



## Codimension 3 bifurcation from orbit-flip homoclinic orbit of weak type

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**Abstract.** This article is devoted to the research of a new codimension 3 homoclinic orbit bifurcation, which is the orbit-flip of weak type. Such kind of homoclinic orbit is a degenerate case of the orbit-flip homoclinic orbit. We show the existence of 1-homoclinic orbit, 1-periodic orbit,  $2^n$ -homoclinic orbit and  $2^n$ -periodic orbit for arbitrary integer  $n$ . Our strategy is based on the local moving coordinates method.

**Keywords:** homoclinic, orbit-flip, weak type, local moving coordinates method.

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### 1 Introduction

In the past decades, multi-round homoclinic bifurcations have developed a lot due to their great applications in spatial dynamics, where they correspond to traveling or standing multi-pulses. Besides, cascades of homoclinic doubling bifurcation can be observed for parameter depending vector fields [8], which are similar to the phenomenon of the period doubling bifurcation for diffeomorphism of maps [6]. Since the codimension-one homoclinic bifurcation with real eigenvalues cannot give birth to multi-round homoclinic orbit, see [21], complicated dynamics of codimension 2 cases need to be considered. In [23], Yanagida studied 3 different kinds of codimension 2 cases, which included inclination-flip bifurcation, resonant bifurcation and the orbit-flip bifurcation. Since then, many research works have been devoted to this subject, see [4, 5, 7, 9, 10, 12, 13] for example.

Except for the above codimension 2 mechanism for the occurrence of homoclinic doubling bifurcation, another strategy is to consider the problem in a more degenerate situation, which is codimension 3. [11] presented the existence of infinitely many homoclinic doubling bifurcation from the inclination-flip homoclinic orbit of weak type, where the bifurcated homoclinic orbit  $\Gamma_N$  of arbitrary order  $N$  were inclination-flip homoclinic orbit. Despite orbit-flip and inclination-flip homoclinic orbits are quite different from their definitions, both of them involve the orientation change of their stable manifolds. So lots of similar bifurcation results

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have been discovered. For example, there exists non-empty interior region, where a suspended horseshoe is discovered. There are parameter curves for the bifurcation of  $N$ -homoclinic orbit, see [7] and [16] for the inclination flip case and [20] for the orbit-flip case. Moreover, the strange attractors are presented both in the unfolding of inclination-flip case and orbit-flip case, see [17] and [18]. A natural question would then be asking whether similar homoclinic doubling bifurcation can occur infinitely many times from the orbit-flip homoclinic orbit of weak type. To answer this question, we consider a smooth system

$$\dot{z} = f(z) + g(z, \mu), \quad (1.1)$$

and its unperturbed system

$$\dot{z} = f(z), \quad (1.2)$$

where  $z \in \mathbb{R}^3$ ,  $\mu \in \mathbb{R}^3$ ,  $0 < |\mu| \ll 1$ ,  $f(0) = 0$ ,  $g(z, 0) = 0$  and  $z = 0$  is a hyperbolic equilibrium. More precisely,  $\text{Spec} df(0)$  is real. Without loss of generality, we suppose  $\text{Spec} df(0) = \{-\alpha, -\beta, 1\}$ , where  $\alpha > \beta > 0$  due to time scaling. We denote the local stable manifold by  $W_{\text{loc}}^s$  and the local unstable manifold by  $W_{\text{loc}}^u$ . Since  $\alpha > \beta$ , one has a local strong stable manifold  $W_{\text{loc}}^{ss}$  corresponding to the eigenvalue  $-\alpha$ . The local strong stable manifold, which is invariant under the flow, belongs to the local stable manifold. We can extend these manifolds by the flow and their extensions are denoted by  $W^s$ ,  $W^u$ ,  $W^{ss}$ .

From now on, we always denote the homoclinic orbit of (1.2) by  $\Gamma = \{r(t), t \in \mathbb{R}\}$ .

**Definition 1.1.**  $\Gamma$  is called an orbit-flip homoclinic orbit if  $\Gamma \cap W^{ss} \neq \{O\}$ .

Before giving the definition of an orbit-flip homoclinic orbit of weak type, we firstly introduce the so called ‘‘weak vector’’.

Let  $\Sigma$  be  $C^1$  cross-section transverse to  $W^{ss}$ . It turns out that  $W^s$  splits  $\Sigma$  into two connected components, say  $\Sigma^+$  and  $\Sigma^-$ . Then the Poincaré return map  $\Phi$  is only defined on a single component of  $\Sigma \setminus W^s$ , we suppose this component is  $\Sigma^+$ . Let  $C = \{C(t), t \in (-1, 1)\}$  be a  $C^1$  curve in  $\Sigma$  transverse to the stable manifold such that  $C(0) = W^{ss} \cap \Sigma = p_\Sigma$ . Since the Poincaré return map is only defined on  $\Sigma^+$ , we put  $\Phi(C) = \Phi\{C(t), t \in (0, 1)\}$ . Set  $C_\Phi(t) = \Phi(C(t))$  and define

$$u_\Sigma = \lim_{t \rightarrow 0} \frac{\frac{d}{dt} C_\Phi(t)}{\left\| \frac{d}{dt} C_\Phi(t) \right\|},$$

then this vector is called the ‘‘weak vector’’ associated with the section  $\Sigma$ .

**Definition 1.2.** We say that an orbit-flip homoclinic orbit is of weak type if for any cross-section  $\Sigma$ ,  $u_\Sigma \in T_{p_\Sigma}(W^s \cap \Sigma)$  where  $p_\Sigma = \Sigma \cap W^u$ , and  $T_{p_\Sigma}(W^s \cap \Sigma)$  is the tangent space of the intersection between  $\Sigma$  and the stable manifold at point  $p_\Sigma$ .

Let  $M_1, M_3$  are Melnikov vectors defined in Section 2. Then, our main theorem is stated as follows.

**Theorem 1.3.** *Assume system (1.2) admits  $\Gamma$  an orbit-flip homoclinic orbit of weak type. The eigenvalues of  $Df(0)$  avoid a finite number of resonances and satisfy  $1 > \beta > \frac{1}{2}$ ,  $\beta + 1 > \alpha > \beta$ ,  $\text{rank}(M_1, M_3) = 2$ , then there exist a 1-homoclinic bifurcation surface  $H^1$ , a 2-fold periodic orbit bifurcation surface  $SN^1$ , a period-doubling bifurcation surface  $P^{2^n}$  of  $2^{n-1}$  periodic orbit and a  $2^n$ -homoclinic bifurcation surface  $H^{2^n}$  for  $\forall n \in \mathbb{N}$ , which share the same normal vector  $M_1$  at  $\mu = 0$ , such that system (1.1) has*

a 1-homoclinic orbit if and only if  $\mu \in H^1$  and  $|\mu| \ll 1$ ;

a 2-fold periodic orbit if and only if  $\mu \in SN^1$ ;

a  $2^{n-1}$ -periodic orbit changing its stability and a  $2^n$ -periodic orbit arising at the same time if and only if  $\mu \in P^{2^n}$ ;

a  $2^n$ -homoclinic orbit if and only if  $\mu \in H^{2^n}$ .

Furthermore, there exist a bifurcation surface  $\Delta_1$  (which is a branch of  $H^1$ ) with codimension 1 and normal vector  $M_1$  such that system (1.1) has a 1-homoclinic orbit as well as a 1-periodic orbit for  $\mu \in \Delta_1$  and  $|\mu| \ll 1$ .

The paper is organized as follows. In Section 2, the local moving coordinates are introduced and the Poincaré return map on a given cross-section is deduced. Section 3 is devoted to proving Theorem 1.3.

## 2 Preliminaries and bifurcation equation

In the following, we assume that the parameter depending vector field (1.1) is locally  $C^2$  linearizable. This condition is not essential but will simplify computations and notations a lot. Such linearization is possible if the eigenvalues  $-\alpha, -\beta$  and 1 avoid a finite number of resonances, see [1, 2, 3, 19, 22] for more details and discussion. As a consequence, there exist a neighborhood  $U$  of 0 in  $\mathbb{R}^3$  and a neighborhood  $V$  of 0 in  $\mathbb{R}^3$ , such that for all  $v \in U$  and all  $\mu \in V$ , (1.1) has the following normal form:

$$\dot{x} = x, \quad \dot{y} = -\beta(\mu)y, \quad \dot{v} = -\alpha(\mu)v. \quad (2.1)$$

**Lemma 2.1.** *Suppose  $\beta + 1 > \alpha > \beta$ . Let  $\Sigma$  be a cross section transverse to the orbit-flip homoclinic orbit  $\Gamma$ , then the weak vector  $\mu_\Sigma$  exists, which is exactly  $\frac{\partial}{\partial v}$ .*

See [18] for details of the proof.

Now we consider the linear variational system of (1.2) and its adjoint system

$$\dot{z} = Df(r(t))z, \quad (2.2)$$

$$\dot{z} = -(Df(r(t)))^*z. \quad (2.3)$$

Denote  $r(t) = (r^x(t), r^y(t), r^v(t))$  and take  $T > 0$  large enough such that  $r(-T) = (\delta, 0, 0)$ ,  $r(T) = (0, 0, \delta)$ , where  $\delta$  is small enough so that  $\{(x, y, v) : |x|, |y|, |v| < 2\delta\} \subset U$ .

**Lemma 2.2.** *Assume system (1.2) admits an orbit-flip homoclinic orbit  $\Gamma$  of weak type and  $\alpha > \beta > 0$ , then there exists a fundamental solution matrix  $Z(t) = (z^1(t), z^2(t), z^3(t))$  for system (2.2) with*

$$\begin{aligned} z^1(t) &\in (T_{r_i(t)}W^u)^c \cap (T_{r_i(t)}W^{ss})^c, \\ z^2(t) &= -\dot{r}(t)/|\dot{r}^v(T)| \in T_{r(t)}W^u \cap T_{r(t)}W^{ss}, \\ z^3(t) &\in T_{r(t)}W^s, \end{aligned}$$

satisfying

$$Z(-T) = \begin{pmatrix} 0 & \omega_{21} & \omega_{31} \\ 1 & 0 & \omega_{32} \\ 0 & 0 & \omega_{33} \end{pmatrix}, \quad Z(T) = \begin{pmatrix} \omega_{11} & 0 & 0 \\ \omega_{12} & 0 & 1 \\ \omega_{13} & 1 & 0 \end{pmatrix},$$

where  $\omega_{21} < 0$ ,  $\omega_{11} \neq 0$ ,  $\omega_{33} \neq 0$ ,  $|\omega_{31}| \ll |\omega_{33}|$ ,  $|\omega_{32}| \ll |\omega_{33}|$ .

*Proof.* Firstly, due to the fact that  $TW_{loc}^{ss} = \text{span}\{(0,0,1)^*\}$  and  $TW_{loc}^u = \text{span}\{(1,0,0)^*\}$ , we can choose  $z^2(T) = (0,0,1)^*$  as an initial value with the definition  $z^2(t) = -\dot{r}(t)/|\dot{r}^v(T)|$  in mind, which implies  $z^2(-T) = (\omega_{21}, 0, 0)^*$ , where  $\omega_{21} < 0$ .

Then, for  $z^3(t) \in T_{r(t)}W^s$  with  $z^3(T) = (0,1,0)^*$ , we have  $z^3(-T) = (\omega_{31}, \omega_{32}, \omega_{33})^*$ . Since system (1.2) admits no inclination-flip, then from the strong-inclination property, it is deduced that  $\omega_{33} \neq 0$  and  $|\omega_{31}| \ll |\omega_{33}|$ ,  $|\omega_{32}| \ll |\omega_{33}|$ .

Finally, note that  $T_{r(-T)}W^u$  is the  $x$ -axis and  $T_{r(T)}W^{ss}$  is the  $v$ -axis, one can check that  $z^1(t) \in (T_{r(t)}W^u)^c \cap (T_{r(t)}W^{ss})^c$  is well defined satisfying  $z^1(-T) = (0,1,0)^*$ . Since  $\det Z(-T) \neq 0$ , then we obtain  $\det Z(T) \neq 0$  thanks to the Liouville formula, which implies  $\omega_{11} \neq 0$ .  $\square$

**Remark 2.3.** If the homoclinic orbit of orbit-flip is weak type, then  $\omega_{32} = 0$ , see Figure 2.1.

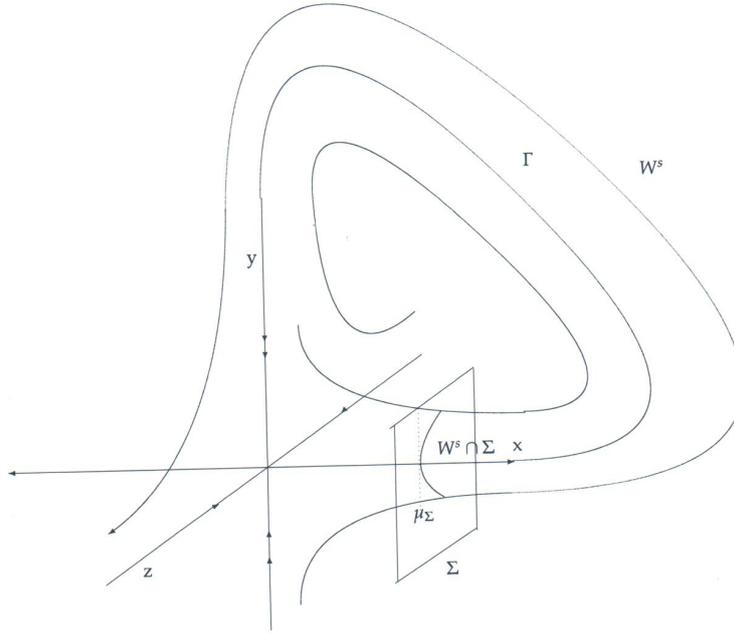


Figure 2.1: Orbit-flip homoclinic orbit of weak type.

As well known from the matrix theory, system (2.3) has a fundamental solution matrix  $\Phi(t) = (Z^{-1}(t))^*$ . We denote  $\Phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t))$ . And for every point  $z = (x(t), y(t), v(t))$  near  $\Gamma$ , introduce the local moving coordinates  $N = (n_1(t), 0, n_3(t))$ . Set

$$z = S(t) = r(t) + Z(t)N^* = r(t) + z^1(t)n_1(t) + z^3(t)n_3(t). \quad (2.4)$$

With this notation, we can choose the cross sections

$$\begin{aligned} S_0 &= \{z = S(T) : |x|, |y|, |v| < 2\delta\} \subset U, \\ S_1 &= \{z = S(-T) : |x|, |y|, |v| < 2\delta\} \subset U. \end{aligned}$$

One can refer [13, 14, 15, 24, 25] for more details and discussions about the local moving coordinates method.

Under the transformation (2.4), system (1.1) has the following form

$$\dot{n}_j = (\phi_j(t))^* g_\mu(r(t), 0)\mu + h.o.t., \quad j = 1, 3, \quad (2.5)$$

which is  $C^2$  and produces the map  $P_1: S_1 \rightarrow S_0$ . Integrating both sides from  $-T$  to  $T$ , we have

$$n_j(T) = n_j(-T) + M_j\mu + h.o.t., \quad j = 1, 3, \quad (2.6)$$

where  $N(T) = (n_1(T), 0, n_3(T))$ ,  $N(-T) = (n_1(-T), 0, n_3(-T))$ , and

$$M_j = \int_{-T}^T (\phi_j(t))^* g_\mu(r(t), 0) dt, \quad j = 1, 3$$

are Melnikov vectors.

**Lemma 2.4.**

$$M_j = \int_{-T}^T \phi_j^*(t) f_\mu(r(t), 0) dt = \int_{-\infty}^{+\infty} \phi_j^*(t) f_\mu(r(t), 0) dt, \quad j = 1, 3. \quad (2.7)$$

*Proof.* To prove (2.7), it is sufficient to verify that  $\phi_j^*(t) f_\mu(r(t), 0) = 0$  for  $t > T$ , and  $t < -T$ ,  $j = 1, 3$ . As  $r(T) = (0, 0, \delta)$ , then  $r(t) = (0, 0, r^v(t))$  for  $t > T$  with  $r^v(t) = O(\delta e^{-\alpha(t-T)})$ . Similarly, we have  $r(t) = (r^x(t), 0, 0)$  with  $r^x(t) = O(\delta e^{t+T})$  for  $t < -T$ , which is due to  $r(-T) = (\delta, 0, 0)$ . According to the flow of linearized system near 0, we have

$$f_\mu(r(t), 0) = (0, 0, O(\delta)) \quad \text{for } t > T, \quad f_\mu(r(t), 0) = (O(\delta), 0, 0) \quad \text{for } t < -T.$$

Denote by  $\phi_j^*(t) = (\phi_j^1(t), \phi_j^2(t), \phi_j^3(t))$ . Since  $\Phi^*(t)Z(t) = I$ , we have  $\phi_j^*(t)z^2(t) = 0$ ,  $j = 1, 3$ . Then  $z^2(T) = (0, 0, 1)^*$  implies that  $\phi_j^3(T) = 0$ ,  $j = 1, 3$ . Thereafter, we have  $\phi_j^3(t) = 0$  for  $t > T$ ,  $j = 1, 3$ . Likewise, we can also obtain  $\phi_j^1(-T) = 0$ ,  $j = 1, 3$ , due to the fact that  $z^2(-T) = (\omega_{21}, 0, 0)^*$ . Consequently,  $\phi_j^1(t) = 0$ , for  $t < -T$ ,  $j = 1, 3$ . Thus, conclusion (2.7) is verified. The proof is completed.  $\square$

Define  $P_0: S_0 \rightarrow S_1$ ,  $q_0 \rightarrow q_1$  induced by the flow of (2.1) in the neighborhood  $U$  of  $z = 0$ . Set the flying time from  $q_0$  to  $q_1$  as  $\tau$  and the Silnikov time  $s = e^{-\tau}$  (see Figure 2.2). Then we have

$$P_0: q_0(x_0, y_0, v_0) \rightarrow q_1(x_1, y_1, v_1), \\ x_0 = sx_1, \quad y_0 = s^\beta y_1, \quad v_0 = s^\alpha v_1,$$

and  $x_1 = \delta$ ,  $v_0 = \delta$ ;

$$n_1^0 = (\omega_{11})^{-1}x_0, \quad n_3^0 = y_0 - \omega_{12}(\omega_{11})^{-1}x_0, \\ n_1^1 = y_1 - \omega_{32}(\omega_{33})^{-1}v_1, \quad n_3^1 = (\omega_{33})^{-1}v_1.$$

From the above, we give the following Poincaré maps:

$$F_1 = P_1 \circ P_0: S_0 \rightarrow S_0, \\ \bar{n}_1^0 = y_0 s^\beta - \omega_{32}(\omega_{33})^{-1} \delta s^\alpha + M_1 \mu + h.o.t., \\ \bar{n}_3^0 = (\omega_{33})^{-1} \delta s^\alpha + M_3 \mu + h.o.t.$$

Now, the successor function is given by  $G(s, y_0) = (G_1, G_3) = (F_1(q_0) - q_0)$  as follows:

$$G_1 = -(\omega_{11})^{-1} \delta s + y_0 s^\beta - \omega_{32}(\omega_{33})^{-1} \delta s^\alpha + M_1 \mu + h.o.t., \\ G_3 = -y_0 + \omega_{12}(\omega_{11})^{-1} \delta s + (\omega_{33})^{-1} \delta s^\alpha + M_3 \mu + h.o.t.$$

By solving  $y_0$  from  $G_3 = 0$  and substituting it into  $G_1 = 0$ , we obtain the bifurcation equation

$$-(\omega_{11})^{-1} \delta s - \omega_{32}(\omega_{33})^{-1} \delta s^\alpha + M_1 \mu + M_3 \mu s^\beta + h.o.t. = 0. \quad (2.8)$$

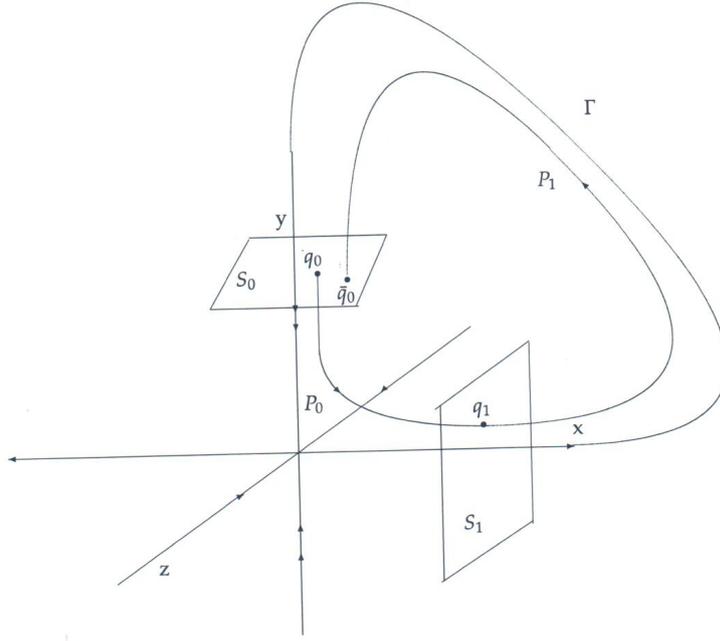


Figure 2.2: Poincaré return map.

### 3 Bifurcation results

In this section, we consider the codimension 3 bifurcation results of the orbit-flip homoclinic bifurcation of weak type, i.e.  $\omega_{32} = 0$ . Then bifurcation equation is:

$$-(\omega_{11})^{-1}\delta s + M_1\mu + M_3\mu s^\beta + h.o.t. = 0. \quad (3.1)$$

**Proposition 3.1.** *Suppose  $1 > \beta > \frac{1}{2}$ , then the following statements hold.*

- (1) *System (1.1) has a unique periodic orbit for  $\mu \in D_-^+ \cup D_+^+$  if  $\omega_{11} > 0$ , and  $\mu \in D_-^- \cup D_+^-$  if  $\omega_{11} < 0$ . Here*

$$D_-^+ = \{\mu : M_1\mu > 0, M_3\mu < 0\}, \quad D_+^+ = \{\mu : M_1\mu > 0, M_3\mu > 0\},$$

$$D_-^- = \{\mu : M_1\mu < 0, M_3\mu > 0\}, \quad D_+^- = \{\mu : M_1\mu < 0, M_3\mu < 0\}.$$

- (2) *There exists a bifurcation surface  $\Delta_1$ :*

$$M_1\mu + h.o.t. = 0, \quad \text{for } \omega_{11}M_3\mu > 0$$

*with normal vector  $M_1$  at  $\mu = 0$  such that system (1.1) has an 1-homoclinic orbit as well as a 1-periodic orbit for  $\mu \in \Delta_1$  and  $|\mu| \ll 1$ .*

- (3) *There exists a unique bifurcation surface  $H^1$ :*

$$M_1\mu + h.o.t. = 0,$$

*with normal vector  $M_1$  at  $\mu = 0$  which coincides with  $\Delta_1$  in the region defined by  $\{\mu : \omega_{11}M_3\mu > 0\}$  such that system (1.1) has a unique 1-homoclinic orbit for  $\mu \in H^1$  and  $|\mu| \ll 1$ .*

(4) There exists a 2-fold periodic orbit bifurcation surface  $SN^1$ :

$$(\omega_{11})^{-1}\delta(\delta^{-1}\beta\omega_{11}M_3\mu)^{\frac{1}{1-\beta}} = M_3\mu(\delta^{-1}\beta\omega_{11}M_3\mu)^{\frac{\beta}{1-\beta}} + M_1\mu + h.o.t.$$

with normal vector  $M_1$  at  $\mu = 0$  such that system (1.1) has a unique 2-fold periodic orbit.

*Proof.* (1) Denote by

$$F(s, \mu) = -(\omega_{11})^{-1}\delta s + M_3\mu s^\beta + M_1\mu + h.o.t.$$

Let  $s^\beta = t$ ,  $L(t, \mu) = M_3\mu t + M_1\mu + h.o.t.$ ,  $N(t, \mu) = (\omega_{11})^{-1}\delta t^{\frac{1}{\beta}} + h.o.t.$  then for  $\mu \in D_+^+$ ,  $\omega_{11} > 0$ ,

$$\begin{aligned} L(0, \mu) &= M_1\mu + h.o.t. > 0, & L'(0, \mu) &= M_3\mu + h.o.t. < 0, \\ N'(t, \mu) &= (\beta\omega_{11})^{-1}\delta t^{\frac{1-\beta}{\beta}} + h.o.t. > 0. \end{aligned}$$

So the line  $W = L(t, \mu)$  and the curve  $W = N(t, \mu)$  intersect at a unique sufficiently small positive point  $\bar{t} < (\delta^{-1}\omega_{11}M_1\mu)^\beta$  and  $F$  has a unique sufficiently small positive zero  $\bar{s} = (\bar{t})^{1/\beta}$ .

If  $\mu \in D_+^+$ ,  $\omega_{11} > 0$ , then

$$\begin{aligned} L(0, \mu) &= M_1\mu + h.o.t. > 0, & L'(t, \mu) &= M_3\mu + h.o.t. > 0, \\ N'(t, \mu) &= (\beta\omega_{11})^{-1}\delta t^{\frac{1-\beta}{\beta}} + h.o.t. > 0, \\ N''(t, \mu) &= (1 - \beta)(\beta^2\omega_{11})^{-1}\delta t^{\frac{1-2\beta}{\beta}} + h.o.t. > 0. \end{aligned}$$

Take  $\bar{t} = [\delta^{-1}\omega_{11}(2M_3\mu + M_1\mu)]^\beta$ , then

$$N(\bar{t}, \mu) - L(\bar{t}, \mu) = 2M_3\mu + M_1\mu - M_3\mu\bar{t} - M_1\mu > M_3\mu > 0.$$

Therefore, based on the fact that  $N(\cdot, \mu)$  is a monotone increasing convex function, we see that the line  $W = L(t, \mu)$  and the curve  $W = N(t, \mu)$  intersect uniquely at  $t^* \in (0, \bar{t})$ , that is,  $F$  has a unique sufficiently small positive zero point  $\bar{s} \in (0, \delta^{-1}\omega_{11}(2M_3\mu + M_1\mu))$ .

(2) Let  $\mu \in \Delta_1 \triangleq \{\mu : F(0, \mu) = M_1\mu + h.o.t. = 0, \omega_{11}M_3\mu > 0\}$ , we have

$$F(s, \mu) = s^\beta [ -(\omega_{11})^{-1}\delta s^{1-\beta} + M_3\mu + h.o.t. ].$$

Consequently, there are two zero points  $s_1 = 0$ ,  $s_2 = (\omega_{11}\delta^{-1}M_3\mu)^{\frac{1}{1-\beta}} + h.o.t.$

(3) For  $\mu \in \{M_1\mu + h.o.t. = 0\}$ , Eq. (3.1) admits  $s = 0$  as its solution. In this case, system (1.1) has a bifurcated homoclinic orbit. And from the above proof, one can easily check that  $H^1$  coincides with  $\Delta_1$  in the region defined by  $\{\mu : \omega_{11}M_3\mu > 0\}$ .

(4) The 2-fold zero point  $\bar{t}$  should satisfy

$$L(t, \mu) = N(t, \mu), \quad L'(t, \mu) = N'(t, \mu). \quad (3.2)$$

The second equation turns out to be

$$(\alpha\omega_{11})^{-1}\delta t^{\frac{1-\beta}{\beta}} + h.o.t. = M_3\mu + h.o.t. \quad (3.3)$$

which forces  $\bar{t} = (\delta^{-1}\beta\omega_{11}M_3\mu)^{\frac{\beta}{1-\beta}} + h.o.t.$  for  $\omega_{11}M_3\mu > 0$  due to (3.3). Then from the first equation of (3.2), we get the corresponding 2-fold periodic orbit bifurcation surface  $SN^1$ :

$$(\omega_{11})^{-1}\delta(\delta^{-1}\beta\omega_{11}M_3\mu)^{\frac{1}{1-\beta}} = M_3\mu(\delta^{-1}\beta\omega_{11}M_3\mu)^{\frac{\beta}{1-\beta}} + M_1\mu + h.o.t.$$

with normal vector  $M_1$  at  $\mu = 0$ .  $\square$

Now, we turn to study on the bifurcation of 2-homoclinic orbit and the period-doubling bifurcation. The second successor function can be given by  $G(s_1, s_2, y_0, y_2) = (G_1^1, G_3^1, G_1^2, G_3^2) = (F_1(q_0) - q_2, F_1(q_2) - q_0)$  as follows:

$$\begin{aligned} G_1^1 &= -(\omega_{11})^{-1}\delta s_2 + y_0 s_1^\beta + M_1\mu + h.o.t., \\ G_3^1 &= -y_2 + \omega_{12}(\omega_{11})^{-1}\delta s_2 + (\omega_{33})^{-1}\delta s_1^\alpha + M_3\mu + h.o.t. \\ G_1^2 &= -(\omega_{11})^{-1}\delta s_1 + y_2 s_2^\beta + M_1\mu + h.o.t., \\ G_3^2 &= -y_0 + \omega_{12}(\omega_{11})^{-1}\delta s_1 + (\omega_{33})^{-1}\delta s_2^\alpha + M_3\mu + h.o.t. \end{aligned}$$

By solving  $y_0$  and  $y_2$  from  $G_3^1 = 0$ ,  $G_3^2 = 0$ , and then by substituting them into  $G_1^1 = 0$ ,  $G_1^2 = 0$ , we obtain the bifurcation equation

$$-(\omega_{11})^{-1}\delta s_2 + (\omega_{33})^{-1}\delta s_1^\beta s_2^\alpha + M_3\mu s_1^\beta + M_1\mu + h.o.t. = 0. \quad (3.4)$$

$$-(\omega_{11})^{-1}\delta s_1 + (\omega_{33})^{-1}\delta s_1^\alpha s_2^\beta + M_3\mu s_2^\beta + M_1\mu + h.o.t. = 0. \quad (3.5)$$

**Proposition 3.2.** *There exists a unique bifurcation surface  $H^2$ :*

$$M_3\mu(\omega_{11}\delta^{-1}M_1\mu)^\beta + M_1\mu + h.o.t. = 0,$$

which is well defined in the region  $\{\mu : \omega_{11}M_1\mu > 0, \omega_{11}M_3\mu < 0\}$  such that system (1.1) has a unique 2-homoclinic orbit for  $\mu \in H^2$ .

*Proof.* Suppose system (1.1) has a unique 2-homoclinic orbit, then (3.4) admits  $s_1 > 0$ ,  $s_2 = 0$  or  $s_1 = 0$ ,  $s_2 > 0$  as its solution. Due to the symmetry of bifurcation equations, we can suppose  $s_1 > 0$ ,  $s_2 = 0$ . Therefore,

$$s_1 = \omega_{11}\delta^{-1}M_1\mu + h.o.t. > 0.$$

So, we can get the 2-homoclinic bifurcation surface  $H^2$ :

$$M_3\mu(\omega_{11}\delta^{-1}M_1\mu)^\beta + M_1\mu + h.o.t. = 0,$$

which is well defined in the region  $\{\mu : \omega_{11}M_1\mu > 0, \omega_{11}M_3\mu < 0\}$  such that system (1.1) has a unique 2-homoclinic orbit for  $\mu \in H^2$ .  $\square$

**Corollary 3.3.** *The 1-homoclinic bifurcation surface  $H^1$  and 2-homoclinic bifurcation surface  $H^2$  have the same normal vector  $M_1$  at  $\mu = 0$ . Then, there is a tongue area bounded by  $H^1$  and  $H^2$ , in which there must be another bifurcation surface  $P^2$  where a period-doubling bifurcation arises.*

Define

$$\begin{aligned} P_0^j &: q_{2j-2}(x_{2j-2}, y_{2j-2}, v_{2j-2}) \rightarrow q_{2j-1}(x_{2j-1}, y_{2j-1}, v_{2j-1}), \\ x_{2j-2} &= s_j x_{2j-1}, \quad y_{2j-1} = s_j^\beta y_{2j-2}, \quad v_{2j-1} = s_j^\alpha v_{2j-2}, \end{aligned}$$

and  $x_{2j-1} = \delta$ ,  $v_{2j-2} = \delta$ ,  $j = 1, 2, \dots$

$$\begin{aligned} n_1^{2j-2} &= (\omega_{11})^{-1} \delta s_j, & n_3^{2j-2} &= y_{2j-2} - \omega_{12} (\omega_{11})^{-1} \delta s_j, \\ n_1^{2j-1} &= y_{2j-1} - \omega_{32} (\omega_{33})^{-1} v_{2j-1}, & n_3^{2j-1} &= (\omega_{33})^{-1} v_{2j-1} = (\omega_{33})^{-1} \delta s_j^\alpha. \end{aligned}$$

From the above, we give the  $n$ -th Poincaré return maps:

$$\begin{aligned} F_1^j &= P_1 \circ P_0^j: S_0 \rightarrow S_0, \quad q_{2j-2} \mapsto \bar{q}_{2j-2}, \\ \bar{n}_1^{2j-2} &= y_{2j-2} s_j^\beta - \omega_{32} (\omega_{33})^{-1} \delta s_j^\alpha + M_1 \mu + h.o.t., \\ \bar{n}_3^{2j-2} &= (\omega_{33})^{-1} \delta s_j^\alpha + M_3 \mu + h.o.t. \end{aligned}$$

Consequently, the associated  $n$ -th successor function is given by

$$\begin{aligned} G^n(s_1, \dots, s_n, v_0, \dots, v_{2n-2}) &= (G_1^1, G_3^1, G_1^2, G_3^2, G_1^3, G_3^3, G_1^4, G_3^4) \\ &= (F_1^1(q_0) - q_2, F_1^2(q_2) - q_4, \dots, F_1^n(q_{2n-2}) - q_0). \end{aligned}$$

Take  $n = 4$  for example.

$$\begin{aligned} G_1^1 &= -(\omega_{11})^{-1} \delta s_2 + y_0 s_1^\beta + M_1 \mu + h.o.t., \\ G_3^1 &= -y_2 + \omega_{12} (\omega_{11})^{-1} \delta s_2 + (\omega_{33})^{-1} \delta s_1^\alpha + M_3 \mu + h.o.t., \\ G_1^2 &= -(\omega_{11})^{-1} \delta s_3 + y_2 s_2^\beta + M_1 \mu + h.o.t., \\ G_3^2 &= -y_4 + \omega_{12} (\omega_{11})^{-1} \delta s_3 + (\omega_{33})^{-1} \delta s_2^\alpha + M_3 \mu + h.o.t., \\ G_1^3 &= -(\omega_{11})^{-1} \delta s_4 + y_4 s_3^\beta + M_1 \mu + h.o.t., \\ G_3^3 &= -y_6 + \omega_{12} (\omega_{11})^{-1} \delta s_4 + (\omega_{33})^{-1} \delta s_3^\alpha + M_3 \mu + h.o.t., \\ G_1^4 &= -(\omega_{11})^{-1} \delta s_1 + y_6 s_4^\beta + M_1 \mu + h.o.t., \\ G_3^4 &= -y_0 + \omega_{12} (\omega_{11})^{-1} \delta s_1 + (\omega_{33})^{-1} \delta s_4^\alpha + M_3 \mu + h.o.t. \end{aligned}$$

By solving  $(y_0, y_2, y_4, y_6)$  from  $(G_3^1, G_3^2, G_3^3, G_3^4) = 0$  and substituting it into  $(G_1^1, G_1^2, G_1^3, G_1^4)$ , then we get the bifurcation equation:

$$-(\omega_{11})^{-1} \delta s_2 + (\omega_{33})^{-1} \delta s_1^\beta s_4^\alpha + s_1^\beta M_3 \mu + M_1 \mu + h.o.t. = 0, \quad (3.6)$$

$$-(\omega_{11})^{-1} \delta s_3 + (\omega_{33})^{-1} \delta s_2^\beta s_1^\alpha + s_2^\beta M_3 \mu + M_1 \mu + h.o.t. = 0, \quad (3.7)$$

$$-(\omega_{11})^{-1} \delta s_4 + (\omega_{33})^{-1} \delta s_3^\beta s_2^\alpha + s_3^\beta M_3 \mu + M_1 \mu + h.o.t. = 0, \quad (3.8)$$

$$-(\omega_{11})^{-1} \delta s_1 + (\omega_{33})^{-1} \delta s_4^\beta s_3^\alpha + s_4^\beta M_3 \mu + M_1 \mu + h.o.t. = 0. \quad (3.9)$$

Thereafter, similarly as in the analysis in Proposition 3.2 for 2-homoclinic bifurcation result, we can get the  $2^2$ -homoclinic bifurcation surface  $H^4$ . By Repeating the above procedure, we can also get the  $2^n$ -homoclinic bifurcation surface  $H^{2^n}$  and the period-doubling bifurcation surface  $P^{2^n}$  for arbitrary  $n \in \mathbb{N}$ . Up to now, it is sufficient to claim that Theorem 1.3 holds.

The bifurcation diagram (see Figure 3.1) is given to better illustrate our main results. In the diagram,  $O$  represents that there is no periodic orbits, while  $P$  (resp.  $P^k$ ) represents that there exists a 1-periodic (resp.  $k$ -periodic) orbit in the corresponding region.

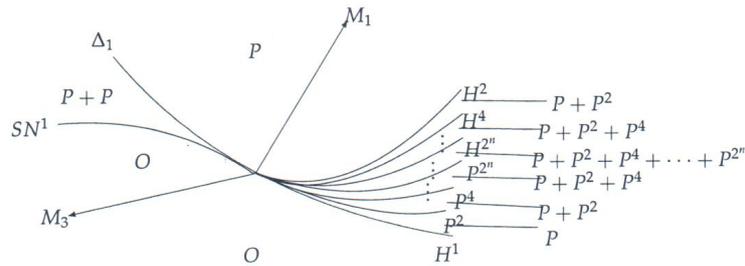


Figure 3.1: Bifurcation diagram.

## 4 Conclusion

This paper is devoted to proving the existence of higher order homoclinic orbits and periodic orbits from the orbit-flip homoclinic orbit of weak type. Such homoclinic orbit is a degenerate version of the so called orbit-flip homoclinic orbit, and it is a new case of codimension 3. The homoclinic orbit of higher order, also named as the multi-round homoclinic orbit, corresponds to the traveling or standing multi-pulse in the spatial dynamics. The method we employ is the local moving coordinates method. The phenomenon of homoclinic doubling bifurcation like we showed in this paper, is just like the cascades of periodic doubling bifurcation found by Feigenbaum and Coullet–Tresser. It is a change of a homoclinic orbit into twice round homoclinic orbit in the neighborhood of the primary homoclinic orbit. More precisely,  $H^1$  is the 1-homoclinic bifurcation surface and  $H^2$  is the 2-homoclinic bifurcation surface as we found, which have the same normal vector  $M_1$  at  $\mu = 0$ . So, there is a tongue area bounded by  $H^1$  and  $H^2$ . In the tongue area, there must be another bifurcation surface  $P^2$  where a period-doubling bifurcation arises. By repeating the similar procedure, we also obtain the  $2^n$ -homoclinic bifurcation surface  $H^{2^n}$  and the period-doubling bifurcation surface  $P^{2^n}$  for arbitrary  $n \in \mathbb{N}$ .

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