Boundedness character of a max-type system of difference equations of second order

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Abstract. The boundedness character of positive solutions of the next max-type system of difference equations

\[ x_{n+1} = \max \left\{ A, \frac{y_n^p}{x_n^p} \right\}, \quad y_{n+1} = \max \left\{ A, \frac{x_n^p}{y_n^p} \right\}, \quad n \in \mathbb{N}_0, \]

with \( \min \{ A, p, q \} > 0 \), is characterized.

Keywords: max-type system of difference equations, positive solutions, bounded solutions, unbounded solutions.

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1 Introduction

Difference equations and systems which do not stem from the differential ones have attracted some attention in last few decades (see, e.g., [1–47]). Some of the systems that are of interest are symmetric or those obtained from symmetric by modifications of their parameters (see, for example, [5,9,13–19,22,23,36,39–44] and the related references therein). Another subarea of interest deals with max-type difference equations and systems (see, for example, [1,7,10–12,17,19,21,28–35,38,40,42,43,45–47] and the related references therein). However, there are only a few papers which belong to both areas (see [17,19,21,40,42,43]). Although majority of the papers in the area treat equations or systems with integer powers of their variables, there are some papers on equations or systems with non-integer powers of their variables (see, for example, [3,4,8,20,27–33,35,47]). Paper [29] is one of the first such papers on max-type difference equations. It studies positive solutions of the difference equation

\[ x_{n+1} = \max \left\{ a, \frac{x_n^p}{x_{n-1}^p} \right\}, \quad n \in \mathbb{N}_0, \]  

(1.1)

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with $\min\{a, p\} > 0$.

Motivated by [29], in [43], S. Stević studied the boundedness character and global attractivity of positive solutions of the following symmetric system of max-type difference equations

$$
x_{n+1} = \max \left\{ a, \frac{y_n^p}{x_{n-1}^q} \right\}, \quad y_{n+1} = \max \left\{ a, \frac{x_n^p}{y_{n-1}^q} \right\}, \quad n \in \mathbb{N}_0, \tag{1.2}
$$

with $\min\{a, p\} > 0$.

For related max-type difference equations see also [28, 30, 33, 35].

Here we continue the line of investigations by studying the boundedness character of positive solutions of the next system of max-type difference equations

$$
x_{n+1} = \max \left\{ A, \frac{y_n^p}{x_{n-1}^q} \right\}, \quad y_{n+1} = \max \left\{ A, \frac{x_n^p}{y_{n-1}^q} \right\}, \quad n \in \mathbb{N}_0, \tag{1.3}
$$

where $\min\{A, p, q\} > 0$.

Two of our results (Theorem 2.2 and Theorem 2.4) are natural extensions of the results on the boundedness character of positive solutions of system (1.2) appearing in [43]. For the other two results (Theorem 2.1 and Theorem 2.3) we need some other methods, different from the ones used in studying system (1.2). Generally speaking, the paper is also a continuation of studying special cases of the next systems of difference equations

$$
x_{n+1} = \max \left\{ A_n, \frac{y_n^p}{x_{n-1}^q} \right\}, \quad y_{n+1} = \max \left\{ A_n, \frac{x_n^p}{y_{n-1}^q} \right\}, \quad n \in \mathbb{N}_0,
$$

where $k, l \in \mathbb{N}$, $\min\{p, q\} > 0$ and $(A_n)_{n \in \mathbb{N}_0}$ is a sequence of positive numbers, as well as special cases of their scalar counterparts

$$
x_{n+1} = A_n + \frac{x_n^p}{x_{n-1}^q}, \quad y_{n+1} = A_n + \frac{x_n^p}{y_{n-1}^q}, \quad n \in \mathbb{N}_0,
$$

where $k, l \in \mathbb{N}$, $\min\{p, q\} > 0$ and $(A_n)_{n \in \mathbb{N}_0}$ is a sequence of positive numbers.

For some results in the area see, for example, [2, 4, 6, 8, 14, 20, 24, 25, 28–30, 33] and the related references therein.

Solution $(x_n, y_n)_{n \geq -1}$ of system (1.3) is bounded if there is an $M \geq 0$ such that

$$
\| (x_n, y_n) \|_2 = \sqrt{x_n^2 + y_n^2} \leq M, \quad n \geq -1. \tag{1.4}
$$

If

$$
\sup_{n \geq -1} \sqrt{x_n^2 + y_n^2} = +\infty
$$

we say that the solution is unbounded.
2 Boundedness character of positive solutions of system (1.3)

In this section we prove the main results of this paper, which give a complete picture for the boundedness character of positive solutions of system (1.3).

**Theorem 2.1.** Assume that \( A > 0, 2\sqrt{q} \leq p < 1 + q \) and \( q \in (0, 1) \). Then all positive solutions of system (1.3) are bounded.

**Proof.** First note that from (1.3) we have

\[
\min \{x_n, y_n\} \geq A, \quad n \in \mathbb{N}.
\]

It is not difficult to see that the conditions \( 2\sqrt{q} \leq p < 1 + q \) and \( q \in (0, 1) \) imply that the polynomial \( P(t) = t^2 - pt + q \) has zeroes \( t_1 \) and \( t_2 \) such that \( 0 < t_2 < t_1 < 1 \).

We have

\[
x_{n+1} = \max \left\{ A, \frac{y_{n+1}}{x_n} \right\},
\]

\[
y_{n+1} = \max \left\{ A, \frac{x_{n+1}}{y_n} \right\}, \quad n \in \mathbb{N},
\]

which along with (2.1) implies that

\[
\frac{x_{n+1}}{y_n} = \max \left\{ A, \frac{y_{n+1}}{x_n} \right\} \leq \max \left\{ A^{1-t_1}, \left( \frac{y_n}{x_n} \right)^{t_2} \right\}
\]

\[
\frac{y_{n+1}}{x_n} = \max \left\{ A, \frac{x_{n+1}}{y_n} \right\} \leq \max \left\{ A^{1-t_1}, \left( \frac{x_n}{y_n} \right)^{t_2} \right\}
\]

for every \( n \in \mathbb{N} \), and consequently

\[
\max \left\{ \frac{x_{n+1}}{y_n}, \frac{y_{n+1}}{x_n} \right\} \leq \max \left\{ A^{1-t_1}, \max \left\{ \frac{x_n}{y_n}, \frac{y_n}{x_n} \right\} \right\}, \quad n \in \mathbb{N}.
\]

Let

\[
u_n = \max \left\{ \frac{x_n}{y_{n-1}}, \frac{y_n}{x_{n-1}} \right\}, \quad n \in \mathbb{N},
\]

and

\[
v_{n+1} = \max \left\{ A^{1-t_1}, v_n^{t_2} \right\}, \quad n \in \mathbb{N},
\]

with

\[
v_1 = u_1.
\]

By induction, we have

\[
u_n \leq v_n, \quad n \in \mathbb{N}.
\]

The fact \( t_2 \in (0, 1) \) implies that the equation \( g(x) = x \), where

\[
g(x) = \max \left\{ A^{1-t_1}, x^{t_2} \right\}, \quad x \in (0, \infty),
\]

implies that...
has a unique fixed point $\bar{x} \geq 1$ and
\[ (g(x) - x)(x - \bar{x}) < 0, \quad x \in \mathbb{R}_+ \setminus \{\bar{x}\}. \quad (2.8) \]

Hence, for $v_1 \in (0, \bar{x}]$, we have
\[ v_n \leq v_{n+1} \leq \bar{x}, \quad n \in \mathbb{N}, \]
and for $v_1 \geq \bar{x}$, we have
\[ v_n \geq v_{n+1} \geq \bar{x}, \quad n \in \mathbb{N}. \]

Hence, $(v_n)_{n \in \mathbb{N}}$ is bounded, which along with (2.6) implies that
\[ u_n \leq L_1, \quad n \in \mathbb{N}_0, \]
for some $L_1 \geq \bar{x} \geq 1$.

Therefore
\[ x_{n+1} \leq L_1 y_n^{l_1}, \quad y_{n+1} \leq L_1 x_n^{l_1}, \quad n \in \mathbb{N}_0. \quad (2.9) \]

From (2.9) we easily get
\[ x_n + y_n \leq 2L_1 (x_{n-1} + y_{n-1})^{l_1}, \quad n \in \mathbb{N}, \quad (2.10) \]
from which it easily follows that
\[ x_n + y_n \leq (2L_1)^{\frac{l_1}{1-p}} (x_0 + y_0)^{l_1^{\frac{1}{p}}} \leq (2L_1)^{\frac{l_1}{1-p}} \max \{1, x_0 + y_0\}. \quad (2.11) \]

From (2.1) and (2.11) the boundedness of sequences $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$, and consequently the theorem follows.

\[ \square \]

**Theorem 2.2.** Assume that $A > 0$, $p > 0$ and $p^2 < 4q$. Then all positive solutions of system (1.3) are bounded.

**Proof.** Let sequence $(p_n)_{n \in \mathbb{N}_0}$ be defined as follows
\[ p_{k+1} = \frac{q}{p - p_k}, \quad p_0 = 0. \quad (2.12) \]

Using (1.3) and (2.12) we have
\[ x_{n+1} = \max \left\{ A, \frac{y_n^p}{x_n^{q/p-1}} \right\} = \max \left\{ A, \left( \frac{y_n}{x_n^{q/p-1}} \right)^p \right\} \]
\[ = \max \left\{ A, \max \left\{ \frac{A}{x_n^{q/p-1}}, \frac{x_n^{p-2}}{y_n^{q/p-1}} \right\}^p \right\} \]
\[ = \max \left\{ A, \max \left\{ \frac{A}{x_n^{q/p-1}}, \left( \frac{x_n^{p-2}}{y_n^{q/p-1}} \right)^p \right\} \right\} \]
\[ = \max \left\{ A, \max \left\{ \frac{A}{x_n^{q/p-1}}, \max \left\{ \frac{A}{y_n^{q/p-1}}, \frac{x_n^{p-2}}{y_n^{q/p-1}} \right\}^p \right\} \right\} \]
\[ = \max \left\{ A, \max \left\{ \frac{A}{x_n^{q/p-1}}, \frac{x_n^{p-2}}{y_n^{q/p-1}} \right\} \right\}. \quad (2.13) \]
\[
\begin{align*}
\text{Max-type system of difference equations} \\
&= \max \left\{ A, \max \left\{ \frac{A}{q/p}, \max \left\{ \frac{A}{q/(p-\frac{q}{p})}, \ldots, \max \left\{ \frac{A}{q/(p-kq)}, \ldots \right\} \right\} \right\} \right\} \\
&= \ldots \\
&= \max \left\{ A, \max \left\{ \frac{A}{x_{n-1}}, \max \left\{ \frac{A}{x_{n-2}}, \ldots, \max \left\{ \frac{A}{x_{n-k}}, \ldots \right\} \right\} \right\} \right\}, \quad (2.14) \\
&= \max \left\{ A, \max \left\{ \frac{A}{x_{n-1}}, \max \left\{ \frac{A}{x_{n-2}}, \ldots, \max \left\{ \frac{A}{x_{n-(k+1)}}, \ldots \right\} \right\} \right\} \right\}, \quad (2.15) \\
&= \max \left\{ A, \max \left\{ \frac{A}{x_{n-1}}, \max \left\{ \frac{A}{x_{n-2}}, \ldots, \max \left\{ \frac{A}{x_{n-(k+2)}}, \ldots \right\} \right\} \right\} \right\}.
\end{align*}
\]

If \( p^2 \leq q \), then by using (2.1) in (2.13), for \( n \geq 3 \), we get
\[
x_{n+1} = \max \left\{ A, \max \left\{ \frac{A}{x_{n-1}}, \frac{x_{n-1}}{x_{n-2}} \right\} \right\} \leq \max \left\{ A, A^{p-q}, \frac{1}{A^{pq}} \right\},
\]
so \((x_n)_{n \geq 1}\) is bounded, in this case.

The monotonicity of \( g(x) = q/(p-x) \) on the interval \((0, p)\) along with the fact \( 0 = p_0 < p_1 = q/p \) implies that \( p_k \) is increasing as far as \( p_k < p \). If \( p_k < p \) for every \( k \in \mathbb{N}_0 \), then there would exist \( \lim_{k \to \infty} p_k := \hat{p} \) and \((\hat{p})^2 - p\hat{p} + q = 0\), but the equation does not have real roots because of the condition \( p^2 < 4q \).

Therefore, there is an \( l_0 \in \mathbb{N} \) such that
\[
p_{l_0-1} < p \quad \text{and} \quad p_{l_0} \geq p.
\]

If \( l_0 = 2k \), then by using (2.1) in (2.14), we get
\[
x_{n+1} = \max \left\{ A, \max \left\{ \frac{A}{x_{n-1}}, \max \left\{ \frac{A}{x_{n-2}}, \ldots, \max \left\{ \frac{A}{x_{n-(k+1)}}, \ldots \right\} \right\} \right\} \right\} \\
\leq \max \left\{ A, \max \left\{ \frac{A}{A^{q/p}}, \max \left\{ \frac{A}{A^{q/(p-\frac{q}{p})}}, \ldots, \max \left\{ \frac{A}{A^{q/(p-kq)}}, \ldots \right\} \right\} \right\} \right\},
\]
for \( n \geq 2k + 2 \), from which the boundedness of \((x_n)_{n \geq 1}\) follows in this case.

If \( l_0 = 2k + 1 \), then by using (2.1) in (2.15), we get
Theorem 2.3. Assume that $A > 0$, $p = 1 + q$, and $q \in (0, 1)$. Then all positive solutions of system (1.3) are bounded.

Proof. First note that by using the change of variables

$$x_n = A\hat{x}_n, \quad y_n = A\hat{y}_n, \quad n \in \mathbb{N}_0,$$

system (1.3), in this case, is reduced to the same system with $A = 1$. Hence we may assume that $A = 1$.

Assume that the sequences $(a_n)_{n \in \mathbb{N}_0}$ and $(b_n)_{n \in \mathbb{N}_0}$ are defined by

$$a_0 = q, \quad b_0 = q + 1,$$
$$a_{2n+1} = (q + 1)b_{2n} - a_{2n}, \quad b_{2n+1} = qb_{2n}, \quad n \in \mathbb{N}_0, \quad a_{2n+2} = qa_{2n+1}, \quad n \in \mathbb{N}_0. \quad (2.16)$$

From this, by using (1.3) and a simple inductive argument, we have

$$x_{n+1} = \max \left\{ \frac{A}{x_n^{p-1}}, \frac{A}{y_n^{p-1}}, \frac{A}{x_n^{p-1}}, \frac{A}{y_n^{p-1}}, \ldots \right\}$$

for $n \geq 2k + 3$, from which the boundedness of $(x_n)_{n \geq 1}$ follows in this case.

Since the system (1.3) is symmetric, the boundedness of $(x_n)_{n \geq 1}$ imply the boundedness of $(y_n)_{n \geq 1}$, finishing the proof of the theorem. □
Max-type system of difference equations

\[ \begin{align*}
&= \max \left\{ \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-1}^{b_1}}, \ldots, \frac{(q+1)b_{2k-2} - a_{2k}}{x_{n-2k-1}^{a_0}}, \ldots, \frac{(q+1)b_{2k-1} - a_{2k}}{y_{n-2k-2}^{b_1}} \right\} \\
&= \max \left\{ \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-1}^{b_1}}, \ldots, \frac{x_{2k-1}^{a_0}}{y_{n-2k-1}^{b_1}} \right\} \\
&= \max \left\{ \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-1}^{b_1}}, \ldots, \frac{1}{y_{n-2k}^{b_{2k-1}}} \right\} \\
&= \max \left\{ \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-1}^{b_1}}, \ldots, \frac{1}{y_{n-2k}^{b_{2k-1}}} \right\} \\
&= \max \left\{ \frac{1}{x_{n-1}^{a_0}}, \frac{1}{y_{n-1}^{b_1}}, \ldots, \frac{1}{y_{n-2k}^{b_{2k-1}}} \right\}, \\
\end{align*} \]

for every \( k \in \mathbb{N} \).

From (2.16) we have that

\[ b_{2n} = \frac{a_{2n+1} + a_{2n}}{q + 1}, \quad n \in \mathbb{N}_0. \]

Applying this to the following relation

\[ b_{2n+2} = (q + 1)a_{2n+1} - qb_{2n}, \quad n \in \mathbb{N}_0, \]

we get

\[ a_{2n+3} + a_{2n+2} - (q^2 + q + 1)a_{2n+1} + qa_{2n} = 0, \quad n \in \mathbb{N}_0. \]

From this and the relation \( a_{2n+2} = qa_{2n+1} \), we get

\[ a_{2n+3} - (q^2 + 1)a_{2n+1} + q^2a_{2n-1} = 0, \quad n \in \mathbb{N}. \]

It is easy to see that the general solution of difference equation (2.19) is

\[ a_{2n+1} = c_1 + c_2q^{2n}, \quad n \in \mathbb{N}_0. \]

From this and since

\[ a_1 = (q + 1)b_0 - a_0 = q^2 + q + 1, \quad b_1 = qb_0 = q^2 + q, \]

\[ a_2 = qa_1 = q^3 + q^2 + q, \quad b_2 = (q + 1)a_1 - b_1 = (q + 1)(q^2 + 1), \]

\[ a_3 = (q + 1)b_2 - a_2 = q^4 + q^3 + q^2 + q + 1, \]

we have that

\[ c_1 + c_2 = q^2 + q + 1, \quad c_1 + c_2q^2 = q^4 + q^3 + q^2 + q + 1 \]

and consequently

\[ c_1 = \frac{1}{1 - q}, \quad c_2 = \frac{q^4 + q^3}{q^2 - 1} = \frac{q^3}{q - 1}. \]

Hence

\[ a_{2n+1} = \frac{1 - q^{2n+3}}{1 - q}, \quad n \in \mathbb{N}_0. \]

Letting \( n \to +\infty \) we get

\[ \lim_{n \to +\infty} a_{2n+1} = \frac{1}{1 - q}. \]
From this and (2.16) we also have that
\[
\lim_{n \to +\infty} a_{2n} = \lim_{n \to +\infty} b_{2n+1} = \frac{q}{1 - q} = q \lim_{n \to +\infty} b_{2n},
\]
(2.22)

Now note that from (2.17) and (2.18) we have that
\[
x_{2n+1} = \max \left\{ \frac{1}{x_{2n-1}}, \frac{1}{y_{2n-2}}, \ldots, \frac{1}{y_{2n-1}}, \frac{1}{x_{2n-2}}, \ldots, \frac{1}{x_{2n-1}} \right\}
\]
and
\[
x_{2n} = \max \left\{ \frac{1}{x_{2n-2}}, \frac{1}{y_{2n-3}}, \ldots, \frac{1}{x_{2n-1}}, \frac{1}{x_{2n-2}}, \ldots, \frac{1}{x_{2n-1}} \right\},
\]
for \( n \in \mathbb{N} \).

From this, since \( \min \{x_n, y_n\} \geq 1 \) for \( n \in \mathbb{N} \), and by using (2.21) and (2.22) the boundedness of the sequence \( (x_n)_{n \geq -1} \) easily follows.

Since system (1.3) is symmetric, the boundedness of \( (x_n)_{n \geq -1} \) imply the boundedness of \( (y_n)_{n \geq -1} \), finishing the proof of the theorem.

The following theorem shows that positive solutions of system (1.3) are unbounded in the other cases.

**Theorem 2.4.** Assume that \( A > 0 \). If \( p^2 \geq 4q \geq 4 \), or \( p > 1 + q \) and \( q \in (0, 1) \), then system (1.3) has positive unbounded solutions.

**Proof.** Assume that \( p^2 \geq 4q \geq 4 \) and \( p \neq 2 \). From (1.3) we have
\[
x_{n+1} \geq \frac{y_n^p}{x_n^{p+q}}, \quad y_{n+1} \geq \frac{x_n^p}{y_n^{p+q}}, \quad n \in \mathbb{N}_0.
\]
(2.23)

Let \( a_n = \ln(x_ny_n) \), \( n \geq 1 \). Then from (2.23), it follows that
\[
a_{n+1} - pa_n + qa_{n-1} \geq 0, \quad n \in \mathbb{N}_0.
\]
(2.24)

The polynomial \( P(t) = t^2 - pt + q \) has the zeroes \( t_{1,2} = (p \pm \sqrt{p^2 - 4q})/2 \), and \( t_1 > 1 \), and \( t_2 > 0 \).

From (2.24) we get
\[
a_{n+1} - t_1a_n - t_2(a_n - t_1a_{n-1}) \geq 0, \quad n \in \mathbb{N}_0,
\]
(2.25)
that is,
\[
\frac{x_{n+1}y_{n+1}}{(x_ny_n)^{t_2}} \geq \left( \frac{x_ny_n}{(x_{n-1}y_{n-1})^{t_2}} \right)^{t_2}, \quad n \in \mathbb{N}_0,
\]
(2.26)

which implies that
\[
\frac{x_{n+1}y_{n+1}}{(x_ny_n)^{t_2}} \geq \left( \frac{x_0y_0}{(x_{-1}y_{-1})^{t_2}} \right)^{t_2}, \quad n \in \mathbb{N}_0.
\]
(2.27)

Let \( x_i, y_i, i \in \{-1, 0\} \) be chosen such that
\[
x_0y_0 > 1 \quad \text{and} \quad x_0y_0 = (x_{-1}y_{-1})^{t_2}.
\]
(2.28)
This, along with (2.27), yields
\[
x_n y_n \geq \left( \frac{x_0 y_0}{(x_{n-1} y_{n-1})^{t_1}} \right)^{t_1} (x_{n-1} y_{n-1})^{t_1} = (x_{n-1} y_{n-1})^{t_1}, \quad n \in \mathbb{N}_0,
\]
from which we get
\[
x_n y_n \geq (x_0 y_0)^{t_1}, \quad n \in \mathbb{N}_0.
\] (2.30)

Letting \( n \to \infty \) in (2.30), using the first assumption in (2.28) and \( t_1 > 1 \), it follows that
\[
x_n y_n \to +\infty \quad \text{as} \quad n \to \infty,
\] (2.31)
which along with the inequality between arithmetic and geometric means implies
\[
\sqrt{x_n^2 + y_n^2} \to +\infty \quad \text{as} \quad n \to \infty,
\] (2.32)
from which it follows that \( (x_n, y_n)_{n \geq -1} \) is unbounded.

The proof in the case \( p > 1 + q \) and \( q \in (0, 1) \) is similar, since then
\[
t_1 = \frac{p + \sqrt{p^2 - 4q}}{2} > 1.
\]
If \( p = q + 1 = 2 \), then \( t_1 = t_2 = 1 \). If we choose \( x_i, y_i, i \in \{-1, 0\} \) such that
\[
x_0 y_0 > x_{-1} y_{-1} > 0,
\] (2.33)
then from (2.27) we get
\[
x_n y_n \geq \frac{x_0 y_0}{x_{-1} y_{-1}} x_{n-1} y_{n-1}, \quad n \in \mathbb{N}_0,
\]
and consequently
\[
x_n y_n \geq \left( \frac{x_0 y_0}{x_{-1} y_{-1}} \right)^n x_0 y_0, \quad n \in \mathbb{N}_0.
\] (2.34)

Letting \( n \to \infty \) in (2.34) we get (2.31) and consequently (2.32), which implies that \( (x_n, y_n)_{n \geq -1} \) is unbounded, finishing the proof of the theorem.

\[\square\]

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