Periodic solutions of second-order systems with subquadratic convex potential

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Abstract. In this paper, we investigate the existence of periodic solutions for the second order systems at resonance:

\[
\begin{align*}
\ddot{u}(t) + m^2 \omega^2 u(t) + \nabla F(t, u(t)) &= 0 \quad \text{a.e. } t \in [0, T], \\
u(0) - u(T) = \dot{u}(0) - \dot{u}(T) &= 0,
\end{align*}
\]

where \( m > 0 \), the potential \( F(t, x) \) is convex in \( x \) and satisfies some general subquadratic conditions. The main results generalize and improve Theorem 3.7 in J. Mawhin and M. Willem [Critical point theory and Hamiltonian systems, Springer-Verlag, New York, 1989].

Keywords: second order Hamiltonian systems, critical points, variational methods, Sobolev’s inequality.

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1 Introduction and main results

Consider the second order Hamiltonian systems

\[
\begin{align*}
\ddot{u}(t) + m^2 \omega^2 u(t) + \nabla F(t, u(t)) &= 0 \quad \text{a.e. } t \in [0, T], \\
u(0) - u(T) = \dot{u}(0) - \dot{u}(T) &= 0,
\end{align*}
\]

(1.1)

where \( T > 0 \), \( \omega = 2\pi/T \) and \( m > 0 \) is an integer. The potential \( F: [0, T] \times \mathbb{R}^N \to \mathbb{R} \) satisfies the following assumption:

(A) \( F(t, x) \) is measurable in \( t \) for every \( x \in \mathbb{R}^N \) and continuously differentiable in \( x \) for a.e. \( t \in [0, T] \), and there exist \( a \in C(\mathbb{R}^+, \mathbb{R}^+) \), \( b \in L^1(0, T; \mathbb{R}^+) \) such that

\[
|F(t, x)| \leq a(|x|)b(t), \quad |\nabla F(t, x)| \leq a(|x|)b(t)
\]

for all \( x \in \mathbb{R}^N \) and a.e. \( t \in [0, T] \).

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If $m = 0$, the non-resonant second order Hamiltonian systems have been extensively investigated during the past two decades. Different solvability hypotheses on the potential are given, such as: the convexity conditions (see [6, 8, 12, 13]); the coercivity conditions (see [1, 5, 10]); the subquadratic conditions (including the sublinear nonlinearity case, see [7, 9, 11–14, 16, 18]); the superquadratic conditions (see [3, 7, 17, 18, 21]) and the asymptotically quadratic conditions (see [19, 21, 24]).

Using the variational principle of Clarke and Ekeland together with an approximate argument of H. Brézis [2], Mawhin and Willem [6] proved an existence theorem for semilinear equations of the form $Lu = \nabla F(x, u)$, where $L$ is a noninvertible linear selfadjoint operator and $F$ is convex with respect to $u$ and satisfies a suitable asymptotic quadratic growth condition. This result was applied to periodic solutions of first order Hamiltonian systems with convex potential. In [5], the authors considered the second order systems (1.1) with $m = 0$. They proved that when the potential $F$ satisfies the following assumptions:

\begin{itemize}
  \item[(A')] $F(t, x)$ is measurable in $t$ for every $x \in \mathbb{R}^N$, and continuously differentiable and convex in $x$ for a.e. $t \in [0, T]$;
  \item[(A1)] There exists $l \in L^1(0, T; \mathbb{R}^N)$ such that
    \[ (l(t), x) \leq F(t, x), \quad \forall x \in \mathbb{R}^N \text{ and a.e. } t \in [0, T]; \]
  \item[(A2')] There exist $\alpha \in (0, \omega^2)$ and $\gamma \in L^2(0, T; \mathbb{R}^+)$ such that
    \[ F(t, x) \leq \frac{1}{2} \alpha |x|^2 + \gamma(t), \quad \forall x \in \mathbb{R}^N \text{ and a.e. } t \in [0, T]; \]
  \item[(A3')] $\int_0^T F(t, x) \, dt \to +\infty$ as $|x| \to \infty$, $x \in \mathbb{R}^N$;
\end{itemize}

then problem (1.1) has at least one solution, see [5, Theorem 3.5]. This result was slightly improved in Tang [8] by relaxing the integrability of $l$ and $\gamma$. In [12], Tang and Wu dealt with the ($\beta, \gamma$)-subconvex case, i.e.,

\[ F(t, \beta(x + y)) \leq \gamma(F(t, x) + F(t, y)), \quad \forall x, y \in \mathbb{R}^N \text{ and a.e. } t \in [0, T] \quad (1.2) \]

for some $\gamma > 0$. Under assumptions $(A)$, $(A2')$ and (1.2) and the subquadratic condition: there exist $0 < \mu < 2$ and $M > 0$ such that

\[ (\nabla F(t, x), x) \leq \mu F(t, x), \quad \forall |x| \geq M \text{ and a.e. } t \in [0, T], \]

they obtained the existence result by taking advantage of Rabinowitz’s saddle point theorem. Recently, Tang and Wu [13] extended a theorem established by A. C. Lazer, E. M. Landesman and D. R. Meyers [4] on the existence of critical points without compactness assumptions, using the reduction method, the perturbation argument and the least action principle. As a main application, they successively studied the existence of periodic solutions of problem (1.1) ($m = 0$) with subquadratic convex potential, with subquadratic $\mu(t)$-convex potential and with subquadratic $k(t)$-concave potential, which unifies and significantly generalizes some earlier results in [5, 8, 15, 22, 23] obtained by other methods.

If $m \neq 0$, it is a resonance case. Using the dual least action principle and the perturbation technique, Mawhin and Willem [5] also obtained the following theorem.
Theorem 1.1. Suppose that $F(t, x)$ satisfies conditions $(A')$, $(A_1)$ and the following:

$(A_2)$ There exist $a \in (0, (2m + 1)\omega^2)$ and $\gamma \in L^2(0, T; \mathbb{R}^+)$ such that

$$F(t, x) \leq \frac{1}{2} a |x|^2 + \gamma(t), \quad \forall x \in \mathbb{R}^N \text{ and a.e. } t \in [0, T].$$

$(A_3)$ \( \int_0^T F(t, a \cos \omega t + b \sin \omega t) \, dt \to +\infty \) as $|a| + |b| \to \infty$, $a, b \in \mathbb{R}^N$.

Then problem (1.1) has at least one solution in $H^1_T$, where

$$H^1_T = \left\{ u : [0, T] \to \mathbb{R}^N \mid \begin{array}{l}
u \text{ is absolutely continuous, } \\
u(0) = u(T) \text{ and } \dot{u} \in L^2(0, T; \mathbb{R}^N) \end{array} \right\}$$

is a Hilbert space with the norm defined by

$$\|u\| = \left( \int_0^T |u(t)|^2 \, dt + \int_0^T |\dot{u}(t)|^2 \, dt \right)^{1/2}.$$

Motivated by the works mentioned above, in this paper, we are interested in problem (1.1), where the potential is convex and satisfies conditions which are more general than $(A_2)$. Applying the abstract critical point theory established in [13], we prove some existence results, which generalize Theorem A and complement the results in [13]. The main results are the following theorems.

Theorem 1.1. Suppose that assumption $(A)$ holds and $F(t, x)$ is convex in $x$ for a.e. $t \in [0, T]$. Assume that $(A_3)$ holds and:

$(A_4)$ There exists $\gamma \in L^1(0, T; \mathbb{R}^+)$ such that

$$F(t, x) \leq \frac{2m + 1}{2} \omega^2 |x|^2 + \gamma(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, and

$$\text{meas} \left\{ t \in [0, T] \mid F(t, x) - \frac{2m + 1}{2} \omega^2 |x|^2 \to -\infty \text{ as } |x| \to \infty \right\} > 0.$$  \hspace{1cm} (1.4)

Then problem (1.1) has at least one solution in $H^1_T$.

Remark 1.2. Theorem 1.1 extends Theorem A, since $(A_4)$ is weaker than $(A_2)$ and assumption $(A)$ holds for functions $F$ in Theorem A (see [13, Remark 1.3] for a proof). There are functions $F$ which match our setting but not satisfying Theorem A. For example, let

$$F(t, x) = \frac{2m + 1}{2} \omega^2 \left( |x|^2 - (1 + |x|^2)^{\frac{2}{3}} \right) + (l(t), x),$$

where $l \in L^3(0, T; \mathbb{R}^N) \setminus L^\infty(0, T; \mathbb{R}^N)$. Then by Young’s inequality, one has

$$-\frac{2m + 1}{2} \omega^2 (1 + |x|^2)^{\frac{2}{3}} + (l(t), x) \leq -\frac{2m + 1}{2} \omega^2 |x|^3 + |l(t)||x|$$

$$\leq -\frac{2m + 1}{2} \omega^2 |x|^3$$

$$+ \frac{2m + 1}{2} \left( \omega^4 |x| \right)^{\frac{1}{3}} + \frac{2m + 1}{4} \left( \frac{4}{3(2m + 1)} \right)^{\frac{1}{3}} \omega^{-4} |l(t)|^3$$

$$\leq \frac{16}{27(2m + 1)^2} \omega^{-4} |l(t)|^3$$
for all \( x \in \mathbb{R}^N \) and a.e. \( t \in [0, T] \). Thus \( F \) satisfies (1.3) with \( \gamma(t) = \frac{16}{27(2m+1)\omega^4} |l(t)|^3 \). Evidently, (A3) and (1.4) are satisfied, and \( F(t, \cdot) \) is convex because

\[
 f(x) := g(h(x))
\]

is convex by the fact that

\[
 g(s) := (s - (1+s)\frac{3}{2}), \quad s > 0
\]

is convex and increasing, and

\[
 h(x) := |x|^2, \quad x \in \mathbb{R}^N
\]

is convex. Hence \( F \) satisfies all the conditions of Theorem 1.1. But it does not satisfy Theorem A, for (A2) does not hold.

Theorem 1.1 yields immediately the following corollary.

**Corollary 1.3.** The conclusion of Theorem 1.1 remains valid if we replace (A4) by

\[
 (A_5) \quad F(t, x) - \frac{2m+1}{2} \alpha^2 |x|^2 \to -\infty \quad \text{as } |x| \to \infty \quad \text{for a.e. } t \in [0, T].
\]

**Remark 1.4.** It is easy to see that (A5) is weaker than (A2). So Corollary 1.3 also generalizes Theorem A.

**Corollary 1.5.** The conclusion of Theorem 1.1 remains valid if we replace (A4) by

\[
 (A_6) \quad \text{There exist } \alpha \in L^\infty(0, T; \mathbb{R}^+) \text{ with } \text{meas } \{ t \in [0, T] : \alpha(t) < (2m+1)\omega^2 \} > 0 \quad \text{and } \alpha(t) \leq (2m+1)\omega^2 \text{ for a.e. } t \in [0, T], \quad \text{and } \gamma \in L^1(0, T; \mathbb{R}^+) \text{ such that}
\]

\[
 F(t, x) \leq \frac{1}{2} \alpha(t) |x|^2 + \gamma(t) \quad \text{for all } x \in \mathbb{R}^N \text{ and a.e. } t \in [0, T]. \quad (1.5)
\]

**Remark 1.6.** Corollary 1.5 also generalizes Theorem A. There are functions \( F \) satisfying our Corollary 1.5 and not satisfying Theorem A and Corollary 1.3. For example, let

\[
 F(t, x) = \frac{1}{2} \beta(t) |x|^2 + (l(t), x),
\]

where \( \beta \in L^\infty(0, T; \mathbb{R}^+) \) with \( \beta(t) \leq (2m+1)\omega^2 \) for a.e. \( t \in [0, T] \), \( \int_0^T \beta(t) dt > 0 \),

\[
 \text{meas } \{ t \in [0, T] : \beta(t) < (2m+1)\omega^2 \} > 0,
\]

and \( l \in L^\infty(0, T; \mathbb{R}^N) \) with \( |l(t)| \leq \frac{1}{2}((2m+1)\omega^2 - \beta(t)) \) for a.e. \( t \in [0, T] \). Then one has

\[
 F(t, x) \leq \frac{1}{2} \beta(t) |x|^2 + |l(t)||x| \leq \frac{1}{2} (\beta(t) + |l(t)||x|^2 + \frac{1}{2}|l(t)|,
\]

which is just (1.5) with \( \alpha = \beta(t) + |l(t)| \) and \( \gamma = |l(t)|/2 \). Hence \( F \) satisfies Corollary 1.5. But in the case that \( \text{meas } \{ t \in [0, T] : \beta(t) = (2m+1)\omega^2 \} > 0 \), \( F \) does not satisfy the conditions of Theorem A and Corollary 1.3.

**Theorem 1.7.** Suppose that assumption (A) holds and \( F(t, x) \) is convex in \( x \) for a.e. \( t \in [0, T] \). Assume that (A3) holds and the following condition is fulfilled.
Remark 1.9. Theorem 1.7 generalizes Theorem A. There are functions \( F \) satisfying (A7) but not (A6). For example, let

\[
F(t, x) = \frac{1}{2} \alpha(t) |x|^2 + |x|^\frac{3}{2}, \quad \forall x \in \mathbb{R}^N \text{ and a.e. } t \in [0, T],
\]

where \( \mu \in L^1(0, T; \mathbb{R}) \) with \( \mu(t) \leq (2m + 1)\omega^2 \) for a.e. \( t \in [0, T] \), \( \int_0^T \mu(t) \, dt > 0 \), and \( \text{meas} \{ t \in [0, T] : \mu(t) < \omega^2 \} > 0 \). Then (A7) holds with \( \alpha = \mu^+(t) \). But \( F \) does not satisfy (A6) if \( \text{meas} \{ t \in [0, T] : \mu(t) = \omega^2 \} > 0 \). On the other hand, there are functions \( F \) satisfying (A6) but not (A7). For example, let

\[
F(t, x) = \frac{1}{3} t^{-\frac{1}{2}} \left( \sqrt{2m + 1} \omega |x| \right)^\frac{3}{2}, \quad \forall x \in \mathbb{R}^N \text{ and a.e. } t \in [0, T].
\]

By Young’s inequality, one has

\[
F(t, x) \leq \frac{1}{3} \left( \frac{3}{4} \left( \sqrt{2m + 1} \omega |x| \right)^2 + \frac{(t^{-\frac{1}{2}})^4}{4} \right) = \frac{(2m + 1)\omega^2}{4} |x|^2 + \frac{t^{-\frac{3}{2}}}{12},
\]

which is just (1.5) with \( \alpha = (2m + 1)\omega^2/2 \) and \( \gamma = t^{-\frac{3}{2}}/12 \). However, \( F(t, x) \) does not satisfy (A7), because

\[
\limsup_{|x| \to \infty} \frac{\frac{1}{3} t^{-\frac{1}{2}} \left( \sqrt{2m + 1} \omega |x| \right)^\frac{3}{2}}{|x|^2} \leq \frac{(2m + 1)\omega^2}{4}
\]

does not uniformly hold for a.e. \( t \in [0, T] \).

Remark 1.8. The conditions (A6) and (A7) are not equivalent in general. There are functions \( F \) satisfying (A7) but not (A6). For example, let

\[
\frac{1}{2} \alpha(t) |x|^2 + |x|^\frac{3}{2} \leq \frac{1}{2} \alpha(t) \quad \text{uniformly for a.e. } t \in [0, T].
\]

Then problem (1.1) has at least one solution in \( H^1_T \).

Theorem 1.10. Suppose that assumption (A) holds and \( F(t, x) \) is convex in \( x \) for a.e. \( t \in [0, T] \). Assume that (A3) holds and:

(A8) There exist \( \alpha \in L^1(0, T; \mathbb{R}^+ \) with \( \int_0^T \alpha(t) \, dt < \frac{12(2m + 1)}{T(m + 1)^2} \) and \( \gamma \in L^1(0, T; \mathbb{R}^+) \) such that

\[
F(t, x) \leq \frac{1}{2} \alpha(t) |x|^2 + \gamma(t), \quad \forall x \in \mathbb{R}^N \text{ and a.e. } t \in [0, T].
\] (1.6)

Then problem (1.1) has at least one solution in \( H^1_T \).
Remark 1.11. There are functions $F$ satisfying our Theorem 1.10 and not satisfying the results mentioned above. For example, let

$$F(t, x) = \frac{1}{2} \beta(t) |x|^2 + (l(t), x),$$

where $\beta \in L^1(0, T; R^+)$ with $0 < \int_0^T \beta(t) dt < \frac{12(2m+1)}{T(m+1)^2}$ and $l \in L^2(0, T; R^N)$. Then one has

$$F(t, x) \leq \frac{1}{2} \beta(t) |x|^2 + |l(t)||x|$$

$$\leq \frac{1}{2} \left( \beta(t) + \frac{12(2m+1) - T(m+1)^2|\beta|}{2T^2(m+1)^2} \right) |x|^2 + \frac{T^2(m+1)^2}{12(2m+1) - T(m+1)^2|\beta|} |l(t)|^2,$$

which is just (1.6) with

$$\alpha = \beta(t) + \frac{12(2m+1) - T(m+1)^2|\beta|}{2T^2(m+1)^2} \quad \text{and} \quad \gamma = \frac{T^2(m+1)^2}{12(2m+1) - T(m+1)^2|\beta|} |l(t)|^2.$$

Thus $F$ satisfies all the conditions of Theorem 1.10. But in the case that

$$\text{meas} \{ t \in [0, T] : \beta(t) > (2m+1)\omega^2 \} > 0,$$

$F$ does not satisfy the conditions of Theorems A, 1.1 and 1.7.

2 Proofs of the theorems

Under assumption (A), the energy functional associated to problem (1.1) given by

$$\varphi(u) = -\frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \frac{m^2 \omega^2}{2} \int_0^T |u(t)|^2 dt + \int_0^T F(t, u(t)) dt$$

is continuously differentiable and weakly upper semi-continuous on $H^1_T$. Furthermore,

$$\langle \varphi'(u), v \rangle = -\int_0^T (\dot{u}(t), \dot{v}(t)) dt + m^2 \omega^2 \int_0^T (u(t), v(t)) dt + \int_0^T (\nabla F(t, u(t)), v(t)) dt$$

for all $u, v \in H^1_T$, and $\varphi'$ is weakly continuous. It is well known that the weak solutions of problem (1.1) correspond to the critical points of $\varphi$ (see [5]).

For $u \in \tilde{H}^1_T = \{ u \in H^1_T : \int_0^T u(t) dt = 0 \}$, we have

$$\|u\|_\infty \leq \frac{T}{12} \int_0^T |\dot{u}(t)|^2 dt \quad \text{(Sobolev's inequality)},$$

which implies that

$$\|u\|_\infty \leq C\|u\|, \quad \forall u \in H^1_T$$

for some $C > 0$, where $\|u\|_\infty = \max_{t \in [0, T]} |u(t)|$ (see [5, Proposition 1.3]).

We recall an abstract critical point theorem which will be used in the sequel.
Lemma 2.2 implies that $\varphi$ is convex in $v$ for all $w \in W$, that is,

$$
\varphi(\lambda v_1 + (1 - \lambda)v_2, w) \leq \lambda \varphi(v_1, w) + (1 - \lambda) \varphi(v_2, w)
$$

for all $\lambda \in [0,1]$ and $v_1, v_2 \in V, w \in W$, and $\varphi'$ is weakly continuous. Assume that

$$
\varphi(0, w) \to -\infty \quad \text{as} \quad \|w\| \to \infty,
$$

and for every $M > 0$, $\varphi(v, w) \to +\infty$ as $\|v\| \to \infty$ uniformly for $\|w\| \leq M$.

Then $\varphi$ has at least one critical point.

Proposition 2.2 ([13, Lemma 5.1]). Assume that $H$ is a real Hilbert space, $f : H \times H \to R$ is a bilinear functional. Then $g : H \to R$ given by

$$
g(u) = f(u, u), \quad \forall u \in H
$$

is convex if and only if $g(u) \geq 0, \quad \forall u \in H$.

For $m > 0$, set

$$
H_m = \left\{ \sum_{j=0}^{m} (a_j \cos j \omega t + b_j \sin j \omega t) : a_j, b_j \in R^N, \ j = 0, \ldots, m \right\},
$$

and denote the orthogonal complement of $H_m$ in $H^1_T$ by $H^1_m$. Applying Proposition 2.2, we obtain the following result.

Lemma 2.3. Assume that $F(t, x)$ is convex in $x$ for a.e. $t \in [0, T]$. Then, for every $w \in H^1_m$, $\varphi(v + w)$ is convex in $v \in H_m$.

Proof. The convexity of $F(t, \cdot)$ implies that $F(t, v + w)$ is convex in $v \in H_m$ for every $w \in H^1_m$, and hence $\int_0^T F(t, v + w)dt$ is convex in $v \in H_m$ for every $w \in H^1_m$. Notice that

$$
-\frac{1}{2} \int_0^T |\dot{v}(t)|^2 dt + \frac{m^2 \omega^2}{2} \int_0^T |v(t)|^2 dt \geq 0, \quad \forall v \in H_m.
$$

Lemma 2.2 implies that

$$
-\frac{1}{2} \int_0^T |\dot{v}(t)|^2 dt + \frac{m^2 \omega^2}{2} \int_0^T |v(t)|^2 dt
$$

is convex in $v \in H_m$. Hence, for each $w \in H^1_m$,

$$
\varphi(v + w) = -\frac{1}{2} \int_0^T |\dot{v}(t) + \dot{w}(t)|^2 dt + \frac{m^2 \omega^2}{2} \int_0^T |v(t) + w(t)|^2 dt + \int_0^T F(t, v(t) + w(t)) dt
$$

$$
= \left( -\frac{1}{2} \int_0^T |\dot{v}(t)|^2 dt + \frac{m^2 \omega^2}{2} \int_0^T |v(t)|^2 dt \right) + \int_0^T F(t, v(t) + w(t)) dt
$$

$$
- \frac{1}{2} \int_0^T |\dot{w}(t)|^2 dt + \frac{m^2 \omega^2}{2} \int_0^T |w(t)|^2 dt
$$

is convex in $v \in H_m$. This completes the proof. \qed
Lemma 2.4. Suppose that assumptions (A) and (A3) hold and $F(t, x)$ is convex in $x$ for a.e. $t \in [0, T]$. Then for every $M > 0$, \[ \varphi(v + w) \to +\infty \quad \text{as} \quad \|w\| \to \infty, \quad v \in H_m, \] uniformly for $w \in H^1_m$ with $\|w\| \leq M$.

Proof. We prove this assertion by contradiction. Suppose that the statement of the theorem does not hold, then there exist $M > 0$, $c_1 > 0$ and two sequences $(v_n) \subset H_m$ and $(w_n) \subset H^1_m$ with $\|v_n\| \to \infty (n \to \infty)$ and $\|w_n\| \leq M$ for all $n$ such that
\[ \varphi(v_n + w_n) \leq c_1, \quad \forall n \in N. \]

For $v \in H_m$, write
\[ v = u + a \cos mw + b \sin mw, \]
where $a, b \in \mathbb{R}^N$ and
\[ u \in H^{-1} \triangleq \left\{ \sum_{j=0}^{m-1} a_j \cos j \omega t + b_j \sin j \omega t \mid a_j, b_j \in \mathbb{R}^N, \ j = 0, 1, \ldots, m - 1 \right\}. \]

Define the function $\bar{F} : \mathbb{R}^{2N} \to \mathbb{R}$ by
\[ \bar{F}(a, b) = \int_0^T F(t, a \cos mw + b \sin mw) \, dt. \]

It follows from the continuous differentiability and the convexity of $F(t, \cdot)$ that $\bar{F}$ is continuously differentiable and convex on $\mathbb{R}^{2N}$, which yields that $\bar{F}$ is weakly lower semi-continuous on $\mathbb{R}^{2N}$. Using (A3), one has
\[ \bar{F}(a, b) = \int_0^T F(t, a \cos mw + b \sin mw) \, dt \to +\infty \quad \text{as} \quad |a| + |b| \to \infty. \]

Hence, by the least action principle [5, Theorem 1.1], $\bar{F}$ has a minimum at some $(a_0, b_0) \in \mathbb{R}^{2N}$ for which
\[ \int_0^T (\nabla F(t, a_0 \cos mw + b_0 \sin mw), \cos mw) \, dt \]
\[ = \int_0^T (\nabla F(t, a_0 \cos mw + b_0 \sin mw), \sin mw) \, dt \]
\[ = 0. \]

By the convexity of $F(t, \cdot)$, we obtain
\[ F(t, v + w) \geq F(t, a_0 \cos mw + b_0 \sin mw) \]
\[ + (\nabla F(t, a_0 \cos mw + b_0 \sin mw), u + w + (a - a_0) \cos mw + (b - b_0) \sin mw), \]
and then, using assumption (A), (2.2) and (2.1),
\[ \int_0^T F(t, v + w) \, dt \geq \int_0^T F(t, a_0 \cos mw + b_0 \sin mw) \, dt \]
\[ + \int_0^T (\nabla F(t, a_0 \cos mw + b_0 \sin mw), u + w) \, dt \]
\[ \geq - \max_{s \in [0, |u| + |b|]} a(s) \int_0^T b(t) \, dt - \max_{s \in [0, |a| + |b|]} a(s) \int_0^T b(t)|u + w| \, dt \]
\[ \geq - \max_{s \in [0, |a| + |b|]} a(s) \int_0^T b(t)dt(1 + \|u\|_{\infty} + \|w\|_{\infty}) \]
\[ \geq - c_2(1 + \|u\|_{\infty}) \]
for all \( w \in H^\perp_m \) with \( \|w\| \leq M \), where \( c_2 = \max_{s \in [0,|a_n|+|b_n|]} a(s) \int_0^T b(t) dt (1 + CM) \). Rewrite \( v_n = u_n + a_n \cos mw + b_n \sin mw \), where \( a_n, b_n \in R^N \) and \( u_n \in H_{m-1} \). Then one has

\[
c_1 \geq \varphi(v_n + w_n) = \frac{1}{2} \int_0^T |u_n|^2 dt + \frac{m^2 \omega^2}{2} \int_0^T |u_n|^2 dt + \frac{1}{2} \int_0^T |\dot{u}_n|^2 dt + \varphi(v_n + w_n) + \frac{m^2 \omega^2}{2} \int_0^T |w_n|^2 dt + \int_0^T F(t, v_n + w_n) dt \geq \frac{1}{2} (m^2 - (m-1)^2) \omega^2 \int_0^T |u_n|^2 dt - \frac{M^2}{2} - c_2 (1 + \|u_n\|_{\infty}) \]

for all \( n \), which implies that \( (u_n) \) is bounded by the equivalence of the norms on the finite-dimensional space \( H_{m-1} \). Combining this with assumption (A), the convexity of \( F(t, \cdot) \) and (2.1), we obtain

\[
c_1 \geq \varphi(v_n + w_n) \geq - c_3 + \int_0^T F(t, v_n + w_n) dt \geq - c_3 + 2 \int_0^T F \left( t, \frac{1}{2} (a_n \cos mw + b_n \sin mw) \right) dt - \int_0^T F(t, -u_n - w_n) dt \geq - c_3 + 2 \int_0^T F \left( t, \frac{1}{2} (a_n \cos mw + b_n \sin mw) \right) dt \geq - \max_{s \in [0,|a_n+w_n|]} a(s) \int_0^T b(t) dt, \]

which yields that the sequences \( (a_n) \) and \( (b_n) \) are also bounded. This contradicts the fact that \( \|v_n\| \to \infty \) as \( n \to \infty \). Therefore the conclusion holds.

Now we are in the position to prove our theorems.

**Proof of Theorem 1.1.** According to Proposition 2.1, it remains to show that

\[
\varphi(w) \to -\infty \quad \text{as} \quad \|w\| \to \infty, \quad w \in H^\perp_m. \tag{2.2}
\]

We follow an argument in [13]. Arguing indirectly, assume that there exists a sequence \( (u_n) \subset H^\perp_m \) satisfying \( \|u_n\| \to \infty \) and

\[
\varphi(u_n) \geq c_4, \quad \forall n \in N \tag{2.3}
\]

for some \( c_4 \in R \). Write \( u_n = a_n \|u_n\| \cos (m+1)\omega t + b_n \|u_n\| \sin (m+1)\omega t + w_n \), where \( a_n, b_n \in R^N \) and \( w_n \in H^\perp_{m+1} \). Then we have, using (1.3),

\[
c_4 \leq \varphi(u_n) \leq \frac{1}{2} \int_0^T |\ddot{u}_n|^2 dt + \frac{m^2 \omega^2}{2} \int_0^T |u_n|^2 dt + \frac{1}{2} \int_0^T |w_n|^2 dt + \int_0^T \gamma(t) dt \leq \frac{1}{2} \left( 1 - \frac{m^2}{(m+2)^2} \right) \int_0^T |\ddot{w}_n|^2 dt + \int_0^T \gamma(t) dt \leq \frac{2m+3}{2(m+2)^2} \int_0^T |w_n|^2 dt + \int_0^T \gamma(t) dt,
\]
which implies that \((w_n)\) is bounded. Taking \(v_n = u_n/\|u_n\|\), then \(\|v_n\| = 1\), and hence the sequences \(\{a_n\}, \{b_n\}\) are bounded. Up to a subsequence, we can assume that

\[ a_n \to a \quad \text{and} \quad b_n \to b \quad \text{as} \quad n \to \infty \]

for some \(a, b \in \mathbb{R}^N\). By the boundedness of \((w_n)\), one has \(w_n/\|u_n\| \to 0\) as \(n \to \infty\). Hence,

\[ v_n \to a \cos(m+1)\omega t + b \sin(m+1)\omega t \quad \text{in} \quad H^1_1, \]

and \(|a| + |b| \neq 0\), which yields that \(v_n(t) \to a \cos(m+1)\omega t + b \sin(m+1)\omega t\) uniformly for a.e. \(t \in [0, T]\) by (2.1). Hence \(|u_n(t)| \to \infty\) as \(n \to \infty\) for a.e. \(t \in [0, T]\), because \(a \cos(m+1)\omega t + b \sin(m+1)\omega t\) only has finite zeros.

Now set

\[ E = \left\{ t \in [0, T] \mid F(t, x) - \frac{(2m+1)}{2} \omega^2 |x|^2 \to -\infty \quad \text{as} \quad |x| \to \infty \right\}. \]

It follows from Fatou’s lemma (see [20]) that

\[
\limsup_{n \to \infty} \varphi(u_n) \leq \limsup_{n \to \infty} \int_0^T \left[ \left( -\frac{(m+1)\omega^2}{2} + \frac{m^2\omega^2}{2} \right) |u_n|^2 + F(t, u_n) \right] dt
\]

\[
= \limsup_{n \to \infty} \int_0^T \left( F(t, u_n) - \frac{(2m+1)\omega^2}{2} |u_n|^2 \right) dt
\]

\[
\leq \limsup_{n \to \infty} \int_E \left( F(t, u_n) - \frac{(2m+1)\omega^2}{2} |u_n|^2 \right) dt + \int_0^T \gamma(t) dt
\]

\[
= -\infty,
\]

a contradiction with (2.3).

A combination of (2.2), Lemmas 2.3, 2.4 and Proposition 2.1 shows that \(\varphi\) has at least a critical point. Consequently, problem (1.1) possesses at least one solution in \(H^1_1\) and the proof is completed.

**Proof of Theorem 1.7.** First, we claim that there exists a constant \(a_0 < \frac{2m+1}{(m+1)^2}\) such that

\[
\int_0^T a(t)|u|^2 \, dt \leq a_0 \int_0^T |\dot{u}|^2 \, dt, \quad \forall u \in H^1_m.
\]

(2.4)

The proof is similar to the first part of [13, Proof of Theorem 3.2], for the convenience of the readers we sketch it here briefly. Arguing indirectly, we assume that there exists a sequence \((u_n) \subset H^1_m\) such that

\[
\int_0^T a(t)|u_n|^2 \, dt > \left( \frac{2m+1}{(m+1)^2} - \frac{1}{n} \right) \int_0^T |\dot{u}_n|^2 \, dt, \quad \forall n \in N,
\]

(2.5)

which implies that \(u_n \neq 0\) for all \(n\). By the homogeneity of the above inequality, we may assume that \(\int_0^T |\dot{u}_n|^2 \, dt = 1\) and

\[
\int_0^T a(t)|u_n|^2 \, dt > \frac{2m+1}{(m+1)^2} - \frac{1}{n}, \quad \forall n \in N.
\]

(2.6)

It follows from the weak compactness of the unit ball of \(H^1_m\) that there exists a subsequence, still denoted by \((u_n)\), such that \(u_n \rightharpoonup u\) in \(H^1_m\), \(u_n \to u\) in \(C(0, T; \mathbb{R}^N)\). This, jointly with (2.6), shows that

\[
\int_0^T a(t)|u|^2 \, dt \geq \frac{2m+1}{(m+1)^2}.
\]
Hence
\[
\frac{2m+1}{(m+1)^2} \geq \frac{2m+1}{(m+1)^2} \int_0^T |\dot{u}|^2 \, dt \geq (2m+1)\omega^2 \int_0^T |u|^2 \, dt \geq \int_0^T \alpha(t) |u|^2 \, dt \geq \frac{2m+1}{(m+1)^2},
\]
and then
\[
1 = \int_0^T |\dot{u}|^2 \, dt = (m+1)\omega^2 \int_0^T |u|^2 \, dt
\]
and
\[
\int_0^T \left( (2m+1)\omega^2 - \alpha(t) \right) |u|^2 \, dt = 0,
\]
which implies that \( u = a \cos(m+1)\omega t + b \sin(m+1)\omega t, \) \( a, b \in \mathbb{R}^N, \) \( u \neq 0 \) and \( u = 0 \) on a positive measure subset. This contradicts the fact that \( u = a \cos(m+1)\omega t + b \sin(m+1)\omega t \) only has finite zeros if \( u \neq 0 \).

It follows from assumptions \((A)\) and \((A_7)\) that, for \( \varepsilon \in (0, \frac{2m+1}{(m+1)^2} - a_0) \), there exists \( M_\varepsilon > 0 \) such that
\[
F(t, x) \leq \frac{1}{2} \left( \alpha(t) + \varepsilon(m+1)^2\omega^2 \right) |x|^2 + \max_{s \in [0, M_\varepsilon]} a(s)b(t)
\]
for all \( x \in \mathbb{R}^N \) and a.e. \( t \in [0, T] \). Combining this with \((2.4)\), we obtain
\[
\varphi(w) \leq -\frac{1}{2} \int_0^T |\ddot{w}|^2 \, dt + \frac{m^2\omega^2}{2} \int_0^T |w|^2 \, dt + \frac{1}{2} \int_0^T \left( \alpha(t) + \varepsilon(m+1)^2\omega^2 \right) w^2 \, dt + c_5
\]
\[
\leq -\frac{1}{2} \left( 1 - \frac{m^2}{(m+1)^2} - a_0 - \varepsilon \right) \int_0^T |\ddot{w}|^2 \, dt + c_5
\]
\[
\leq -\frac{1}{2} \left( \frac{2m+1}{(m+1)^2} - a_0 - \varepsilon \right) \int_0^T |\ddot{w}|^2 \, dt + c_5
\]
for \( w \in H^\perp_m \), where \( c_5 = \max_{s \in [0, M_\varepsilon]} a(s) \int_0^T b(t) \, dt \), which implies that
\[
\varphi(w) \to -\infty \quad \text{as } \|w\| \to \infty \quad \text{on } H^\perp_m,
\]
by the equivalence of the \( L^2 \)-norm of \( \dot{w} \) and the \( H^\perp_m \)-norm on \( H^\perp_m \). This, jointly with Lemmas 2.3, 2.4 and Proposition 2.1, yields that \( \varphi \) possesses at least one critical point, and hence problem \((1.1)\) has at least one solution in \( H^\perp_7 \). This concludes the proof. \( \square \)

**Proof of Theorem 1.10.** By \((A_8)\) and Sobolev’s inequality, we have
\[
\varphi(w) \leq -\frac{1}{2} \left( 1 - \frac{m^2}{(m+1)^2} \right) \int_0^T |\ddot{w}|^2 \, dt + \frac{1}{2} \int_0^T \alpha(t) |w|^2 \, dt + \int_0^T \gamma(t) \, dt
\]
\[
\leq -\frac{2m+1}{2(m+1)^2} \int_0^T |\ddot{w}|^2 \, dt + \frac{1}{2} \int_0^T \alpha(t) \, dt \cdot \|w\|_\infty^2 + \int_0^T \gamma(t) \, dt
\]
\[
\leq -\frac{2m+1}{2(m+1)^2} \int_0^T |\ddot{w}|^2 \, dt + \frac{1}{2} \int_0^T \alpha(t) \, dt \cdot \frac{T}{12} \int_0^T |\dot{w}|^2 \, dt + \int_0^T \gamma(t) \, dt
\]
\[
\leq -\frac{1}{2} \left( \frac{2m+1}{(m+1)^2} - \frac{T}{12} \int_0^T \alpha(t) \, dt \right) \int_0^T |\ddot{w}|^2 \, dt + \int_0^T \gamma(t) \, dt
\]
for all \( w \in H^\perp_m \). Noting \( \int_0^T \alpha(t) \, dt < \frac{12(2m+1)}{T(m+1)^2} \), the last inequality implies that
\[
\varphi(w) \to -\infty \quad \text{as } \|w\| \to \infty, \ w \in H^\perp_m.
\]
Consequently, Theorem 1.10 follows from Lemmas 2.3, 2.4 and Proposition 2.1. This completes the proof. \( \square \)
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References


