Existence of global solution for a nonlocal parabolic problem

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Abstract

In this paper, we study a non-local initial boundary-value problem arising in Ohmic heating. By using a dynamical systems approach, some existence and uniqueness results are proved and the existence of a compact attractor is shown.

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1 Introduction

In this paper, we shall deal with the following nonlocal parabolic problem

\[
\frac{\partial u}{\partial t} - \Delta u = \lambda \frac{f(u)}{\left( \int_{\Omega} f(u) \, dx \right)^2}, \quad \text{in } \Omega \times [0; T],
\]

\[
u = 0 \quad \text{on } \partial \Omega \times [0; T],
\]

\[
u/\tau = u_0 \quad \text{in } \Omega,
\]

where \( T > 0, \Omega \) is a regular open bounded subset of \( \mathbb{R}^N, N \geq 1 \), \( \lambda \) is a positive parameter and \( f \) a function from \( \mathbb{R} \) to \( \mathbb{R} \) satisfying the hypotheses \((H_1) - (H_2)\) below. Problem (1.1) \(- (1.3)\) represents, for example, static material such as thermistors [3, 6, 14, 15] and arises by reducing the system of two equations

\[
u_t = \nabla \cdot (k(u) \nabla u) + \sigma(u)|\nabla \varphi|^2,
\]

\[
\nabla (\sigma(u) \nabla \varphi) = 0,
\]

to a simple but realistic equation (see [8]). More precisely, \( u \) represents the temperature produced by an electric current flowing through a conductor, \( \varphi \) the electric potential, \( \sigma(u) \) is the electrical conductivity and \( k(u) \) is the thermal conductivity. Taking the latter to be constant, problem (1.4) \(- (1.5)\) can then be reduced to the single nonlocal equation (1.1), where \( f(u) = \sigma(u) \) and \( \lambda = \frac{I^2}{|\Omega|} \geq 0, I \) is the electric current which is supposed to be constant and \( |\Omega| \) is the measure of \( \Omega \).
Let us denote by $\| \cdot \|$ the norm in $L^2(\Omega)$. We adopt the following weak formulation for (1.1) - (1.3):
$u$ is a solution of (1.1) – (1.3) if and only if

$$u \in L^\infty(\tau, +\infty, H^1_0(\Omega) \cap L^\infty(\Omega)) \text{ with } \frac{\partial u}{\partial t} \in L^2(\tau, +\infty, L^2(\Omega))$$

for any $\tau > 0$, and satisfying

$$\int_0^T \int_\Omega u \frac{\partial \phi}{\partial t} - \nabla u \nabla \phi \, dx \, dt = \int_0^T \left( \frac{\lambda}{\int_\Omega f(u) \, dx} \int_\Omega f(u) \phi \, dx \right) \, dt,$$

for any $\phi \in C^\infty((0, \infty), \Omega)$.

Now, we state our main result.

**Theorem 2.1.** Let hypotheses $(H_1) – (H_2)$ be satisfied. Assume that $u_0 \in L^{k_0+2}(\Omega)$ with $k_0$ such that

$$k_0 \geq \max \left( 0, \frac{\alpha N}{2} - 2 \right).$$

Then, there exists $d_0 > 0$ such that if $\|u_0\|_{k_0+2} < d_0$, the problem (1.1) admits a solution $u$ verifying for all $\tau > 0$

$$u \in L^\infty(\tau, +\infty, L^{k_0+2}(\Omega)), \quad |u|^\gamma u \in L^\infty(\tau, +\infty, H^1_0(\Omega)), \text{ with } \gamma = \frac{k_0}{2}.$$  

Moreover, if $u_0 \in L^\infty(\Omega)$, then $u \in L^\infty(\tau, +\infty, L^\infty(\Omega))$ and is unique.

**Remark.** The value of $d_0$ will be given in the course of the proof.

**Proof.** We use a Faedo-Galerkin method see [10]. Let $u_m \subseteq D(\Omega)$ be such that $u_{0m} \to u_0$ in $H^1_0(\Omega)$ and let $(w_j)_j \subseteq H^1_0(\Omega)$ a special basis. We seek $u$ to be the limit of a sequence $(u_m)_m$ such that

$$u_m(t) = \sum_{j=1}^m g_{jm}(t)w_j,$$

where $g_{jm}$ is the solution of the following ordinary differential system

$$\begin{cases}
\left\langle u'_m, w_j \right\rangle + \left\langle u_m, w_j \right\rangle = \frac{\lambda}{\int_\Omega f(u_m) \, dx} \left\langle f(u_m), w_j \right\rangle, \quad 1 \leq j \leq m, \\
u_m(0) = u_{0m}.
\end{cases}$$

(2.2)

It is easy to see that (2.2) has a unique solution $u_m$ according to hypotheses $(H_1) – (H_2)$ and Cartan’s existence theorem concerning ordinary differential equations (see [5]). This solution is shown to exist on a maximal interval $[0; t_m]$. The following estimates enable us to assert that it can be continued on the whole interval $[0; T]$.

We shall denote by $C_i$ different positive constants, depending on data, but not on $m$. 

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Lemma 2.2. For any $\tau > 0$, there exists a constant $c_3(\tau), c_4(\tau)$ such that
\[
\|u_m(t)\|_{k_0+2} \leq c_3(\tau), \forall t \geq \tau, \tag{2.3}
\]
\[
\|u_m(t)\|_{\infty} \leq c_4(\tau), \forall t \geq \tau. \tag{2.4}
\]

Proof. (i) Multiplying the first equation of (2.2) by $|u_m|^k g_j m$, integrating on $\Omega$, adding from $j = 1$ to $m$ and using $(H_1) - (H_2)$, yields
\[
\frac{1}{k + 2} \frac{d}{dt} \|u_m\|_{k+2}^k + \frac{4}{(k + 2)^2} \|\nabla|u_m|^2 u_m\|_2^2 \leq c_3 \|u_m\|_{k+\alpha+2} + c_6. \tag{2.5}
\]

By using well-known Sobolev and Gagliardo-Nirenberg’s inequalities, we have
\[
\|u_m\|_{k_0+\alpha+2} \leq c_7 \|u_m\|_{k_0+2}^{\alpha} \|\nabla|u_m|^2 u_m\|_2^2, \tag{2.6}
\]
Thus, from (2.5) and (2.6), we obtain
\[
\frac{1}{k_0 + 2} \frac{d}{dt} \|u_m\|_{k_0+2}^k \leq \left(c_8 \|u_m\|_{k_0+2}^{\alpha} - \frac{4}{(k_0 + 2)^2}\right) \|\nabla|u_m|^2 u_m\|_2^2 + c_6. \tag{2.7}
\]
We shall make the following compatibility condition on $u_0$
\[
\|u_0\|_{k_0+2} < \left(\frac{4}{c_8(k_0 + 2)^2}\right)^{\frac{1}{\alpha}} = d_0. \tag{2.8}
\]
Then, there exists a small $\tau > 0$ such that
\[
\|u_m(t)\|_{k_0+2} < d_0 \text{ for } t \in [0, \tau]. \tag{2.9}
\]

Hence
\[
\frac{1}{k_0 + 2} \frac{d}{dt} \|u_m\|_{k_0+2}^k + c_9 \|\nabla|u_m|^2 u_m\|_2^2 \leq c_6 \quad \forall \quad 0 < t < \tau. \tag{2.10}
\]

By Poincaré’s inequality and after integrating, it follows that
\[
\|u_m(t)\|_{k_0+2} \leq c_{10}, \quad \forall \quad 0 < t < \tau,
\]
Therefore, relation (2.3) is achieved by iterating successively the same process on intervals of period $\tau$ such as $[0, \tau], [\tau, t + \tau], ....$

(ii) By using Hölder’s inequality, we get
\[
\|u_m\|_{k+\alpha+2} \leq c_{11} \|u_m\|_{k+2}^{\theta_1} \|u_m\|_{k_0+2}^{\theta_2} \|u_m\|_q^{\theta_3}, \tag{2.11}
\]
with $\theta_1, \theta_2$ and $\theta_3$ satisfying
\[
\frac{\theta_1}{k + 2} + \frac{\theta_2}{k_0 + 2} + \frac{\theta_3}{q} = 1 \quad \text{and} \quad \theta_1 + \theta_2 + \theta_3 = k + \alpha + 2.
\]

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We require moreover
\[ \frac{\theta_1}{k+2} + \frac{\theta_3}{2(\gamma + 1)} = 1. \]

Using the boundedness of \(\|u_m\|_{k+2}\), the choice of \(q\), Sobolev and Young’s inequalities and relation (2.11), we derive that
\[
c_5\|u_m\|_{k+2} \leq c_{12}\|u_m\|_{k+2}^{\theta_1}\|\nabla u_m\|^\gamma u_m \leq 2c_{13}(k+2)\theta_4\|u_m\|_{k+2}^{k+2} + \frac{2}{(k+2)^2}\|\nabla u_m\|^\gamma u_m^2,
\]

where \(\theta_4\) is some positive constant. Hence (2.5) becomes
\[
\frac{1}{k+2} \frac{d}{dt}\|u_m\|_{k+2} + \frac{c_{14}}{(k+2)^2}\|\nabla u_m\|^\gamma u_m^2 \leq c_{15}(k+2)\theta_4\|u_m\|_{k+2}^{k+2} + c_5.
\]

Therefore, by applying lemma 4 ([7]) we conclude to (2.4).

Passage to the limit in (2.2) as \(m \to \infty\). Multiplying the jth equation of system (2.2) by \(g_{jm}(t)\), adding these equations for \(j = 1, \ldots, m\) and integrating with respect to the time variable, we deduce the existence of a subsequence of \(u_m\) such that
\[
\begin{align*}
 u_m &\to u \text{ weak star in } L^\infty(0, T; L^2(\Omega)), \\
u_m &\to u \text{ weak in } L^2(0, T; H_0^1(\Omega)), \\
u_{mt} &\to u_t \text{ weak in } L^2(0, T; H^{-1}(\Omega)), \\
u_m &\to u \text{ strongly in } L^2(0, T; L^2(\Omega)) \text{ and a.e in } Q_T.
\end{align*}
\]

Straightforward standard compactness arguments allow us to assert that \(u\) is a solution of problem (1.1)

Uniqueness. Consider \(u_1\) and \(u_2\) two weak solutions of the problem (1.1) and define \(w = u_1 - u_2\). Subtracting the equations verified by \(u_1\) and \(u_2\), we obtain
\[
\frac{dw}{dt} - \Delta w = \frac{\lambda}{(\int_{\Omega} f(u_1) \, dx)^2} \left( f(u_1) - f(u_2) \right) \left( f(u_1) + f(u_2) \right) + \lambda \left( \frac{\int_{\Omega} f(u_1) \, dx}{(\int_{\Omega} f(u_1) \, dx)^2} \left( \int_{\Omega} f(u_2) \, dx \right) \right) f(u_2). \tag{2.12}
\]

Taking the inner product of (2.12) by \(w\) and using \((H_1)\) and (2.4), we get
\[
\frac{1}{2} \frac{d}{dt}\|w(t)\|^2 \leq c_{16}\|w(t)\|^2.
\]

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which implies that \( w = 0 \). Hence the solution is unique. \( \square \)

b) We denote by \( \{T(t), t \geq 0\} \) the continuous semi-group generated by (1.1) and defined by

\[
T(t) : L^\infty(\Omega) \to L^\infty(\Omega) \\
u_0 \to T(t)u_0 = u(t, \cdot).
\]

In this part, we refer to [12] for used concepts.

**Theorem 2.3.** Assume that \((H_1)-(H_2)\) are satisfied, Then \( T(t) \) possesses a maximal attractor which is bounded in \( H_0^1(\Omega) \cap L^\infty(\Omega) \), compact and connected in \( L^\infty(\Omega) \).

**Proof.**

(i) (2.4) imply that there exists an absorbing set in \( L^k(\Omega), \ 1 \leq k \leq \infty \).

(ii) To obtain existence of absorbing sets in \( H_0^1(\Omega) \) and the uniform compactness of \( T(t) \), multiply (2.2) by \( g_{jm}(t) \), add from \( j = 1 \) to \( m \) and integrate on \( \Omega \) by using Young’s inequality, it follows therefore that, for any \( t \geq \tau > 0 \)

\[
\int_{\Omega} \left( \frac{\partial u_m}{\partial t} \right)^2 dx + \frac{d}{dt} \| \nabla u_m \|_2^2 \leq c_{17}(\tau),
\]

which gives

\[
\frac{d}{dt} \| \nabla u_m \|_2^2 \leq c_{17}(\tau), \forall t \geq \tau > 0.
\]

On the other hand, multiplying (2.2) by \( g_{jm} \), adding and integrating on \( \Omega \times [t, t + \tau] \) we get

\[
\int_t^{t+\tau} \| \nabla u_m(s) \|_2^2 ds \leq c_{18}(\tau), \forall t \geq \tau > 0.
\]

Then, by the uniform Gronwall’s lemma (see [12], p.89) and the lower semi-continuity of the norm, we have

\[
\| \nabla u(t) \|_2^2 \leq c_{19}(\tau), \forall t \geq \tau.
\]

Therefore, the open ball \( B(0, c_{19}(\tau)) \) is an absorbing set in \( H_0^1(\Omega) \).

Hence, by theorem (1.1)( [12], p.23), we conclude to the results of theorem (2.3).

**Theorem 2.4.** We suppose \((H_1)-(H_2)\) and

(H3) \( f \in C^1(\mathbb{R}) \).
Then, we have
\[ y(t) \equiv ||u_t||^2 \leq c_{20}(\tau), \text{ for any } t \geq \tau > 0. \]

**Proof.** Differentiating equation (1.1) with respect to time (the justification of the formal derivatives can be done as in [5]), we get
\[
 u_t - \Delta u_t = \frac{\lambda f'(u) u_t}{(\int_{\Omega} f(u) \, dx)^2} - 2\lambda f(u) \int_{\Omega} f'(u) u_t \, dx.
\] (2.17)

Multiplying (2.17) by \( u_t \), integrating over \( \Omega \) and using the \( L^\infty \) estimate of \( u \) and H"older's inequality, yields
\[
 \frac{1}{2} y'(t) \leq c_{21}(\tau) y(t).
\] (2.18)

On the other hand, taking the scalar product of (1.1) with \( u_t \), using Young's inequality, integrating on \([t, t + \tau]\) and using estimate (2.16), then gives
\[
 \int_t^{t+\tau} y(s) ds \leq c_{23}(\tau), \text{ for any } t \geq \tau.
\] (2.19)

From (2.18) and the uniform Gronwall’s lemma, we have
\[
 y(t) \leq c_{23}(\tau), \text{ for any } t \geq \tau.
\]

Therefore,
\[
 u_t \in L^\infty(\tau, \infty, L^2(\Omega)).
\]

By (1.1), we then get
\[
 -\Delta u \in L^\infty(\tau, \infty, L^2(\Omega)),
\]
that is,
\[
 u(t) \text{ is in a bounded subset of } H^2(\Omega).
\]

Hence the existence of an absorbing set in \( H^2(\Omega) \) is shown.

**References**


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