Product-type system of difference equations of second-order solvable in closed form

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Abstract. This paper presents solutions to the following product-type second-order system of difference equations

\[ x_{n+1} = \frac{y_n}{x_{n-1}^b}, \quad y_{n+1} = \frac{z_n}{x_{n-1}^d}, \quad z_{n+1} = \frac{x_n}{y_{n-1}^g}, \quad n \in \mathbb{N}_0, \]

where \( a, b, c, d, f, g \in \mathbb{Z} \), and \( x_{-i}, y_{-i}, z_{-i} \in \mathbb{C} \setminus \{0\}, i \in \{0, 1\} \), in closed form.

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1 Introduction

Recently there has been some renewed interest in solving difference equations and systems of difference equations and their applications (see, e.g., [1–4, 7, 8, 14, 18, 20, 23–38, 41–45]), especially after the publication of note [18] in which a method for solving a nonlinear difference equation of second-order was presented. Since the end of the 1990s there has been also some interest in concrete systems of difference equations (see, e.g., [8–13, 15–17, 23, 24, 26–28, 30–40, 42, 43, 45]). In the line of our investigations [5, 6, 19, 21, 22] (see also the references therein) we studied the long-term behavior of several classes of difference equations related to the product-type ones. Somewhat later we studied some systems which are extensions of these equations [39, 40].

Long-term behavior of positive solutions to the following system

\[ x_{n+1} = \max \left\{ c, \frac{y_n^p}{x_{n-1}^q} \right\}, \quad y_{n+1} = \max \left\{ c, \frac{z_n^p}{y_{n-1}^q} \right\}, \quad z_{n+1} = \max \left\{ c, \frac{x_n^p}{y_{n-1}^q} \right\}, \quad (1.1) \]
An interesting feature of system (1.2) is that it can be solved in closed form in the case of positive initial values. Namely, a simple inductive argument shows that in this case
\[ \min\{x_n, y_n, z_n\} > 0, \quad \text{for every } n \geq -1. \]
Hence, it is legitimate to take the logarithm of all the three equations in (1.2) and by using the change of variables
\[ u_n = \ln x_n, \quad v_n = \ln y_n, \quad w_n = \ln z_n, \quad n \geq -1, \quad (1.3) \]
the system is transformed into the following linear one
\[ u_{n+1} = pv_n - pw_{n-1}, \quad v_{n+1} = pw_n - pu_{n-1}, \quad n \in \mathbb{N}_0. \quad (1.4) \]
Using the third equation of (1.4) in the first and second ones we obtain the following system of difference equations
\[ u_{n+1} = pv_n - p^2 u_{n-2} + p^3 v_{n-3}, \quad n \in \mathbb{N}_0. \quad (1.5) \]
Using (1.6) in (1.5) we get
\[ v_{n+3} - (p^3 - 3p^2)v_n + p^3 v_{n-3} = 0, \quad n \in \mathbb{N}_0. \quad (1.7) \]
This is a linear difference equation which can be solved in closed form, from which along with (1.6) and the third equation in (1.4) closed form formulas for \( u_n, v_n \) and \( w_n \) are obtained, and consequently by using (1.3) we obtain formulas for \( x_n, y_n \) and \( z_n \). We leave the details to the reader as a simple exercise.

A natural question is whether system (1.2) can be solved in closed form if initial values \( x_{-i}, y_{-i}, z_{-i}, i \in \{0, 1\} \), are complex numbers and for which values of parameter \( p \).

Motivated by all above mentioned and by our recent paper [37], here we will study the solvability of the following system of difference equations
\[ x_{n+1} = \frac{y_n^p}{z_n^p}, \quad y_{n+1} = \frac{z_n^p}{x_n^p}, \quad z_{n+1} = \frac{x_n^p}{y_n^p}, \quad n \in \mathbb{N}_0, \quad (1.8) \]
where \( a, b, c, d, f, g \in \mathbb{Z} \), and when initial values \( x_{-i}, y_{-i}, z_{-i}, i \in \{0, 1\} \), are complex numbers different from zero (it is easy to see that a solution to the system is well-defined if and only if all initial values are different from zero). We present a constructive method for solving the system. Condition \( a, b, c, d, f, g \in \mathbb{Z} \) is naturally posed, in order not to deal with multi-valued sequences.
2 Main result

Here we present our main result in this paper.

Theorem 2.1. Consider system (1.8) with \( a, b, c, d, f, g \in \mathbb{Z} \). If \( x_{-i}, y_{-i}, z_{-i} \in \mathbb{C} \setminus \{0\}, i \in \{0,1\} \), then the system is solvable in closed form.

Proof. Let

\[
\begin{align*}
a_1 &= a, \quad b_1 &= b, \quad c_1 &= c, \quad d_1 &= d, \quad f_1 &= f, \quad g_1 &= g. \\
\end{align*}
\]

(2.1)

By using the equations in (1.8) we obtain

\[
\begin{align*}
x_{n+1} &= \frac{y_n}{z_{n-1}} = \frac{z_{n-1} - b_1}{x_{n-2}} = \frac{z_{n-1}}{x_{n-2}}, \\
y_{n+1} &= \frac{x_n}{y_{n-1}} = \frac{f_{c_1} - d_1}{y_{n-2}} = \frac{x_{n-1}}{y_{n-2}}, \\
z_{n+1} &= \frac{x_{n-1}}{y_{n-1}} = \frac{f_1}{z_{n-2}} = \frac{z_{n-2}}{x_{n-2}}.
\end{align*}
\]

(2.2), (2.3), (2.4)

where we define \( a_2, b_2, c_2, d_2, f_2 \) and \( g_2 \) as follows

\[
\begin{align*}
a_2 &= ca_1 - b_1, \quad b_2 := da_1, \quad c_2 := fc_1 - d_1, \quad d_2 := gc_1, \quad f_2 := af_1 - g_1, \quad g_2 := bf_1.
\end{align*}
\]

By using (2.2), (2.3), (2.4) and the equations in (1.8), it follows that

\[
\begin{align*}
x_{n+1} &= \frac{z_{n-1}}{x_{n-2}} = \frac{x_{n-2} - b_2}{y_{n-3}} = \frac{x_{n-2}}{y_{n-3}}, \\
y_{n+1} &= \frac{x_{n-1}}{y_{n-2}} = \frac{y_{n-2} - d_2}{z_{n-3}} = \frac{y_{n-2}}{z_{n-3}}, \\
z_{n+1} &= \frac{f_{n-1}}{z_{n-2}} = \frac{z_{n-2} - g_2}{x_{n-3}} = \frac{z_{n-2}}{x_{n-3}}.
\end{align*}
\]

(2.5), (2.6), (2.7)

where we define \( a_3, b_3, c_3, d_3, f_3 \) and \( g_3 \) as follows

\[
\begin{align*}
a_3 &= af_2 - b_2, \quad b_3 := ga_2, \quad c_3 := ac_2 - d_2, \quad d_3 := bc_2, \quad f_3 := cf_2 - g_2, \quad g_3 := df_2.
\end{align*}
\]

By using (2.5), (2.6), (2.7) and the equations in (1.8), we further get

\[
\begin{align*}
x_{n+1} &= \frac{z_{n-1}}{y_{n-3}} = \frac{y_{n-3} - b_3}{z_{n-4}} = \frac{y_{n-3}}{z_{n-4}}, \\
y_{n+1} &= \frac{x_{n-2}}{z_{n-3}} = \frac{z_{n-3} - d_3}{x_{n-4}} = \frac{z_{n-3}}{x_{n-4}}, \\
z_{n+1} &= \frac{f_{n-3}}{x_{n-3}} = \frac{x_{n-3} - g_3}{y_{n-4}} = \frac{x_{n-3}}{y_{n-4}}.
\end{align*}
\]

(2.8), (2.9), (2.10)
where we define $a_4, b_4, c_4, d_4, f_4$ and $g_4$ as follows

$$a_4 := a_3 - b_3, \quad b_4 := b a_3, \quad c_4 := c c_3 - d_3, \quad d_4 := d c_3, \quad f_4 := f f_3 - g_3, \quad g_4 := g f_3.$$  

Let

$$x_{n+1} = \frac{x^{3k-1}_{n-3k+2}}{x^{3k-1}_{n-3k+1}}, \quad y_{n+1} = \frac{y^{3k-1}_{n-3k+2}}{y^{3k-1}_{n-3k+1}}, \quad z_{n+1} = \frac{z^{3k-1}_{n-3k+2}}{z^{3k-1}_{n-3k+1}},$$

where

$$a_{3k-1} := c a_{3k-2} - b_{3k-2}, \quad b_{3k-1} := d a_{3k-2}, \quad c_{3k-1} := f c_{3k-2} - d_{3k-2},$$

$$d_{3k-1} := g c_{3k-2}, \quad f_{3k-1} := a f_{3k-2} - g_{3k-2}, \quad g_{3k-1} := b f_{3k-2},$$

$$x_{n+1} = \frac{x_{n-3k+1}}{y_{n-3k}}, \quad y_{n+1} = \frac{y_{n-3k+1}}{z_{n-3k}}, \quad z_{n+1} = \frac{z_{n-3k+1}}{x_{n-3k}},$$

where

$$a_{3k} := f a_{3k-1} - b_{3k-1}, \quad b_{3k} := g a_{3k-1}, \quad c_{3k} := a c_{3k-1} - d_{3k-1},$$

$$d_{3k} := b c_{3k-1}, \quad f_{3k} := c f_{3k-1} - g_{3k-1}, \quad g_{3k} := d f_{3k-1},$$

for some $k \in \mathbb{N}$ such that $n \geq 3k$.

From the relations in (2.11) and by using the equations in (1.8) we obtain

$$x_{n+1} = \frac{y^{3k+1}_{n-3k}}{z^{3k+1}_{n-3k+1}}, \quad y_{n+1} = \frac{z^{3k+1}_{n-3k}}{x^{3k+1}_{n-3k+1}}, \quad z_{n+1} = \frac{x^{3k+1}_{n-3k}}{y^{3k+1}_{n-3k+1}},$$

where we define $a_{3k+2}, b_{3k+2}, c_{3k+2}, d_{3k+2}, f_{3k+2}$ and $g_{3k+2}$ as follows

$$a_{3k+2} := c a_{3k+1} - b_{3k+1}, \quad b_{3k+2} := d a_{3k+1}, \quad c_{3k+2} := f c_{3k+1} - d_{3k+1},$$

$$d_{3k+2} := g c_{3k+1}, \quad f_{3k+2} := a f_{3k+1} - g_{3k+1}, \quad g_{3k+2} := b f_{3k+1}.$$
By using (2.12), (2.13), (2.14) and the equations in (1.8), it follows that

\[
\begin{align*}
x_{n+1} &= \frac{x_{n+1}}{x_{n+1}} = \frac{a_{3k+3} - b_{3k+3}}{a_{3k+3} - b_{3k+3}} = x_{n-3k-2}, \\
y_{n+1} &= \frac{y_{n+1}}{y_{n+1}} = \frac{y_{n+1}}{y_{n+1}} = y_{n-3k-3}, \\
z_{n+1} &= \frac{z_{n+1}}{z_{n+1}} = \frac{z_{n+1}}{z_{n+1}} = z_{n-3k-3}.
\end{align*}
\]

(2.15)

(2.16)

(2.17)

where we define \(a_{3k+3}, b_{3k+3}, c_{3k+3}, d_{3k+3}, f_{3k+3}\) and \(g_{3k+3}\) as follows

\[
\begin{align*}
a_{3k+3} &:= f_{a_{3k+2}} - b_{3k+2}, & b_{3k+3} &:= g_{a_{3k+2}}, & c_{3k+3} &:= a_{c_{3k+2}} - d_{3k+2}, \\
d_{3k+3} &:= b_{c_{3k+2}}, & f_{3k+3} &:= f_{a_{3k+2}} - g_{3k+2}, & g_{3k+3} &:= d_{f_{3k+2}}.
\end{align*}
\]

By using (2.15), (2.16), (2.17) and the equations in (1.8) we further get

\[
\begin{align*}
x_{n+1} &= \frac{x_{n+1}}{x_{n+1}} = \frac{x_{n+1}}{x_{n+1}} = x_{n-3k-2}, \\
y_{n+1} &= \frac{y_{n+1}}{y_{n+1}} = \frac{y_{n+1}}{y_{n+1}} = y_{n-3k-3}, \\
z_{n+1} &= \frac{z_{n+1}}{z_{n+1}} = \frac{z_{n+1}}{z_{n+1}} = z_{n-3k-3}.
\end{align*}
\]

(2.18)

(2.19)

(2.20)

where we define \(a_{3k+4}, b_{3k+4}, c_{3k+4}, d_{3k+4}, f_{3k+4}\) and \(g_{3k+4}\) as follows

\[
\begin{align*}
a_{3k+4} &:= a_{a_{3k+3}} - b_{3k+3}, & b_{3k+4} &:= ba_{a_{3k+3}}, & c_{3k+4} &:= c_{c_{3k+3}} - d_{3k+3}, \\
d_{3k+4} &:= b_{c_{3k+3}}, & f_{3k+4} &:= f_{a_{3k+3}} - g_{3k+3}, & g_{3k+4} &:= g_{f_{3k+3}}.
\end{align*}
\]

Hence, this inductive argument shows that sequences \((a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}, (c_n)_{n \in \mathbb{N}}, (d_n)_{n \in \mathbb{N}}, (f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}}\), satisfy the following recurrent relations

\[
\begin{align*}
a_{3k+2} &= ca_{3k+1} - b_{3k+1}, & a_{3k+3} &= f_{a_{3k+2}} - b_{3k+2}, & a_{3k+4} &= a_{a_{3k+3}} - b_{3k+3}, \\
b_{3k+2} &= da_{3k+1}, & b_{3k+3} &= g_{a_{3k+2}}, & b_{3k+4} &= b_{a_{3k+3}}, \\
c_{3k+2} &= fc_{3k+1} - d_{3k+1}, & c_{3k+3} &= ac_{3k+2} - d_{3k+2}, & c_{3k+4} &= c_{c_{3k+3}} - d_{3k+3}, \\
d_{3k+2} &= gc_{3k+1}, & d_{3k+3} &= bc_{3k+2}, & d_{3k+4} &= d_{c_{3k+3}}, \\
f_{3k+2} &= af_{3k+1} - g_{3k+1}, & f_{3k+3} &= cf_{3k+2} - g_{3k+2}, & f_{3k+4} &= f_{a_{3k+3}} - g_{3k+3}, \\
g_{3k+2} &= bf_{3k+1}, & g_{3k+3} &= d_{f_{3k+2}}, & g_{3k+4} &= g_{f_{3k+3}}.
\end{align*}
\]

(2.21)

(2.22)

(2.23)

(2.24)

(2.25)

(2.26)

for \(k \in \mathbb{N}_0\).

From (2.12)–(2.20) we easily obtain

\[
\begin{align*}
x_{3n+1} &= \frac{x_{3n+1}}{x_{3n+1}} = \frac{x_{3n+1}}{x_{3n+1}} = x_{n+3}, \\
y_{3n+1} &= \frac{y_{3n+1}}{y_{3n+1}} = \frac{y_{3n+1}}{y_{3n+1}} = y_{n+3}, \\
z_{3n+1} &= \frac{z_{3n+1}}{z_{3n+1}} = \frac{z_{3n+1}}{z_{3n+1}} = z_{n+3}.
\end{align*}
\]

(2.27)

(2.28)

(2.29)

\(n \in \mathbb{N}_0\).
From (2.21) and (2.22) we have that

\[ a_{3k+2} = ca_{3k+1} - ba_{3k}, \quad a_{3k+3} = fa_{3k+2} - da_{3k+1}, \quad a_{3k+4} = aa_{3k+3} - ga_{3k+2}, \]

\( k \in \mathbb{N}_0 \), (here we regard that \( a_0 = 1 \) if \( b \neq 0 \), due to the relation \( b_{3k+4} = ba_{3k+3} \) with \( k = -1 \)).

Using the first equation in the second and third ones it follows that

\[ a_{3k+3} - (cf - d)a_{3k+1} + bfa_{3k} = 0, \quad k \in \mathbb{N}_0, \tag{2.30} \]

and

\[ a_{3k+4} - aa_{3k+3} + cga_{3k+1} - bg a_{3k} = 0, \quad k \in \mathbb{N}_0. \tag{2.31} \]

Using (2.30) into (2.31) it follows that

\[ a_{3k+6} + (ad + bf + cg - acf)a_{3k+3} + bdga_{3k} = 0, \quad k \in \mathbb{N}_0. \tag{2.32} \]

Relation (2.32) means that the sequence \( (a_{3k})_{k \in \mathbb{N}_0} \) annihilate the linear operator

\[ \mathcal{L}(x_n) = x_{n+2} + (ad + bf + cg - acf)x_{n+1} + bdgx_n. \]

From this along with (2.30) it follows that sequence \( (a_{3k+1})_{k \in \mathbb{N}_0} \) annihilate the operator too. Using these facts and the relation \( a_{3k+2} = ca_{3k+1} - ba_{3k} \), it follows that that sequence \( (a_{3k+2})_{k \in \mathbb{N}_0} \) also annihilate the operator. This along with the relations in (2.22) implies that sequences \( (b_{3k+i})_{k \in \mathbb{N}_0}, \ i = 0, 1, 2, \) annihilate the operator too. Since sequences \( (c_{3k+2+i})_{k \in \mathbb{N}_0} \) and \( (d_{3k+2+i})_{k \in \mathbb{N}_0} \), that is, \( (f_{3k+1+i})_{k \in \mathbb{N}_0} \) and \( (g_{3k+1+i})_{k \in \mathbb{N}_0}, \ i = 0, 1, 2, \) satisfy the relations in (2.21) and (2.22) it follows that they also annihilate the operator.

Case \( b = 0 \). In this case we have that sequences \( (a_{3k+i})_{k \in \mathbb{N}_0}, (c_{3k+i})_{k \in \mathbb{N}_0}, (f_{3k+i})_{k \in \mathbb{N}_0} \), satisfy the recurrent relation

\[ x_{n+1} = (acf - ad - cg)x_n, \quad n \in \mathbb{N}. \]

From this and since \( a_1 = a, c_1 = c, f_1 = f, a_2 = ac - b, c_2 = cf - d, f_2 = af - g, a_3 = acf - bf - ad, c_3 = acf - ad - cg, f_3 = acf - bf - cg \), and by using the condition \( b = 0 \), it follows that

\[ a_{3k+1} = a(acf - ad - cg)^k, \tag{2.33} \]

\[ a_{3k+2} = ac(acf - ad - cg)^k, \tag{2.34} \]

\[ a_{3k+3} = a(cf - d)(acf - ad - cg)^k, \tag{2.35} \]

\[ c_{3k+1} = cacf - ad - cg)^k, \tag{2.36} \]

\[ c_{3k+2} = (cf - d)(acf - ad - cg)^k, \tag{2.37} \]

\[ c_{3k+3} = (acf - ad - cg)^{k+1}, \tag{2.38} \]

\[ f_{3k+1} = f(acf - ad - cg)^k, \tag{2.39} \]

\[ f_{3k+2} = (af - g)(acf - ad - cg)^k, \tag{2.40} \]

\[ f_{3k+3} = c(af - g)(acf - ad - cg)^k, \quad k \in \mathbb{N}_0. \tag{2.41} \]
Using (2.33)–(2.41) into (2.22), (2.24) and (2.26), as well as the condition \( b = 0 \), it follows that

\[
\begin{align*}
  b_{3k+1} &= 0, \quad (2.42) \\
  b_{3k+2} &= ad(acf - ad - cg)^k, \quad (2.43) \\
  b_{3k+3} &= acg(acf - ad - cg)^k, \quad (2.44) \\
  d_{3k+1} &= d(acf - ad - cg)^k, \quad (2.45) \\
  d_{3k+2} &= cg(acf - ad - cg)^k, \quad (2.46) \\
  d_{3k+3} &= 0, \quad (2.47) \\
  g_{3k+1} &= cg(af - g)(acf - ad - cg)^{k-1}, \quad (2.48) \\
  g_{3k+2} &= 0, \quad (2.49) \\
  g_{3k+3} &= d(acf - ad - cg)^k, \quad k \in \mathbb{N}_0. \quad (2.50)
\end{align*}
\]

Employing (2.33)–(2.50) into (2.27), (2.28) and (2.29) we obtain that the well-defined solutions to system (1.8) in this case are given by the following formulas

\[
\begin{align*}
  x_{3n+1} &= y_0^{n(acf - ad - cg)}, \quad (2.51) \\
  x_{3n+2} &= c_0^{ac(acf - ad - cg)^n - ad(acf - ad - cg)^n} x_{n-1}, \quad (2.52) \\
  x_{3n+3} &= x_0^{a(cf - d)(acf - ad - cg)^n - acg(acf - ad - cg)^n} y_{n-1}, \quad (2.53) \\
  y_{3n+1} &= c_0^{c(acf - ad - cg)^n - d(acf - ad - cg)^n} x_{n-1}, \quad (2.54) \\
  y_{3n+2} &= x_0^{(cf - d)(acf - ad - cg)^n - cg(acf - ad - cg)^n} y_{n-1}, \quad (2.55) \\
  y_{3n+3} &= y_0^{(acf - ad - cg)^{n+1}}, \quad (2.56) \\
  z_{3n+1} &= x_0^{f(acf - ad - cg)^n - cg(af - g)(acf - ad - cg)^n} x_{n-1}, \quad (2.57) \\
  z_{3n+2} &= y_0^{(af - g)(acf - ad - cg)^n}, \quad (2.58) \\
  z_{3n+3} &= c_0^{c(af - g)(acf - ad - cg)^n - d(af - g)(acf - ad - cg)^n} x_{n-1}, \quad n \in \mathbb{N}_0. \quad (2.59)
\end{align*}
\]

Case \( d = 0 \). In this case we have that sequences \((a_{3k+i})_{k \in \mathbb{N}_0}, (c_{3k+i})_{k \in \mathbb{N}_0}, (f_{3k+i})_{k \in \mathbb{N}_0}\), satisfy the recurrent relation

\[
  x_{n+1} = (acf - bf - cg)x_n, \quad n \in \mathbb{N}.
\]

From this and since \( a_1 = a, c_1 = c, f_1 = f, a_2 = ac - b, c_2 = cf - d, f_2 = af - g, a_3 = acf - bf - ad, c_3 = acf - ad - cg, f_3 = acf - bf - cg \), and by using the condition \( d = 0 \), it follows that

\[
\begin{align*}
  a_{3k+1} &= a(acf - bf - cg)^k, \quad (2.60) \\
  a_{3k+2} &= (ac - b)(acf - bf - cg)^k, \quad (2.61) \\
  a_{3k+3} &= f(ac - b)(acf - bf - cg)^k, \quad (2.62) \\
  c_{3k+1} &= c(acf - bf - cg)^k, \quad (2.63) \\
  c_{3k+2} &= cf(acf - bf - cg)^k, \quad (2.64) \\
  c_{3k+3} &= c(af - g)(acf - bf - cg)^k, \quad (2.65) \\
  f_{3k+1} &= f(acf - bf - cg)^k. \quad (2.66)
\end{align*}
\]
\[ f_{3k+2} = (af - g)(acf - bf - cg)^k, \quad \text{(2.67)} \]
\[ f_{3k+3} = (acf - bf - cg)^{k+1}, \quad k \in \mathbb{N}_0. \quad \text{(2.68)} \]

Using (2.60)–(2.68) into (2.22), (2.24) and (2.26), as well as the condition \( d = 0 \), it follows that

\[ b_{3k+1} = bf(ac - b)(acf - bf - cg)^{k-1}, \quad \text{(2.69)} \]
\[ b_{3k+2} = 0, \quad \text{(2.70)} \]
\[ b_{3k+3} = g(ac - b)(acf - bf - cg)^k, \quad \text{(2.71)} \]
\[ d_{3k+1} = 0, \quad \text{(2.72)} \]
\[ d_{3k+2} = cg(acf - bf - cg)^k, \quad \text{(2.73)} \]
\[ d_{3k+3} = bcf(acf - bf - cg)^k, \quad \text{(2.74)} \]
\[ g_{3k+1} = g(acf - bf - cg)^k, \quad \text{(2.75)} \]
\[ g_{3k+2} = bf(acf - bf - cg)^k, \quad \text{(2.76)} \]
\[ g_{3k+3} = 0, \quad k \in \mathbb{N}_0. \quad \text{(2.77)} \]

Employing (2.60)–(2.77) into (2.27), (2.28) and (2.29) we obtain that the well-defined solutions of system (1.8) in this case are given by the following formulas

\[ x_{3n+1} = \frac{y_0(aacf - bf - cg)^n - bf(ac - b)(acf - bf - cg)^n - 1}{z_1}, \quad \text{(2.78)} \]
\[ x_{3n+2} = \frac{z_0(ac - b)(acf - bf - cg)^n}{z_1}, \quad \text{(2.79)} \]
\[ x_{3n+3} = \frac{x_0(f(ac - b)(acf - bf - cg)^n - g(ac - b)(acf - bf - cg)^n)}{y_1}, \quad \text{(2.80)} \]
\[ y_{3n+1} = \frac{z_0(cacf - bf - cg)^n}{z_1}, \quad \text{(2.81)} \]
\[ y_{3n+2} = \frac{x_0(cf(acf - bf - cg)^n - cg(acf - bf - cg)^n)}{z_1}, \quad \text{(2.82)} \]
\[ y_{3n+3} = \frac{y_0(cacf - bf - cg)^n - bcf(acf - bf - cg)^n}{z_1}, \quad \text{(2.83)} \]
\[ z_{3n+1} = \frac{x_0(f(acf - bf - cg)^n - g(acf - bf - cg)^n)}{z_1}, \quad \text{(2.84)} \]
\[ z_{3n+2} = \frac{y_0(af - g)(acf - bf - cg)^n - bf(acf - bf - cg)^n}{y_1}, \quad \text{(2.85)} \]
\[ z_{3n+3} = \frac{z_0(acf - bf - cg)^n + 1}{z_1}, \quad n \in \mathbb{N}_0. \quad \text{(2.86)} \]

Case \( g = 0 \). In this case we have that sequences \((a_{3k+i})_{k \in \mathbb{N}_0}, (c_{3k+i})_{k \in \mathbb{N}_0}, (f_{3k+i})_{k \in \mathbb{N}_0}\), satisfy the recurrent relation

\[ x_{n+1} = (acf - ad - bf)x_n, \quad n \in \mathbb{N}. \]

From this and since \( a_1 = a, c_1 = c, f_1 = f, a_2 = ac - b, c_2 = cf - d, f_2 = af - g, a_3 = acf - bf - ad, c_3 = acf - ad - cg, f_3 = acf - bf - cg, \) and by using the condition \( g = 0 \), it follows that

\[ a_{3k+1} = a(acf - ad - bf)^k, \quad \text{(2.87)} \]
\[ a_{3k+2} = (ac - b)(acf - bf - ad)^k, \quad \text{(2.88)} \]
\[ a_{3k+3} = (acf - bf - ad)^{k+1}, \quad \text{(2.89)} \]
\[ c_{3k+1} = c(acf - bf - ad)^k, \quad \text{(2.90)} \]
\[ c_{3k+2} = (cf - d)(acf - bf - ad)^k, \quad \text{(2.91)} \]
that solutions of system (1.8) in this case are given by the following formulas

\[ c_{3k+3} = (ac - d)(acf - bf - ad)^k, \quad (2.92) \]
\[ f_{3k+1} = f(acf - bf - ad)^k, \quad (2.93) \]
\[ f_{3k+2} = af(acf - bf - ad)^k, \quad (2.94) \]
\[ f_{3k+3} = f(ac - b)(acf - bf - ad)^k, \quad k \in \mathbb{N}_0. \quad (2.95) \]

Using (2.87)–(2.95) into (2.22), (2.24) and (2.26), as well as the condition \( g = 0 \), it follows that

\[ b_{3k+1} = b(acf - bf - ad)^k, \quad (2.96) \]
\[ b_{3k+2} = ad(acf - ad - bf)^k, \quad (2.97) \]
\[ b_{3k+3} = 0, \quad (2.98) \]
\[ d_{3k+1} = ad(cf - d)(acf - bf - ad)^{k-1}, \quad (2.99) \]
\[ d_{3k+2} = 0, \quad (2.100) \]
\[ d_{3k+3} = b(cf - d)(acf - bf - ad)^k, \quad (2.101) \]
\[ g_{3k+1} = 0, \quad (2.102) \]
\[ g_{3k+2} = bf(acf - bf - ad)^k, \quad (2.103) \]
\[ g_{3k+3} = adf(acf - bf - ad)^k, \quad k \in \mathbb{N}_0. \quad (2.104) \]

Employing (2.87)–(2.104) into (2.27), (2.28) and (2.29) we obtain that the well-defined solutions of system (1.8) in this case are given by the following formulas

\[ x_{3n+1} = y_0^{(acf - ad - bf)^n} z_{-1}^{-b(acf - bf - ad)^n}, \quad (2.105) \]
\[ x_{3n+2} = z_0^{(acf - ad - bf)^n} x_{-1}^{-ad(acf - ad - bf)^n}, \quad (2.106) \]
\[ x_{3n+3} = x_0^{(acf - bf - ad)^{n+1}}, \quad (2.107) \]
\[ y_{3n+1} = z_0^{(acf - bf - ad)^n} x_{-1}^{-ad(cf - d)(acf - bf - ad)^{n-1}}, \quad (2.108) \]
\[ y_{3n+2} = x_0^{(cf - d)(acf - bf - ad)^n}, \quad (2.109) \]
\[ y_{3n+3} = y_0^{(cf - d)(acf - bf - ad)^n} z_{-1}^{-b(cf - d)(acf - bf - ad)^n}, \quad (2.110) \]
\[ z_{3n+1} = x_0^{f(acf - bf - ad)^n}, \quad (2.111) \]
\[ z_{3n+2} = y_0^{f(acf - bf - ad)^n} z_{-1}^{-bf(acf - bf - ad)^n}, \quad (2.112) \]
\[ z_{3n+3} = z_0^{f(ac - b)(acf - bf - ad)^n} x_{-1}^{-adf(acf - bf - ad)^n}, \quad n \in \mathbb{N}_0. \quad (2.113) \]

Case \( bdg \neq 0 \). Let \( \lambda_{1,2} \) be the roots of the characteristic polynomial

\[ P(\lambda) = \lambda^2 - (acf - ad - bf - cg)\lambda + bdg, \]

of difference equation

\[ u_{n+2} - (acf - ad - bf - cg)u_{n+1} + bdgu_n = 0, \quad n \in \mathbb{N}. \quad (2.114) \]

It is known that the general solution of equation (2.114) has the following form

\[ u_n = a_1\lambda_1^n + a_2\lambda_2^n, \quad n \in \mathbb{N}, \]
if \((acf - ad - bf - cg)^2 \neq 4bdg\), where \(a_1\) and \(a_2\) are arbitrary constants, while in the case \((acf - ad - bf - cg)^2 = 4bdg\), the general solution has the following form

\[ u_n = (\beta_1 n + \beta_2)\lambda^2_n, \quad n \in \mathbb{N}, \]

where \(\beta_1\) and \(\beta_2\) are arbitrary constants.

By some calculation and using the values for \(a_i, b_i, c_i, d_i, f_i, g_i\) for \(i \in \{1, 2, 3, 4\}\), if \((acf - ad - bf - cg)^2 \neq 4bdg\), we have that

\[ a_{3k+1} = \frac{(a\lambda_1 + bg)\lambda^k_1 - (bg + a\lambda_2)\lambda^k_2}{\lambda_1 - \lambda_2} \]

(2.115)

\[ a_{3k+2} = \frac{(ac - b)\lambda^{k+1}_1 - \lambda^{k+1}_2}{\lambda_1 - \lambda_2} \]

(2.116)

\[ a_{3k+3} = \frac{(acf - bf - ad - \lambda_2)\lambda^{k+1}_1 + (\lambda_1 - acf + bf + ad)\lambda^{k+1}_2}{\lambda_1 - \lambda_2} \]

(2.117)

\[ b_{3k+1} = \frac{b(acf - bf - ad - \lambda_2)\lambda^k_1 + (\lambda_1 - acf + bf + ad)\lambda^k_2}{\lambda_1 - \lambda_2} \]

(2.118)

\[ b_{3k+2} = \frac{d(a\lambda_1 + bg)\lambda^k_1 - (bg + a\lambda_2)\lambda^k_2}{\lambda_1 - \lambda_2} , \]

(2.119)

\[ b_{3k+3} = \frac{g(ac - b)\lambda^{k+1}_1 - \lambda^{k+1}_2}{\lambda_1 - \lambda_2} , \]

(2.120)

\[ c_{3k+1} = \frac{(c\lambda_1 + bd)\lambda^k_1 - (bd + c\lambda_2)\lambda^k_2}{\lambda_1 - \lambda_2} \]

(2.121)

\[ c_{3k+2} = \frac{(cf - d)\lambda^{k+1}_1 - \lambda^{k+1}_2}{\lambda_1 - \lambda_2} \]

(2.122)

\[ c_{3k+3} = \frac{(acf - ad - cg - \lambda_2)\lambda^{k+1}_1 + (\lambda_1 - acf + ad + cg)\lambda^{k+1}_2}{\lambda_1 - \lambda_2} \]

(2.123)

\[ d_{3k+1} = \frac{d(acf - ad - cg - \lambda_2)\lambda^k_1 + (\lambda_1 - acf + ad + cg)\lambda^k_2}{\lambda_1 - \lambda_2} \]

(2.124)

\[ d_{3k+2} = \frac{g(c\lambda_1 + bd)\lambda^k_1 - (bd + c\lambda_2)\lambda^k_2}{\lambda_1 - \lambda_2} , \]

(2.125)

\[ d_{3k+3} = \frac{b(cf - d)\lambda^{k+1}_1 - \lambda^{k+1}_2}{\lambda_1 - \lambda_2} , \]

(2.126)

\[ f_{3k+1} = \frac{(\lambda_1 + dg)\lambda^k_1 - (dg + f\lambda_2)\lambda^k_2}{\lambda_1 - \lambda_2} \]

(2.127)

\[ f_{3k+2} = \frac{(af - g)\lambda^{k+1}_1 - \lambda^{k+1}_2}{\lambda_1 - \lambda_2} \]

(2.128)

\[ f_{3k+3} = \frac{(acf - bf - cg - \lambda_2)\lambda^{k+1}_1 + (\lambda_1 - acf + bf + cg)\lambda^{k+1}_2}{\lambda_1 - \lambda_2} \]

(2.129)

\[ g_{3k+1} = \frac{(acf - bf - cg - \lambda_2)\lambda^k_1 + (\lambda_1 - acf + bf + cg)\lambda^k_2}{\lambda_1 - \lambda_2} \]

(2.130)

\[ g_{3k+2} = \frac{b(\lambda_1 + dg)\lambda^k_1 - (dg + f\lambda_2)\lambda^k_2}{\lambda_1 - \lambda_2} , \]

(2.131)

\[ g_{3k+3} = \frac{d(af - g)\lambda^{k+1}_1 - \lambda^{k+1}_2}{\lambda_1 - \lambda_2} , \]

(2.132)
for \( k \in \mathbb{N}_0 \).

By using (2.115)–(2.132) in (2.27), (2.28) and (2.29) we obtain that the well-defined solutions of system (1.8) in this case are given by the following formulas

\[
\begin{align*}
x_{3n+1} &= Y_0 \frac{\lambda_1^k k + a}{\lambda_1 - \lambda_2} x_1 - y \left( a c f - b f - a d - \frac{\lambda_1}{\lambda_1} \right) k + 1 + 1 \lambda_1^{k+1}, \\
x_{3n+2} &= Z_0 \frac{(ac-b)^{\lambda_1+1}}{\lambda_1 - \lambda_2} x_1, \\
x_{3n+3} &= X_0 \frac{(acf - bf - ad - \frac{\lambda_1}{\lambda_1} \right) k + 1 + 1 \lambda_1^{k+1}}{\lambda_1 - \lambda_2} y, \\
y_{3n+1} &= Z_0 \frac{\lambda_1^k k + a}{\lambda_1 - \lambda_2} x_1 - y \left( a c f - b f - a d - \frac{\lambda_1}{\lambda_1} \right) k + 1 + 1 \lambda_1^{k+1}, \\
y_{3n+2} &= X_0 \frac{(ac-b)^{\lambda_1+1}}{\lambda_1 - \lambda_2} x_1, \\
y_{3n+3} &= Y_0 \frac{(acf - bf - ad - \frac{\lambda_1}{\lambda_1} \right) k + 1 + 1 \lambda_1^{k+1}}{\lambda_1 - \lambda_2} y, \\
z_{3n+1} &= X_0 \frac{(acf - bf - ad - \frac{\lambda_1}{\lambda_1} \right) k + 1 + 1 \lambda_1^{k+1}}{\lambda_1 - \lambda_2} x_1, \\
z_{3n+2} &= Y_0 \frac{(acf - bf - ad - \frac{\lambda_1}{\lambda_1} \right) k + 1 + 1 \lambda_1^{k+1}}{\lambda_1 - \lambda_2} y, \\
z_{3n+3} &= Z_0 \frac{(acf - bf - ad - \frac{\lambda_1}{\lambda_1} \right) k + 1 + 1 \lambda_1^{k+1}}{\lambda_1 - \lambda_2} x_1,
\end{align*}
\]

If \((a c f - a d - b f - c g)^2 = 4 b d g\), that is, if \(\lambda_1 = \lambda_2 = (a c f - a d - b f - c g)/2\), we have that

\[
\begin{align*}
a_{3k+1} &= \left( \left( k + \frac{\lambda_1}{d} \right) k + a \right) \lambda_1^{k+1}, \\
a_{3k+2} &= (ac-b)(k+1)\lambda_1^{k+1}, \\
a_{3k+3} &= \left( \frac{a c f - b f - a d - \lambda_1}{\lambda_1} \right) k + 1 + 1 \lambda_1^{k+1}, \\
b_{3k+1} &= b \left( \left( k + \frac{\lambda_1}{d} \right) k + a \right) \lambda_1^{k+1}, \\
b_{3k+2} &= d \left( \left( k + \frac{\lambda_1}{d} \right) k + a \right) \lambda_1^{k+1}, \\
b_{3k+3} &= g(a c - b)(k+1)\lambda_1^{k}, \\
c_{3k+1} &= \left( \left( \frac{c + \lambda_1}{g} \right) k + c \right) \lambda_1^{k}, \\
c_{3k+2} &= (c f - d)(k + 1)\lambda_1^{k}, \\
c_{3k+3} &= \left( \frac{a c f - a d - c g - \lambda_1}{\lambda_1} \right) k + 1 + 1 \lambda_1^{k+1}, \\
d_{3k+1} &= d \left( \left( \frac{a c f - a d - c g - \lambda_1}{\lambda_1} \right) k + 1 \right) \lambda_1^{k}, \\
d_{3k+2} &= g \left( \left( \frac{c + \lambda_1}{g} \right) k + c \right) \lambda_1^{k}, \\
d_{3k+3} &= b(c f - d)(k + 1)\lambda_1^{k}, \\
f_{3k+1} &= \left( \left( \frac{a c f - a d - c g - \lambda_1}{\lambda_1} \right) k + 1 \right) \lambda_1^{k}, \\
f_{3k+2} &= \left( \left( \frac{c + \lambda_1}{g} \right) k + c \right) \lambda_1^{k}, \\
f_{3k+3} &= (c f - d)(k + 1)\lambda_1^{k},
\end{align*}
\]
\[ f_{3k+2} = (af - kg)(k + 1)\lambda_1^k \]  
\[ f_{3k+3} = \left( \frac{acf - bg - cg - \lambda_1^k}{\lambda_1} (k + 1) + 1 \right) \lambda_1^{k+1} \]  
\[ g_{3k+1} = g \left( \frac{acf - bg - cg - \lambda_1^k}{\lambda_1} k + 1 \right) \lambda_1^k \]  
\[ g_{3k+2} = b \left( f + \frac{\lambda_1^k}{b} \right) k + f \lambda_1^k \]  
\[ g_{3k+3} = d(af - kg)(k + 1)\lambda_1^k \]  
for \( k \in \mathbb{N}_0 \).

By using (2.142)–(2.159) into (2.27), (2.28) and (2.29) we obtain that the well-defined solutions of system (1.8) in this case are given by the following formulas

\[ x_{3n+1} = y_0 \left( (a+\frac{\lambda_1^k}{\lambda_1})n + a \lambda_1^k - b \left( \frac{af-bf-cg-\lambda_1^k}{\lambda_1} n + 1 \right) \lambda_1^k \right) \]  
\[ x_{3n+2} = z_0 \left( (ac-b)(n+1) \lambda_1^k - d \left( (a+\frac{\lambda_1^k}{\lambda_1})n + a \right) \lambda_1^k \right) \]  
\[ x_{3n+3} = x_0 \left( \frac{af-bf-cg-\lambda_1(n+1)+1}{\lambda_1} \lambda_1^{n+1} - g(ac-b)(n+1) \lambda_1^n \right) \]  
\[ y_{3n+1} = y_0 \left( (c+\frac{\lambda_1^k}{\lambda_1})n + c \lambda_1^k - d \left( (a+\frac{\lambda_1^k}{\lambda_1})n + a \right) \lambda_1^k \right) \]  
\[ y_{3n+2} = x_0 \left( (cf-d)(n+1) \lambda_1^n - g \left( (c+\frac{\lambda_1^k}{\lambda_1})n + c \right) \lambda_1^n \right) \]  
\[ y_{3n+3} = y_0 \left( \frac{af-bf-cg-\lambda_1(n+1)+1}{\lambda_1} \lambda_1^{n+1} - g \left( (cf-d)(n+1) \lambda_1^n \right) \right) \]  
\[ z_{3n+1} = z_0 \left( (f+\frac{\lambda_1^k}{\lambda_1})n + f \lambda_1^k - b \left( f + \frac{\lambda_1^k}{\lambda_1} \right) n + f \lambda_1^k \right) \]  
\[ z_{3n+2} = y_0 \left( (af-g)(n+1) \lambda_1^n - b \left( (f+\frac{\lambda_1^k}{\lambda_1})n + f \right) \lambda_1^n \right) \]  
\[ z_{3n+3} = z_0 \left( \frac{af-bf-cg-\lambda_1(n+1)+1}{\lambda_1} \lambda_1^{n+1} - (af-g)(n+1) \lambda_1^n \right), \quad n \in \mathbb{N}_0 \]

finishing the proof of the theorem. \( \square \)

**Remark 2.2.** To get formulas (2.115)–(2.132) and (2.142)–(2.159) we need to know values of \( a_i, b_i, c_i, d_i, f_i, g_i \), \( i = \overline{1,6} \). Since the expressions of these initial values become more and more complicated when index \( i \) increases, if we want to get the formulas by hand then the process is time-consuming because of much calculations. Hence, we suggest the following procedure which facilitates getting the formulas. Namely, recurrent relations (2.21)–(2.26) can be used in a natural way to calculate also \( a_i, b_i, c_i, d_i, f_i, g_i \), for \( i \in \{-2,-1,0\} \). This is possible since relations (2.21)–(2.26) define values of \( a_i, b_i, c_i, d_i, f_i, g_i \), for all non-positive \( i \), when \( bdg \neq 0 \). For example, if in the relation \( b_{3k+4} = bd_{3k+3} \), we choose \( k = -1 \), then we get \( b_1 = ba_0 \), from which it follows that \( a_0 = 1 \) (here the assumption \( b \neq 0 \) is used). Using this fact in the relation \( a_{3k+4} = aa_{3k+3} - b_{3k+3} \), with \( k = -1 \), we obtain that \( a_1 = a - 1 \), from which it follows that \( b_0 = 1 \). From this and the relation \( b_{3k+3} = ga_{3k+2} \), with \( k = -1 \), we get \( a_{-1} = 0 \) (here the assumption \( g \neq 0 \) is used). Continuing in this way it can be obtained that

\[ a_{-2} = -\frac{1}{d}, \quad b_{-2} = -\frac{c}{d}, \quad c_{-2} = -\frac{1}{g}, \quad d_{-2} = -\frac{f}{g}, \quad f_{-2} = -\frac{1}{b}, \quad g_{-2} = -\frac{a}{b} \]

\[ a_{-1} = 0, \quad b_{-1} = -1, \quad c_{-1} = 0, \quad d_{-1} = -1, \quad f_{-1} = 0, \quad g_{-1} = -1 \]

\[ a_0 = 1, \quad b_0 = 0, \quad c_0 = 1, \quad d_0 = 0, \quad f_0 = 1, \quad g_0 = 0. \]
Using these “initial values” along with $a_i, b_i, c_i, d_i, f_i, g_i, i = 1,3$, all the calculations in getting formulas (2.115)–(2.132) and (2.142)–(2.159) become somewhat simpler.

From the proof of Theorem 2.1 we obtain the following corollary.

**Corollary 2.3.** Consider system (1.8) with $a, b, c, d, f, g \in \mathbb{Z}$. Assume that $x_{-i}, y_{-i}, z_{-i}, \in \mathbb{C} \setminus \{0\}, i \in \{0,1\}$. Then the following statement are true.

(a) If $b = 0$, then the general solution to system (1.8) is given by (2.51)–(2.59).

(b) If $d = 0$, then the general solution to system (1.8) is given by (2.78)–(2.86).

(c) If $g = 0$, then the general solution to system (1.8) is given by (2.105)–(2.113).

(d) If $bdg \neq 0$ and $(ad + bf + cg - acf)^2 \neq 4bdg$, then the general solution to system (1.8) is given by (2.133)–(2.141).

(e) If $bdg \neq 0$ and $(ad + bf + cg - acf)^2 = 4bdg$, then the general solution to system (1.8) is given by (2.160)–(2.168).

**Remark 2.4.** Formulas (2.51)–(2.59), (2.78)–(2.86), (2.105)–(2.113), (2.133)–(2.141) and (2.160)–(2.168) can be used in describing the long term behavior of well-defined solutions of system (1.8). We leave the formulations and proofs of these results to the reader as some exercises.

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