[0, \pi], \ k = 1, 2; \right. \]
\[
\left. y_1(0) \cos \alpha + y_2(0) \sin \alpha = 0, \quad y_1(\pi) = 0 \right\}
\]

where $AC[0, \pi]$ is the set of absolutely continuous functions on $[0, \pi]$ (see, e.g. [13,16]). It is well known (see [1,5,9]) that under these conditions the spectra of the operator $L(p, q, \alpha)$ is purely discrete and consists of simple, real eigenvalues, which we denote by $\lambda_n = \lambda_n(p, q, \alpha) = \ldots$
λ_n(Ω, α), n ∈ ℤ, to emphasize the dependence of λ_n on quantities p, q and α. It is also well known (see, e.g. [1, 5, 9]) that the eigenvalues form a sequence, unbounded below as well as above. So we will enumerate it as λ_k < λ_{k+1}, k ∈ ℤ, λ_k > 0, when k > 0 and λ_k < 0, when k < 0, and the nearest to zero eigenvalue we will denote by λ_0. If there are two nearest to zero eigenvalue, then by λ_0 we will denote the negative one. With this enumeration it is proved (see [1, 5, 9]), that the eigenvalues have the asymptotics:

\[ \lambda_n(Ω, α) = n - \frac{α}{π} + r_n, \quad r_n = o(1), \quad n \to \pm∞. \quad (1.4) \]

In what follows, writing Ω ∈ A will mean p, q ∈ A. If Ω ∈ L^2_R[0, π], then we know, (see, e.g. [9]), that instead of r_n = o(1) we have:

\[ \sum_{n=-∞}^{∞} r_n^2 < ∞. \quad (1.5) \]

Let Φ(x, λ) = Φ(x, λ, α, Ω) be the solution of the Cauchy problem

\[ \ell Φ = λ Φ, \quad Φ(0, λ) = \begin{pmatrix} \sin α \\ -\cos α \end{pmatrix}. \quad (1.6) \]

Since the differential expression \( \ell \) self-adjoint, then the components \( Φ_1(x, λ) \) and \( Φ_2(x, λ) \) of the vector-function \( Φ(x, λ) \) we can choose real-valued for real \( λ \). By \( a_n = a_n(Ω, α) \) we denote the squares of the \( L^2 \)-norm of the eigenfunctions \( Φ_n(x, Ω) = Φ(x, λ_n(Ω, α), α, Ω) \):

\[ a_n = ||Φ_n||^2 = ∫_0^π |Φ_n(x, Ω)|^2 dx, \quad n ∈ ℤ. \]

The numbers \( a_n \) are called norming constants. And by \( h_n(x, Ω) \) we will denote normalized eigenfunctions (i.e. \( ||h_n(x)|| = 1 \)):

\[ h_n(x, Ω) = h_n(x) = \frac{Φ_n(x, Ω)}{\sqrt{a_n(Ω, α)}}. \quad (1.7) \]

It is known (see [5, 9]) that in the case of \( Ω ∈ L^2_R[0, π] \) the norming constants have an asymptotic form:

\[ a_n(Ω) = π + c_n, \quad ∑_{n=-∞}^{∞} c_n^2 < ∞. \quad (1.8) \]

**Definition 1.1.** Two Dirac operators \( L(Ω, α) \) and \( L(Ω, ˜α) \) are said to be isospectral, if \( λ_n(Ω, α) = λ_n(Ω, ˜α) \), for every \( n ∈ ℤ \).

**Lemma 1.2.** Let \( Ω, ˜Ω ∈ L^2_R[0, π] \) and the operators \( L(Ω, α) \) and \( L(Ω, ˜α) \) are isospectral. Then \( ˜α = α \).

**Proof.** The proof follows from the asymptotics (1.4):

\[ \frac{α}{π} = \lim_{n→∞} \left(n - λ_n(Ω, α)\right) = \lim_{n→∞} \left(n - λ_n(Ω, ˜α)\right) = \frac{˜α}{π}. \]

So, instead of isospectral operators \( L(Ω, α) \) and \( L(Ω, ˜α) \), we can talk about “isospectral potentials” \( Ω \) and \( ˜Ω \).
Theorem 1.3 (Uniqueness theorem). The map

\[(\Omega, \alpha) \in L^2_{\mathbb{R}}[0, \pi] \times \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \leftrightarrow \{\lambda_n(\Omega, \alpha), a_n(\Omega, \alpha); n \in \mathbb{Z}\}\]

is one-to-one.

Remark 1.4. It is natural to call this a Marchenko theorem, since it is an analogue of the famous theorem of V. A. Marchenko [15], in the case for Sturm–Liouville problem. The proof of this theorem for the case \(p, q \in AC[0, \pi]\) there is in the paper [18]. The detailed proof for the case \(p, q \in L^2_{\mathbb{R}}[0, \pi]\) there is in [7] (see also [4–6, 8, 10, 19]).

Let us fix some \(\Omega \in L^2_{\mathbb{R}}[0, \pi]\) and consider the set of all canonical potentials \(\tilde{\Omega} = \left( \tilde{p}, \tilde{q}; \tilde{q} - \tilde{p} \right)\), with the same spectra as \(\Omega\):

\[M^2(\Omega) = \{\tilde{\Omega} \in L^2_{\mathbb{R}}[0, \pi]: \lambda_n(\tilde{\Omega}, \tilde{\alpha}) = \lambda_n(\Omega, \alpha), n \in \mathbb{Z}\}\].

Our main goal is to give the description of the set \(M^2(\Omega)\) as explicit as it possible.

From the uniqueness theorem the next corollary easily follows.

Corollary 1.5. The map

\[\tilde{\Omega} \in M^2(\Omega) \leftrightarrow \{a_n(\tilde{\Omega}), n \in \mathbb{Z}\}\]

is one-to-one.

Since \(\tilde{\Omega} \in M^2(\Omega)\), then \(a_n(\tilde{\Omega})\) have similar to (1.8) asymptotics. Since \(a_n(\Omega)\) and \(a_n(\tilde{\Omega})\) are positive numbers, there exist real numbers \(t_n = t_n(\tilde{\Omega})\), such that \(\frac{a_n(\Omega)}{a_n(\tilde{\Omega})} = e^{t_n}\). From the latter equality and from (1.8) follows that

\[e^{t_n} = 1 + d_n, \quad \sum_{n=-\infty}^{\infty} d_n^2 < \infty. \tag{1.9}\]

It is easy to see, that the sequence \(\{t_n; n \in \mathbb{Z}\}\) is also from \(l^2\), i.e. \(\sum_{n=-\infty}^{\infty} t_n^2 < \infty\). Since all \(a_n(\Omega)\) are fixed, then from the corollary 1.5 and the equality \(a_n(\tilde{\Omega}) = a_n(\Omega)e^{-t_n}\) we will get the following corollary.

Corollary 1.6. The map

\[\tilde{\Omega} \in M^2(\Omega) \leftrightarrow \{t_n(\tilde{\Omega}), n \in \mathbb{Z}\} \in l^2\]

is one-to-one.

Thus, each isospectral potential is uniquely determined by a sequence \(\{t_n; n \in \mathbb{Z}\}\). Note, that the problem of description of isospectral Sturm–Liouville operators was solved in [3, 11, 12, 17].

For Dirac operators the description of \(M^2(\Omega)\) is given in [8]. This description has a “recurrent” form, i.e. at the first in [8] is given the description of a family of isospectral potentials \(\Omega(x,t), t \in \mathbb{R}\), for which only one norming constant \(a_m(\Omega(\cdot,t))\) different from \(a_m(\Omega)\) (namely, \(a_m(\Omega(\cdot,t)) = a_m(\Omega)e^{-t\lambda}\)) while the others are equal, i.e. \(a_m(\Omega(\cdot,t)) = a_m(\Omega)\), when \(n \neq m\).
Theorem 1.7 ([8]). Let \( t \in \mathbb{R}, a \in \left( -\frac{n}{2}, \frac{n}{2} \right) \) and

\[
\Omega(x,t) = \Omega(x) + \frac{e^t - 1}{\theta_m(x, t, \Omega)} \{ Bh_m(x, \Omega) h_m^*(x, \Omega) - h_m(x, \Omega) h_m^*(x, \Omega) B \},
\]

where \( \theta_n(x, t, \Omega) = 1 + (e^t - 1) \int_0^t |h_n(s, \Omega)|^2 ds \), and \(*\) is a sign of transposition, e.g. \( h_m^* = \left(h_{n_1}^*, h_{n_2}^*\right) = (h_m^*, h_m^*) \). Then, for arbitrary \( t \in \mathbb{R}, \lambda_n(\Omega, t) = \lambda_n(\Omega) \) for all \( n \in \mathbb{Z}, a_n(\Omega, t) = a_n(\Omega) \) for all \( n \in \mathbb{Z} \setminus \{m\} \) and \( a_m(\Omega, t) = a_m(\Omega)e^t \). The normalized eigenfunctions of the problem \( L(\Omega(\cdot, t), \alpha) \) are given by the formulae:

\[
h_m(x, \Omega(\cdot, t)) = \begin{cases} 
\frac{e^{-t/2}}{\theta_m(x, t, \Omega)} h_m(x, \Omega), & \text{if } n = m, \\
h_n(x, \Omega) - \frac{(e^t - 1) \int_0^t h_m^*(s, \Omega) h_n(s, \Omega) ds}{\theta_m(x, t, \Omega)} h_m(x, \Omega), & \text{if } n \neq m.
\end{cases}
\]

Theorem 1.7 shows that it is possible to change exactly one norming constant, keeping the others. As examples of isospectral potentials \( \Omega \) and \( \tilde{\Omega} \) we can present \( \Omega(x) \equiv 0 = \left( \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right) \) and

\[
\tilde{\Omega}(x) = \Omega_m(t) = \frac{\pi (e^t - 1)}{\pi + (e^t - 1)x} \begin{pmatrix} \sin 2mx & -\cos 2mx \\ \cos 2mx & \sin 2mx \end{pmatrix},
\]

where \( t \in \mathbb{R} \) is an arbitrary real number and \( m \in \mathbb{Z} \) is an arbitrary integer.

Changing successively each \( a_m(\Omega) \) by \( a_m(\Omega)e^{-t_m} \), we can obtain any isospectral potential, corresponding to the sequence \( \{t_m; m \in \mathbb{Z}\} \in \ell^2 \). It follows from the uniqueness Theorem 1.3 that the sequence, in which we change the norming constants, is not important.

In [8] were used the following designations:

\[
T_{-1} = \{\ldots, 0, \ldots\}, \\
T_0 = \{\ldots, 0, t_0, 0, \ldots\}, \\
T_1 = \{\ldots, 0, 0, t_0, t_1, 0, \ldots\}, \\
T_2 = \{\ldots, 0, 0, t_0, t_1, 0, \ldots\}, \\
\vdots \\
T_{2n} = \{\ldots, 0, 0, t_{-n}, \ldots, t_1, t_0, \ldots, t_{n-1}, t_n, 0, \ldots\}, \\
T_{2n+1} = \{\ldots, 0, t_{-n}, t_{-n+1}, \ldots, t_1, t_0, \ldots, t_n, 0, \ldots\}, \\
\vdots
\]

Let \( \Omega(x, T_{-1}) \equiv \Omega(x) \) and

\[
\Omega(x, T_m) = \Omega(x, T_{m-1}) + \Delta \Omega(x, T_m), \quad m = 0, 1, 2, \ldots,
\]

where

\[
\Delta \Omega(x, T_m) = \frac{e^{t_m} - 1}{\theta_m(x, t_m, \Omega(\cdot, T_{m-1}))} \left[ Bh_m(x, \Omega(\cdot, T_{m-1})) h_m^*(\cdot) - h_m(\cdot) h_m^*(\cdot) B \right],
\]

where \( m = \frac{m+1}{2} \), if \( m \) is odd and \( m = -\frac{m}{2} \), if \( m \) is even. The arguments in others \( h_m(\cdot) \) and \( h_m^*(\cdot) \) are the same as in the first. And after that in [8] the following theorem was proved.
Theorem 1.8 ([8]). Let \( T = \{ t_n, n \in \mathbb{Z} \} \in l^2 \) and \( \Omega \in L^2_{\mathbb{R}}[0, \pi] \). Then

\[
\Omega(x, T) \equiv \Omega(x) + \sum_{m=0}^{\infty} \Delta \Omega(x, T_m) \in M^2(\Omega). \tag{1.10}
\]

We see, that each potential matrix \( \Delta \Omega(x, T_m) \) defined by normalized eigenfunctions \( h_{m_n}(x, \Omega(x, T_{m-1})) \) of the previous operator \( L(\Omega', T_{m-1}, \alpha) \). This approach we call “recursive” description.

In this paper, we want to give a description of the set \( M^2(\Omega) \) only in terms of eigenfunctions \( h_n(x, \Omega) \) of the initial operator \( L(\Omega, \alpha) \) and sequence \( T \in l^2 \). With this aim, let us denote by \( N(T_m) \) the set of the positions of the numbers in \( T_m \), which are not necessary zero, i.e.

\[
N(T_0) = \{ 0 \}, \\
N(T_1) = \{ 0, 1 \}, \\
N(T_2) = \{ -1, 0, 1 \}, \\
\vdots \\
N(T_{2n}) = \{ -n, -(n - 1), \ldots, 0, \ldots, n - 1, n \}, \\
N(T_{2n+1}) = \{ -n, -(n - 1), \ldots, 0, \ldots, n, n + 1 \}, \\
\vdots
\]

in particular \( N(T) \equiv \mathbb{Z} \). By \( S(x, T_m) \) we denote the \((m + 1) \times (m + 1)\) square matrix

\[
S(x, T_m) = \begin{pmatrix}
\delta_{ij} + (e^i - 1) & \int_0^x h_i^*(s)h_j(s)ds \\
\end{pmatrix}_{i,j \in N(T_m)} \tag{1.11}
\]

where \( \delta_{ij} \) is a Kronecker symbol. By \( S_p^{(k)}(x, T_m) \) we denote a matrix which is obtained from the matrix \( S(x, T_m) \) by replacing the \( k \)th column of \( S(x, T_m) \) by \( H_p(x, T_m) = \{ -(e^i - 1)h_{k_p}(x) \}_{k \in N(T_m)} \) column, \( p = 1, 2, \) Now we can formulate our result as follows.

Theorem 1.9. Let \( T = \{ t_k \}_{k \in \mathbb{Z}} \in l^2 \) and \( \Omega \in L^2_{\mathbb{R}}[0, \pi] \). Then the isospectral potential from \( M^2(\Omega) \), corresponding to \( T \), is given by the formula

\[
\Omega(x, T) = \Omega(x) + G(x, x, T)B - BG(x, x, T) = \begin{pmatrix}
p(x, T) & q(x, T) \\
q(x, T) & -p(x, T)
\end{pmatrix}, \tag{1.12}
\]

where

\[
G(x, x, T) = \frac{1}{\det S(x, T)} \sum_{k \in \mathbb{Z}} \begin{pmatrix}
\det S_1^{(k)}(x, T) \\
\det S_2^{(k)}(x, T)
\end{pmatrix} h_k^*(x),
\]

and \( \det S(x, T) = \lim_{m \to \infty} \det S(x, T_m) \) (the same for \( \det S_p^{(k)}(x, T), p = 1, 2 \).

In addition, for \( p(x, T) \) and \( q(x, T) \) we get explicit representations:

\[
p(x, T) = p(x) - \frac{1}{\det S(x, T)} \sum_{k \in \mathbb{Z}} \sum_{p=1}^{2} \det S_p^{(k)}(x, T)h_{k_{\alpha-p}}(x),
\]

\[
q(x, T) = q(x) + \frac{1}{\det S(x, T)} \sum_{k \in \mathbb{Z}} \sum_{p=1}^{2} (-1)^{p-1} S_p^{(k)}(x, T)h_{k_p}(x).
\]
2 Proof of Theorem 1.9

The spectral function of an operator $L(\Omega, \alpha)$ defined as

$$\rho(\lambda) = \begin{cases} 
\sum_{0<\lambda_n \leq \lambda} \frac{1}{a_n(\Omega)^2} & \lambda > 0, \\
- \sum_{\lambda < \lambda_n \leq 0} \frac{1}{a_n(\Omega)^2} & \lambda < 0,
\end{cases}$$

i.e. $\rho(\lambda)$ is left-continuous, step function with jumps in points $\lambda = \lambda_n$ equals $\frac{1}{a_n}$ and $\rho(0) = 0$.

Let $\Omega, \hat{\Omega} \in L^2[0, \pi]$ and they are isospectral. It is known (see [1, 2, 6, 13]), that there exists a function $G(x, y)$ such that:

$$\varphi(x, \lambda, \alpha, \hat{\Omega}) = \varphi(x, \lambda, \alpha, \Omega) + \int_0^x G(x, s) \varphi(s, \lambda, \alpha, \Omega) ds.$$  \hfill (2.1)

It is also known (see, e.g. [1, 6, 13]), that the function $G(x, y)$ satisfies to the Gelfand–Levitan integral equation:

$$G(x, y) + F(x, y) + \int_0^x G(x, s)F(s, y) ds = 0, \quad 0 \leq y \leq x, \hfill (2.2)$$

where

$$F(x, y) = \int_{-\infty}^{\infty} \varphi(x, \lambda, \alpha, \Omega) \varphi^*(y, \lambda, \alpha, \Omega) d[\tilde{\rho}(\lambda) - \rho(\lambda)]. \hfill (2.3)$$

If the potential $\hat{\Omega}$ from $M^2(\Omega)$ is such that only finite norming constants of the operator $L(\hat{\Omega}, \alpha)$ are different from the norming constants of the operator $L(\Omega, \alpha)$, i.e. $a_n(\hat{\Omega}) = a_n(\Omega)e^{-i\lambda_n}$, $n \in N(T_m)$ and the others are equal, then it means, that

$$d\tilde{\rho}(\lambda) - d\rho(\lambda) = \sum_{k \in N(T_m)} \left( \frac{1}{\tilde{a}_k} - \frac{1}{a_k} \right) \delta(\lambda - \lambda_k) d\lambda = \sum_{k \in N(T_m)} \left( \frac{e^{i\lambda_k} - 1}{a_k} \right) \delta(\lambda - \lambda_k) d\lambda, \hfill (2.4)$$

where $\delta$ is Dirac $\delta$-function. In this case the kernel $F(x, y)$ can be written in a form of a finite sum (using notation (1.7)):

$$F(x, y) = F(x, y, T_m) = \sum_{k \in N(T_m)} (e^{i\lambda_k} - 1)h_k(x, \Omega)h_k^*(y, \Omega), \hfill (2.5)$$

and consequently, the integral equation (2.2) becomes to an integral equation with degenerated kernel, i.e. it becomes to a system of linear equations and we will look for the solution in the following form:

$$G(x, y, T_m) = \sum_{k \in N(T_m)} g_k(x)h^*_k(y), \hfill (2.6)$$

where $g_k(x) = \left( \frac{g_{1k}(x)}{g_{2k}(x)} \right)$ is an unknown vector-function. Putting the expressions (2.5) and (2.6) into the integral equation (2.2) we will obtain a system of algebraic equations for determining the functions $g_k(x)$:
The systems (2.8) might be written in matrix form 

\[ g_k(x) + \sum_{i \in N(T_m)} s_{ik}(x)g_i(x) = -(e^{ih} - 1)h_k(x), \quad k \in N(T_m), \tag{2.7} \]

where

\[ s_{ik}(x) = (e^{ih} - 1) \int_0^x h_i^*(s)h_k(s)ds. \]

It would be better if we consider the equations (2.7) for the vectors \( g_k = \begin{pmatrix} g_{k_1} \\ g_{k_2} \end{pmatrix} \) by coordinates \( g_{k_1} \) and \( g_{k_2} \) to be a system of scalar linear equations:

\[ g_{k_p}(x) + \sum_{i \in N(T_m)} s_{ik}(x)g_{ip}(x) = -(e^{ih} - 1)h_{kp}(x), \quad k \in N(T_m), \quad p = 1, 2. \tag{2.8} \]

The systems (2.8) might be written in matrix form

\[ S(x, T_m)g_p(x, T_m) = H_p(x, T_m), \quad p = 1, 2, \tag{2.9} \]

where the column vectors \( g_p(x, T_m) = \{g_{kp}(x, T_m)\}_{k \in N(T_m)}, \quad p = 1, 2, \) and the solution can be found in the form (Cramer’s rule):

\[ g_{kp}(x, T_m) = \frac{\det S_p^{(k)}(x, T_m)}{\det S(x, T_m)}, \quad k \in N(T_m), \quad p = 1, 2. \]

Thus we have obtained for \( g_k(x) \) the following representation:

\[ g_k(x, T_m) = \frac{1}{\det S(x, T_m)} \begin{pmatrix} \det S_1^{(k)}(x, T_m) \\ \det S_2^{(k)}(x, T_m) \end{pmatrix} \tag{2.10} \]

and then by putting (2.10) into (2.6) we find the \( G(x, y, T_m) \) function. If the potential \( \Omega \) is from \( L^1_{K_r} \) then such is also the kernel \( G(x, x, T_m) \) (see [8]), and the relation between them gives as follows:

\[ \Omega(x, T_m) = \Omega(x) + G(x, x, T_m)B - BG(x, x, T_m). \tag{2.11} \]

On the other hand we have

\[ \Omega(x, T_m) = \Omega(x) + \sum_{k=0}^m \Delta \Omega(x, T_k). \tag{2.12} \]

So, using the Theorem 1.8 and the equality (2.12) we can pass to the limit in (2.11), when \( m \to \infty \):

\[ \Omega(x, T) = \Omega(x) + G(x, x, T)B - BG(x, x, T). \tag{2.13} \]

The potentials \( \Omega(x, T) \) in (1.10) and (2.13) have the same spectral data \( \{\lambda_n(T), a_n(T)\}_{n \in \mathbb{Z}} \), and therefore they are the same and \( \Omega(\cdot, T) \) defined by (2.13) is also from \( M^2(\Omega) \).

Using (2.6) and (2.10) we calculate the expression \( G(x, x, T_m)B - BG(x, x, T_m) \) and pass to the limit, obtaining for the \( p(x, T) \) and \( q(x, T) \) the representations:
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\[ p(x, T) = p(x) - \frac{1}{\det S(x, T)} \sum_{k \in \mathbb{N}(T)} \sum_{p=1}^{2} \det S_p^{(k)}(x, T) h_{k(3-p)}(x), \]

\[ q(x, T) = q(x) + \frac{1}{\det S(x, T)} \sum_{k \in \mathbb{N}(T)} \sum_{p=1}^{2} (-1)^{p-1} S_p^{(k)}(x, T) h_{k_p}(x). \]

Theorem 1.9 is proved.

For example, when we change just one norming constant (e.g. for \( T_0 \)) we get two independent linear equations:

\[ (1 + s_{00}(x)) g_0_1(x) = -(e^{t_0} - 1) h_{0_1}(x), \]

\[ (1 + s_{00}(x)) g_0_2(x) = -(e^{t_0} - 1) h_{0_2}(x). \]

For the solutions we get:

\[ g_0_1(x) = \frac{-(e^{t_0} - 1) h_{0_1}(x)}{1 + s_{00}(x)}, \]

\[ g_0_2(x) = \frac{-(e^{t_0} - 1) h_{0_2}(x)}{1 + s_{00}(x)}, \]

and for the potentials \( p(x, T_0) \) and \( q(x, T_0) \):

\[ p(x, T_0) = p(x) + \frac{e^{t_0} - 1}{1 + s_{00}(x)} (2h_{0_1}(x) h_{0_2}(x)), \]

\[ q(x, T_0) = q(x) + \frac{e^{t_0} - 1}{1 + s_{00}(x)} (h_{0_2}^2(x) - h_{0_1}^2(x)). \]

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References


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