On Lyapunov-type inequality for a class of quasilinear systems

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Abstract. In this paper, we establish a new Lyapunov-type inequality for quasilinear systems. Our result in special case reduces to the result of Watanabe et al. [J. Inequal. Appl. 242(2012), 1–8]. As an application, we also obtain lower bounds for the eigenvalues of corresponding systems.

Keywords: Lyapunov-type inequality, quasilinear system, lower bound.

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1 Introduction

In 1907, Lyapunov [23] obtained the following remarkable inequality

\[ \frac{4}{b-a} \leq \int_{a}^{b} f_1(z) \, dz, \tag{1.1} \]

if Hill’s equation

\[ u_1'' + f_1(x) u_1 = 0 \tag{1.2} \]

has a real nontrivial solution \( u_1(x) \) such that the Dirichlet boundary conditions

\[ u_1(a) = 0 = u_1(b) \tag{1.3} \]

hold, where \( a, b \in \mathbb{R} \) with \( a < b \) consecutive zeros, \( u_1 \) is not identically zero on \([a, b]\), and \( f_1 \) is a real-valued positive continuous function defined on \( \mathbb{R} \). We know that the constant \( 4 \) on the left-hand side of inequality (1.1) cannot be replaced by a larger number (see [19, p. 345]).

Since the appearance of Lyapunov’s fundamental paper, various proofs and generalizations or improvements have appeared in the literature under the Dirichlet boundary conditions. For example, for authors who are interested in the Lyapunov-type inequalities, we refer to Eliaison [16], Harris and Kong [18], Hartman [19], Kwong [21], and Reid [33]. We should also mention here that inequality (1.1) has been generalized to second order nonlinear differential
where holds, where \( f \) holds for \( k \) with the clamped-free boundary conditions

\[
\alpha_k, \alpha_k^+ = 0, (1.7)
\]

which hold for \( k = 1, 2, \ldots, n \), \( u_k \) for \( k = 1, 2, \ldots, n \) are not identically zero on \([-s, s]\), \( 1 < p_k < \infty \) and \( \alpha_k \) for \( k, i = 1, 2, \ldots, n \) are nonnegative constants.

As an application, we have also investigated the lower bounds on the generalized eigenvalue \( (\lambda_1, \lambda_2, \ldots, \lambda_n) \) of the following problem

\[
(r_k(x) \phi_{p_k} (u_k'))' + \lambda kr(x) \phi_{a_{ik}} (u_k) \prod_{i=1}^{n} |u_i|^{a_{i}} = 0
\]
with the boundary conditions \( (1.8) \) for \( k = 1, 2, \ldots, n \) and \( r(x) \in C([-s,s], \mathbb{R}) \).

As usual, it is easier to find upper bounds for eigenvalues than lower bounds. In fact, they can be obtained by using elementary inequalities. Finding the estimated lower bounds is based on giving a suitable Lyapunov inequality for the corresponding systems. For readers who are interested in the existence of the generalized eigenvalues for the special case of system \((1.9)\), we refer to the paper by Napoli and Pinasco \([24]\).

Note that if we take \( \alpha_{ik} = p_{ik}, k = 1, 2, \ldots, n, \) and for \( i \neq k, \alpha_{ki} = 0 \) for \( i = 1, 2, \ldots, n, \) then we obtain uncoupled equations, i.e. the half-linear second order differential equations

\[
(r_k(x) \phi_{p_k}(u'_k))^' + f_k(x) \phi_{p_k}(u_k) = 0 \tag{1.10}
\]

for \( k = 1, 2, \ldots, n \) from system \((1.7)\). However, the equation \((1.4)\), which was considered by Watanabe et al. \([43]\), does not reduce to the equation \((1.10)\). Moreover, when \( n = 1 \) in the problem \((1.7)-(1.8)\) with \( r_1(x) = 1 \) and \( p_1 = 2 \) or \((1.4)-(1.5)\), we have the following linear problem

\[
\begin{cases}
  u''_1 + f_1(x)u_1 = 0, \\
  u_1(-s) = 0 = u'_1(s).
\end{cases} \tag{1.11}
\]

Thus, we obtain the following inequality

\[
\frac{1}{2s} < \int_s^s f_1^+(z)dz \tag{1.12}
\]

from Theorem A with \( n = 1 \) given by Watanabe et al. \([43]\).

In this paper, our motivation comes from the recent papers of Çakmak and Tiryaki \([9]\), Yang et al. \([40]\), and Watanabe et al. \([43]\). We prove a new Lyapunov-type inequality for the quasilinear systems of differential equations, we shall assume the existence of the nontrivial solution of system \((1.7)\) with the boundary conditions \((1.8)\).

Since our attention is restricted to the Lyapunov-type inequality for the quasilinear systems of differential equations, we shall assume the existence of the nontrivial solution of system \((1.7)\). For readers who are interested in the existence of the solution of this type of systems, we refer to the paper by Afrouzi and Heidarkhani \([1]\).

## 2 Main results

We prove a lemma which we will use in the proof of our main result.

**Lemma 2.1.** If \((u_1(x), u_2(x), \ldots, u_n(x))\) is a nontrivial solution of system \((1.7)\) satisfying the condition \(u_k(-s) = 0 = u'_k(s)\) for \( k = 1, 2, \ldots, n, \) then we have

\[
|u_k(z)| < \left( \int_{-s}^{s} r_k^{1/(p_k-1)}(v) \, dv \right)^{1/p_k} \left( \int_{-s}^{s} r_k(v) |u'_k(v)|^{p_k} \, dv \right)^{1/p_k} \tag{2.1}
\]

for \( z \in [-s,s] \) and \( k = 1, 2, \ldots, n. \)

**Proof.** Let \( u_k(-s) = 0 = u'_k(s) \) for \( k = 1, 2, \ldots, n \) where \( n \in \mathbb{N} \) and \( u_k \) for \( k = 1, 2, \ldots, n \) are not identically zero on \([-s,s]\). By using \( u_k(-s) = 0 \) and Hölder’s inequality, we get

\[
|u_k(z)| = \left| \int_{-s}^{z} u'_k(v) \, dv \right| \leq \int_{-s}^{z} |u'_k(v)| \, dv \leq \int_{-s}^{s} |u'_k(v)| \, dv = \int_{-s}^{s} r_k^{-1/p_k}(v) r_k^{1/p_k}(v) |u'_k(v)| \, dv \leq \left( \int_{-s}^{s} r_k^{1/(p_k-1)}(v) \, dv \right)^{1/(p_k-1)} \left( \int_{-s}^{s} r_k(v) |u'_k(v)|^{p_k} \, dv \right)^{1/p_k} \tag{2.2}
\]
for \( z \in [-s, s] \) and \( k = 1, 2, \ldots, n \). We claim that
\[
|u_k(z)|^{p_k} < \left( \int_{-s}^{s} r_k^{-1/(p_k-1)}(v) \, dv \right)^{p_k-1} \left( \int_{-s}^{s} r_k(v) |u_k'(v)|^{p_k} \, dv \right)
\]
for \( z \in [-s, s] \) and \( k = 1, 2, \ldots, n \). In fact, if (2.3) is not true, then it follows from (2.2) that
\[
\left( \int_{-s}^{s} |u_k'(v)|^{p_k} \, dv \right)^{p_k} = \left( \int_{-s}^{s} r_k^{-1/(p_k-1)}(v) \, dv \right)^{p_k-1} \left( \int_{-s}^{s} r_k(v) |u_k'(v)|^{p_k} \, dv \right), \quad k = 1, 2, \ldots, n,
\]
which, together with the Hölder’s inequality, implies that there exists a constant \( c \) such that
\[
r_k(x) |u_k'(x)|^{p_k} = cr_k^{-1/(p_k-1)}(x)
\]
for \( -s \leq x \leq s \) and \( k = 1, 2, \ldots, n \). If \( c = 0 \), then \( u_k'(x) = 0 \) for \( x \in [-s, s] \), it follows from (2.2) that \( u_k(z) = 0 \), which contradicts the fact that \( u_k(z) \neq 0 \) for \( z \in [-s, s] \) and \( k = 1, 2, \ldots, n \). If \( c \neq 0 \), then \( |u_k'(x)| > 0 \) for \( x \in [-s, s] \), it follows that \( u_k'(z) \neq 0 \) for \( z \in [-s, s] \) and \( k = 1, 2, \ldots, n \), which contradicts the fact that \( u_k'(s) = 0 \) for \( k = 1, 2, \ldots, n \). Therefore, the inequality (2.1) for \( z \in [-s, s] \) and \( k = 1, 2, \ldots, n \) holds.

Now, we give the main result of this paper.

**Theorem 2.2.** Assume that there exist nontrivial solutions \((e_1, e_2, \ldots, e_n)\) of the following linear homogeneous system
\[
e_k \left( 1 - \frac{\alpha_{kk}}{p_k} \right) - \sum_{i \neq k} \frac{\alpha_{ki}}{p_k} e_i = 0,
\]
where \( e_k \geq 0 \) for \( k = 1, 2, \ldots, n \). If \( f_k \in C([-s, s], \mathbb{R}) \) for \( k = 1, 2, \ldots, n \) and \((u_1(x), u_2(x), \ldots, u_n(x))\) is a nontrivial solution on \([-s, s]\) for problem (1.7)-(1.8), then the inequality
\[
1 < \prod_{k=1}^{n} \left[ \int_{-s}^{s} f_k^+(z) \prod_{i=1}^{n} \left( \int_{-s}^{s} r_i^{1/(1-p_i)}(v) \, dv \right)^{\alpha_{ki}(p_i-1)/p_i} \, dz \right]^{q_k}
\]
holds, where \( f_k^+(x) = \max \{0, f_k(x)\} \) for \( k = 1, 2, \ldots, n \).

**Proof.** Let \( u_k(-s) = 0 = u_k'(s) \) for \( k = 1, 2, \ldots, n \) where \( n \in \mathbb{N} \) and \( u_k \) for \( k = 1, 2, \ldots, n \) are not identically zero on \([-s, s]\). Multiplying the \( k \)-th equation of system (1.7) by \( u_k \), integrating from \(-s\) to \( s\), and by using boundary conditions (1.8), we get
\[
\int_{-s}^{s} r_k(z) |u_k'(z)|^{p_k} \, dz = \int_{-s}^{s} f_k(z) \prod_{i=1}^{n} |u_i(z)|^{p_{ki}} \, dz
\]
for \( k = 1, 2, \ldots, n \). By using the inequality (2.1) in (2.8), we obtain
\[
\int_{-s}^{s} r_k (z) |u_k'(z)|^{p_k} \, dz \\
\leq \int_{-s}^{s} f_k^+(z) \prod_{i=1}^{n} |u_i (z)|^{a_{i i}} \, dz \\
< \int_{-s}^{s} f_k^+(z) \prod_{i=1}^{n} \left( \left( \int_{-s}^{s} r_i^1(1-1/p_i)(v) \, dv \right)^{a_{i i}(p_i - 1)/p_i} \left( \int_{-s}^{s} r_i (v) |u_i'(v)|^{p_i} \, dv \right)^{a_{i i} / p_i} \right) \, dz \\
= \left[ \prod_{i=1}^{n} \left( \int_{-s}^{s} r_i (z) |u_i'(z)|^{p_i} \, dz \right)^{a_{i i}/p_i} \right] \times \left[ \int_{-s}^{s} f_k^+(z) \prod_{i=1}^{n} \left( \int_{-s}^{s} r_i^1(1-1/p_i)(v) \, dv \right)^{a_{i i}(p_i - 1)/p_i} \, dz \right]
\tag{2.9}
\]
for \( k = 1, 2, \ldots, n \). Now, we prove that \( 0 < \int_{-s}^{s} r_k (z) |u_k'(z)|^{p_k} \, dz \) for \( k = 1, 2, \ldots, n \). If the inequality \( 0 < \int_{-s}^{s} r_k (z) |u_k'(z)|^{p_k} \, dz \) is not true, then \( \int_{-s}^{s} r_k (z) |u_k'(z)|^{p_k} \, dz = 0 \) for \( k = 1, 2, \ldots, n \). If \( \int_{-s}^{s} r_k (z) |u_k'(z)|^{p_k} \, dz = 0 \), then it follows that
\[
u_k(x) \equiv 0
\tag{2.10}
\]
for \( -s \leq x \leq s \) and \( k = 1, 2, \ldots, n \). Combining (2.2) with (2.10), we obtain that \( u_k(z) = 0 \), which contradicts \( u_k(z) \neq 0 \) for \( z \in [-s, s] \) and \( k = 1, 2, \ldots, n \). Therefore,
\[
0 < \int_{-s}^{s} r_k (z) |u_k'(z)|^{p_k} \, dz
\tag{2.11}
\]
for \( k = 1, 2, \ldots, n \) holds. Thus, from (2.9) and (2.11), we have
\[
\left( \int_{-s}^{s} r_k (z) |u_k'(z)|^{p_k} \, dz \right)^{1 - \frac{a_{i k}}{p_k}} < \left[ \prod_{i=1, i \neq k}^{n} \left( \int_{-s}^{s} r_i (z) |u_i'(z)|^{p_i} \, dz \right)^{a_{i i} / p_i} \right] \times \left[ \int_{-s}^{s} f_k^+(z) \prod_{i=1}^{n} \left( \int_{-s}^{s} r_i^1(1-1/p_i)(v) \, dv \right)^{a_{i i}(p_i - 1)/p_i} \, dz \right]
\tag{2.12}
\]
for \( k = 1, 2, \ldots, n \). Raising both sides of the inequality (2.12) to the power \( \epsilon_k \) for each \( k = 1, 2, \ldots, n \), respectively, and multiplying the resulting inequalities side by side, we obtain
\[
\prod_{k=1}^{n} \left( \int_{-s}^{s} r_k (z) |u_k'(z)|^{p_k} \, dz \right)^{1 - \frac{a_{i k}}{p_k}} \epsilon_k < \prod_{k=1}^{n} \left[ \prod_{i=1, i \neq k}^{n} \left( \int_{-s}^{s} r_i (z) |u_i'(z)|^{p_i} \, dz \right)^{a_{i i} / p_i} \right] \times \prod_{k=1}^{n} \left[ \int_{-s}^{s} f_k^+(z) \prod_{i=1}^{n} \left( \int_{-s}^{s} r_i^1(1-1/p_i)(v) \, dv \right)^{a_{i i}(p_i - 1)/p_i} \, dz \right] \epsilon_k
\tag{2.13}
\]
and hence
\[
\prod_{k=1}^{n} \left( \int_{-s}^{s} r_k (z) |u_k'(z)|^{p_k} \, dz \right)^{1 - \frac{a_{i k}}{p_k}} \epsilon_k < \prod_{k=1}^{n} \left[ \prod_{i=1, i \neq k}^{n} \left( \int_{-s}^{s} r_k (z) |u_k'(z)|^{p_k} \, dz \right)^{\sum_{i=1, i \neq k}^{n} \frac{a_{i k}}{p_k} \epsilon_i} \right] \times \prod_{k=1}^{n} \left[ \int_{-s}^{s} f_k^+(z) \prod_{i=1}^{n} \left( \int_{-s}^{s} r_i^1(1-1/p_i)(v) \, dv \right)^{a_{i i}(p_i - 1)/p_i} \, dz \right] \epsilon_k.
\tag{2.14}
\]
Thus, we have
\[
\prod_{k=1}^{n} \left( \int_{-s}^{s} r_k(z) |u'_k(z)|^{p_k} \, dz \right)^{\theta_k} 
< \prod_{k=1}^{n} \left[ \int_{-s}^{s} f^+_k(z) \prod_{i=1}^{n} \left( \int_{-s}^{s} r_i^{1/(1-p_i)}(v) \, dv \right)^{a_i(p_i-1)/p_i} \, dz \right]^{c_k},
\]  
(2.15)

where
\[
\theta_k = e_k \left( 1 - \frac{\alpha_{kk}}{p_k} \right) - \sum_{i=1, i \neq k}^{n} \frac{\alpha_{ik}}{p_k} e_i
\]
for \( k = 1, 2, \ldots, n \). By assumption, system (2.6) has nontrivial solutions \((e_1, e_2, \ldots, e_n)\) such that \( \theta_k = 0 \) for \( k = 1, 2, \ldots, n \), where \( e_k \geq 0 \) for \( k = 1, 2, \ldots, n \) and at least one \( e_j > 0 \) for \( j = 1, 2, \ldots, n \). Choosing one of the solutions \((e_1, e_2, \ldots, e_n)\), we obtain the inequality (2.7) from (2.15). This completes the proof.

The proof of the following result proceeds along the lines of that of Corollary 1 in Yang et al. [40] and hence is omitted.

**Corollary 2.3.** Assume that
\[
\sum_{i=1}^{n} \alpha_{ik} = p_k \tag{2.16}
\]
for \( k = 1, 2, \ldots, n \). If \( f_k \in C([-s, s], \mathbb{R}) \) for \( k = 1, 2, \ldots, n \) and \((u_1(x), u_2(x), \ldots, u_n(x))\) is a nontrivial solution on \([-s, s]\) for problem (1.7)–(1.8), then the inequality
\[
1 < \prod_{k=1}^{n} \left[ \int_{-s}^{s} f^+_k(z) \prod_{i=1}^{n} \left( \int_{-s}^{s} r_i^{1/(1-p_i)}(v) \, dv \right)^{a_i(p_i-1)/p_i} \, dz \right] \tag{2.17}
\]
holds, where \( f^+_k(x) = \max\{0, f_k(x)\} \) for \( k = 1, 2, \ldots, n \).

**Remark 2.4.** If we take \( n = 1 \) and \( \alpha_{11} = p_1 \) in the problem (1.7)–(1.8), then we obtain the following half-linear problem
\[
\begin{cases}
(r_1(x) \phi_{p_1}(u'_1))' + f_1(x) \phi_{p_1}(u_1) = 0, \\
u_1(-s) = 0 = u'_1(s).
\end{cases}
\]  
(2.18)

Thus, we have the following inequality
\[
\left( \int_{-s}^{s} r_1^{1/(1-p_1)}(v) \, dv \right)^{1-p_1} < \int_{-s}^{s} f^+_1(z) \, dz
\]  
(2.19)

from the inequality (2.17) in Corollary 2.3. In addition to this, if we take \( p_1 = 2 \) and \( r_1(x) = 1 \) in the problem (2.18), then the inequality (2.19) reduces to the inequality (1.12) given by Watanabe et al. [43].

Now, we present an application of the Lyapunov-type inequality obtained for system (1.7).

We obtain the following result which gives lower bounds for the \( n \)-th component of any generalized eigenvalue \((\lambda_1, \lambda_2, \ldots, \lambda_n)\) of problem (1.9)–(1.8). The proof of the following theorem is based on above generalization of the Lyapunov-type inequality, as in that of Theorem 9 of Çakmak and Tiryaki [9] and hence is omitted.
**Theorem 2.5.** Assume that there exist nontrivial solutions \((e_1, e_2, \ldots, e_n)\) of system (2.6). Then there exists a function \(h_1(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})\) such that

\[
h_1(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) < |\lambda_n| \tag{2.20}
\]

for any generalized eigenvalue \((\lambda_1, \lambda_2, \ldots, \lambda_n)\) of problem (1.9)–(1.8), where

\[
h_1(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) = \left\{ \prod_{k=1}^{n-1} |\lambda_k|^{e_k} \left[ \prod_{k=1}^{n} \left( \frac{\int_{s-k}^{s} |r(z)| \prod_{i=1}^{n} \left( \int_{s-k}^{s} |r_i^{1/(1-p_i)} (v) dv \right)^{a_{ki}(p_i-1)/p_i} dz \right)^{e_k} \right] \right\}^{-\frac{1}{n}}. \tag{2.21}
\]

**Remark 2.6.** Since \(h_1\) is a continuous function, then \(h_1(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \to +\infty\) as any component of eigenvalue \(\lambda_k \to 0\) for \(k = 1, 2, \ldots, n - 1\). Therefore, there exists a ball centered in the origin such that the generalized spectrum is contained in its exterior. Also, by rearranging terms in (2.20) we obtain

\[
\prod_{k=1}^{n} \left( \frac{\int_{s-k}^{s} |r(z)| \prod_{i=1}^{n} \left( \int_{s-k}^{s} |r_i^{1/(1-p_i)} (v) dv \right)^{a_{ki}(p_i-1)/p_i} dz \right)^{e_k} < \prod_{k=1}^{n} |\lambda_k|^{e_k}. \tag{2.22}
\]

It is clear that when the interval collapses, left-hand side of (2.22) goes to infinity.

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**References**


