Existence results for a two point boundary value problem involving a fourth-order equation

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Abstract. We study the existence of non-zero solutions for a fourth-order differential equation with nonlinear boundary conditions which models beams on elastic foundations. The approach is based on variational methods. Some applications are illustrated.

Keywords: fourth-order equations, critical points, variational methods

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1 Introduction

In this paper, we consider the following fourth-order problem

\[
\begin{aligned}
&u^{(iv)}(x) = \lambda f(x, u(x)) \quad \text{in } [0, 1], \\
u(0) = u'(0) = 0, \\
u''(1) = 0, \quad u'''(1) = \mu g(u(1)),
\end{aligned}
\]

where \( f: [0, 1] \times \mathbb{R} \to \mathbb{R} \) is an \( L^1 \)-Carathéodory function, \( g: \mathbb{R} \to \mathbb{R} \) is a continuous function and \( \lambda, \mu \) are positive parameters. The problem \((P_{\lambda,\mu})\) describes the static equilibrium of a flexible elastic beam of length 1 when, along its length, a load \( f \) is added to cause deformation. Precisely, conditions \( u(0) = u'(0) = 0 \) mean that the left end of the beam is fixed and conditions \( u''(1) = 0, \ u'''(1) = \mu g(u(1)) \) mean that the right end of the beam is attached to a bearing device, given by the function \( g \).

Existence and multiplicity results for this kinds of problems has been extensively studied. In particular, by using a variational approach, the existence of three solutions for the problems \((P_{\lambda,1})\) and \((P_{\lambda,\lambda})\) has been established respectively in [6] and in [4]. Moreover, in [8] the author obtained the existence of at least two positive solutions for the problem \((P_{1,1})\). Finally, we point out that the problem \((P_{\lambda,\mu})\) can be also studied by iterative methods (see for instance [7])

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and, for fourth order equations subject to conditions of different type, we refer, for instance, to \cite{3, 5} and references therein.

In this paper we will deal with the existence of one non-zero solution for the problem \((P_{\lambda, \mu})\). Precisely, using a variational approach, under conditions involving the antiderivatives of \(f\) and \(g\), we will obtain two precise intervals of the parameters \(\lambda\) and \(\mu\) for which the problem \((P_{\lambda, \mu})\) admits at least one non-zero classical solution (see Theorem 3.1). As a way of example, we present here a special case of our results.

**Theorem 1.1.** Let \(f: \mathbb{R} \to \mathbb{R}\) be a nonnegative continuous function. Then, for each \(\lambda \in \left[0, \frac{1}{\int_0^1 f(t) \, dt}\right]\), the problem

\[
\begin{cases}
\ u^{(iv)}(x) = \lambda f(u(x)) \quad \text{in } [0, 1], \\
\ u(0) = u'(0) = 0, \\
\ u''(1) = 0, \quad u'''(1) = \sqrt{|u(1)|}
\end{cases}
\]

admits at least one non-zero classical solution.

We explicitly observe that in Theorem 1.1, assumptions on the behavior of \(f\), as for instance asymptotic conditions at zero or at infinity, are not requested, whereby \(f\) is a totally arbitrary function.

The paper is arranged as follows. In Section 2, we recall some basic definitions and our main tool (Theorem 2.2), which is a local minimum theorem established in \cite{1}. Finally, Section 3 is devoted to our main results. Precisely, under a suitable behaviour of \(f\) and for parameters \(\mu\) small enough, the existence of a non-zero solution for \((P_{\lambda, \mu})\) is obtained (Theorem 3.1) and a variant is highlighted (Theorem 3.3). Moreover, some consequences are pointed out (Corollaries 3.4 and 3.5) and a concrete example of application is given (Example 3.7).

## 2 Basic definitions and preliminary results

We consider the space

\[ X := \{ u \in H^2([0, 1]) : u(0) = u'(0) = 0 \} \]

where \(H^2([0, 1])\) is the Sobolev space of all functions \(u: [0, 1] \to \mathbb{R}\) such that \(u\) and its distributional derivative \(u'\) are absolutely continuous and \(u''\) belongs to \(L^2([0, 1])\). \(X\) is a Hilbert space with inner product

\[ \langle u, v \rangle := \int_0^1 u''(t)v''(t) \, dt \]

and norm

\[ \|u\| := \left( \int_0^1 (u''(t))^2 \, dt \right)^{\frac{1}{2}}, \]

which is equivalent to the usual norm \(\int_0^1 (|u(t)|^2 + |u'(t)|^2 + |u''(t)|^2) \, dt\). Moreover, the inclusion \(X \hookrightarrow C^1([0, 1])\) is compact (see \cite{6}) and it results

\[ \|u\|_{C^1([0, 1])} := \max \{ \|u\|_\infty, \|u'\|_\infty \} \leq \|u\| \]

(2.1)

for each \(u \in X\). We consider the functionals \(\Phi, \Psi_{\lambda, \mu}: X \to \mathbb{R}\) defined by

\[ \Phi(u) := \frac{1}{2} \|u\|^2 \]

and \(\Psi_{\lambda, \mu}(u) := \int_0^1 \frac{1}{2} (u''(t))^2 + \lambda u(t)^2 \, dt\). When \(\lambda = 0\), the problem \((P_{\lambda, \mu})\) reduces to the fourth order boundary value problem

\[ \begin{cases}
\ u^{(iv)}(x) = \mu g(u(x)) \quad \text{in } [0, 1], \\
\ u(0) = u'(0) = 0, \\
\ u''(1) = 0, \quad u'''(1) = \sqrt{|u(1)|}
\end{cases} \]

for which we also refer to \cite{6}.
and

\[ \Psi_{\lambda, \mu}(u) := \int_0^1 F(x, u(x)) \, dx + \frac{\mu}{\lambda} G(u(1)) \]

for each \( u \in X \) and for each \( \lambda, \mu > 0 \) where \( F(x, \xi) := \int_0^1 f(x, t) \, dt \) and \( G(\xi) := \int_0^1 g(t) \, dt \) for each \( x \in [0, 1], \xi \in \mathbb{R} \). By standard arguments, \( \Phi \) is sequentially weakly lower semicontinuous and coercive. Moreover, \( \Phi \) and \( \Psi_{\lambda, \mu} \) are in \( C^1(X) \) and their Fréchet derivatives are respectively

\[ \langle \Phi'(u), v \rangle = \int_0^1 u''(x)v''(x) \, dx \]

and

\[ \langle \Psi'_{\lambda, \mu}(u), v \rangle = \int_0^1 f(x, u(x))v(x) \, dx + \frac{\mu}{\lambda} g(u(1))v(1) \]

for each \( u, v \in X \). In [6] the authors proved that \( \Phi' \) admits a continuous inverse on \( X^* \) and \( \Psi' \) is compact. In particular, in Lemma 2.1 of [6] it has been shown that, for each \( \lambda, \mu > 0 \), the critical points of the functional

\[ I_{\lambda, \mu} := \Phi - \lambda \Psi_{\lambda, \mu} \]

are solutions for problem \((P_{\lambda, \mu})\).

In order to obtain solutions for the problem \((P_{\lambda, \mu})\), we make use of a recent critical point result, where a novel type of Palais–Smale condition is applied (see Theorem 3.1 of [1]). We recall it.

**Definition 2.1.** Let \( \Phi \) and \( \Psi \) two continuously Gâteaux differentiable functionals defined on a real Banach space \( X \) and fix \( r \in \mathbb{R} \). The functional \( I = \Phi - \Psi \) is said to verify the Palais–Smale condition cut off upper at \( r \) (in short \((P.S.)^r\)) if any sequence \( \{u_n\}_{n \in \mathbb{N}} \) in \( X \) such that

(a) \( \{I(u_n)\} \) is bounded;

(b) \( \lim_{n \to +\infty} \|I'(u_n)\|_{X^*} = 0 \);

(c) \( \Phi(u_n) < r \) for each \( n \in \mathbb{N} \);

has a convergent subsequence.

The following theorem is a particular case of Theorem 5.1 of [1] and it is the main tool of the next section.

**Theorem 2.2** (Theorem 2.3 of [2]). Let \( X \) be a real Banach space, \( \Phi, \Psi: X \to \mathbb{R} \) be two continuously Gâteaux differentiable functionals such that \( \inf_{x \in X} \Phi(x) = \Phi(0) = \Psi(0) = 0 \). Assume that there exist \( r > 0 \) and \( \bar{x} \in X \), with \( 0 < \Phi(\bar{x}) < r \), such that:

(a1) \( \frac{\sup_{\Phi(x) \leq r} \Psi(x)}{r} \frac{\Phi(\bar{x})}{\Psi(\bar{x})} < 1 \)

(a2) for each

\[ \lambda \in \left[ \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x) \leq r} \Psi(x)} \right] \]

the functional \( I_\lambda := \Phi - \lambda \Psi \) satisfies \((P.S.)^r\) condition.

Then, for each

\[ \lambda \in \Lambda_r := \left[ \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x) \leq r} \Psi(x)} \right] \]

there is \( x_{0, \lambda} \in \Phi^{-1}([0, r]) \) such that \( I'_\lambda(x_{0, \lambda}) = \partial X^* \) and \( I_\lambda(x_{0, \lambda}) \leq I_\lambda(x) \) for all \( x \in \Phi^{-1}([0, r]) \).
3 Existence of one solution

Before introducing the main result, we define some notation. With \( \alpha \geq 0 \), we put

\[
F^\alpha := \int_0^1 \max_{|\xi| \leq \alpha} F(x, \xi) \, dx
\]

and

\[
G^\alpha := \max_{|\xi| \leq \alpha} G(\xi).
\]

**Theorem 3.1.** Assume that

\((f_1)\) there exist \( \delta, \gamma \in \mathbb{R} \), with \( 0 < \delta < \gamma \), such that

\[
\frac{F^\gamma}{\gamma^2} < \frac{1}{8\pi^4} \left( \frac{3}{2} \right)^3 \frac{\int_0^1 F(x, \delta) \, dx}{\delta^2}
\]

\((f_2)\) \( F(x, t) \geq 0 \) for almost every \( x \in [0, 1] \) and for all \( t \in [0, \delta] \).

Then, for each

\[
\lambda \in \Lambda_{\delta, \gamma} := \left\{ \lambda \in I_{\delta, \gamma} : \left( 4\pi^4 \left( \frac{2}{3} \right)^3 \frac{\delta^2}{\int_0^1 F(x, \delta) \, dx} \right) \leq \frac{\gamma^2}{2F^\gamma} \right\},
\]

and for each \( g : \mathbb{R} \rightarrow \mathbb{R} \) continuous, there exists \( \eta_{\lambda, g} > 0 \), where

\[
\eta_{\lambda, g} = \begin{cases} \frac{\gamma^2 - 2\lambda F^\gamma}{2G^\gamma} & \text{if } G(\delta) \geq 0, \\ \min \left\{ \frac{\gamma^2 - 2\lambda F^\gamma}{2G^\gamma}, \frac{4\pi^4\delta^2 - \lambda \left( \frac{3}{2} \right)^3 \int_0^1 F(x, \delta) \, dx}{\left( \frac{3}{2} \right)^3 G(\delta)} \right\} & \text{if } G(\delta) < 0, \end{cases}
\]

such that for each \( \mu \in [0, \eta_{\lambda, g}] \) the problem \((P_{\lambda, \mu})\) admits at least one non-zero solution \( u_\lambda \) such that \( \|u_\lambda\|_{\infty} \|u'_\lambda\|_{\infty} < \gamma \).

**Proof.** Fix \( \lambda \in \Lambda_{\delta, \gamma} \). We observe that \( \eta_{\lambda, g} > 0 \). Indeed, if \( G(\delta) \geq 0 \), then \( G^\gamma \geq 0 \) and by \( \lambda \in \Lambda_{\delta, \gamma} \) it follows that \( \gamma^2 - 2\lambda F^\gamma > 0 \). Hence \( \eta_{\lambda, g} > 0 \). Let \( G(\delta) < 0 \). We have by \( \lambda \in \Lambda_{\delta, \gamma} \) that

\[
4\pi^4 \left( \frac{2}{3} \right)^3 \frac{\delta^2}{\int_0^1 F(x, \delta) \, dx} < \lambda,
\]

which implies

\[
4\pi^4\delta^2 - \lambda \left( \frac{3}{2} \right)^3 \int_0^1 F(x, \delta) \, dx < 0.
\]

Hence \( \eta_{\lambda, g} > 0 \), in this case as well.

Now, fix \( g : \mathbb{R} \rightarrow \mathbb{R} \) continuous, \( \mu \in [0, \eta_{\lambda, g}] \) and consider the space \( X \). Our aim is to apply Theorem 2.2 to the functionals \( \Phi, \Psi_{\lambda, \mu} \) defined above. To this end, we fix \( r = \frac{2}{3} \).

The properties of the functionals \( \Phi \) and \( \Psi_{\lambda, \mu} \) ensure that the functional \( I_{\lambda, \mu} = \Phi - \lambda \Psi_{\lambda, \mu} \) verifies \((P.S.)\) condition for each \( r, \lambda, \mu > 0 \) (see Proposition 2.1 of [1]) and so condition \((a_2)\) of Theorem 2.2 is verified.

Denote by \( \varphi \) the function of \( X \) defined by

\[
\varphi(x) = \begin{cases} 0 & x \in \left[0, \frac{3}{8}\right), \\ \delta \cos^2 \left( \frac{4\pi x}{3} \right) & x \in \left[\frac{3}{8}, \frac{3}{4}\right), \\ \delta & x \in \left[\frac{3}{4}, 1\right], \end{cases}
\]
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for which it results

\[ \Phi(\bar{v}) = 4\pi^4\delta^2 \left(\frac{2}{3}\right)^3. \]  

(3.3)

Taking into account that \( \bar{v}(x) \in [0, \delta] \) for each \( x \in \left[\frac{3}{8}, \frac{3}{4}\right] \), condition \((f_2)\) ensures that

\[ \int_0^\frac{3}{4} F(x, \bar{v}(x)) \, dx \geq 0 \]

and

\[ \int_\frac{1}{3}^1 F(x, \delta) \, dx \geq 0, \]

which implies

\[ \Psi_{\lambda, \mu}(\bar{v}) = \int_0^1 F(x, \bar{v}(x)) \, dx + \frac{\mu}{\lambda} G(\delta) \geq \int_\frac{1}{3}^1 F(x, \delta) \, dx + \frac{\mu}{\lambda} G(\delta). \]

This ensures that

\[ \frac{\Psi_{\lambda, \mu}(\bar{v})}{\Phi(\bar{v})} \geq \frac{\int_\frac{1}{3}^1 F(x, \delta) \, dx + \frac{\mu}{\lambda} G(\delta)}{4\pi^4\delta^2 \left(\frac{2}{3}\right)^3}. \]  

(3.4)

For each \( u : \Phi(u) = \frac{|u|^2}{2} \leq r \), by \((2.1)\) one has

\[ ||u|| \leq \gamma = \sqrt{2r} \]

and

\[ ||u||_\infty \leq \gamma. \]

It results

\[ \Psi_{\lambda, \mu}(u) = \int_0^1 F(x, u(x)) \, dx + \frac{\mu}{\lambda} G(u(1)) \leq F^\gamma + \frac{\mu}{\lambda} G^\gamma \]

for each \( u \in \Phi^{-1}([-\infty, r]) \). This leads to

\[ \frac{1}{r} \sup_{u \in \Phi^{-1}([-\infty, r])} \Psi_{\lambda, \mu}(u) \leq \frac{2}{\gamma^2} F^\gamma + \frac{2}{\gamma^2} \frac{\mu}{\lambda} G^\gamma. \]  

(3.5)

Now, taking into account \((f_1)\), if \( G(\delta) \geq 0 \), then, it results

\[ \frac{2}{\gamma^2} F^\gamma + \frac{2}{\gamma^2} \frac{\mu}{\lambda} G^\gamma \leq \frac{2}{\gamma^2} F^\gamma + \frac{2}{\gamma^2} \frac{\eta_{\lambda, \delta}}{\lambda} G^\gamma = \frac{1}{\lambda} \]

and

\[ \frac{1}{\lambda} \leq \frac{1}{4\pi^4\delta^2} \left(\frac{3}{2}\right)^3 \int_\frac{1}{3}^1 F(x, \delta) \, dx \leq \frac{1}{4\pi^4\delta^2} \left(\frac{3}{2}\right)^3 \left( \int_\frac{1}{3}^1 F(x, \delta) \, dx + \frac{\mu}{\lambda} G(\delta) \right). \]

If \( G(\delta) < 0 \), taking into account that

\[ \mu < \eta_{\lambda, \delta} = \min \left\{ \frac{\gamma^2 - 2\lambda F^\gamma}{2G^\gamma}, \frac{4\pi^4\delta^2 - \lambda \left(\frac{3}{2}\right)^3 \int_\frac{1}{3}^1 F(x, \delta) \, dx}{\left(\frac{3}{2}\right)^3 \frac{G(\delta)}{G^\gamma}} \right\}, \]  

(3.6)

it results

\[ \frac{2}{\gamma^2} F^\gamma + \frac{2}{\gamma^2} \frac{\mu}{\lambda} G^\gamma \leq \frac{2}{\gamma^2} F^\gamma + \frac{2}{\gamma^2} \frac{\eta_{\lambda, \delta}}{\lambda} G^\gamma \leq \frac{1}{\lambda}. \]
if $G > 0$, and $\frac{\gamma}{F} + \frac{\mu}{G} < \frac{1}{2}$ if $G = 0$.
Moreover, again from (3.6),
\[
\frac{1}{\lambda} < \frac{1}{4\pi^2 \delta^2} \left(\frac{3}{2}\right)^3 \int_{\frac{1}{4}}^1 F(x, \delta) \, dx + \frac{\mu}{\lambda} \frac{1}{4\pi^2 \delta^2} \left(\frac{3}{2}\right)^3 G(\delta).
\]
In all cases, taking into account (3.4) and (3.5), we have
\[
\frac{1}{\lambda} \sup_{u \in \Phi^{-1}([-\infty, r])} \Psi_{\lambda, \mu}(u) < \frac{1}{\lambda} < \frac{\Psi_{\lambda, \mu}(\delta)}{\Phi(\delta)}.
\]
Moreover, we observe that from $\delta < \gamma$, taking $(f_1)$ into account, we obtain
\[
\sqrt{8\pi^4 \left(\frac{2}{3}\right)^3} \delta < \gamma.
\]
In fact, arguing by a contradiction, if we assume $\delta < \gamma \leq \sqrt{8\pi^4 \left(\frac{2}{3}\right)^3} \delta$, we obtain
\[
\frac{F(\gamma)}{\gamma^2} \geq \frac{1}{\pi^4 \left(\frac{3}{4}\right)^3} \int_{\frac{1}{4}}^1 F(x, \delta) \, dx \delta^2
\]
and this is an absurd by $(f_1)$. Therefore, we have $\Phi(\delta) = 4\pi^4 \delta^2 \left(\frac{3}{2}\right)^3 < \frac{\gamma^2}{2} = r$ and the condition $(a_1)$ of Theorem 2.2 is verified.
Moreover, since $\lambda \in \Lambda_{\delta, \gamma} \subseteq \left\{ \Phi(\delta), \sup_{\Phi(u) \leq r} \Psi_{\lambda, \mu}(u) \right\}$,

Theorem 2.2 guarantees the existence of a local minimum point $u_\lambda$ for the functional $I_\lambda$ such that
\[
0 < \Phi(u_\lambda) < r
\]
and so $u_\lambda$ is a nontrivial classical solution of problem $(P_{\lambda, \mu})$ such that $\|u_\lambda\|_{\infty}, \|u_\lambda'\|_{\infty} < \gamma$. □

**Remark 3.2.** We observe that in Theorem 3.1 we read $\frac{\gamma^2 - 2\lambda F}{2G} = +\infty$ when $G = 0$.

By reversing the roles of $\lambda$ and $\mu$, we obtain the following result.

**Theorem 3.3.** Assume that

$(g_1)$ there exist $\delta, \gamma \in \mathbb{R}$ with $0 < \delta < \gamma$:
\[
\frac{G^\gamma}{\gamma^2} < \frac{1}{8\pi^4 \left(\frac{3}{2}\right)^3} \frac{G(\delta)}{\delta^2}.
\]

Then for each $\mu \in \Gamma_{\delta, \gamma} := \left\{ 4\pi^4 \left(\frac{3}{2}\right)^3 \frac{\delta^2}{G(\delta)} \right\},$ and for each $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ $L^1$-Carathéodory function verifying condition $(f_2)$ of Theorem 3.1, there exists $\theta_{\mu, f} > 0$, where
\[
\theta_{\mu, f} := \frac{\gamma^2 - 2\mu G^\gamma}{2F^\gamma},
\]
such that for each $\lambda \in ]0, \theta_{\mu, f}[ \text{ the problem } (P_{\lambda, \mu}) \text{ admits at least one non-zero solution } u \text{ such that } \|u\|_{\infty}, \|u'\|_{\infty} < \gamma.$
Proof. Fix \( \mu \in \Gamma_{\delta, \gamma} \) and \( \lambda \in [0, \theta_{\mu, f}] \). Put

\[
\tilde{\Psi}_{\lambda, \mu}(u) := \frac{\lambda}{\mu} \int_0^1 F(x, u(x)) \, dx + G(u(1)), \quad \tilde{I}_{\lambda, \mu}(u) := \Phi(u) - \mu \tilde{\Psi}_{\lambda, \mu}(u),
\]

for all \( u \in X \). Clearly, one has \( \tilde{I}_{\lambda, \mu} = I_{\lambda, \mu} \).

Now, let \( \vartheta \) the function as given in (3.2) and \( r = \frac{\gamma^2}{\pi} \). Arguing as in the proof of Theorem 3.1 (see (3.4) and (3.5)) we obtain

\[
\frac{\tilde{\Psi}_{\lambda, \mu}(\vartheta)}{\Phi(\vartheta)} \geq \frac{\lambda}{\mu} \frac{\int_0^1 F(x, \delta) \, dx + G(\delta)}{4\pi^2 \delta^2 \left( \frac{2}{3} \right)^3} > \frac{1}{\mu},
\]

(3.7)

and

\[
\frac{1}{r} \sup_{u \in \Phi^{-1}(\gamma, r)} \tilde{\Psi}_{\lambda, \mu}(u) \leq \frac{2}{\gamma^2} \frac{\mu}{\lambda} F' + \frac{2}{\gamma^2} G' = \frac{1}{\mu}.
\]

(3.8)

Therefore, from (3.7) we obtain

\[
\frac{\tilde{\Psi}_{\lambda, \mu}(\bar{v})}{\Phi(\bar{v})} \geq \frac{G(\delta)}{4\pi^2 \delta^2 \left( \frac{2}{3} \right)^3} > \frac{1}{\mu}
\]

and from (3.8) it follows that

\[
\frac{1}{r} \sup_{u \in \Phi^{-1}(\gamma, r)} \tilde{\Psi}_{\lambda, \mu}(u) < \frac{2}{\gamma^2} \frac{\mu}{\lambda} F' + \frac{2}{\gamma^2} G' = \frac{1}{\mu}.
\]

Moreover, from \((g_1)\), arguing as in the proof of Theorem 3.1, one has \( \Phi(\vartheta) < r \). So, assumption \((a_1)\) of Theorem 2.2 is verified and

\[
\mu \in \left[ \frac{\Phi(\bar{v})}{\tilde{\Psi}_{\lambda, \mu}(\bar{v})}, \frac{r}{\sup_{\Phi(u) \leq r} \tilde{\Psi}_{\lambda, \mu}(u)} \right],
\]

for which \( \Phi - \mu \tilde{\Psi}_{\lambda, \mu} \) admits a non-zero critical point and the conclusion is obtained. \( \Box \)

Now, we present some consequences of previous results.

**Corollary 3.4.** Assume that \( f : \mathbb{R} \to \mathbb{R} \) is a continuous and non negative function such that

\[
(f''') \limsup_{t \to 0^+} \frac{F(t)}{t^2} = +\infty.
\]

Then, for each \( \gamma > 0 \), \( \lambda \in \left[ 0, \frac{\gamma^2}{2F(\gamma)} \right] \) for each \( g : \mathbb{R} \to \mathbb{R} \) continuous and nonnegative and for each \( \mu \in \left[ 0, \frac{\gamma^2 - 2F(\gamma) \lambda}{2\alpha(\gamma)} \right] \), the problem

\[
\begin{cases}
  u^{(iv)}(x) = \lambda f(u(x)) \quad \text{in } [0,1], \\
  u(0) = u'(0) = 0, \\
  u''(1) = 0, \quad u'''(1) = \mu g(u(1))
\end{cases}
\]

\((\tilde{P}_{\lambda, \mu})\)

admits at least one non-zero classical solution \( u \) such that \( \|u\|_{\infty, r}, \|u'\|_{\infty} < \gamma \).
Proof. Fix $\gamma > 0$, $\lambda \in ]0, \frac{\gamma^2}{2f'(\gamma)}[$, $g : \mathbb{R} \to \mathbb{R}$ continuous and nonnegative and $\mu \in ]0, \frac{\gamma^2 - 2F(\gamma)\lambda}{2G(\gamma)}[$.

Condition $(f_2)$ of Theorem 3.1 is verified. Moreover, by $(f_1'')$, there exists $0 < \delta < \gamma$ such that

$$\frac{F(\delta)}{\delta^2} > \frac{16\pi^4\left(\frac{3}{2}\right)^3}{\lambda}.$$ 

Taking into account that $\lambda \in ]0, \frac{\gamma^2}{2f'(\gamma)}[$, it results

$$\frac{F(\gamma)}{\gamma^2} < \frac{1}{2\lambda} < \frac{F(\delta)}{\delta^2} \left(\frac{3}{2}\right)^3 \frac{1}{16\pi^4}$$

and so condition $(f_1)$ of Theorem 3.1 is verified. Since $g$ is nonnegative, $\eta_{\lambda, g} = \frac{\gamma^2 - 2F(\gamma)\lambda}{2G(\gamma)}$ and the conclusion follows easily.

Clearly, arguing as in the proof of Corollary 3.4, from Theorem 3.3 we obtain the following result.

Corollary 3.5. Let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that $\lim_{t \to 0^+} \frac{g(t)}{t} = +\infty$. Then, for each $\gamma > 0$, for each $\mu \in ]0, \frac{\gamma^2}{2G(\gamma)}[$, for each nonnegative continuous function $f : \mathbb{R} \to \mathbb{R}$ and for each $\lambda \in ]0, \frac{\gamma^2 - 2\mu G(\gamma)}{2F(\gamma)}[$, the problem $(P_{\lambda, \mu})$ admits at least one non-zero classical solution $u$ such that $\|u\|_\infty, \|u\|_{\infty} < \gamma$.

Remark 3.6. Theorem 1.1 in the Introduction is an immediate consequence of Corollary 3.5. Indeed, it is enough to pick $g(t) = \sqrt{|t|}$ for all $t \in \mathbb{R}$ and $\gamma = 2$, so that one has $\lim_{t \to 0^+} \frac{g(t)}{t} = +\infty$, $\mu = 1 < \frac{\gamma^2}{G(2)}$ and $\lambda < \frac{1}{10F(2)} < \frac{12 - 8\sqrt{2}}{6F(2)} = \frac{\gamma^2 - 2\mu G(\gamma)}{2F(\gamma)}$.

Example 3.7. Let us take $\delta = 1/2$, $\gamma = 22$ and $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(u) := \begin{cases} 0, & u < 0, \\ u - u^2, & 0 \leq u \leq 1, \\ 0, & u > 1. \end{cases}$$

Then, by Theorem 3.1, for each $\lambda \in ]1385.4, 1452[$ and each $g : \mathbb{R} \to \mathbb{R}$ continuous there exists $\eta_{\lambda, g} > 0$ such that for each $\mu \in ]0, \eta_{\lambda, g}[$, the problem $(P_{\lambda, \mu})$ admits at least one non-zero solution $u_\lambda$ with $\|u\|_\infty, \|u\|_{\infty} < 22$.

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References

Existence results for a two point boundary value problem


