Regularity in Orlicz spaces for nondivergence elliptic operators with potentials satisfying a reverse Hölder condition*

Kelei Zhang†

Department of Applied Mathematics, Northwestern Polytechnical University,
Xi’an, Shaanxi, 710129, PR China

Abstract: The purpose of this paper is to obtain the global regularity in Orlicz spaces for nondivergence elliptic operators with potentials satisfying a reverse Hölder condition.

Keywords: nondivergence elliptic operator; regularity, Orlicz space; potential; reverse Hölder condition.


1 Introduction

In this paper we consider the following nondivergence elliptic operator

$$Lu \equiv Au + Vu \equiv - \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + Vu,$$ (1.1)

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n (n \geq 3)$, and establish the regularity in Orlicz spaces for (1.1). It will be assumed that the following assumptions on the coefficients of the operator $A$ and the potential $V$ are satisfied

$(H_1) \ a_{ij} \in L^\infty(\mathbb{R}^n)$ and $a_{ij} = a_{ji}$ for all $i, j = 1, 2, \ldots, n$, and there exists a positive constant $\Lambda$ such that

$$\Lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j \leq \Lambda |\xi|^2$$

*This work was supported by the National Natural Science Foundation of China (Grant Nos. 11271299, 11001221) and Natural Science Foundation Research Project of Shaanxi Province (Grant No. 2012JM1014).

†E-mail address: eaststonezhang@126.com
for any \( x \in \mathbb{R}^n \) and \( \xi \in \mathbb{R}^n \);

\((H_2)\) \( a_{ij}(x) \in VMO(\mathbb{R}^n) \), which means that for \( i, j = 1, 2, \ldots, n \),

\[
\eta_{ij}(r) = \sup_{\rho \leq r} \sup_{x \in \mathbb{R}^n} \left( |B_\rho(x)|^{-1} \int_{B_\rho(x)} |a_{ij}(y) - a_{ij}^B| \, dy \right) \to 0, \quad r \to 0^+,
\]

where \( a_{ij}^B = |B_\rho(x)|^{-1} \int_{B_\rho(x)} a_{ij}(y) \, dy \);

\((H_3)\) \( V \in B_q \) for \( n/2 \leq q < \infty \), which means that \( V \in L^q_{\text{loc}}(\mathbb{R}^n), V \geq 0 \), and there exists a positive constant \( c_1 \) such that the reverse Hölder inequality

\[
\left( |B|^{-1} \int_B V(x)^q \, dx \right)^{1/q} \leq c_1 \left( |B|^{-1} \int_B V(x) \, dx \right)
\]

holds for every ball \( B \) in \( \mathbb{R}^n \).

Note that when we say \( V \in B_\infty \), it means

\[
\sup_B V(x) \leq c_1 \left( |B|^{-1} \int_B V(x) \, dx \right).
\]

In fact, if \( V \in B_\infty \), then it implies that \( V \in B_q \) for \( 1 < q < \infty \).

Regularity theory for elliptic operators with potentials satisfying a reverse Hölder condition has been studied by many authors (see [4], [9]–[12], [14], [15]). When \( A \) is the Laplace operator and \( V \in B_q \) \( (n/2 \leq q < \infty) \), Shen [10] derived \( L^p \) boundedness for \( 1 < p \leq q \) and showed that the range of \( p \) is optimal. If \( A \) is the Laplace operator and \( V \in B_\infty \), an extension of \( L^p \) estimates to the global Orlicz estimates was given by Yao [14] with modifying the iteration-covering method introduced by Acerbi and Mingione [1]. For \( a_{ij} \in C^1(\mathbb{R}^n) \) and \( V \in B_\infty \), regularity theory in Orlicz spaces for the operators \( \sum_{i,j=1}^n \partial_{x_i} (a_{ij} \partial_{x_j}) + V \) was proved by Yao [15]. Recently, under the assumptions \((H_1)-(H_3)\), the global \( L^p(\mathbb{R}^n) \) estimates for \( L \) in (1.1) has been deduced by Bramanti et al [4].

In this paper we will establish global estimates in Orlicz spaces for \( L \) which extends results in [4] to the case of the general Orlicz spaces. Our approach is based on an iteration-covering lemma (Lemma 3.1), the technique of “S. Agmon’s idea” (see [3], p. 124) and an approximation procedure.

The definitions of Yong functions \( \phi \), Orlicz spaces \( L^\phi(\mathbb{R}^n) \), Orlicz–Sobolev spaces \( W^2 L^\phi(\mathbb{R}^n), W^2_{\text{loc}} L^\phi(\mathbb{R}^n) \), and their properties will be described in Section 2.

We now state the main result of this paper.
Theorem 1.1 Let $\phi$ be a Young function and satisfy the global $\Delta_2 \cap \nabla_2$ condition. Assume that the operator $L$ satisfies the assumptions $(H_1)$, $(H_2)$ and $(H_3)$ for $q \geq \max \{n/2, \alpha_1\}$, $f \in L^\phi(\mathbb{R}^n)$. If $u \in W^2_V L^\phi(\mathbb{R}^n)$ satisfies

$$Lu - \mu u = f, \quad x \in \mathbb{R}^n, \quad (1.2)$$

then there exists a constant $C > 0$ such that for any $\mu \gg 1$ large enough, we have

$$\mu^{\alpha_2} \int_{\mathbb{R}^n} \phi(|u|) dx + \mu^{\alpha_2/2} \int_{\mathbb{R}^n} \phi(|Du|) dx + \int_{\mathbb{R}^n} \phi(|Vu|) dx + \int_{\mathbb{R}^n} \phi(|D^2u|) dx \leq C \int_{\mathbb{R}^n} \phi(|f|) dx, \quad (1.3)$$

where the constants $\alpha_1$ and $\alpha_2$ appear in Orlicz spaces, see (2.4), $C$ depends only on $n$, $q$, $\Lambda$, $c_1$, $\alpha_1$, $\alpha_2$ and the VMO moduli of the leading coefficients $a_{ij}$.

The proof of Theorem 1.1 is based on the following result.

Theorem 1.2 Under the same assumptions on $\phi$, $a_{ij}$, $V$, $q$, $f$ as in Theorem 1.1, let $u \in C_0^\infty(\mathbb{R}^n)$ satisfy $Lu = f$ in $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} \phi(|Du|) dx + \int_{\mathbb{R}^n} \phi(|Vu|) dx \leq C \left\{ \int_{\mathbb{R}^n} \phi(|f|) dx + \int_{\mathbb{R}^n} \phi(|u|) dx \right\}, \quad (1.4)$$

where $C$ depends only on $n$, $q$, $\Lambda$, $c_1$, $a$, $K$ and the VMO moduli of $a_{ij}$.

Note that Theorem 1.2 and Definition 2.9 easily imply the following result by using the monotonicity, convexity of $\phi$, (2.2) and Remark 2.7.

Corollary 1.3 Under the same assumptions on $\phi$, $a_{ij}$, $V$, $q$, $f$ as in Theorem 1.1, let $u \in W^2_V L^\phi(\mathbb{R}^n)$ satisfy $Lu = f$ in $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} \phi(|D^2u|) dx + \int_{\mathbb{R}^n} \phi(|Vu|) dx \leq C \left\{ \int_{\mathbb{R}^n} \phi(|f|) dx + \int_{\mathbb{R}^n} \phi(|u|) dx \right\},$$

where $C$ depends only on $n$, $q$, $\Lambda$, $c_1$, $a$, $K$ and the VMO moduli of $a_{ij}$.

Remark 1.4 When we take $\phi(t) = t^p$, $t \geq 0$ for $1 < p < \infty$, then (1.4) is reduced to the classical $L^p$ estimates (see [4, Theorem 1]).

This paper will be organized as follows. In Section 2 some basic facts about Orlicz spaces and Orlicz–Sobolev spaces are recalled. In Section 3 we prove Theorem 1.2 by describing an iteration-covering lemma (Lemma 3.1) and using the
results in [4]. Section 4 is devoted to the proof of Theorem 1.1. We first assume
\( u \in C_0^\infty(B_{R_0}/2) \) satisfying (1.2) and prove that (1.3) is valid by using Theorem
1.2 and “S. Agmon’s idea” (see [3], p. 124); then we show that the assumption
\( u \in C_0^\infty(B_{R_0}/2) \) can be removed by an approximation procedure and a covering
lemma in [5].

**Dependence of constants.** Throughout this paper, the letter \( C \) denotes a posi-
tive constant which may vary from line to line.

## 2 Preliminaries

We collect here some facts about Orlicz spaces and Orlicz–Sobolev spaces which
will be needed in the following. For more properties, we refer the readers to [2] and
[8].

We use the following notation:

\[
\Phi = \{ \phi : [0, +\infty) \to [0, +\infty) \mid \phi \text{ is increasing and convex} \}.
\]

**Definition 2.1** A function \( \phi \in \Phi \) is said to be a Young function if

\[
\phi(0) = 0, \quad \lim_{t \to +\infty} \frac{\phi(t)}{t} = +\infty, \quad \lim_{t \to 0^+} \frac{\phi(t)}{t} = \lim_{t \to +\infty} \frac{t}{\phi(t)} = 0. \tag{2.1}
\]

**Definition 2.2** A Young function \( \phi \) is said to satisfy the global \( \Delta_2 \) condition de-
noted by \( \phi \in \Delta_2 \), if there exists a positive constant \( K \) such that for any \( t > 0 \),

\[
\phi(2t) \leq K \phi(t). \tag{2.2}
\]

**Definition 2.3** A Young function \( \phi \) is said to satisfy the global \( \nabla_2 \) condition de-
noted by \( \phi \in \nabla_2 \), if there exists a positive constant \( a > 1 \) such that for any \( t > 0 \),

\[
\phi(at) \geq 2a\phi(t). \tag{2.3}
\]

The following result was obtained in [7].

**Lemma 2.4** If \( \phi \in \Delta_2 \cap \nabla_2 \), then for any \( t > 0 \) and \( 0 < \theta_2 \leq 1 \leq \theta_1 < +\infty \),

\[
\phi(\theta_1 t) \leq K \theta_1^{\alpha_1} \phi(t) \quad \text{and} \quad \phi(\theta_2 t) \leq 2 a \theta_2^{\alpha_2} \phi(t), \tag{2.4}
\]

where \( \alpha_1 = \log_2 K, \alpha_2 = \log_2 a + 1 \) and \( \alpha_1 \geq \alpha_2 \).

**Definition 2.5** (Orlicz spaces) Given a Young function \( \phi \), we define the Orlicz
class \( K^\phi(\mathbb{R}^n) \) which consists of all the measurable functions \( g : \mathbb{R}^n \to \mathbb{R} \) satisfying

\[
\int_{\mathbb{R}^n} \phi(|g|) dx < \infty
\]

and the Orlicz space \( L^\phi(\mathbb{R}^n) \) which is the linear hull of \( K^\phi(\mathbb{R}^n) \).

EJQTDE, 2013 No. 78, p. 4
In the Orlicz spaces $L^\phi(\mathbb{R}^n)$, we use the following Luxembourg norm
\[
\|u\|_{L^\phi(\mathbb{R}^n)} = \inf \left\{k > 0 : \int_{\mathbb{R}^n} \phi(|u|/k) \, dx \leq 1 \right\}.
\] (2.5)
The space $L^\phi(\mathbb{R}^n)$ equipped with the Luxembourg norm $\|\cdot\|_{L^\phi(\mathbb{R}^n)}$ is a Banach space. In general, $K^\phi \subset L^\phi$. Moreover, if $\phi$ satisfies the global $\Delta_2$ condition, then $K^\phi = L^\phi$ and $C^\infty_0$ is dense in $L^\phi$ (see [2], pp. 266–274).

Definition 2.6 (Convergence in mean) A sequence $\{u_k\}$ of functions in $L^\phi(\mathbb{R}^n)$ is said to converge in mean to $u \in L^\phi(\mathbb{R}^n)$ if
\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \phi(|u_k(x) - u(x)|) \, dx = 0.
\]

Remark 2.7 (see [2], p. 270)
(i) The norm convergence in $L^\phi(\mathbb{R}^n)$ implies the mean convergence.
(ii) If $\phi \in \Delta_2$, then the mean convergence implies the norm convergence.

Definition 2.8 (Orlicz–Sobolev spaces) The Orlicz–Sobolev space $W^{2,L^\phi}(\mathbb{R}^n)$ is the set of all functions $u$ which satisfy $|D^\alpha u(x)| \in L^\phi(\mathbb{R}^n)$ for $0 \leq |\alpha| \leq 2$. The norm is defined by
\[
\|u\|_{W^{2,L^\phi}(\mathbb{R}^n)} = \|u\|_{L^\phi(\mathbb{R}^n)} + \|Du\|_{L^\phi(\mathbb{R}^n)} + \|D^2u\|_{L^\phi(\mathbb{R}^n)},
\]
where $Du(x) = \{u_{x_i}\}_{i=1}^n$, $D^2u(x) = \{u_{x_i x_j}\}_{i,j=1}^n$, $\|Du\|_{L^\phi(\mathbb{R}^n)} = \sum_{i=1}^n \|u_{x_i}\|_{L^\phi(\mathbb{R}^n)}$, $\|D^2u\|_{L^\phi(\mathbb{R}^n)} = \sum_{i,j=1}^n \|u_{x_i x_j}\|_{L^\phi(\mathbb{R}^n)}$.

The following definition is analogous to the definition of the space $W^{2,p}_V(\mathbb{R}^n)$ introduced by Bramanti, Brandolini, Harboure and Viviani in [4].

Definition 2.9 The space $W^{2,L^\phi}_V(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ in the norm
\[
\|u\|_{W^{2,L^\phi}_V(\mathbb{R}^n)} = \|u\|_{W^{2,L^\phi}(\mathbb{R}^n)} + \|Vu\|_{L^\phi(\mathbb{R}^n)}.
\]

Remark 2.10 (see e.g. [13]) If $g \in L^\phi(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} \phi(|g|) \, dx$ can be easily rewritten in an integral form
\[
\int_{\mathbb{R}^n} \phi(|g|) \, dx = \int_0^\infty \{|x \in \mathbb{R}^n : |g| > t\} d[\phi(t)].
\] (2.6)
As usual, we denote by $B_R(x)$ the open ball in $\mathbb{R}^n$ of radius $R$ centered at $x$ and $B_R = B_R(0)$. 

EJQTDE, 2013 No. 78, p. 5
3 Proof of Theorem 1.2

Before the proof of Theorem 1.2, some notions and two useful lemmas are given. Let us introduce the notation

\[ p = \frac{1 + \alpha_2}{2} > 1. \]

For \( u \in C_0^\infty(\mathbb{R}^n) \) satisfying \( Lu = f \), set

\[ \lambda_0^p = \int_{\mathbb{R}^n} |Vu|^p dx + \varepsilon^{-p} \left( \int_{\mathbb{R}^n} |f|^p dx + \int_{\mathbb{R}^n} |u|^p dx \right), \]

where \( \varepsilon \in (0, 1) \) is a small enough constant to be determined later. Let

\[ u_\lambda = \frac{u}{\lambda_0^p} \quad \text{and} \quad f_\lambda = \frac{f}{\lambda_0^p}, \quad \text{for any } \lambda > 0. \]

Then \( u_\lambda \) satisfies \( Lu_\lambda = f_\lambda \). For any ball \( B \) in \( \mathbb{R}^n \), we use the notations

\[ J_\lambda[B] = \frac{1}{|B|} \int_B |Vu_\lambda|^p dx + \frac{1}{\varepsilon^p |B|} \left( \int_B |f_\lambda|^p dx + \int_B |u_\lambda|^p dx \right) \]

and

\[ E_\lambda(1) = \{ x \in \mathbb{R}^n : |Vu_\lambda| > 1 \}. \]

The following lemma is just an analogous version of the result given in [15, Lemma 2.2]. Here the selection of \( \lambda_0 \) and the condition of \( V \) are different from [15].

**Lemma 3.1 (Iteration-covering lemma)** For any \( \lambda > 0 \), there exists a family of disjoint balls \( \{ B_{\rho_{x_i}}(x_i) \} \) with \( x_i \in E_\lambda(1) \) and \( \rho_{x_i} = \rho(x_i, \lambda) > 0 \) such that

\[ J_\lambda[B_{\rho_{x_i}}(x_i)] = 1, \quad J_\lambda[B_{\rho'}(x_i)] < 1 \quad \text{for any } \rho > \rho_{x_i}, \quad (3.1) \]

and

\[ E_\lambda(1) \subset \bigcup_{i \geq 1} B_{5\rho_{x_i}}(x_i) \bigcup F, \quad (3.2) \]

where \( F \) is a zero measure set. Moreover,

\[ |B_{\rho_{x_i}}(x_i)| \leq \frac{3^{p-1}}{3^{p-1} - 1} \left\{ \int_{\{ x \in B_{\rho_{x_i}}(x_i) : |Vu_\lambda| \geq \frac{1}{4} \}} |Vu_\lambda|^p dx \right. \]

\[ + \varepsilon^{-p} \int_{\{ x \in B_{\rho_{x_i}}(x_i) : |f_\lambda| \geq \frac{1}{4} \}} |f_\lambda|^p dx + \varepsilon^{-p} \int_{\{ x \in B_{\rho_{x_i}}(x_i) : |u_\lambda| \geq \frac{1}{4} \}} |u_\lambda|^p dx \left. \right\}. \quad (3.3) \]
We omit the proof of Lemma 3.1 because it is actually similar to that of [15, Lemma 2.2].

In analogy with [4, Theorem 13], the following lemma holds by using [4, Theorem 2, Theorem 3], and standard techniques involving cutoff functions and the interpolation inequality (see e.g. [6]).

**Lemma 3.2** Under the assumptions \((H_1)–(H_3)\), for any \(\gamma \in (1,q]\), there exists a positive constant \(C\) such that for any \(x_i, \rho_{x_i}\) as in Lemma 3.1 and \(u \in C_0^\infty(\mathbb{R}^n)\),

\[
\int_{B_{5\rho_{x_i}}(x_i)} |Vu|^\gamma dx \leq C \left\{ \int_{B_{10\rho_{x_i}}(x_i)} |Lu|^\gamma dx + \int_{B_{10\rho_{x_i}}(x_i)} |u|^\gamma dx \right\},
\]

where \(C\) depends only on \(n, \gamma, q, c_1, \Lambda\) and the VMO moduli of \(a_{ij}\).

**Proof of Theorem 1.2.** In order to prove (1.4), the first step is to check the following estimate

\[
\int_{\mathbb{R}^n} \phi (|Vu|) dx \leq C \left( \int_{\mathbb{R}^n} \phi (|f|) dx + \int_{\mathbb{R}^n} \phi (|u|) dx \right). \tag{3.4}
\]

Since \(u \in C_0^\infty(\mathbb{R}^n)\), then there exists some constant \(R_0 > 0\) such that \(u\) is compactly supported in \(B_{R_0}\). It follows from \(q \geq \max\{n/2, \alpha_1\}\) and (2.4) that

\[
\begin{align*}
\int_{\mathbb{R}^n} \phi (|Vu|) dx &= \int_{\{x \in \mathbb{R}^n : |Vu| \geq 1\}} \phi (|Vu|) dx + \int_{\{x \in \mathbb{R}^n : |Vu| \leq 1\}} \phi (|Vu|) dx \\
&\leq K \phi (1) \int_{\mathbb{R}^n} |Vu|^\alpha_1 dx + 2a \phi (1) \int_{\mathbb{R}^n} |Vu|^\alpha_2 dx \\
&\leq C \left( \sup_{B_{R_0}} |u|^\alpha_1 + \sup_{B_{R_0}} |u|^\alpha_2 \right) \left( \int_{B_{R_0}} |V|^\alpha_1 dx + \int_{B_{R_0}} |V|^\alpha_2 dx \right) \\
&< \infty,
\end{align*}
\]

that is \(|Vu| \in L^\phi(\mathbb{R}^n)\). Hence by (2.6), it yields

\[
\int_{\mathbb{R}^n} \phi (|Vu|) dx = \int_0^\infty |\{x \in \mathbb{R}^n : |Vu| > \lambda \lambda_0 \}| d[\phi(\lambda \lambda_0)].
\]

Due to (3.2),

\[
|\{x \in \mathbb{R}^n : |Vu| > \lambda \lambda_0 \}| \leq \sum_{i=1}^\infty \left| \{x \in B_{5\rho_{x_i}}(x_i) : |Vu| > 1 \} \right|.
\]

EJQTDE, 2013 No. 78, p. 7
Thus the key is to estimate \(|\{x \in B_{5\rho_i}(x_i) : |Vu_\lambda| > 1\}|\). Applying Lemma 3.2, (3.1) and (3.3) we deduce

\[
\left|\{x \in B_{5\rho_i}(x_i) : |Vu_\lambda| > 1\}\right| \\
\leq \int_{B_{5\rho_i}(x_i)} |Vu_\lambda|^p dx \\
\leq C \left\{ \int_{B_{10\rho_i}(x_i)} |f_\lambda|^p dx + \int_{B_{10\rho_i}(x_i)} |u_\lambda|^p dx \right\} \\
\leq \varepsilon^p C(p, n) \left| B_{\rho_i}(x_i) \right| \\
\leq C(p, n) \left\{ \varepsilon^p \int_{\{x \in B_{\rho_i}(x_i) : |Vu_\lambda| > \frac{1}{3}\}} |Vu_\lambda|^p dx + \int_{\{x \in B_{\rho_i}(x_i) : |f_\lambda| > \frac{1}{3}\}} |f_\lambda|^p dx \\
+ \int_{\{x \in B_{\rho_i}(x_i) : |u_\lambda| > \frac{1}{3}\}} |u_\lambda|^p dx \right\}.
\]

Set \(\tilde{\lambda} = \lambda_0 \lambda\) and observe that

\[
\int_{\mathbb{R}^n} \phi(|Vu|) dx = \int_0^\infty \left| \left\{ x \in \mathbb{R}^n : |Vu| > \tilde{\lambda} \right\} \right| d[\phi(\tilde{\lambda})] \\
\leq C(p, n) \varepsilon^p \int_0^\infty \tilde{\lambda}^{-p} \left\{ \int_{\{x \in \mathbb{R}^n : |Vu| > \frac{1}{3}\}} |Vu|^p dx \right\} d[\phi(\tilde{\lambda})] \\
+ C(p, n) \int_0^\infty \tilde{\lambda}^{-p} \left\{ \int_{\{x \in \mathbb{R}^n : |f| > \frac{1}{3}\}} |f|^p dx \right\} d[\phi(\tilde{\lambda})] \\
+ C(p, n) \int_0^\infty \tilde{\lambda}^{-p} \left\{ \int_{\{x \in \mathbb{R}^n : |u| > \frac{1}{3}\}} |u|^p dx \right\} d[\phi(\tilde{\lambda})] \\
= : C(p, n) (\varepsilon^p I_1 + I_2 + I_3).
\]

By Fubini’s theorem, integration by parts and (2.4), it implies that

\[
I_1 = \int_{\mathbb{R}^n} |Vu|^p \left\{ \int_0^{3|Vu|} \frac{d\phi(\tilde{\lambda})}{\tilde{\lambda}^p} \right\} dx \\
= \frac{1}{3^p} \int_{\mathbb{R}^n} \phi(3|Vu|) dx + p \int_{\mathbb{R}^n} |Vu|^p \left\{ \int_0^{3|Vu|} \frac{d\phi(\tilde{\lambda})}{\tilde{\lambda}^{p+1}} \right\} dx \\
\leq \frac{1}{3^p} \int_{\mathbb{R}^n} \phi(3|Vu|) dx + \frac{2ap}{3^p(\alpha_2 - p)} \int_{\mathbb{R}^n} \phi(3|Vu|) dx \\
\leq C(n, p, a, K) \int_{\mathbb{R}^n} \phi(|Vu|) dx.
\]

Similarly,

\[
I_2 \leq C(n, p, a, K) \varepsilon^{p-\alpha_1} \int_{\mathbb{R}^n} \phi(|f|) dx
\]

EJQTDE, 2013 No. 78, p. 8
and
\[ I_3 \leq C(n, p, a, K)\varepsilon^{p-\alpha_1} \int_{\mathbb{R}^n} \phi(|u|)dx. \]

Therefore,
\[ \int_{\mathbb{R}^n} \phi(|Vu|)dx \leq C \left\{ \varepsilon^p \int_{\mathbb{R}^n} \phi(|Vu|)dx + \varepsilon^{p-\alpha_1} \int_{\mathbb{R}^n} \phi(|f|)dx + \varepsilon^{p-\alpha_1} \int_{\mathbb{R}^n} \phi(|u|)dx \right\}. \]

Choosing a suitable \( \varepsilon \) such that \( C(n, p, a, K)\varepsilon^p < \frac{1}{2} \), (3.4) is obtained.

Next, taking into account [16, Theorem 2.8], the convexity of \( \phi \), (2.2) and (3.4), we have
\[
\int_{\mathbb{R}^n} \phi\left(\left|D^2 u\right|\right)dx \leq C \int_{\mathbb{R}^n} \phi\left(|f - Vu|\right)dx \\
\leq \frac{C}{2} \int_{\mathbb{R}^n} \phi\left(|f|\right)dx + \frac{C}{2} \int_{\mathbb{R}^n} \phi\left(|Vu|\right)dx \\
\leq \frac{KC}{2} \int_{\mathbb{R}^n} \phi\left(|f|\right)dx + \frac{KC}{2} \int_{\mathbb{R}^n} \phi\left(|Vu|\right)dx \\
\leq C \left\{ \int_{\mathbb{R}^n} \phi\left(|f|\right)dx + \int_{\mathbb{R}^n} \phi\left(|u|\right)dx \right\}. \tag{3.5}
\]

Thus, (3.5) implies (1.4). The proof is finished. \( \square \)

4 Proof of Theorem 1.1

By the technique of “S. Agmon’s idea” (see [3], p. 124) and Theorem 1.2, we first prove the following lemma.

**Lemma 4.1** Under the same assumptions on \( \phi, a_{ij}, V, q, f \) as in Theorem 1.1, let \( u \in C_0^\infty(B_{R_0/2}) \) satisfy the following equation
\[ Lu - \mu u = f, \quad x \in \mathbb{R}^n. \]

Then for any \( \mu \gg 1 \) large enough,
\[
\mu^{\alpha_2} \int_{\mathbb{R}^n} \phi\left(|u|\right)dx + \mu^{\alpha_2/2} \int_{\mathbb{R}^n} \phi\left(|Du|\right)dx + \int_{\mathbb{R}^n} \phi\left(|Vu|\right)dx + \int_{\mathbb{R}^n} \phi\left(|D^2 u|\right)dx \\
\leq C \int_{\mathbb{R}^n} \phi\left(|Lu - \mu u|\right)dx = C \int_{\mathbb{R}^n} \phi\left(|f|\right)dx, \tag{4.1}
\]

where the constant \( C \) is independent of \( \mu \), and \( R_0, \alpha_2 \) are the constants in the proofs of Theorem 1.2 and (2.4), respectively.
Proof Let $\xi \in C_0^\infty(-R_0/2,R_0/2)$ be a cutoff function (not identically zero) and set

$$\tilde{u}(z) = \tilde{u}(x,t) = \xi(t) \cos(\sqrt{\mu}t)u(x)$$

and

$$\tilde{L}\tilde{u}(z) = L\tilde{u} + \tilde{u}_{tt},$$

where $\mu \geq 1$ will be chosen later, then $\tilde{u}(z) \in C_0^\infty(B_{R_0/2} \times (-R_0/2,R_0/2))$. It is easy to verify that the coefficients matrix

$$\begin{pmatrix}
(a_{ij})_{n \times n} & 0 \\
0 & 1
\end{pmatrix}
$$

of the operator $\tilde{L}$ still satisfies the assumptions $(H_1)$ and $(H_2)$. Furthermore, in view of (4.2) and (4.3) we find that

$$\tilde{L}\tilde{u}(z) = \tilde{f}(z),$$

where

$$\tilde{f}(z) = \xi(t) \cos(\sqrt{\mu}t)(Lu - \mu u) + (\xi''(t) \cos(\sqrt{\mu}t) - 2\sqrt{\mu}\xi'(t) \sin(\sqrt{\mu}t))u.$$  

For the sake of convenience, we use the following notation

$$D^2_{zz}\tilde{u}(z) = \{D^2_{xx}\tilde{u}(z), \tilde{u}_{xt}(z), \tilde{u}_{tt}(z)\},$$

where

$$D^2_{xx}\tilde{u}(z) = \{\tilde{u}_{x_i x_j}\}_{i,j=1}^n \quad \text{and} \quad \tilde{u}_{xt} = \{\tilde{u}_{x_i t}\}_{i=1}^n.$$

Applying Theorem 1.2 to (4.4),

$$\int_{\mathbb{R}^{n+1}} \phi \left( |D^2_{zz}\tilde{u}| \right) dxdt + \int_{\mathbb{R}^{n+1}} \phi \left( |V\tilde{u}| \right) dxdt \leq C \left\{ \int_{\mathbb{R}^{n+1}} \phi \left( |\tilde{f}| \right) dxdt + \int_{\mathbb{R}^{n+1}} \phi \left( |\tilde{u}| \right) dxdt \right\}. \quad (4.6)$$

If $|\xi(t) \cos(\sqrt{\mu}t)| > 0$, by (2.4) we have

$$\phi \left( |D^2u(x)| \right) = \phi \left( \left| \left( \xi(t) \cos(\sqrt{\mu}t) \right)^{-1} \xi(t) \cos(\sqrt{\mu}t) D^2u(x) \right| \right)$$

$$\leq K |\xi(t) \cos(\sqrt{\mu}t)|^{-\alpha_1} \phi \left( |\xi(t) \cos(\sqrt{\mu}t) D^2u(x)| \right).$$

EJQTDE, 2013 No. 78, p. 10
This and (4.2) yield
\[
\int_{\mathbb{R}^n} \phi (|D^2 u(x)|) \, dx \\
= \left( \int_{\mathbb{R}} K^{-1} |\xi(t) \cos(\sqrt{\mu}t)|^{-1} \int_{\mathbb{R}^{n+1}} K^{-1} |\xi(t) \cos(\sqrt{\mu}t)|^{\alpha_1} \phi (|D^2 u(x)|) \, dx \, dt \right)^{-1} \\
\leq C \int_{\mathbb{R}^{n+1}} K^{-1} |\xi(t) \cos(\sqrt{\mu}t)|^{\alpha_1} \phi (|D^2 u(x)|) \, dx \, dt \\
= C \int_{\mathbb{R}^{n+1}} \{ (x,t) \in \mathbb{R}^{n+1} | |\xi(t) \cos(\sqrt{\mu}t)| > 0 \} K^{-1} |\xi(t) \cos(\sqrt{\mu}t)|^{\alpha_1} \phi (|D^2 u(x)|) \, dx \, dt \\
\leq C \int_{\mathbb{R}^{n+1}} \phi (|D^2_2 \tilde{u}(z)|) \, dx \, dt.
\] (4.7)

Similarly to (4.7) we get
\[
\int_{\mathbb{R}^n} \phi (|Vu|) \, dx \leq C \int_{\mathbb{R}^{n+1}} \phi (|V\tilde{u}(z)|) \, dx \, dt. \tag{4.8}
\]

Using (2.4),
\[
\phi (|Du(x)|) \leq K |\xi(t) \sin(\sqrt{\mu}t)|^{-\alpha_1} \phi (|\xi(t) \sin(\sqrt{\mu}t) Du|).
\]

Thus,
\[
\int_{\mathbb{R}^n} \phi (|Du(x)|) \, dx \leq C \int_{\mathbb{R}^{n+1}} \phi (|\xi(t) \sin(\sqrt{\mu}t) Du|) \, dx \, dt \\
\leq C \sum_{i=1}^{n} \int_{\mathbb{R}^{n+1}} \phi (\mu^{-1/2} |\xi'(t) \cos(\sqrt{\mu}t) u_{x_i} - \tilde{u}_{x_i,t}|) \, dx \, dt \\
\leq C \mu^{-\alpha_2/2} \left( \int_{\mathbb{R}^n} \phi (|Du|) \, dx + \int_{\mathbb{R}^{n+1}} \phi (|\tilde{u}_{xt}|) \, dx \, dt \right).
\]

By choosing \( \mu \gg 1 \) large enough, we obtain the following
\[
\mu^{\alpha_2/2} \int_{\mathbb{R}^n} \phi (|Du(x)|) \, dx \leq C \int_{\mathbb{R}^{n+1}} \phi (|\tilde{u}_{xt}(z)|) \, dx \, dt \\
\leq C \int_{\mathbb{R}^{n+1}} \phi (|D^2_2 \tilde{u}(z)|) \, dx \, dt. \tag{4.9}
\]

Since
\[
-\mu \xi(t) \cos(\sqrt{\mu}t) u(x) = \tilde{u}_{tt}(z) - (\xi''(t) \cos(\sqrt{\mu}t) - 2\sqrt{\mu} \xi'(t) \sin(\sqrt{\mu}t)) u(x),
\]

EJQTDE, 2013 No. 78, p. 11
we get
\[ \mu^{α2} \int_{\mathbb{R}^n} \phi(|u(x)|)dx \leq C \int_{\mathbb{R}^{n+1}} \phi(|\tilde{u}_t(z)|)dxdt \]
\[ \leq C \int_{\mathbb{R}^{n+1}} \phi\left(|D_{zz}^2 \tilde{u}(z)|\right)dxdt. \quad (4.10) \]
Combining (4.5)–(4.10) and noting that
\[ -\sqrt{\mu} \xi'(t) \sin(\sqrt{\mu}t) u(x) = (\xi'(t) \cos(\sqrt{\mu}t))_t - \xi''(t) \cos(\sqrt{\mu}t) u(x), \]
we immediately find that
\[ \mu^{α2} \int_{\mathbb{R}^n} \phi(|u|)dx + \mu^{α2/2} \int_{\mathbb{R}^n} \phi(|Du|)dx + \int_{\mathbb{R}^n} \phi(|Vu|)dx + \int_{\mathbb{R}^n} \phi\left(|D^2u|\right)dx \]
\[ \leq C \left\{ \int_{\mathbb{R}^{n+1}} \phi\left(|D_{zz}^2 \tilde{u}|\right)dxdt + \int_{\mathbb{R}^{n+1}} \phi\left(|V\tilde{u}|\right)dxdt \right\} \]
\[ \leq C \left\{ \int_{\mathbb{R}^{n+1}} \phi\left(|\tilde{f}|\right)dxdt + \int_{\mathbb{R}^{n+1}} \phi(|\tilde{u}|)dxdt \right\} \]
\[ \leq C \left( \int_{\mathbb{R}^n} \phi(|Lu - \mu u|)dx + \int_{\mathbb{R}^n} \phi(|u|)dx \right). \]
The desired estimate (4.1) follows by taking \( \mu \gg 1 \) large enough. The lemma is proved. □

Furthermore, we shall show that the assumption \( C_0^\infty(B_{R_0/2}) \) can be removed.
A covering lemma in a locally invariant quasimetric space was proved by Bramanti et al. in [5]. Since the Euclidean space \( \mathbb{R}^n \) is a special locally invariant quasimetric space, the covering lemma also holds in \( \mathbb{R}^n \). For the convenience to readers, we describe it as follows.

**Lemma 4.2** For given \( R_0 \) and any \( \kappa > 1 \), there exist \( R_1 \in (0, R_0/2) \), a positive integer \( M \) and a sequence of points \( \{x_i\}_{i=1}^\infty \subset \mathbb{R}^n \) such that
\[ \mathbb{R}^n = \bigcup_{i=1}^\infty B_{R_1}(x_i); \]
\[ \sum_{i=1}^\infty \chi_{B_{\kappa R_1}(x_i)}(y) \leq M \quad \text{for any} \quad y \in \mathbb{R}^n, \]
where \( \chi_{B_{\kappa R_1}(x_i)}(y) \) is the characteristic function of \( B_{\kappa R_1}(x_i) \), that is, the function equal to 1 in \( B_{\kappa R_1}(x_i) \) and 0 in \( \mathbb{R}^n \setminus B_{\kappa R_1}(x_i) \).
Proof of Theorem 1.1. Let $\rho(x)$ be a cutoff function on $B_{R_0/2}$ relative to $B_{R_1}$, namely, $\rho(x) \in C_0^\infty(B_{R_0/2}), 0 \leq \rho(x) \leq 1$ and $\rho(x) \equiv 1$ on $B_{R_1}$, where $R_1$ is as in Lemma 4.2. For any fixed $x_0 \in \mathbb{R}^n$, we set

$$u^0(x) = u(x)\rho(x - x_0) =: u(x)\rho^0(x) \quad (4.11)$$

and observe that

$$Lu^0(x) - \mu u^0(x) = f\rho^0 - 2a_{ij}u_{x_i}\rho^0_{x_j} - a_{ij}u\rho^0_{x_i}x_j =: f^0.$$  

By Definition 2.9, there exists a sequence $\{u^k\}$ of functions in $C_0^\infty(\mathbb{R}^n)$ such that

$$\|u_k - u\|_{W^2L^\phi(\mathbb{R}^n)} + \|Vu_k - Vu\|_{L^\phi(\mathbb{R}^n)} \to 0, \text{ as } k \to \infty. \quad (4.12)$$

It follows from Remark 2.7 that

$$\int_{\mathbb{R}^n} \phi(|u_k - u|)dx + \int_{\mathbb{R}^n} \phi(|D(u_k - u)|)dx + \int_{\mathbb{R}^n} \phi(|D^2(u_k - u)|)dx$$

$$+ \int_{\mathbb{R}^n} \phi(V|u_k - u|)dx \to 0, \text{ as } k \to \infty. \quad (4.13)$$

Let $u^0_k = u_k\rho^0$. Then using the properties of $\rho$, the monotonicity, convexity of $\phi$, (4.13), (2.4) and Remark 2.7, we obtain

$$\|u^0_k - u^0\|_{W^2L^\phi(\mathbb{R}^n)} + \|Vu^0_k - Vu^0\|_{L^\phi(\mathbb{R}^n)} \to 0, \text{ as } k \to \infty. \quad (4.14)$$

Set

$$f_k = Lu_k - \mu u_k \text{ and } f^0_k = Lu^0_k - \mu u^0_k.$$  

It follows by $(H_1)$ and (4.12) that

$$\|f^0_k - f^0\|_{L^\phi(\mathbb{R}^n)}$$

$$\leq \|Lu^0_k - Lu^0\|_{L^\phi(\mathbb{R}^n)} + \mu\|u^0_k - u^0\|_{L^\phi(\mathbb{R}^n)} \to 0, \text{ as } k \to \infty. \quad (4.15)$$

Hence, by (4.14), (4.15), Lemma 4.1 and Remark 2.7 we have

$$\mu^{\alpha_2}\int_{\mathbb{R}^n} \phi(|u^0|)dx + \mu^{\alpha_2/2}\int_{\mathbb{R}^n} \phi(|Du^0|)dx + \int_{\mathbb{R}^n} \phi(|Vu^0|)dx$$

$$+ \int_{\mathbb{R}^n} \phi(|D^2u^0|)dx$$

$$\leq C\int_{\mathbb{R}^n} \phi(|f^0|)dx$$

$$\leq C\left\{\int_{B_{R_0/2}(x_0)} \phi(|f|)dx + \int_{B_{R_0/2}(x_0)} \phi(|u|)dx + \int_{B_{R_0/2}(x_0)} \phi(|Du|)dx\right\}. \quad (4.16)$$

EJQTDE, 2013 No. 78, p. 13
Note that (4.11) and (2.4) yield
\[ \int_{\mathbb{R}^n} \phi(|\rho^0 Du|) \, dx \leq C \left\{ \int_{\mathbb{R}^n} \phi(|Du^0|) \, dx + \int_{\mathbb{R}^n} \phi(|uD\rho^0|) \, dx \right\} \] (4.17)
and
\[ \int_{\mathbb{R}^n} \phi(|\rho^0 D^2 u|) \, dx \leq C \left\{ \int_{\mathbb{R}^n} \phi(|D^2 u^0|) \, dx + \int_{\mathbb{R}^n} \phi(|uD^2 \rho^0|) \, dx \right. \\
+ \left. \int_{\mathbb{R}^n} \phi(|D \cdot D\rho^0|) \, dx \right\}. \] (4.18)

Then combining (4.16), (4.17) and (4.18) implies that
\[ \mu^{\alpha_2} \int_{B_{R_0/2}(x_0)} \phi(|\rho^0 u|) \, dx + \mu^{\alpha_2/2} \int_{B_{R_0/2}(x_0)} \phi(|\rho^0 Du|) \, dx \\
+ \int_{B_{R_0/2}(x_0)} \phi(|\rho^0 Vu|) \, dx + \int_{B_{R_0/2}(x_0)} \phi(|\rho^0 D^2 u|) \, dx \leq C \left\{ \int_{B_{R_0/2}(x_0)} \phi(|f|) \, dx + \mu^{\alpha_2/2} \int_{B_{R_0/2}(x_0)} \phi(|u|) \, dx + \int_{B_{R_0/2}(x_0)} \phi(|Du|) \, dx \right\}. \]

Therefore, by the above inequality and Lemma 4.2 we deduce that
\[ \mu^{\alpha_2} \int_{\mathbb{R}^n} \phi(|u|) \, dx + \mu^{\alpha_2/2} \int_{\mathbb{R}^n} \phi(|Du|) \, dx + \int_{\mathbb{R}^n} \phi(|Vu|) \, dx + \int_{\mathbb{R}^n} \phi(|D^2 u|) \, dx \leq C \left\{ \int_{B_{R_1}(x_i)} \phi(|\rho^0 u|) \, dx + \mu^{\alpha_2/2} \int_{B_{R_1}(x_i)} \phi(|\rho^0 Du|) \, dx \\
+ \int_{B_{R_1}(x_i)} \phi(|\rho^0 Vu|) \, dx + \int_{B_{R_1}(x_i)} \phi(|\rho^0 D^2 u|) \, dx \right\} \\
\leq C \sum_{i=1}^{\infty} \left\{ \int_{B_{R_0/2}(x_i)} \phi(|f|) \, dx + \mu^{\alpha_2/2} \int_{B_{R_0/2}(x_i)} \phi(|u|) \, dx \\
+ \int_{B_{R_0/2}(x_i)} \phi(|Du|) \, dx \right\} \leq C \left\{ \int_{\mathbb{R}^n} \phi(|f|) \, dx + \mu^{\alpha_2/2} \int_{\mathbb{R}^n} \phi(|u|) \, dx + \int_{\mathbb{R}^n} \phi(|Du|) \, dx \right\}. \]

(1.3) is obtained by taking \( \mu \gg 1 \) large enough. The theorem is proved. \( \square \)

**Acknowledgments.** The author thanks the anonymous referee for offering valuable suggestions which have improved the presentation.
References


(Received June 27, 2013)