A generalized Picard–Lindelöf theorem

Stefan Siegmund 1, Christine Nowak 2 and Josef Diblík 3, 4

1 Institute for Analysis & Center for Dynamics, Department of Mathematics, Technische Universität Dresden, 01062 Dresden, Germany
2 Institute for Mathematics, University of Klagenfurt, 9020 Klagenfurt, Austria
3 Brno University of Technology, Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, 602 00 Brno, Czech Republic
4 Brno University of Technology, Department of Mathematics, Faculty of Electrical Engineering and Communication, 616 00 Brno, Czech Republic

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Abstract. We generalize the Picard–Lindelöf theorem on the unique solvability of initial value problems \( \dot{x} = f(t, x), \ x(t_0) = x_0 \), by replacing the sufficient classical Lipschitz condition of \( f \) with respect to \( x \) with a more general Lipschitz condition along hyperspaces of the \((t, x)\)-space. A comparison with known results is provided and the generality of the new criterion is shown by an example.

Keywords: Picard–Lindelöf theorem, initial value problem, generalized Lipschitz condition, unique solvability.

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1 Introduction

We consider the initial value problem

\[ \dot{x} = f(t, x), \quad x(t_0) = x_0, \tag{1.1} \]

where \( f : D \to \mathbb{R}^n \) is defined on an open set \( D \subseteq \mathbb{R} \times \mathbb{R}^n \) and \((t_0, x_0) \in D\). We assume throughout the paper that \( f \) is continuous. Problem (1.1) is called locally uniquely solvable if there exists an open interval \( I \) containing \( t_0 \) such that (1.1) has exactly one solution on \( I \).

The unique solvability problem of (1.1) is not fully solved up to now as simple examples show (see [2] and the references therein, see also [1]). The classical Lipschitz condition measures the vector field differences with respect to the \( x \) variable and is assumed in the classical Picard–Lindelöf theorem to prove unique solvability for (1.1). By introducing a Lipschitz condition along a hyperspace of the extended state space \( \mathbb{R} \times \mathbb{R}^n \), we establish a new uniqueness theorem which generalizes the classical Picard–Lindelöf theorem and Theorem 3.2 in the paper by Cid [2]. It is also an \( n \)-dimensional generalization of the scalar criterion in [6] and of the uniqueness theorem in [3] if the functions \( \varphi \) and \( \psi \) are constants. The advantage of our result is shown by an example.

\( \odot \) Corresponding author. Email: stefan.siegmund@tu-dresden.de
Definition 1.1 (Lipschitz continuity along a hyperspace). Let $D \subseteq \mathbb{R} \times \mathbb{R}^n$ be open, $f: D \to \mathbb{R}^n$ be continuous and let $V \subset \mathbb{R} \times \mathbb{R}^n$ be a hyperspace, i.e. $V$ is an $n$-dimensional linear subspace of $\mathbb{R}^{1+n}$. We say that $f$ is Lipschitz continuous along $V$ on an open set $U \subseteq D$ if there exists a constant $L \geq 0$ such that for all $(t,x),(s,y) \in U$

$$\|f(t,x) - f(s,y)\| \leq L\|(t,x) - (s,y)\| \quad \text{if} \quad (t,x) - (s,y) \in V.$$ 

2 Main result

In the following let $F(t,x) = (1, f(t,x))^T$ be the vector of the direction field of (1.1) determined by $f$ at the point $(t,x) \in D$.

Theorem 2.1 (Generalized Picard–Lindelöf theorem). Consider the initial value problem (1.1), let $V \subset \mathbb{R} \times \mathbb{R}^n$ be a hyperspace and assume that the following two conditions hold:

(A1) Transversality condition: $F(t_0,x_0) \notin V$,

(A2) Generalized Lipschitz condition: $f$ is Lipschitz continuous along $V$ on an open neighborhood $U \subseteq D$ of $(t_0,x_0)$.

Then (1.1) is locally uniquely solvable.

The proof of Theorem 2.1 uses only Peano’s theorem and the implicit function theorem. Since the classical Picard–Lindelöf theorem is a special case of Theorem 2.1, the following proof also offers an alternative proof of Picard–Lindelöf’s theorem.

Proof. Let $\| \cdot \|$ denote the Euclidean norm and its induced matrix norm, respectively. Since $V$ is a hyperspace in $\mathbb{R}^{1+n}$, there exist linearly independent vectors $v^{(1)}, \ldots, v^{(n)} \in \mathbb{R}^{1+n}$ with $V = \text{span}\{v^{(1)}, \ldots, v^{(n)}\} \subseteq \mathbb{R}^{1+n}$. Write

$$v^{(i)} = (v^{(i)}_1, v^{(i)}_2, \ldots, v^{(i)}_n)^T \quad \text{for} \quad i = 1, \ldots, n,$$

and define $v_i := (v^{(1)}_i, \ldots, v^{(n)}_i) \in \mathbb{R}^n$, $v^{(i)}_x := (v^{(i)}_1, \ldots, v^{(i)}_n)^T \in \mathbb{R}^n$, $V_x := (v^{(1)}_x | \cdots | v^{(n)}_x) \in \mathbb{R}^{n \times n}$. Then for

$$V := (v^{(1)} | \cdots | v^{(n)}) = \begin{pmatrix} v^{(1)}_1 & \cdots & v^{(1)}_n \\ \vdots & \ddots & \vdots \\ v^{(n)}_1 & \cdots & v^{(n)}_n \end{pmatrix} = \begin{pmatrix} v^{(1)}_t \\ \vdots \\ v^{(n)}_t \end{pmatrix} = \begin{pmatrix} v_t \\ V_x \end{pmatrix}$$

we have $V \in \mathbb{R}^{(1+n) \times n}$ and rank $V = n$. Peano’s theorem guarantees that (1.1) has at least one solution $x: [t_0 - \alpha, t_0 + \alpha] \to \mathbb{R}^n$ for some $\alpha > 0$. By shrinking $\alpha > 0$ if necessary, we can assume that graph $x \subset U$ and, by assumption (A1) and continuity of $f$, $F(t,x(t)) \notin V$ for all $t \in I := (t_0 - \alpha, t_0 + \alpha)$. To prove that (1.1) is locally uniquely solvable with solution $x$ on $I$, assume to the contrary that there exists a solution $y: I \to \mathbb{R}^n$ of (1.1) and $x \neq y$ on $[t_0, t_0 + \alpha]$ (the case $x \neq y$ on $(t_0 - \alpha, t_0)$ is treated similarly). For $t_1 := \sup\{t \in [t_0, t_0 + \alpha) : x(s) = y(s) \text{ for } s \in [t_0, t]\}$ we have $t_1 \in [t_0, t_0 + \alpha)$, $x(t_1) = y(t_1) =: x_1$ by continuity and $F(t_1, x_1) \notin V$. 


We show that the equation
\[ y(t + v_t k(t)) = x(t) + V_t k(t) \] (2.1)
is uniquely solvable with respect to \( k = k(t) = (k_1(t), \ldots, k_n(t))^T \) on a subinterval of \( I \) which contains \( t_1 \). The problem suggests to apply the implicit function theorem. Choose \( \epsilon > 0 \) such that
\[ H(t, k) := y(t + v_t k) - x(t) - V_t k \]
is well-defined on \([t_1 - \epsilon, t_1 + \epsilon] \times [-\epsilon, \epsilon]^n\). Then \( H(t_1, 0) = 0 \),
\[ \frac{\partial H}{\partial k}(t, k) = (f_j(t + v_t k, y(t + v_t k)) - v^{(j)}_i)_{i,j=1,\ldots,n} \]
and therefore \( \partial H(t_1, 0) / \partial k = WV \) with
\[ W := \begin{pmatrix} f(t_1, x_1) & -1 & \cdot & \cdot & -1 \end{pmatrix} \in \mathbb{R}^n \times (1+n). \]

By the rank-nullity theorem (see e.g. [4, p. 199]) \( \dim \text{im}(V) + \dim \ker(V) = n \) and, using the fact that \( \dim \text{im}(V) = \text{rank} V = n \), we get \( \ker V = \{0\} \). Assume that \( WV \) is not invertible. Then there exists \( v \in \mathbb{R}^n \setminus \{0\} \) such that \( WVv = 0 \). Hence \( v := Vv \neq 0 \) and \( v \in V \), as well as \( w = w = \text{span}\{F(t_1, x_1)\} \). Therefore \( F(t_1, x_1) \in V \) leads to a contradiction, proving that \( WV \) is invertible.

The implicit function theorem (cf. e.g. [5, Theorem 9.28]) yields a unique \( C^1 \) function \( k: J \rightarrow [-\epsilon, \epsilon]^n \) on an open interval \( J \subseteq I \) containing \( t_1 \) such that \( k(t_1) = 0 \) and \( H(t, k(t)) = 0 \) for all \( t \in J \). Using the fact that \( \partial H(t_1, 0) / \partial k \) is invertible, we get by shrinking \( J \) if necessary, that \( (\partial H(t, k(t)) / \partial k)^{-1} \) exists and is bounded for \( t \) in \( J \), i.e. there exists \( \eta \geq 0 \) such that
\[ \left\| \frac{\partial H(t, k(t))}{\partial k} \right\|^{-1} \leq \eta \quad \text{for } t \in J. \]

Since \( \partial H(t, k(t)) / \partial t = f(t + v_t k, y(t + v_t k)) - f(t, x(t)) \), (A2) implies, together with (2.1) and \( Vk(t) \in V \), that
\[ \left\| \frac{\partial H}{\partial t}(t, k(t)) \right\| \leq L\|Vk(t)\|. \]

Now we consider \( u(t) := \|k(t)\|^2 = \langle k(t), k(t) \rangle \). We get
\[ \dot{u}(t) = \frac{d}{dt} \langle k(t), k(t) \rangle = 2\langle k(t), \dot{k}(t) \rangle. \]

Using the fact that
\[ k(t) = -\frac{\partial H}{\partial k}(t, k(t))^{-1} \frac{\partial H}{\partial t}(t, k(t)), \]
we conclude that
\[ \dot{u}(t) \leq \left\| 2k(t)^T \frac{\partial H}{\partial k}(t, k(t))^{-1} \frac{\partial H}{\partial t}(t, k(t)) \right\| \leq 2\|k(t)\| \eta L \|V\| \|k(t)\| \]
and hence
\[ \dot{u}(t) \leq 2\eta L \|V\| u(t) \]
which is equivalent to
\[
\frac{d}{dt} \left[ e^{-2\eta \|V\|(t-t_1)} u(t) \right] \leq 0.
\]
Since \( u(t_1) = \|k(t_1)\|^2 = 0 \), we get \( u(t) = \|k(t)\|^2 \equiv 0 \), and hence from (2.1) we conclude \( x(t) \equiv y(t) \) on \( J \), which contradicts the definition of \( t_1 \).

**Remark 2.2.** (a) The classical Picard–Lindelöf theorem which requires a Lipschitz condition with respect to \( x \) is a special case of Theorem 2.1 with
\[
V = \begin{pmatrix} v_1 \\ V_x \end{pmatrix}, \quad v_1 = 0 \in \mathbb{R}^n \quad \text{and} \quad V_x = I_n, \tag{2.2}
\]
where \( I_n \) denotes the \( n \times n \) identity matrix. Cid [2] introduces the notion of Lipschitz continuity when fixing component \( i_0 \in \{0, 1, \ldots, n\} \) where the component \( i_0 = 0 \) corresponds to the variable \( t \), i.e. Lipschitz continuity when fixing \( i_0 = 0 \) is equivalent to Lipschitz continuity with respect to \( x \). Lipschitz continuity when fixing another component is defined similarly. Under the assumption that \( f \) is Lipschitz continuous when fixing a component \( i_0 \), Cid can show uniqueness provided that either \( i_0 = 0 \) or \( f(i_0) \neq 0 \). Thus Theorem 3.2 by Cid can be interpreted as a special case of our Theorem 2.1 with matrices \( V \) of the form (2.2) where in the case of \( i_0 \neq 0 \) the corresponding column of \( V \) is replaced by a vector \( v^{(i_0)} \) with \( v^{(i_0)}_0 = 1 \) and all other components equal 0. Note that [3, Theorem 1] is a special case of Theorem 2.1 for \( n = 1 \) if the functions \( \varphi \) and \( \psi \) are constants.

(b) Let \( V = \text{span}\{v^{(1)}, \ldots, v^{(n)}\} \subset \mathbb{R}^{1+n} \) and \( U \subseteq D \) be a convex open neighborhood of \( (t_0, x_0) \in D \subseteq \mathbb{R} \times \mathbb{R}^n \). If the directional derivatives
\[
\frac{\partial f}{\partial v}(t, x) = \lim_{h \to 0} \frac{f((t, x) + hv) - f(t, x)}{h\|v\|}, \quad v \in V,
\]
exist and are continuous and bounded on \( U \), then \( f \) is Lipschitz continuous along \( V \) on \( U \).

**Proof.** With \( (t, x) = (s, y) + v, v \in V, \) and \( g(\tau) := f((s, y) + \tau v) \) we get
\[
\begin{align*}
f(t, x) - f(s, y) &= g(1) - g(0) = \int_0^1 g'(\tau) d\tau \\
&= \int_0^1 \lim_{h \to 0} \frac{g(\tau + h) - g(\tau)}{h} d\tau \\
&= \int_0^1 \lim_{h \to 0} \frac{f((s, y) + (\tau + h)v) - f((s, y) + \tau v)}{h} d\tau \\
&= \int_0^1 \left( \lim_{h \to 0} \frac{f((s, y) + (\tau + h)v) - f((s, y) + \tau v)}{h\|v\|} \right) \|v\| d\tau \\
&= \int_0^1 \frac{\partial f}{\partial v}(s, y + \tau v) \|v\| d\tau
\end{align*}
\]
and therefore
\[
\|f(t, x) - f(s, y)\| \leq L\|v\|, \quad L := \sup_{\tau \in [0,1]} \frac{\partial f}{\partial v}(s, y + \tau v).
\]

**Example 2.3.** Consider the 2-dimensional initial value problem
\[
\dot{x} = f(t, x), \quad x(0) = 0,
\]
where \( f(t, x) = (f_1(t, x_1, x_2), f_2(t, x_1, x_2))^T \) with
\[
\begin{align*}
  f_1(t, x_1, x_2) &= \begin{cases} 
    x_1 + g(x_2), & x_1 < t, \\
    x_1 + g(x_2) + \sqrt{x_1 - t}, & x_1 \geq t,
  \end{cases} \\
  f_2(t, x_1, x_2) &= 1 + h(x_1),
\end{align*}
\]
\( g(x_2) \) and \( h(x_1) \) are Lipschitz continuous functions and \( g(0) \neq 1 \). The classical Lipschitz condition is not fulfilled, and we cannot show uniqueness with the hyperspace \( V \) being the \((t, x_1)\)-plane or \((t, x_2)\)-plane. Therefore the result by Cid cannot be applied.

With the basis vectors \( v^{(1)} = (1, 1, 0)^T, v^{(2)} = (0, 0, 1)^T \) and \( V = \text{span}\{v^{(1)}, v^{(2)}\} \) we can show uniqueness of the given problem.

(A1) is satisfied, as \((1, g(0), 1 + h(0))^T \notin V \) if \( g(0) \neq 1 \). The only numbers \( \alpha, \beta, \gamma, \) satisfying \( \alpha(1, f(0, 0))^T + \beta v^{(1)} + \gamma v^{(2)} = 0 \) are \( \alpha = \beta = \gamma = 0 \) if \( g(0) \neq 1 \).

Now (A2) is shown. With \( v_t = (1, 0) \) and \( V_x = \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) \) we have to show that
\[
\|f(t + v_t k, x + V_x k) - f(t, x)\| = \|f(t + k_1, x_1 + k_1, x_2 + k_2) - f(t, x_1, x_2)\| \leq L\|v_t k, V_x k\|^T
\]
with \( k = (k_1, k_2)^T \). For \( x_1 < t \) we get
\[
\left\| \begin{pmatrix} x_1 + k_1 + g(x_2 + k_2) - x_1 - g(x_2) \\
   1 + h(x_1 + k_1) - 1 - h(x_1)
\end{pmatrix} \right\|
\]
which can be estimated by \( L\|(k_1, k_1, k_2)^T\| \) with \( L \geq 0 \). For \( x_1 \geq t \) we get
\[
\left\| \begin{pmatrix} x_1 + k_1 + g(x_2 + k_2) + \sqrt{x_1 + k_1 - t - k_1} - x_1 - g(x_2) - \sqrt{x_1 - t} \\
   1 + h(x_1 + k_1) - 1 - h(x_1)
\end{pmatrix} \right\|
\]
which can also be estimated by \( L\|(k_1, k_1, k_2)^T\| \) with \( L \geq 0 \).

3 Alternative proof

We provide an alternative proof for Theorem 2.1 by transforming (1.1) into a system to which the classical Picard–Lindelöf theorem can be applied.

Alternative proof of Theorem 2.1. Choose a unit vector \( a_0 \in \mathbb{R}^{1+n} \) such that \( V = a_0^\top \) and also \( \langle a_0, F(t_0, x_0) \rangle > 0 \), which is possible due to assumption (A1). Since \( \mathbb{R}^{1+n} = \langle a_0 \rangle \oplus V \) is the direct sum of \( \langle a_0 \rangle = \{s a_0 \in \mathbb{R}^{1+n} : s \in \mathbb{R}\} \) and \( V \), there exist unique \( s_0 \in \mathbb{R} \) and \( v_0 \in V \) with \( (t_0, x_0) = s_0 a_0 + v_0 \). We divide the proof into three steps.

Step 1: We show that the nonautonomous initial value problem on \( V \)
\[
\frac{dv}{ds} = g(s, v) := \frac{F(s a_0 + v) - \sigma(s, v) a_0}{\sigma(s, v)}, \quad v(s_0) = v_0,
\]
with \( \sigma(s, v) := \langle a_0, F(s a_0 + v) \rangle \) is well-posed and locally uniquely solvable.

The function
\[
\sigma : \mathbb{R} \times V \to \mathbb{R}, \quad (s, v) \mapsto \sigma(s, v) = \langle a_0, F(s a_0 + v) \rangle
\]
is continuous and satisfies \( \sigma(s_0, v_0) = \langle a_0, F(s_0a_0 + v_0) \rangle = \langle a_0, F(t_0, x_0) \rangle > 0 \). As a consequence there exists an \( \eta > 0 \) and a bounded open neighborhood \( U \subseteq \mathbb{R} \times \mathcal{V} \) of \((s_0, v_0)\) such that \( \sigma(s, v) \geq \eta \) for all \((s, v) \in U\).

Using assumption (A2) and by shrinking \( U \) if necessary, we can w.l.o.g. assume that \( f \) is Lipschitz continuous along \( \mathcal{V} \) on the open neighborhood \{\( s_0 + v \in \mathbb{R}^{1+n} : (s, v) \in U \)\} of \((t_0, x_0)\). Using this fact, we get for \((s, v), (s, \bar{v}) \in U\)

\[
|\sigma(s, v) - \sigma(s, \bar{v})| = |\langle a_0, F(s_0 + v) - F(s_0 + \bar{v}) \rangle| \\
= |\langle a_0, F(s_0 + v) - F(s_0 + \bar{v}) \rangle| \leq \|a_0\| \cdot \|F(s_0 + v) - F(s_0 + \bar{v})\| \\
= \|F(s_0 + v) - F(s_0 + \bar{v})\| = \|f(s_0 + v) - f(s_0 + \bar{v})\| \\
\leq L\|v - \bar{v}\|,
\]

proving that \( \sigma \) is Lipschitz continuous on \( U \). With \( \sigma \) also the quotient \( 1/\sigma \) is Lipschitz continuous with respect to \( v \). Thus we get

\[
\|g(s, v) - g(s, \bar{v})\| = \left\| \frac{F(s_0 + v)}{\sigma(s, v)} - \frac{F(s_0 + \bar{v})}{\sigma(s, \bar{v})} \right\| \\
\leq \left| \frac{1}{\sigma(s, v)} \right| \cdot \|F(s_0 + v) - F(s_0 + \bar{v})\| \\
+ \left| \frac{1}{\sigma(s, v)} - \frac{1}{\sigma(s, \bar{v})} \right| \cdot \|F(s_0 + \bar{v})\|.
\]

By shrinking \( U \) again if necessary, we can assume w.l.o.g. that \( \bar{U} \subseteq D \). Then boundedness of \( F \) and of \( 1/\sigma \) on \( \bar{U} \) imply Lipschitz continuity of \( g \) with respect to \( v \) on the neighborhood \( U \) of \((s_0, v_0)\). Since \( \mathcal{V} \) is isomorphic to \( \mathbb{R}^n \), the classical Picard–Lindelöf theorem can be applied to (3.1) to prove local unique solvability.

**Step 2:** We show that the autonomous initial value problem on \( \mathbb{R} \times \mathcal{V} \)

\[
\begin{align*}
\dot{s} &= \sigma(s, v), & s(t_0) &= s_0, \\
\dot{v} &= F(s_0 + v) - \sigma(s, v)a_0, & v(t_0) &= v_0,
\end{align*}
\]

(3.2)
is locally uniquely solvable.

By Peano’s theorem (3.2) admits a solution. Assume that \((\dot{s}_1, \dot{v}_1), (\dot{s}_2, \dot{v}_2) : J \to \mathbb{R} \times \mathcal{V} \) are two solutions of (3.2) on an open interval \( J \) containing \( t_0 \). Then the solution identities

\[
\begin{align*}
\dot{s}_i(t) &= \sigma(s_i(t), \dot{v}_i(t)), \\
\dot{v}_i(t) &= F(s_i(t)a_0 + \dot{v}_i(t)) - \sigma(s_i(t), \dot{v}_i(t))a_0
\end{align*}
\]

(3.3)
for \( t \in J \) and the initial conditions

\[
\begin{align*}
\dot{s}_i(t_0) &= s_0, & \dot{v}_i(t_0) &= v_0
\end{align*}
\]

(3.4)
are fulfilled for \( i = 1, 2 \). By shrinking \( J \) if necessary, we can w.l.o.g. assume that \((\dot{s}_i(t), \dot{v}_i(t)) \in U \) and therefore \( \dot{s}_i(t) = \sigma(\dot{s}_i(t), \dot{v}_i(t)) \geq \eta \) for \( t \in J \). As a consequence the functions \( \dot{s}_i : J \to \mathbb{R} \) are strictly monotonically increasing, and hence the inverse functions \( \dot{s}_i^{-1} : \dot{s}_i(J) \to J \) exist and satisfy

\[
\dot{s}_i^{-1}(s_0) = t_0
\]

(3.5)
for \(i = 1, 2\). With the bijection \(t = \hat{s}_i^{-1}(s)\) both solution curves through \((s_0,v_0)\) can be reparametrized in the form

\[
\{(\hat{s}_i(t),\hat{v}_i(t)) : t \in J\} = \{(\hat{s}_i(\hat{s}_i^{-1}(s)),\hat{v}_i(\hat{s}_i^{-1}(s)) : s \in \hat{s}_i(J)\} = \{(s,\hat{v}_i(\hat{s}_i^{-1}(s)) : s \in \hat{s}_i(J)\}
\]

for \(i = 1, 2\). Then

\[
v_i : \hat{s}_i(J) \to V, \quad v_i(s) := \hat{v}_i(\hat{s}_i^{-1}(s)),
\]

solve (3.1) for \(i = 1, 2\), since

\[
\frac{dv_i}{ds}(s) = \frac{\hat{v}_i(\hat{s}_i^{-1}(s))}{\hat{s}_i(\hat{s}_i^{-1}(s))} F(sa_0 + v_i) - \sigma(s,v_i)a_0 \quad (3.3)
\]

and

\[
v_i(s_0) = \hat{v}_i(\hat{s}_i^{-1}(s_0)) = \hat{v}_i(t_0) \quad (3.4).
\]

By shrinking \(J\) if necessary, we can apply Step 1 to conclude that \(v_1 = v_2\) on \(J\) and hence \(\hat{v}_1(\hat{s}_1^{-1}(s)) = \hat{v}_2(\hat{s}_2^{-1}(s))\) for all \(s \in \hat{s}_1(J) \cap \hat{s}_2(J)\), proving that \(\hat{s}_1 = \hat{s}_2\) and \(\hat{v}_1 = \hat{v}_1\) on \(J\).

**Step 3:** We show that (1.1) is locally uniquely solvable.

By Peano’s theorem (1.1) admits a solution. Assume that \(x_1, x_2 : I \to \mathbb{R}^n\) are two solutions of (1.1). For \(t \in I\) we have \(X_i(t) := (1, x_i(t)) \in \mathbb{R}^{1+n} = \langle a_0 \rangle \oplus V\) and therefore there exist unique functions \(s_i : I \to \mathbb{R}\) and \(v_i : I \to V\) such that

\[
X_i(t) = s_i(t)a_0 + v_i(t).
\]

Moreover, \((s_i(t_0), v_i(t_0)) = (s_0, v_0)\), and using the fact that \(\|a_0\| = 1\) and \(a_0^\perp = V\), \(s_i(t) = \langle a_0, X_i(t) \rangle\) and \(v_i(t) = X_i(t) - s_i(t)a_0\) for \(t \in I\) and \(i = 1, 2\). Now \((s_i, v_i) : I \to \mathbb{R} \times V\) solve (3.2), since

\[
\hat{s}_i(t) = \langle a_0, \dot{X}_i(t) \rangle = \langle a_0, F(t, x_i(t)) \rangle = \langle a_0, F(s_i(t)a_0 + v_i(t)) \rangle = \sigma(s_i(t),v_i(t)),
\]

\[
\dot{v}_i(t) = \dot{X}_i(t) - \langle a_0, \dot{X}_i(t) \rangle a_0 = F(t, x_i(t)) - \langle a_0, F(t, x_i(t)) \rangle a_0 = F(s_i(t)a_0 + v_i(t)) - \langle a_0, F(s_i(t)a_0 + v_i(t)) \rangle a_0 = F(s_i(t)a_0 + v_i(t)) - \sigma(s_i(t),v_i(t))a_0.
\]

for \(t \in I\) and \(i = 1, 2\). By shrinking \(I\) if necessary, we can apply Step 2 to conclude that \(s_1 = s_2\) and \(v_1 = v_2\) on \(I\), proving that \(x_1 = x_2\).

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