Existence of solutions for a fourth-order boundary value problem on the half-line via critical point theory

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Abstract. In this paper, a fourth-order boundary value problem on the half-line is considered and existence of solutions is proved using a minimization principle and the mountain pass theorem.

Keywords: fourth-order BVPs, unbounded interval, critical point, minimization principle, mountain-pass theorem.

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1 Introduction

We consider the existence of solutions for the following fourth-order boundary value problem set on the half-line

\[
\begin{align*}
&u^{(4)}(t) - u''(t) + u(t) = f(t, u(t)), \quad t \in [0, +\infty), \\
&u(0) = u(+\infty) = 0, \\
&u''(0) = u''(+\infty) = 0,
\end{align*}
\]  

(1.1)

where \( f \in C([0, +\infty) \times \mathbb{R}, \mathbb{R}) \).

Many authors used critical point theory to establish the existence of solutions for fourth-order boundary value problems on bounded intervals (see for example \([8,9,13]\)), but there are only a few papers that consider the above problem on the half-line using critical point theory. We cite \([5]\) where the authors consider the existence of solutions for a particular fourth-order BVP on the half-line using critical point theory.

We endow the following space

\[
H^2_0(0, +\infty) = \left\{ u \in L^2(0, +\infty), u' \in L^2(0, +\infty), u'' \in L^2(0, +\infty), u(0) = 0, u'(0) = 0 \right\}
\]

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with its natural norm

\[ \|u\| = \left( \int_0^{+\infty} u''(t)^2 dt + \int_0^{+\infty} u'(t)^2 dt + \int_0^{+\infty} u^2(t)dt \right)^{\frac{1}{2}}. \]

Note that if \( u \in H_0^2(0, +\infty) \), then \( u(+\infty) = 0, u'(+\infty) = 0 \), (see [3, Corollary 8.9]). Let \( p, q : [0, +\infty) \to (0, +\infty) \) be two continuously differentiable and bounded functions with

\[ M_1 = \max(||p||_{L^2}, ||p'||_{L^2}) < +\infty, \quad M_2 = \max(||q||_{L^2}, ||q'||_{L^2}) < +\infty. \]

We also consider the following spaces

\[ C_{l,p}[0, +\infty) = \left\{ u \in C([0, +\infty), \mathbb{R}) : \lim_{t \to +\infty} p(t)u(t) \text{ exists} \right\} \]

dowered with the norm

\[ \|u\|_{\infty,p} = \sup_{t \in [0, +\infty)} p(t)|u(t)|, \]

and

\[ C_{l,p,q}^1[0, +\infty) = \left\{ u \in C^1([0, +\infty), \mathbb{R}) : \lim_{t \to +\infty} p(t)u(t), \lim_{t \to +\infty} q(t)u'(t) \text{ exist} \right\} \]

dowered with the natural norm

\[ \|u\|_{\infty,p,q} = \sup_{t \in [0, +\infty)} p(t)|u(t)| + \sup_{t \in [0, +\infty)} q(t)|u'(t)|. \]

Let

\[ C_l[0, +\infty) = \left\{ u \in C([0, +\infty), \mathbb{R}) : \lim_{t \to +\infty} u(t) \text{ exists} \right\} \]

dowered with the norm \( \|u\|_{\infty} = \sup_{t \in [0, +\infty)} |u(t)| \).

To prove that \( H_0^2(0, +\infty) \) embeds compactly in \( C_{l,p,q}^1(0, +\infty), \mathbb{R} \), we need the following Corduneanu compactness criterion.

**Lemma 1.1** ([4]). Let \( D \subset C_l([0, +\infty), \mathbb{R}) \) be a bounded set. Then \( D \) is relatively compact if the following conditions hold:

(a) \( D \) is equicontinuous on any compact sub-interval of \( \mathbb{R}^+ \), i.e.

\[ \forall J \subset [0, +\infty) \text{ compact}, \forall \epsilon > 0, \exists \delta > 0, \forall t_1, t_2 \in J : |t_1 - t_2| < \delta \implies |u(t_1) - u(t_2)| \leq \epsilon, \forall u \in D; \]

(b) \( D \) is equiconvergent at \(+\infty\) i.e.,

\[ \forall \epsilon > 0, \exists T = T(\epsilon) > 0 \text{ such that } \forall t : t \geq T(\epsilon) \implies |u(t) - u(+\infty)| \leq \epsilon, \forall u \in D. \]

Similar reasoning as in [6] yields the following compactness criterion in the space \( C_{l,p,q}^1([0, +\infty), \mathbb{R}) \).
Lemma 1.2. Let $D \subset C_{l,p,q}^{1}([0, +\infty), \mathbb{R})$ be a bounded set. Then $D$ is relatively compact if the following conditions hold:
(a) $D$ is equicontinuous on any compact sub-interval of $[0, +\infty)$, i.e.
\[
\forall J \subset [0, +\infty) \text{ compact, } \forall \varepsilon > 0, \exists \delta > 0, \forall t_1, t_2 \in J:
\]
\[
|t_1 - t_2| < \delta \implies |p(t_1)u(t_1) - p(t_2)u(t_2)| \leq \varepsilon, \forall u \in D,
\]
\[
|t_1 - t_2| < \delta \implies |q(t_1)u'(t_1) - q(t_2)u'(t_2)| \leq \varepsilon, \forall u \in D;
\]

(b) $D$ is equiconvergent at $+\infty$ i.e.,
\[
\forall \varepsilon > 0, \exists T = T(\varepsilon) > 0 \text{ such that }
\]
\[
\forall t : t \geq T(\varepsilon) \implies |p(t)u(t) - (pu)(+\infty)| \leq \varepsilon, \forall u \in D,
\]
\[
\forall t : t \geq T(\varepsilon) \implies |q(t)u'(t) - (qu')(+\infty)| \leq \varepsilon, \forall u \in D.
\]

Now we recall some essential facts from critical point theory (see [1, 2, 10]).

Definition 1.3. Let $X$ be a Banach space, $\Omega \subset X$ an open subset, and $J : \Omega \rightarrow \mathbb{R}$ a functional. We say that $J$ is Gâteaux differentiable at $u \in \Omega$ if there exists $A X^*$ such that
\[
\lim_{t \rightarrow 0} \frac{J(u + tv) - J(u)}{t} = Av,
\]
for all $v \in X$. Now $A$, which is unique, is denoted by $A = J'_G(u)$.

The mapping which sends to every $u \in \Omega$ the mapping $J'_G(u)$ is called the Gâteaux differential of $J$ and is denoted by $J'_G$.

We say that $J \in C^1$ if $J$ is Gâteaux differential on $\Omega$ and $J'_G$ is continuous at every $u \in \Omega$.

Definition 1.4. Let $X$ be a Banach space. A functional $J : \Omega \rightarrow \mathbb{R}$ is called coercive if, for every sequence $(u_k)_{k \in \mathbb{N}} \subset X$,
\[
\|u_k\| \rightarrow +\infty \implies |J(u_k)| \rightarrow +\infty.
\]

Definition 1.5. Let $X$ be a Banach space. A functional $J : X \rightarrow (-\infty, +\infty]$ is said to be sequentially weakly lower semi-continuous (swlsc for short) if
\[
J(u) \leq \liminf_{n \rightarrow +\infty} J(u_n)
\]
as $u_n \rightharpoonup u$ in $X$, $n \rightarrow \infty$.

Lemma 1.6 (Minimization principle [2]). Let $X$ be a reflexive Banach space and $J$ a functional defined on $X$ such that
\begin{enumerate}
\item[(1)] $\lim\|u\| \rightarrow +\infty J(u) = +\infty$ (coercivity condition),
\item[(2)] $J$ is sequentially weakly lower semi-continuous.
\end{enumerate}

Then $J$ is lower bounded on $X$ and achieves its lower bound at some point $u_0$.

Definition 1.7. Let $X$ be a real Banach space, $J \in C^1(X, \mathbb{R})$. If any sequence $(u_n) \subset X$ for which $(J(u_n))$ is bounded in $\mathbb{R}$ and $J'(u_n) \rightarrow 0$ as $n \rightarrow +\infty$ in $X'$ possesses a convergent subsequence, then we say that $J$ satisfies the Palais–Smale condition (PS condition for brevity).
Lemma 1.8 (Mountain Pass Theorem, [11, Theorem 2.2], [12, Theorem 3.1]). Let $X$ be a Banach space, and let $J \in C^1(X, \mathbb{R})$ satisfy $J(0) = 0$. Assume that $J$ satisfies the $(PS)$ condition and there exist positive numbers $\rho$ and $\alpha$ such that

1. $J(u) \geq \alpha$ if $\|u\| = \rho$,
2. there exists $u_0 \in X$ such that $\|u_0\| > \rho$ and $J(u_0) < \alpha$.

Then there exists a critical point. It is characterized by

$$J'(u) = 0, \quad J(u) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$

where

$$\Gamma = \{ \gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = u_0 \}.$$

1.1 Variational setting

Take $v \in H^2_0(0, +\infty)$, and multiply the equation in Problem (1.1) by $v$ and integrate over $(0, +\infty)$, so we get

$$\int_0^{+\infty} (u^{(4)}(t) - u''(t) + u(t))v(t)dt = \int_0^{+\infty} f(t, u(t))v(t)dt.$$

Hence

$$\int_0^{+\infty} (u''(t)v''(t) + u'(t)v'(t) + u(t)v(t))dt = \int_0^{+\infty} f(t, u(t))v(t)dt.$$

This leads to the natural concept of a weak solution for Problem (1.1).

Definition 1.9. We say that a function $u \in H^2_0(0, +\infty)$ is a weak solution of Problem (1.1) if

$$\int_0^{+\infty} (u''(t)v''(t) + u'(t)v'(t) + u(t)v(t))dt = \int_0^{+\infty} f(t, u(t))v(t)dt,$$

for all $v \in H^2_0(0, +\infty)$.

In order to study Problem (1.1), we consider the functional $J : H^2_0(0, +\infty) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\|u\|^2 - \int_0^{+\infty} F(t, u(t))dt,$$

where

$$F(t, u) = \int_0^u f(t, s)ds.$$

2 Some embedding results

We begin this section by proving some continuous and compact embeddings. Here $p$ and $q$ (and $M_1, M_2$) are as in Section 1.

Lemma 2.1. $H^2_0(0, +\infty)$ embeds continuously in $C^1_{1,p,q}[0, +\infty)$. 

Proof. For $u \in H^2_0(0, +\infty)$, we have

\begin{align*}
|p(t)u(t)| &= |p(+\infty)u(+\infty) - p(t)u(t)| \\
&= \left| \int_t^{+\infty} (pu)'(s)ds \right| \\
&\leq \left| \int_t^{+\infty} p'(s)u(s)ds \right| + \left| \int_t^{+\infty} p(s)u'(s)ds \right| \\
&\leq \left( \int_0^{+\infty} p^2(s)ds \right)^{\frac{1}{2}} \left( \int_t^{+\infty} u^2(s)ds \right)^{\frac{1}{2}} + \left( \int_0^{+\infty} p^2(s)ds \right)^{\frac{1}{2}} \left( \int_t^{+\infty} u^2(s)ds \right)^{\frac{1}{2}} \\
&\leq \max(\|p\|_{L^2}, \|p\|_{L^2})\|u\| \\
&\leq M_1\|u\|,
\end{align*}

and

\begin{align*}
|q(t)u'(t)| &= |q(+\infty)u'(+\infty) - q(t)u'(t)| \\
&= \left| \int_t^{+\infty} (qu')'(s)ds \right| \\
&\leq \left| \int_t^{+\infty} q'(s)u'(s)ds \right| + \left| \int_t^{+\infty} q(s)u''(s)ds \right| \\
&\leq \left( \int_0^{+\infty} q^2(s)ds \right)^{\frac{1}{2}} \left( \int_t^{+\infty} u^2(s)ds \right)^{\frac{1}{2}} + \left( \int_0^{+\infty} q^2(s)ds \right)^{\frac{1}{2}} \left( \int_t^{+\infty} u^2(s)ds \right)^{\frac{1}{2}} \\
&\leq \max(\|q\|_{L^2}, \|q\|_{L^2})\|u\| \\
&\leq M_2\|u\|.
\end{align*}

Hence $\|u\|_{C_t, \infty, p, q} \leq M\|u\|$, with $M = \max(M_1, M_2)$. \hfill \Box

The following compactness embedding is an important result.

**Lemma 2.2.** The embedding $H^2_0(0, +\infty) \hookrightarrow C^1_{t, p, q}[0, +\infty)$ is compact.

**Proof.** Let $D \subset H^2_0(0, +\infty)$ be a bounded set. Then it is bounded in $C^1_{t, p, q}[0, +\infty)$ by Lemma 2.1. Let $R > 0$ be such that for all $u \in D$, $\|u\| \leq R$. We will apply Lemma 1.2.

(a) $D$ is equicontinuous on every compact interval of $[0, +\infty)$. Let $u \in D$ and $t_1, t_2 \in J \subset [0, +\infty)$ where $J$ is a compact sub-interval. Using the Cauchy–Schwarz inequality, we have

\begin{align*}
|p(t_1)u(t_1) - p(t_2)u(t_2)| &= \left| \int_{t_2}^{t_1} (pu)'(s)ds \right| \\
&= \left| \int_{t_2}^{t_1} (p'(s)u(s) + u'(s)p(s))ds \right| \\
&\leq \left( \int_{t_2}^{t_1} p^2(s)ds \right)^{\frac{1}{2}} \left( \int_{t_2}^{t_1} u^2(s)ds \right)^{\frac{1}{2}} + \left( \int_{t_2}^{t_1} p^2(s)ds \right)^{\frac{1}{2}} \left( \int_{t_2}^{t_1} u^2(s)ds \right)^{\frac{1}{2}} \\
&\leq \max \left[ \left( \int_{t_2}^{t_1} p^2(s)ds \right)^{\frac{1}{2}}, \left( \int_{t_2}^{t_1} p^2(s)ds \right)^{\frac{1}{2}} \right] \|u\| \\
&\leq R \max \left[ \left( \int_{t_2}^{t_1} p^2(s)ds \right)^{\frac{1}{2}}, \left( \int_{t_2}^{t_1} p^2(s)ds \right)^{\frac{1}{2}} \right] \to 0,
\end{align*}
as \(|t_1 - t_2| \to 0\), and
\[
|q(t_1)u'(t_1) - q(t_2)u'(t_2)| = \left| \int_{t_2}^{t_1} (qu')'(s)ds \right|
\leq \left( \int_{t_2}^{t_1} q'^2(s)ds \right)^{1/2} \left( \int_{t_2}^{t_1} u'^2(s)ds \right)^{1/2}
+ \left( \int_{t_2}^{t_1} q'^2(s)ds \right)^{1/2} \left( \int_{t_2}^{t_1} u''^2(s)ds \right)^{1/2}
\leq \max \left[ \left( \int_{t_2}^{t_1} q'^2(s)ds \right)^{1/2}, \left( \int_{t_2}^{t_1} u'^2(s)ds \right)^{1/2} \right] \|u\|
\leq R \max \left[ \left( \int_{t_2}^{t_1} q'^2(s)ds \right)^{1/2}, \left( \int_{t_2}^{t_1} u'^2(s)ds \right)^{1/2} \right] \to 0,
\]
as \(|t_1 - t_2| \to 0\).

(b) \(D\) is equiconvergent at \(+\infty\). For \(t \in [0, +\infty)\) and \(u \in D\), using the fact that \((pu)(+\infty) = 0, (qu')(+\infty) = 0\) (note that \(u(\infty) = 0, u'(\infty) = 0\) and \(p, q\) are bounded) and using the Cauchy–Schwarz inequality, we have
\[
|(pu)(t) - (pu)(+\infty)| = \left| \int_{t}^{+\infty} (pu)'(s)ds \right|
= \left| \int_{t}^{+\infty} (p'(s)u(s) + u'(s)p(s)) ds \right|
\leq \max \left[ \left( \int_{t}^{+\infty} p'^2(s)ds \right)^{1/2}, \left( \int_{t}^{+\infty} p^2(s)ds \right)^{1/2} \right] \|u\|
\leq R \max \left[ \left( \int_{t}^{+\infty} p'^2(s)ds \right)^{1/2}, \left( \int_{t}^{+\infty} p^2(s)ds \right)^{1/2} \right] \to 0,
\]
as \(t \to +\infty\), and
\[
|(qu')(t) - (qu')(+\infty)| = \left| \int_{t}^{+\infty} (qu')'(s)ds \right|
= \left| \int_{t}^{+\infty} (q'(s)u'(s) + q(s)u''(s)) ds \right|
\leq \max \left[ \left( \int_{t}^{+\infty} q'^2(s)ds \right)^{1/2}, \left( \int_{t}^{+\infty} q^2(s)ds \right)^{1/2} \right] \|u\|
\leq R \max \left[ \left( \int_{t}^{+\infty} q'^2(s)ds \right)^{1/2}, \left( \int_{t}^{+\infty} q^2(s)ds \right)^{1/2} \right] \to 0,
\]
as \(t \to +\infty\). \qed

**Corollary 2.3.** \(C_{l,p,q}^1[0, +\infty)\) embeds continuously in \(C_{l,p}[0, +\infty)\).

**Corollary 2.4.** The embedding \(H^2_0(0, +\infty) \hookrightarrow C_{l,p}[0, +\infty)\) is continuous and compact.
3 Existence results

Here \( p \) (and \( M_1 \)) are as in Section 1.

**Theorem 3.1.** Assume that \( F \) satisfy the following conditions.

\( (F1) \) There exist two constants \( 1 < \alpha < \beta < 2 \) and two functions \( a, b \) with \( \frac{a}{p^\alpha} \in L^1([0, +\infty), [0, +\infty)), \)
\( \frac{b}{p^\beta} \in L^1([0, +\infty), [0, +\infty)) \) such that

\[
|F(t, x)| \leq a(t)|x|^\alpha, \quad \forall (t, x) \in [0, +\infty) \times \mathbb{R}, |x| \leq 1
\]

and

\[
|F(t, x)| \leq b(t)|x|^\beta, \quad \forall (t, x) \in [0, +\infty) \times \mathbb{R}, |x| > 1.
\]

\( (F2) \) There exist an open bounded set \( I \subset [0, +\infty) \) and two constants \( \eta > 0 \) and \( 0 < \gamma < 2 \) such that

\[
F(t, x) \geq \eta |x|^\gamma, \quad \forall (t, x) \in I \times \mathbb{R}, |x| \leq 1.
\]

Then Problem (1.1) has at least one nontrivial weak solution.

**Proof.**

**Claim 1.** We first show that \( J \) is well defined.

Let

\[
\Omega_1 = \{ t \geq 0, |u(t)| \leq 1 \}, \quad \Omega_2 = \{ t \geq 0, |u(t)| > 1 \}.
\]

Given \( u \in H^2_0(0, +\infty) \), it follows from (F1) and Corollary 2.4 that

\[
\int_0^{+\infty} |F(t, u(t))| dt = \int_{\Omega_1} |F(t, u(t))| dt + \int_{\Omega_2} |F(t, u(t))| dt
\]

\[
\leq \int_{\Omega_1} a(t)|u(t)|^\alpha dt + \int_{\Omega_2} b(t)|u(t)|^\beta dt
\]

\[
\leq \int_{\Omega_1} \frac{a(t)}{p^\alpha(t)}|p(t)u(t)|^\alpha dt + \int_{\Omega_2} \frac{b(t)}{p^\beta(t)}|p(t)u(t)|^\beta dt
\]

\[
\leq \left| \frac{a}{p^\alpha} \right|_{L^1} \|u\|^\alpha_{p, \infty} + \left| \frac{b}{p^\beta} \right|_{L^1} \|u\|^\beta_{p, \infty}
\]

\[
\leq M_1^\alpha \left| \frac{a}{p^\alpha} \right|_{L^1} \|u\|^\alpha + M_1^\beta \left| \frac{b}{p^\beta} \right|_{L^1} \|u\|^\beta.
\]

Thus

\[
|J(u)| \leq \frac{1}{2} \|u\|^2 + M_1^\alpha \left| \frac{a}{p^\alpha} \right|_{L^1} \|u\|^\alpha + M_1^\beta \left| \frac{b}{p^\beta} \right|_{L^1} \|u\|^\beta < +\infty.
\]

**Claim 2.** \( J \) is coercive.

From (F1) and Corollary 2.4, we have

\[
J(u) = \frac{1}{2} \|u\|^2 - \int_{\Omega_1} F(t, u(t)) dt - \int_{\Omega_2} F(t, u(t)) dt
\]

\[
\geq \frac{1}{2} \|u\|^2 - M_1^\alpha \left| \frac{a}{p^\alpha} \right|_{L^1} \|u\|^\alpha - M_1^\beta \left| \frac{b}{p^\beta} \right|_{L^1} \|u\|^\beta. \tag{3.1}
\]

Now since \( 0 < \alpha < \beta < 2 \), then (3.1) implies that

\[
\lim_{\|u\| \to +\infty} J(u) = +\infty.
\]
Consequently, $J$ is coercive.

**Claim 3.** $J$ is sequentially weakly lower semi-continuous.

Let $(u_n)$ be a sequence in $H_0^2(0, +\infty)$ such that $u_n \rightharpoonup u$ as $n \to +\infty$ in $H_0^2(0, +\infty)$. Then there exists a constant $A > 0$ such that $\|u_n\| \leq A$, for all $n \geq 0$ and $\|u\| \leq A$. Now (see Corollary 2.4) $(p(t)u_n(t))$ converges to $(p(t)u(t))$ as $n \to +\infty$ for $t \in [0, +\infty)$. Since $F$ is continuous, we have $F(t,u_n(t)) \to F(t,u(t))$ as $n \to +\infty$, and using (F1) we have

$$|F(t,u_n(t))| \leq a(t)|u_n(t)|^\alpha + b(t)|u_n(t)|^{\beta}$$

$$\leq \frac{a(t)}{p^a(t)}|p(t)u_n(t)|^\alpha + \frac{b(t)}{p^\beta(t)}|p(t)u_n(t)|^{\beta}$$

$$\leq \frac{a(t)}{p^a(t)}\|u_n\|_{\alpha,p}^\alpha + \frac{b(t)}{p^\beta(t)}\|u_n\|_{\alpha,p}^{\beta}$$

$$\leq \frac{a(t)}{p^a(t)}M^\alpha_1\|u_n\|^\alpha + \frac{b(t)}{p^\beta(t)}M^\beta_1\|u_n\|^{\beta}$$

$$\leq \frac{a(t)}{p^a(t)}M^\alpha_1A^\alpha + \frac{b(t)}{p^\beta(t)}M^\beta_1A^\beta,$$

so from the Lebesgue Dominated Convergence Theorem we have

$$\lim_{n \to +\infty} \int_0^{+\infty} F(t,u_n(t))dt = \int_0^{+\infty} F(t,u(t))dt.$$  

The norm in the reflexive Banach space is sequentially weakly lower semi-continuous, so

$$\liminf_{n \to +\infty} \|u_n\| \geq \|u\|.$$  

Thus one has

$$\liminf_{n \to +\infty} J(u_n) = \liminf_{n \to +\infty} \left(\frac{1}{2}\|u_n\|^2 - \int_0^{+\infty} F(t,u_n(t))dt\right)$$

$$\geq \frac{1}{2}\|u\|^2 - \int_0^{+\infty} F(t,u(t))dt = J(u).$$

Then, $J$ is sequentially weakly lower semi-continuous.

From Lemma 1.6, $J$ has a minimum point $u_0$ which is a critical point of $J$.

**Claim 4.** We show that $u_0 \neq 0$.

Let $u_1 \in H_0^2(0, +\infty) \setminus \{0\}$ and $|u_1(t)| \leq 1$, for all $t \in I$. Then from (F2), we have

$$J(su_1) = \frac{s^2}{2}\|u_1\|^2 - \int_0^{+\infty} F(t,su_1(t))dt$$

$$\leq \frac{s^2}{2}\|u_1\|^2 - \int_\gamma |su_1(t)|^\gamma dt$$

$$\leq \frac{s^2}{2}\|u_1\|^2 - s^\gamma \int_\gamma |u_1(t)|^\gamma dt, \quad 0 < s < 1.$$  

Since $0 < \gamma < 2$, it follows that $J(su_1) < 0$ for $s > 0$ small enough. Hence $J(u_0) < 0$, and therefore $u_0$ is a nontrivial critical point of $J$.

Finally, it is easy to see that under (F1), the functional $J$ is Gâteaux differentiable and the Gâteaux derivative at a point $u \in X$ is

$$\langle f'(u), v \rangle = \int_0^{+\infty} \left(u''(t)v''(t) + u'(t)v'(t) + u(t)v(t)\right)dt - \int_0^{+\infty} f(t,u(t))v(t)dt,$$  \hspace{1cm}(3.2)
for all \( v \in H^2_0(0, +\infty) \). Therefore \( u \) is a weak solution of Problem (1.1).

**Theorem 3.2.** Assume that \( f \) satisfies the following assumptions.

(F3) There exist nonnegative functions \( \varphi, g \) such that \( g \in C([R, 0, +\infty]) \) with

\[
|f(t, x)| \leq \varphi(t)g(x), \text{ for all } t \in [0, +\infty) \text{ and all } x \in \mathbb{R},
\]

and for any constant \( R > 0 \) there exists a nonnegative function \( \psi_R \) with \( \varphi \psi_R \in L^1(0, +\infty) \) and

\[
\sup \left\{ g \left( \frac{y}{p(t)} \right) : y \in [-R, R] \right\} \leq \psi_R(t) \quad \text{for a.e. } t \geq 0.
\]

(F4)

\[
\frac{1}{a(t)} F(t, \frac{1}{p(t)}x) = o(|x|^2) \quad \text{as } x \to 0
\]

uniformly in \( t \in [0, +\infty) \) for some function \( a \in L^1(0, +\infty) \cap C[0, +\infty) \).

(F5) There exists a positive function \( c_1 \) and a nonnegative function \( c_2 \) with \( c_1, c_2 \in L^1(0, \infty) \), and \( \mu > 2 \) such that

(a) \( F(t, x) \geq c_1(t)|x|^\mu - c_2(t), \) for \( t \geq 0, \forall x \in \mathbb{R} \setminus \{0\} \),

(b) \( \mu F(t, x) \leq x f(t, x), \) for \( t \geq 0, \forall x \in \mathbb{R} \).

Then Problem (1.1) has at least one nontrivial weak solution.

**Proof.** We have \( J(0) = 0 \).

Claim 1. \( J \) satisfies the (PS) condition.

Assume that \( (u_n)_{n \in \mathbb{N}} \subset H^2_0(0, +\infty) \) is a sequence such that \( (J(u_n))_{n \in \mathbb{N}} \) is bounded and \( J'(u_n) \to 0 \) as \( n \to +\infty \). Then there exists a constant \( d > 0 \) such that

\[
|J(u_n)| \leq d, \quad \|J'(u_n)\|_{E'} \leq d \mu, \quad \forall n \in \mathbb{N}.
\]

From (F5)(b) we have

\[
2d + 2d\|u_n\| \geq 2J(u_n) - \frac{2}{\mu} (J'(u_n), u_n)
\]

\[
\geq \left( 1 - \frac{2}{\mu} \right) \|u_n\|^2 + 2 \left[ \int_0^{+\infty} \left( \frac{1}{\mu} u_n(t) f(t, u_n(t)) - F(t, u_n(t)) \right) dt \right]
\]

\[
\geq \left( 1 - \frac{2}{\mu} \right) \|u_n\|^2.
\]

Since \( \mu > 2 \), then \( (u_n)_{n \in \mathbb{N}} \) is bounded in \( H^2_0(0, +\infty) \).

Now, we show that \( (u_n) \) converges strongly to some \( u \) in \( H^2_0(0, +\infty) \). Since \( (u_n) \) is bounded in \( H^2_0(0, +\infty) \), there exists a subsequence of \( (u_n) \) still denoted by \( (u_n) \) such that \( (u_n) \) converges weakly to some \( u \) in \( H^2_0(0, +\infty) \). There exists a constant \( c > 0 \) such that \( \|u_n\| \leq c \). Now (see Corollary 2.4) \( (p(t)u_n(t)) \) converges to \( p(t)u(t) \) on \([0, +\infty)\). We have \( f(t, u_n(t)) \to f(t, u(t)) \) and

\[
|f(t, u_n(t))| = \left| f(t, \frac{1}{p(t)} p(t)u_n(t)) \right|
\]

\[
\leq \varphi(t)g \left( \frac{1}{p(t)} p(t)u_n(t) \right)
\]

\[
\leq \varphi(t) \psi_{\ell M_1}(t),
\]

Fourth-order BVP via critical point theory
and using the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n \to +\infty} \int_0^{+\infty} (f(t, u_n(t)) - f(t, u(t))) (u_n(t) - u(t)) \, dt = 0. \tag{3.3}$$

Since \( \lim_{n \to +\infty} f'(u_n) = 0 \) and \((u_n)\) converges weakly to some \( u \), we have

$$\lim_{n \to +\infty} \langle f'(u_n) - f'(u), u_n - u \rangle = 0. \tag{3.4}$$

It follows from (3.2) that

$$\langle f'(u_n) - f'(u), u_n - u \rangle = \|u_n - u\|^2 - \int_0^{+\infty} (f(t, u_n(t)) - f(t, u(t))) (u_n(t) - u(t)) \, dt.$$}

Hence \( \lim_{n \to +\infty} \|u_n - u\| = 0 \). Thus \( (u_n) \) converges strongly to \( u \) in \( H_0^1(0, +\infty) \), so \( f \) satisfies the (PS) condition.

**Claim 2.** \( f \) satisfies assumption (1) of Lemma 1.8. Let \( 0 < \varepsilon < \frac{1}{|a|_{L^1}} \). From (F4), there exists \( 0 < \delta < 1 \) such that

$$\left| \frac{1}{a(t)} F(t, \frac{1}{p(t)} x) \right| \leq \frac{\varepsilon}{2} |x|^2, \quad \text{for } t \in [0, +\infty) \text{ and } |x| \leq \delta.$$}

Using Corollary 2.4, we have

$$\int_0^{+\infty} |F(t, u(t))| \, dt = \int_0^{+\infty} \left| F\left( t, \frac{1}{p(t)} u(t) \right) \right| \, dt \leq \int_0^{+\infty} \frac{\varepsilon}{2} |a(t)| p^2(t) |u(t)|^2 \, dt \leq \frac{\varepsilon}{2} M_1^2 |a|_{L^1} \|u\|^2,$$

whenever \( \|u\|_{\infty, p} \leq \delta \).

Let \( 0 < \rho \leq \frac{\delta}{\alpha} \) and \( \alpha = \frac{1}{2} (1 - \varepsilon |a|_{L^1} M_1^2) \rho^2 \). Then for \( \|u\| = \rho \) (note \( \|u\|_{\infty, p} \leq \delta \)), we have

$$f(u) = \frac{1}{2} \|u\|^2 - \int_0^{+\infty} F(t, u(t)) \, dt \geq \frac{1}{2} (1 - \varepsilon |a|_{L^1} M_1^2) \|u\|^2 = \alpha,$$

so assumption (1) in Lemma 1.8 is satisfied.

**Claim 3.** \( f \) satisfies assumption (2) of Lemma 1.8. By (F5)(a) we have for some \( v_0 \in H_0^1(0, +\infty), \ v_0 \neq 0, \)

$$f(\xi v_0) = \frac{1}{2} \xi^2 \|v_0\|^2 - \int_0^{+\infty} F(t, \xi v_0(t)) \, dt \leq \frac{1}{2} \xi^2 \|v_0\|^2 - |\xi|^\mu \int_0^{+\infty} c_1(t) \|v_0(t)\|^\mu \, dt + \int_0^{+\infty} c_2(t) \, dt.$$

Now since \( \mu > 2 \), then for \( u_0 = \xi v_0, \; f(u_0) \leq 0, \; \text{as } \xi \to +\infty, \) so assumption (2) in Lemma 1.8 is satisfied. From Lemma 1.8, \( f \) possesses a critical point which is a nontrivial weak solution of Problem (1.1).
As an example of the above theorem, take \( f(t, x) = \frac{5}{2} \exp(-t)|x|^\frac{1}{2}x \). To see this take
\[
c_1(t) = \exp(-t), \quad c_2(t) = 0,
\]
\[
\mu = \frac{5}{2}, \quad a(t) = \frac{1}{(1+t)^2}, \quad p(t) = \frac{1}{1+t},
\]
\[
\phi(t) = \frac{5}{2}e^{-t}, \quad g(x) = |x|^\frac{3}{2} \quad \text{and} \quad \psi_R(t) = (1+t)^\frac{3}{2}R^\frac{3}{2}.
\]

References


