On the stability of a fractional-order differential equation with nonlocal initial condition

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Abstract

The topic of fractional calculus (integration and differentiation of fractional-order), which concerns singular integral and integro-differential operators, is enjoying interest among mathematicians, physicists and engineers (see [1]-[2] and [5]-[14] and the references therein). In this work, we investigate initial value problem of fractional-order differential equation with nonlocal condition. The stability (and some other properties concerning the existence and uniqueness) of the solution will be proved.

Key words: Fractional calculus; Banach contraction fixed point theorem; Nonlocal condition; Stability.

1 Introduction

Let $L_1[a,b]$ denote the space of all Lebesgue integrable functions on the interval $[a,b]$, $0 \leq a < b < \infty$, with the $L_1$-norm $||x||_{L_1} = \int_a^b |x(t)| \, dt$.

Definition 1.1 The fractional (arbitrary) order integral of the function $f \in L_1[a,b]$ of order $\beta \in \mathbb{R}^+$ is defined by (see [11]-[14])

$$I_\beta^a f(t) = \int_a^t \frac{(t - s)^{\beta - 1}}{\Gamma(\beta)} f(s) \, ds,$$

where $\Gamma(.)$ is the gamma function.

Definition 1.2 The (Caputo) fractional-order derivative $D^\alpha$ of order $\alpha \in (0,1]$ of the function $g(t)$ is defined as (see [12]-[14])

$$D^\alpha_a g(t) = I_1^{1-\alpha} \frac{d}{dt} g(t), \quad t \in [a,b].$$

Now the following theorem (some properties of the fractional-order integration and the fractional-order differentiation) can be easily proved.
Theorem 1.1 Let $\beta, \gamma \in \mathbb{R}^+$ and $\alpha \in (0, 1]$. Then we have:

(i) $I^\beta_a : L_1 \to L_1$, and if $f(t) \in L_1$, then $I^\beta_a I^\gamma_a f(t) = I^{\gamma+\beta}_a f(t)$.

(ii) $\lim_{\beta \to n} I^\beta_a f(t) = I^n_a f(t)$, $n = 1, 2, 3, \ldots$ uniformly.

If $f(t)$ is absolutely continuous on $[a, b]$, then

(iii) $\lim_{\alpha \to 1} D^\alpha_a f(t) = D f(t)$

(iv) If $f(t) = k \neq 0$, $k$ is a constant, then $D^\alpha_a k = 0$.

In ([3]) the nonlocal initial value problem for first-order differential inclusions:

\[
\begin{cases}
x'(t) \in F(t, x(t)), & t \in (0, 1], \\
x(0) + \sum_{k=1}^{m} a_k x(t_k) = x_0,
\end{cases}
\]

was studied, where $F : J \times \mathbb{R} \to 2^\mathbb{R}$ is a set-valued map, $J = [0, 1]$, $x_0 \in \mathbb{R}$ is given, $0 < t_1 < t_2 < \cdots < t_m < 1$, and $a_k \neq 0$ for all $k = 1, 2, \cdots, m$.

Our objective in this paper is to investigate, by using the Banach contraction fixed point theorem, the existence of a unique solution of the following fractional-order differential equation:

\[
D^\alpha_a x(t) = c(t) f(x(t)) + b(t),
\]

with the nonlocal condition:

\[
x(0) + \sum_{k=1}^{m} a_k x(t_k) = x_0,
\]

where $x_0 \in \mathbb{R}$ and $0 < t_1 < t_2 < \cdots < t_m < 1$, and $a_k \neq 0$ for all $k = 1, 2, \cdots, m$. Then we will prove that this solution is uniformly stable.

2 Existence of solution

Here the space $C[0, 1]$ denotes the space of all continuous functions on the interval $[0, 1]$ with the supremum norm $||y|| = \sup_{t \in [0, 1]} |y(t)|$.

To facilitate our discussion, let us first state the following assumptions:

(i) $\left| \frac{\partial f}{\partial x} \right| \leq k$,

(ii) $c(t)$ is a function which is absolutely continuous,

(iii) $b(t)$ is a function which is absolutely continuous.
Definition 2.1 By a solution of the initial value Problem (1) - (2) we mean a function $x \in C[0, 1]$ with $\frac{dx}{dt} \in L_1[0, 1]$.

Theorem 2.1 If the above assumptions (i) - (iii) are satisfied such that

$$1 + \sum_{k=1}^{m} a_k \neq 0 \quad \text{and} \quad A < \frac{\Gamma(1 + \alpha)}{k \|c\|},$$

where $A = 1 + |a| \sum_{k=1}^{m} |a_k|$ and $a = \left(1 + \sum_{k=1}^{m} a_k\right)^{-1}$,

then the initial value Problem (1) - (2) has a unique solution.

Proof: For simplicity let $c(t)f(x(t)) + b(t) = g(t, x(t))$.

If $x(t)$ satisfies (1) - (2), then by using the definitions and properties of the fractional-order integration and fractional-order differentiation equation (1) can be written as

$$I^{1 - \alpha} x'(t) = g(t, x(t)).$$

Operating by $I^\alpha$ on both sides of the last equation, we obtain

$$x(t) - x(0) = I^\alpha g(t, x(t)),$$

by substituting for the value of $x(0)$ from (2), we get

$$x(t) = x_0 - \sum_{k=1}^{m} a_k x(t_k) + I^\alpha g(t, x(t)).$$

If we put $t = t_k$ in (3), we obtain

$$x(t_k) = x_0 - \sum_{k=1}^{m} a_k x(t_k) + I^\alpha g(t, x(t))|_{t=t_k}. \quad (4)$$

Then subtract (3) from (4) to get

$$x(t_k) = x(t) - I^\alpha g(t, x(t)) + I^\alpha g(t, x(t))|_{t=t_k}. \quad (5)$$

Substitute from (5) in (3), we get

$$x(t) = x_0 + I^\alpha g(t, x(t))$$

$$- \sum_{k=1}^{m} a_k \left( x(t) - I^\alpha g(t, x(t)) + I^\alpha g(t, x(t))|_{t=t_k} \right)$$

$$= x_0 + I^\alpha g(t, x(t))$$

$$- \sum_{k=1}^{m} a_k x(t) + \sum_{k=1}^{m} a_k I^\alpha g(t, x(t)) - \sum_{k=1}^{m} a_k I^\alpha g(t, x(t))|_{t=t_k},$$

$$\left(1 + \sum_{k=1}^{m} a_k\right) x(t) = x_0 - \sum_{k=1}^{m} a_k I^\alpha g(t, x(t))|_{t=t_k} + \left(1 + \sum_{k=1}^{m} a_k\right) I^\alpha g(t, x(t)),$$

$$x(t) = a \left( x_0 - \sum_{k=1}^{m} a_k I^\alpha g(t, x(t))|_{t=t_k} \right) + I^\alpha g(t, x(t)).$$

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Now define the operator \( T : C \rightarrow C \) by

\[
Tx(t) = a \left( x_0 - \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s)f(x(s)) + b(s) \, ds \right) + I^\alpha \{ c(s)f(x(s)) + b(s) \}.
\]

Let \( x, y \in C \), then

\[
Tx(t) - Ty(t) = -a \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s)f(x(s)) \, ds
+ a \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s)f(y(s)) \, ds
+ \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) \{ f(x(s)) - f(y(s)) \} \, ds,
\]

\[
|Tx(t) - Ty(t)| \leq k |a| \sum_{k=1}^{m} |a_k| \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} |c(s)||x(s) - y(s)| \, ds
+ k \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} |c(s)||x(s) - y(s)| \, ds
\leq k |a| \sum_{k=1}^{m} |a_k| \sup_t |c(t)| \sup_t |x(t) - y(t)| \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} \, ds
+ k \sup_t |c(t)| \sup_t |x(t) - y(t)| \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \, ds
\leq k |a| \sum_{k=1}^{m} |a_k| ||c|| \|x - y\| \frac{t_k}{\Gamma(1 + \alpha)}
+ k ||c|| \|x - y\| \frac{t^\alpha}{\Gamma(1 + \alpha)}
\leq \frac{k}{\Gamma(1 + \alpha)} \left( 1 + |a| \sum_{k=1}^{m} |a_k| \right) ||c|| \|x - y\|
\leq \frac{kA ||c||}{\Gamma(1 + \alpha)} \|x - y\| = K \|x - y\|.
\]

but since \( K = \frac{kA ||c||}{\Gamma(1 + \alpha)} < 1 \), then we get

\[
||Tx - Ty|| < K ||x - y||,
\]

which proves that the map \( T : C \rightarrow C \) is contraction. Applying the Banach contraction fixed point theorem we deduce that (7) has a unique fixed point \( x \in C[0,1] \).
Now, differentiate (6) to obtain

\[ x'(t) = \frac{d}{dt} I^\alpha (c(t) f(x(t)) + b(t)) = (c(t) f(x(t)) + b(t)) \bigg|_{t=0}^{t=1} - \Gamma(1 + \alpha) \frac{d}{dt} (c(t) f(x(t)) + b(t)) = K_1 \Gamma(1 + \alpha) + I^\alpha \left( c'(t) f(x(t)) + \frac{\partial f}{\partial x} x'(t) c(t) + b'(t) \right), \]

\[
\int_0^1 |x'(t)| \, dt \leq \frac{K_1}{\Gamma(1 + \alpha)} c^\alpha 1 + \frac{K_1}{\Gamma(1 + \alpha)} \int_0^1 \left| c'(s) f(x(s)) + \frac{\partial f}{\partial x} x'(s) c(s) + b'(s) \right| ds dt
\]

\[
\leq \frac{K_1}{\Gamma(1 + \alpha)} + \frac{1}{\Gamma(1 + \alpha)} \left( ||c'||_{L_1} ||f|| + k ||x'||_{L_1} ||c|| + ||b'||_{L_1} \right),
\]

\[
\left(1 - \frac{k ||c||}{\Gamma(1 + \alpha)}\right) ||x'||_{L_1} \leq \frac{K_1}{\Gamma(1 + \alpha)} + \frac{1}{\Gamma(1 + \alpha)} \left( ||c'||_{L_1} ||f|| + ||b'||_{L_1} \right),
\]

\[
||x'||_{L_1} \leq \left(1 - \frac{k ||c||}{\Gamma(1 + \alpha)}\right)^{-1} \left( \frac{K_1}{\Gamma(1 + \alpha)} + \frac{1}{\Gamma(1 + \alpha)} \left( ||c'||_{L_1} ||f|| + ||b'||_{L_1} \right) \right).
\]

Therefore we obtain that \( x' \in L_1[0,1]. \)

To complete the equivalence of equation (6) with the initial value problem (1) - (2), let \( x(t) \) be a solution of (6), differentiate both sides, and get

\[
x'(t) = \frac{d}{dt} I^\alpha g(t, x(t)) = g(t, x(t)) \bigg|_{t=0}^{t=1} - \Gamma(1 + \alpha) \frac{d}{dt} g(t, x(t)) = \frac{K_1}{\Gamma(1 + \alpha)} + \frac{1}{\Gamma(1 + \alpha)} \left( ||c'||_{L_1} ||f|| + ||b'||_{L_1} \right).
\]

Then operate by \( I^{1-\alpha} \) on both sides to obtain

\[
D^\alpha x(t) = g(t, x(t)).
\]

And if \( t = 0 \) we find that the nonlocal condition (2) is satisfied. Which proves the equivalence. \( \Box \)

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3 Stability

In this section we study the uniform stability (see [1], [4] and [6]) of the solution of the initial-value problem (1) - (2).

**Theorem 3.1** The solution of the initial-value problem (1) - (2) is uniformly stable

**Proof:** Let \( x(t) \) be a solution of

\[
x(t) = a \left( x_0 - \sum_{k=1}^{m} a_k \int_{0}^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} \{ c(s)f(x(s)) + b(s) \} \, ds \right) + I^\alpha \{ c(s)f(x(s)) + b(s) \}
\]

and let \( \tilde{x}(t) \) be a solution of equation (8) such that \( \tilde{x}(0) = \tilde{x}_0 - \sum_{k=1}^{m} a_k \tilde{x}(t_k) \). Then

\[
x(t) - \tilde{x}(t) = a \left( x_0 - \tilde{x}_0 \right) - a \sum_{k=1}^{m} a_k \int_{0}^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(x(s)) \, ds
\]

\[
+ a \sum_{k=1}^{m} a_k \int_{0}^{t} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(\tilde{x}(s)) \, ds
\]

\[
+ \int_{0}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) \{ f(x(s)) - f(\tilde{x}(s)) \} \, ds,
\]

\[
|x(t) - \tilde{x}(t)| \leq |a| |x_0 - \tilde{x}_0|
\]

\[
+ |a| \sum_{k=1}^{m} |a_k| \int_{0}^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} |c(s)| |f(x(s)) - f(\tilde{x}(s))| \, ds
\]

\[
+ \int_{0}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} |c(s)| |f(x(s)) - f(\tilde{x}(s))| \, ds
\]

\[
\leq |a| |x_0 - \tilde{x}_0|
\]

\[
+ k |a| \sum_{k=1}^{m} |a_k| \sup_{t} |c(t)| \int_{0}^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} |x(s) - \tilde{x}(s)| \, ds
\]

\[
+ k \sup_{t} |c(t)| \int_{0}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} |x(s) - \tilde{x}(s)| \, ds
\]

\[
\leq |a| |x_0 - \tilde{x}_0|
\]

\[
+ k |a| |c| \sum_{k=1}^{m} |a_k| \sup_{t} |x(t) - \tilde{x}(t)| \int_{0}^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} \, ds
\]

\[
+ k |c| \sup_{t} |x(t) - \tilde{x}(t)| \int_{0}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \, ds,
\]

\[
\|x - \tilde{x}\| \leq |a| |x_0 - \tilde{x}_0| + k |a| |c| \sum_{k=1}^{m} |a_k| |x - \tilde{x}| \frac{t_k^\alpha}{\Gamma(1 + \alpha)}
\]

\[
+ k |c| \|x - \tilde{x}\| \frac{t^\alpha}{\Gamma(1 + \alpha)}
\]

\[
\leq |a| |x_0 - \tilde{x}_0| + k |c| \|x - \tilde{x}\| \frac{t^\alpha}{\Gamma(1 + \alpha)} \left( 1 + |a| \sum_{k=1}^{m} |a_k| \right) \|x - \tilde{x}\|
\]

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\[ a |x_0 - \tilde{x}_0| \leq k A \|c\| \frac{\|x - \tilde{x}\|}{\Gamma(1 + \alpha)}, \]
\[ (1 - k A \|c\| \frac{1}{\Gamma(1 + \alpha)}) \|y - \tilde{y}\| \leq (1 - k A \|c\| \frac{1}{\Gamma(1 + \alpha)})^{-1} a |y_0 - \tilde{y}_0|. \]

Therefore, if \( |x_0 - \tilde{x}_0| < \delta(\varepsilon) \), then \( \|y - \tilde{y}\| < \varepsilon \), which complete the proof of the theorem.

**References**


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