Oscillatory solutions of nonlinear fourth order differential equations with a middle term

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Abstract. We study the oscillation of a fourth order nonlinear differential equation with a middle term. Using a certain energy function, we describe the properties of oscillatory solutions. The paper extends oscillation criteria stated for equations with the operator $x^{(4)} + x^{''}$ and completes the results stated for super-linear and sub-linear case. Oscillation results are new also for the linear equation.

Keywords: fourth order nonlinear differential equation, oscillatory solution, oscillation.

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1 Introduction

Consider the fourth order nonlinear differential equation

$$x^{(4)}(t) + q(t)x''(t) + r(t)f(x(t)) = 0 \quad (1.1)$$

under the following assumptions:

(i) $q \in C(\mathbb{R}_+), q(t) > 0$ for large $t$, $r \in C(\mathbb{R}_+), r(t) > 0$ for large $t$ and $\mathbb{R}_+ = [0, \infty)$;

(ii) $f \in C(\mathbb{R})$ satisfies $f(u)u > 0$ for $u \neq 0$ and either

$$|f(u)| \geq |u| \quad \text{for } u \in \mathbb{R} \quad (1.2)$$

or there exists $0 < \lambda < 1$ such that

$$|f(u)| \geq |u|^\lambda \quad \text{for } u \in \mathbb{R}, \quad (1.3)$$

where $\mathbb{R} = (-\infty, \infty)$.

A special case of (1.1) is the equation

$$x^{(4)}(t) + q(t)x''(t) + r(t)|x(t)|^\lambda \text{sgn} x(t) = 0, \quad (1.4)$$
where \( \lambda \leq 1 \).

By a solution of (1.1) we mean a function \( x \in C^4[0, \infty) \), which satisfies (1.1) on \([0, \infty)\). A solution is said to be nonoscillatory if \( x(t) \neq 0 \) for large \( t \), otherwise is said to be oscillatory. A solution is said to be proper if it is nontrivial in any neighbourhood of infinity. Equation (1.1) is oscillatory if all its solutions are oscillatory.

The oscillatory behavior of fourth order differential equations enjoys a great deal of interest, see [1–4, 6, 10] and references contained therein. The important role in the investigation of (1.1) is played by the fact whether the associated second order linear equation

\[
h''(t) + q(t)h(t) = 0
\]

is oscillatory or nonoscillatory. For example, if (1.5) is nonoscillatory, then (1.4) can be written as a two-term equation, see [3], or as a four-dimensional Emden–Fowler differential system, see [10], and oscillation criteria for (1.4) can be obtained by this approach.

If (1.5) is oscillatory and \( \lambda \geq 1 \), then (1.1) and (1.4) have been investigated in [3]. Here conditions determining that all nonoscillatory solutions are vanishing at infinity have been given, and the oscillation theorem for (1.4) has been proved in the case \( \lambda > 1 \).

The natural problem is to study oscillation of (1.1) and (1.4) when \( \lambda \leq 1 \). If \( \lambda = 1 \) and \( q(t) \equiv 1 \), then (1.4) is the linear equation

\[
x^{(4)}(t) + x''(t) + r(t)x(t) = 0
\]

and the following well-known result holds, see, e.g., [8, Corollary 1.3].

**Theorem A.** Let (1.2) hold. If either

\[
\lim \inf_{t \to \infty} t \int_{t}^{\infty} r(s) \, ds > \frac{1}{4} \quad \text{or} \quad \lim \sup_{t \to \infty} t \int_{t}^{\infty} r(s) \, ds > 1,
\]

then (1.6) is oscillatory.

If \( \lambda < 1 \) and (1.5) is oscillatory, the following oscillation criterion for (1.4) has been proved in [4, Theorem 2].

**Theorem B.** Let \( \lambda < 1 \) and (1.5) be oscillatory. Assume that

\[
q(t) \geq q_0 > 0, \quad q'(t) \leq 0, \quad q''(t) \geq 0 \quad \text{for large } t,
\]

and

\[
\lim_{t \to \infty} t^{2(\lambda-1)} r(t) = \infty.
\]

Then (1.4) is oscillatory.

Motivated by these results, we study oscillation of (1.1), and properties of zeros of oscillatory solutions. We allow that the function \( q \) can tend to zero or to infinity as \( t \to \infty \) and both cases that the corresponding second order equation (1.5) is nonoscillatory/oscillatory are considered. Our approach is based on a suitable energy function for (1.1) and a comparison method for (1.1) and (1.4). Our results are applicable to the equation

\[
x^{(4)}(t) + kx''(t) + r(t)f(x(t)) = 0, \quad (k > 0),
\]

studied in [7]. If \( f \) is a locally Lipschitz function, then this equation is known as the Swift–Hohenberg equation.
2 Classification of solutions

We start with the possible types of nonoscillatory solutions of (1.1). Due to the sign-condition on \( f \), we can focus on eventually positive solutions of (1.1).

To this aim, a function \( g \), defined in a neighborhood of infinity, is said to change its sign, if there exists a sequence \( \{t_k\} \to \infty \) such that \( g(t_k)g(t_{k+1}) < 0 \).

Lemma 2.1. Every eventually positive solution \( x \) of (1.1) is one of the following type:

- Type (a): \( x(t) > 0, x'(t) > 0, x''(t) \leq 0 \) for large \( t \),
- Type (b): \( x(t) > 0, x'(t) > 0, x''(t) > 0, x'''(t) > 0 \) for large \( t \),
- Type (c): \( x'' \) changes sign.

Moreover, if (1.5) is nonoscillatory, then \( x \) is of Type (a) or (b), and if (1.5) is oscillatory, then \( x \) is of Type (a) or (c).

Proof. From Theorem 2 and Theorem 2’ in [3] it follows that if (1.5) is nonoscillatory, then every eventually positive solution \( x \) satisfies \( x'(t) > 0 \) and \( x'' \) is of one sign for large \( t \), whereby if (1.5) is oscillatory, then every eventually positive solution \( x \) satisfies either \( x''(t) \leq 0 \) or \( x'' \) changes sign.  

Assume that \( x(t) > 0 \) and \( x''(t) \leq 0 \) for large \( t \). If \( x'(t) \leq 0 \), then \( x \) is nonincreasing and concave, which is a contradiction with the positivity of \( x \).

Assume that \( x(t) > 0, x'(t) > 0 \) and \( x''(t) > 0 \) for large \( t \). Then \( x^{(4)}(t) < 0 \) and so \( x''' \) is of one sign for large \( t \). If \( x'''(t) \leq 0 \), then \( x'' \) is positive nonincreasing and concave function, which is a contradiction with the positivity of \( x'' \).

Finally, if (1.5) is oscillatory, then the last conclusion follows from Theorem 2, part (b) in [3].

In the sequel, we consider equation (1.4) with \( \lambda \leq 1 \).

Lemma 2.2. Let (1.5) be nonoscillatory. If there exists \( \lambda \leq 1 \) such that

\[
\int_0^\infty t^{2\lambda}r(t)\,dt = \infty, \tag{2.1}
\]

then (1.4) has no solution of Type (b).

Proof. Let (1.5) be nonoscillatory and (2.1) hold for \( \lambda \leq 1 \). Assume that (1.4) has a solution \( x \) of Type (b), i.e., there exists \( t_0 \geq 0 \) such that \( x(t) > 0, x'(t) > 0, x''(t) > 0 \) and \( x'''(t) > 0 \) for \( t \geq t_0 \). Then from (1.4), \( x^{(4)}(t) < 0 \) for \( t \geq t_0 \). Thus there exists \( t_1 \geq t_0 \) such that \( x''' \) is positive and decreasing for \( t \geq t_1 \) and there exist \( C > 0 \) and \( t_2 \geq t_1 \) such that \( x''(t) \geq C \) and \( x(t) \geq Ct^2 \) for \( t \geq t_2 \). From here, integrating (1.4) from \( t_2 \) to \( t \), we get

\[
x'''(t_2) - x'''(t) \geq -\int_{t_2}^t x^{(4)}(s)\,ds = \int_{t_2}^t \left( q(s)x''(s) + r(s)x'(s) \right)\,ds \\
\geq C^\lambda \int_{t_2}^t r(s)s^{2\lambda} \,ds.
\]

Letting \( t \to \infty \), we get a contradiction to the boundedness of \( x''' \).
3 Oscillation theorems

In this section we state two oscillation theorems for (1.1).

**Theorem 3.1.** Let (1.2) hold. Assume that
\[
\lim_{t \to \infty} r(t) = \infty, \quad (3.1)
\]
\[
q^2(t) \leq 4r(t) \quad \text{for large } t, \quad (3.2)
\]
and, in addition if (1.5) is nonoscillatory, that
\[
\int_0^\infty t^2 r(t) \, dt = \infty. \quad (3.3)
\]
Then (1.1) is oscillatory.

To prove this result, we introduce the following energy function used for (1.4) in [4].

**Definition 3.2.** Let \(x\) be a solution (possibly oscillatory or nonoscillatory) of (1.1). Define the function \(F\) as
\[
F(t) = -x'''(t) x(t) + x'(t) x''(t), \quad t \in \mathbb{R}_+. \quad (3.4)
\]
If \(x(t) \neq 0\), then by (1.2) and (3.2)
\[
F'(t) = \left( \sqrt{r(t)} \sqrt{f(x(t))} x(t) \text{sgn } x(t) + \frac{q(t)}{2 \sqrt{r(t)}} x''(t) \sqrt{x(t)/f(x(t))} \right)^2
\]
\[+ (x''(t))^2 \left(1 - \frac{q^2(t)}{4r(t)} \frac{x(t)}{f(x(t))} \right) \geq 0.
\]
If \(x(t) \neq 0\) at some \(\bar{t} > 0\), then \(F'(t) \geq 0\) in a neighbourhood of \(\bar{t}\). By (3.4), \(F'\) is continuous for \(t > 0\) and thus \(F'(t) \geq 0\) for large \(t\) and we get the monotonicity of \(F\) for large \(t\).

Let \(x(t) > 0\) for \(t \geq T_1 \geq 0\) and by contradiction, suppose that \(x\) is of Type (c), i.e., \(x''\) changes sign. Let \(\{t_k\}_{k=1}^\infty\) and \(\{\tau_k\}_{k=1}^\infty\), \(T_1 \leq t_k < \tau_k < t_{k+1}, k = 1, 2, \ldots\) be sequences of zeros of \(x''\) tending to \(\infty\) such that
\[
x''(t) > 0 \quad \text{on } (t_k, \tau_k), \quad k = 1, 2, \ldots \quad (3.5)
\]
Then (1.4) implies \(x^{(4)}(t) < 0\) on \([t_k, \tau_k]\) and, hence, \(x'''\) is decreasing. According to (3.5) and the fact that \(x''(t_k) = x''(\tau_k) = 0\), numbers \(\xi_k \in (t_k, \tau_k)\) exist such that \(x'''(\xi_k) = 0, k = 1, 2, \ldots\). From this and from the fact that \(x''\) is decreasing, we have
\[
x'''(t_k) > 0 \quad \text{and } x'''(\tau_k) < 0, \quad k = 1, 2, \ldots
\]
Hence,
\[
F(t_k) = -x'''(t_k) x(t_k) < 0, \quad F(\tau_k) = -x'''(\tau_k) x(\tau_k) > 0, \quad k = 1, 2, \ldots
\]
In view of the monotonicity of \(F\), we get a contradiction. Thus \(x''\) does not change sign and this proves the lemma. \qed
Proof of Theorem 3.1. Step 1. We prove first the statement for the linear equation

\[ x^{(4)}(t) + q(t)x''(t) + r(t)x(t) = 0. \]  

(3.6)

Let \( T > 0 \) be such that (3.2) holds for \( t \geq T \). Without loss of generality, consider a solution \( x \) of (3.6) such that \( x(t) > 0 \) for \( t \geq T \). Using Lemma 3.3, the function \( F \) is nondecreasing for large \( t \), and in view of Lemmas 2.1, 2.2 and 3.3, \( x \) is of Type (a), i.e., \( x'(t) > 0, x''(t) \leq 0 \). Then either \( x''' \) oscillates or \( x'''(t) > 0 \) for large \( t \); observe that the case \( x'''(t) < 0 \) for large \( t \) is impossible as \( x' \) would change sign. Consider a sequence \( \{t_k\} \) such that \( t_1 \geq T, \lim_{t \to \infty} t_k = \infty \) and \( x'''(t_k) = 0 \) in case \( x'' \) oscillates; otherwise it can be arbitrary. In both cases we have \( F(t_k) < 0 \) for \( k = 1, 2, \ldots \). According to Lemma 3.3, \( F \) is nondecreasing, so \( F(t) < 0 \) for \( t \geq t_1 \). Define the function

\[ Z(t) = -x''(t)x(t) + (x'(t))^2 \]

for \( t \geq t_1 \geq T \). Then \( Z'(t) = F(t) < 0 \) and taking into account that \( x''(t) \leq 0 \), we have \( Z(t) \geq 0 \). Thus,

\[ 0 \leq -x''(t)x(t) \leq Z(t_1), \quad x(t) \geq K, \]

for \( t \geq t_1 \) and \( K = x(t_1) \). Hence, there exists a constant \( M > 0 \) such that \( |x''(t)| \leq M \) for \( t \geq t_1 \). From this and (3.6),

\[ x^{(4)}(t) = -q(t)x''(t) - r(t)x(t) \leq Mq(t) - Kr(t) \]

for \( t \geq t_1 \) and (3.1) implies the existence of \( \tau \geq t_1 \) such that

\[ x^{(4)}(t) \leq -Cr(t) < 0 \quad \text{for } t \geq \tau \quad \text{and} \quad C = K^\lambda/2. \]  

(3.7)

Since \( x''' \) is decreasing for \( t \geq \tau \), there exists \( \tau_1 \geq \tau \) such that \( x'''(t) > 0 \) for \( t \geq \tau_1 \). From this and the fact that \( x'(t) > 0 \) and \( x''(t) \leq 0 \), we have \( \lim_{t \to \infty} x^{(j)}(t) = 0 \) for \( j = 2, 3 \). Therefore,

\[ |x^{(j)}(t)| = \int_t^\infty |x^{(j+1)}(s)| \, ds, \quad j = 2, 3, \]

and using (3.7), for \( t \geq \tau_1 \) we have

\[ x'''(t) = \int_t^\infty |x^{(4)}(s)| \, ds \geq C \int_t^\infty r(s) \, ds, \]

so \( r \in L^1(\mathbb{R}_+) \). Proceeding in the same way, \( |x''(t)| = \int_t^\infty |x'''(s)| \, ds \) and thus

\[ x'(t) - x'({\tau_1}) = \int_{\tau_1}^t |x''(s)| \, ds \geq C \int_{\tau_1}^t s^2 r(s) \, ds \]

Since \( x' \) is bounded, letting \( t \to \infty \) we get a contradiction to (3.3). Thus, a solution of Type (a) does not exist and equation (3.6) is oscillatory.

Step 2. Consider nonlinear equation (1.1) and assume, by contradiction, that (1.1) has a solution \( x(t) > 0 \) for \( t \geq T \). Then \( y = x \) is the solution of the linear equation

\[ y^{(4)} + q(t)y'' + R(t)y = 0, \]  

(3.8)

where

\[ R(t) = \frac{r(t)f(x(t))}{x(t)}. \]
According to (1.2), we have \( R(t) \geq r(t) \) for \( t \geq T \). Thus, using (3.1), (3.2) and (3.3), we get
\[
4R(t) \geq q^2(t), \quad \lim_{t \to \infty} \frac{R(t)}{q(t)} = \infty, \quad \int_0^\infty t^2 R(t) \, dt = \infty.
\]

According to the first part of the proof, equation (3.8) is oscillatory. This is a contradiction to the fact that \( x \) is a nonoscillatory solution.

Our next result extends Theorem A to (1.1).

**Theorem 3.4.** Let (1.3) hold. If (1.7) and (1.8) hold, then (1.1) is oscillatory.

**Proof.** Assume, by contradiction, that (1.1) has a solution \( x(t) > 0 \) for \( t \geq T \). Since (1.7) holds, (1.5) is oscillatory, and by Lemma 2.1, \( x \) is of Type (a) or (c). Moreover, \( y = x \) is a solution of the equation
\[
y^{(4)} + q(t)y'' + R(t)|y(t)|^\lambda \sgn y(t) = 0
\]
for \( t \geq T \), where
\[
R(t) = \frac{r(t)f(x(t))}{x^\lambda(t)} \geq r(t).
\]
From here and (1.8) we have
\[
\lim_{t \to \infty} \frac{t^2(\lambda - 1) R(t)}{t^2} = \infty.
\]
Applying Theorem A to (3.9), the oscillation of (3.9) follows. This is a contradiction to the fact that \( x \) is a nonoscillatory solution.

The following examples illustrate our results.

**Example 3.5.** Consider the equation
\[
x^{(4)}(t) + \frac{c}{t^2}x''(t) + \frac{1}{t^2-\varepsilon}f(x(t)) = 0 \quad (t \geq 1),
\]
where \( c > 0, \varepsilon > 0 \), and
\[
f(u) = \begin{cases} 
\frac{4}{\pi} \arctan u & \text{for } |u| \leq 1, \\
u & \text{for } |u| > 1.
\end{cases}
\]
By Theorem 3.1, (3.10) is oscillatory.

**Example 3.6.** Consider the equation
\[
x^{(4)}(t) + \left(1 + \frac{1}{t}\right)x''(t) + t \ln(t + 1) f(x(t)) = 0, \quad (t \geq 1),
\]
where
\[
f(u) = \begin{cases} 
\sqrt{u} & \text{for } |u| \leq 1, \\
u & \text{for } |u| > 1.
\end{cases}
\]
By Theorem 3.4, (3.11) is oscillatory.
4 Existence and zeros of oscillatory solutions

We start with the existence of oscillatory solutions for (1.4).

**Proposition 4.1.** Assume (1.2) and
\[
\limsup_{u \to \infty} \frac{f(u)}{u} < \infty.
\]

If (1.5) is oscillatory and
\[
q^2(t) \leq 4r(t) \quad \text{for } t \in \mathbb{R}_+,
\]
then (1.1) has proper oscillatory solutions.

**Proof.** According to [8, Theorem 11.5], all solutions of (1.1) are defined on $\mathbb{R}_+$. By Lemmas 2.1 and 3.3, we have that any solution of (1.4) is either proper oscillatory, or trivial in a neighbourhood of infinity, or of Type (a).

Consider the function $F$ from Definition 3.2. If $x$ is of type (a), then $F(t) < 0$ for large $t$, and by Lemma 3.3, $F(t) < 0$ for $t \in \mathbb{R}_+$. If $x(t) \equiv 0$ for large $t$, then $F(t) \equiv 0$ for large $t$. Hence, any solution of (1.1) with the initial condition $F(0) > 0$ is proper oscillatory.

In the sequel, we describe zeros of proper oscillatory solutions $x$ of (1.1) and of their derivatives. As a motivation, consider equation (1.1) with $q(t) \equiv 0$. Then any oscillatory solution has the following properties in the neighbourhood of infinity: any zero of $x$ and $x'$ is simple (i.e. not double or triple), and zeros of $x$ and $x'$ separate each other, i.e., between two zeros of $x$ [x'] there exists exactly one zero of $x'$ [x]. Here we prove that the same properties remain to hold for (1.1).

**Theorem 4.2.** Assume (1.2) and (3.2). Then for any proper oscillatory solution $x$ of (1.1) there exists $T > 0$ such that all zeros of $x$ and $x'$ are simple, and between two zeros of $x$ [x'] there exists exactly one zero of $x'$ [x] on $[T, \infty)$.

**Proof.** Let $x$ be a proper solution of (1.1) such that $x(t_k) = 0$, where $\{t_k\}_{k=1}^\infty$ tends to infinity. By Lemma 3.3, the function $F$ is nondecreasing for $t \geq T$.

If $F(t) \equiv 0$ for large $t$, then $Z(t) \equiv 0$ for $t \geq T_1 > T$ and from the definition of $Z$ we have $x''(t)x(t) \geq 0$ and
\[
0 \equiv F(t) = r(t)x(t)f(x(t)) + q(t)x''(t)x(t) + (x''(t))^2 \geq r(t)x(t)f(x(t)) \geq 0.
\]

Since $r(t) > 0$ and $f(u)u > 0$ for $u \neq 0$, we get $x(t) \equiv 0$ for large $t$, which is a contradiction to the fact that $x$ is proper.

Define the function
\[
Z(t) = -x''(t)x(t) + (x'(t))^2
\]
for $t \geq t_1 \geq T$. Then $Z(t) = F(t)$ and $Z(t_k) \geq 0$. If $F(t) > 0$ ( $F(t) < 0$) for large $t$, then $Z$ is increasing (decreasing) and taking into account that $Z(t_k) \geq 0$, we have
\[
Z(t) > 0 \quad \text{for } t \geq T_1 > T. \quad (4.3)
\]

If $\tau \geq T_1$ is such that $x'(\tau) = 0$, then, from (4.3), $x''(\tau)x(\tau) < 0$, and so $\tau$ is a simple zero of $x$.

If $\tau_1 \geq T_1$ is such that $x(\tau_1) = 0$, then again from (4.3) we have $x'(\tau_1) \neq 0$ and $\tau_1$ is a simple zero of $x$. 
Let $\tau_2, \tau_3$, where $T_1 \leq \tau_2 < \tau_3$ be two successive zeros of $x'$ such that $x'(t) > 0$ on $(\tau_2, \tau_3)$. Then, from (4.3), we have

$$x''(\tau_2)x(\tau_2) < 0 \quad \text{and} \quad x''(\tau_3)x(\tau_3) < 0.$$ 

Since $x''(\tau_2) > 0$ and $x''(\tau_3) < 0$, we get $x(\tau_2) < 0$ and $x(\tau_3) > 0$, and $x$ has a zero on $(\tau_2, \tau_3)$. Since $x$ is increasing on $(\tau_2, \tau_3)$, $x$ has a simple zero. From above we get that between two successive zeros of $x'$ there exists exactly one zero of $x$.

Let $\tau_4, \tau_5$, where $T_1 \leq \tau_4 < \tau_5$ be two successive zeros of $x$ such that $x(t) > 0$ on $(\tau_4, \tau_5)$. According to Rolle’s theorem, $x'$ has a zero $\tau_6$ in $(\tau_4, \tau_5)$. The fact that $\tau_6$ is the only zero of $x'$ in $(\tau_4, \tau_5)$ follows from the fact that between two zeros of $x'$ there exists exactly one zero of $x$.

Remark 4.3. If (4.2) holds, then Theorem 4.2 is valid with $T = 0$, i.e., for all zeros of a proper oscillatory solution. For instance, equations (3.10) with $c = 1$ and (3.11) have by Proposition 4.1 and Theorem 4.2 proper oscillatory solutions $x$ such that zeros of $x$ and $x'$ are simple and separate each other.

Example 4.4. Consider equation (1.9) where $f$ satisfies (1.2) and (4.1), and $r(t) \geq k^2/4$ for $t \in \mathbb{R}_+$. By Proposition 4.1 and Theorem 4.2, (1.9) has proper oscillatory solutions and zeros of $x$ and $x'$ are simple and separate each other.

We conclude this paper with the following open question: *Is it possible to relax the assumptions (1.7) and (1.8) of Theorem 3.4 in the sub-linear case, i.e., $f$ satisfies (1.3)?*

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