

Large deviations for processes in random environments with jumps

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Abstract

A deterministic walk in a random environment can be understood as a general random process with finite-range dependence that starts repeating a loop once it reaches a site it has visited before. Such process lacks the Markov property. We study the exponential decay of the probabilities that the walk will reach sites located far away from the origin. We also study a similar problem for the continuous analogue: the process that is a solution to an ODE with random coefficients. In this second model the environment also has “teleports” which are the regions from where the process can make discontinuous jumps. .

Key words: large deviations, processes in random environments, deterministic walks in random environments.

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1 Introduction

A deterministic process in a random environment is a solution to a differential equation of the form $\frac{dX_t}{dt} = b(X_t, \omega)$, $X_0 = 0$, where $b : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ is a vector field defined on some probability space (Ω, \mathbb{P}) . We are interested in large deviation properties of the solution X_t . This process is called deterministic because once the environment is sampled from the probability space Ω , the entire process depends only on the initial position X_0 .

We will consider a modified version of the previous process, in which we assume certain randomness in time. The environment is placed in a Poisson point process, and small neighborhoods of the Poisson points serve as teleports. Once the particle spends sufficient time within a teleport, it is subject to a jump. However, whether a jump occurs or not depends on an additional time-dependent random sequence.

One motivation for studying such processes is a simplified model for the spread of viruses. Assume that the primary mode of infection spread is a local contact of members of a group. However, the mutation of other viruses into the virulent strain could appear in isolated areas where they are not introduced through direct contact. We model this phenomena by random jumps.

The model we study is also related to killed Brownian motions in Poisson potentials. The killed Brownian motion evolves according to a stochastic ODE, but once it spends too much time in a neighborhood of a Poisson point, it dies. Such processes are discussed in [14] where certain large deviation results are obtained. Our process continues to live, but reappears at another location.

Under assumptions on b and the environment, we will establish the following result about the moment generating functions of X_t .

Theorem 1.1. *Under the assumptions from Section 7 there exists a convex function $\Lambda : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ such that*

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{E} \left[e^{\lambda \cdot X_t} \right] = \Lambda(\lambda).$$

Here \mathbb{E} denotes the expected value with respect to (Ω, \mathbb{P}) . This is known as an *annealed* or *averaged* process.

To get an idea on how to approach the problem we first consider a discrete analogue – a deterministic walk in a random environment. We consider a random sequence X_n defined recursively as $X_{n+1} - X_n = b(X_n, \omega)$, for a suitable function $b : \mathbb{Z}^d \times \Omega \rightarrow \mathbb{Z}^d$. In this case we may assume that $b(x, \omega)$ is of the form $b(x, \omega) = \eta_x$, where η is a random field on \mathbb{Z}^d . The field η itself could be understood as the random environment in which the walk occurs. When the particular realization of the environment is fixed, the walk becomes deterministic. Our aim is to establish the following theorem.

Theorem 1.2. *Let $(\eta_z)_{z \in \mathbb{Z}^d}$ be a stationary \mathbb{Z}^d -valued random field that satisfies the assumptions (i)–(iii) (we refer to the assumptions from Section 2). Assume that the sequence $(X_n)_{n=1}^\infty$ of random variables is defined recursively as $X_0 = 0$, $X_{n+1} = X_n + \eta_{X_n}$. Then there exists a convex function $\Lambda : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ such that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} \left[e^{\lambda \cdot X_n} \right] = \Lambda(\lambda).$$

In order to make connections of this model with the one of the random walk in a random environment, we start from the representation of X_n as $X_{n+1} - X_n = b(X_n, \omega, \pi)$. Here (Ω, \mathbb{P}) and (Π, P)

are probability spaces and $b : \mathbb{Z}^d \times \Omega \times \Pi \rightarrow \mathbb{Z}^d$ is a random variable. In this model, (Ω, \mathbb{P}) is the environment, and for each fixed $\omega \in \Omega$, the walk X_n could be understood as a random walk on probability space (Π, P) . Recent works [16] and [7] have established the quenched and annealed large deviations for the random walk in a random environment under certain non-degeneracy assumptions on (Π, P) . The articles [11] and [17] found variational formulas for the rate functions and some connections between the two rate functions. The model we are studying is related to the annealed (averaged) case studied in the mentioned papers. Our probability space (Π, P) is very degenerate.

The main idea in our study is the adaptation of the subadditive argument. However, we need some work to find a quantity that happens to be subadditive. This quantity is found in Section 4 where we prove that it has the same exponential rate of decay as our large deviation probabilities. After that we are ready for the proofs of the main theorems which are presented in Section 5. Section 6 deals with consequences of large deviations such as the law of large numbers and the existence of a loop. The law of large numbers with limiting velocity 0 for the deterministic walk in random environment is not very surprising. Intuitively, the expected time until the occurrence of the loop is finite which means that the walk can't go too far.

We will restrict ourselves to environments with finite-range dependence. The special cases are iid environments where many of the presented arguments can be simplified. We are not assuming that the walk is nearest-neighbor. We do assume however, certain *ellipticity* conditions on the environment: There is a *nice* set of vectors such that using only these vectors the walk can move in any direction. Our ellipticity condition is the requirement that the probability that the walk at any position of the environment takes any particular vector from the nice set is uniformly positive. The existence of the nice set guarantees that there are no forbidden directions for the walk.

In the next section we state the assumptions we impose to the discrete model and state Theorem 2.1 which is the main focus of our study. In order to prove it we will need some deterministic lemmas that establish uniform bounds on probabilities that the walk can go in any direction. These statements are stated and proved in Section 3. Subadditive quantities are found in Section 4 and in Section 5 we use them to prove Theorem 5.2. The law of large numbers and the existence of a loop are proved in Section 6. In the last section we discuss the generalization of this approach to the continuous setting.

Many of the arguments here involve the consideration of the position of the particle at the time of the first self-intersection. Other interesting results regarding self-intersection times of random walks can be found in [1] and [2]. The hitting times and back-tracking times were a useful tool in establishing the law of large numbers for ballistic random diffusions (see [12]).

2 Definitions and Assumptions

Throughout the paper we will use \mathbb{R}_+ to denote the set of positive real numbers and $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. For $v \in \mathbb{R}^d$ we will use $|v|$ to denote its norm. The indicator function for a measurable set B will be denoted by $1(B)$. The ball with center x and radius r will be denoted by $B_r(x)$, or B_r for brevity if the center is clear from the context. For $x \in \mathbb{R}$ we will denote by $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) the largest integer not greater than x (resp. the smallest integer not smaller than x). The complement of a set Q will be denoted by Q^C .

Definition 2.1. The set of vectors $\{u_1, \dots, u_m\} \subseteq \mathbb{Z}^d$ is called nice if for every $l \in \mathbb{R}^d \setminus \{0\}$ there exists $i \in \{1, 2, \dots, m\}$ such that $l \cdot u_i > 0$.

Let $(\eta_z)_{z \in \mathbb{Z}^d}$ be a stationary \mathbb{Z}^d -valued random field that satisfies the following conditions:

- (i) There exists a positive real number L such that $|\eta_z| \leq L$ for all $z \in \mathbb{Z}^d$.
- (ii) There exists a real number M such that η_z is independent of the environment outside of the ball with center z and radius M .
- (iii) There exist a nice set of vectors $\{u_1, \dots, u_m\} \in \mathbb{Z}^d$ and a constant $c > 0$ such that $\mathbb{P}(\eta_z = u_i | \mathcal{F}_z) > c$ for all $i \in \{1, 2, \dots, m\}$, where \mathcal{F}_z is a sigma-algebra generated by all η_w for $w \in \mathbb{Z}^d$ such that $0 < |w - z| \leq M$.

The last assumption implies the existence of a loop in any half-space with a positive probability (see Theorem 3.1). It also implies that there exists a constant $c > 0$ such that $\mathbb{P}(\eta_z \cdot l > 0 | \mathcal{F}_z) > c$ for every z and every $l \in \mathbb{R}^d \setminus \{0\}$.

A special case is the iid environment when in condition (ii) we require $M < 1$. The condition (iii) is then replaced by $\mathbb{P}(\eta_z = u_i) > c$ for all i .

The random variable X_n is defined recursively as $X_0 = 0$ and

$$X_{n+1} = X_n + \eta_{X_n} \quad \text{for } n \geq 1.$$

We will use the following equivalent interpretation of X_n . The process X_n behaves like a random walk until the first self intersection. The increments of the random walk are sampled at every step according to the law of the random field η_z . After the self intersection occurs, the walk becomes deterministic and repeats the loop.

Here is the precise definition of the walk X_n that we will use. Let ξ_n be a \mathbb{Z}^d -valued random sequence defined recursively: $\xi_0 = 0$, and

$$\xi_n = \eta_{\xi_0 + \dots + \xi_{n-1}} \quad \text{for } n \geq 1. \tag{1}$$

Let us define $Y_n = \sum_{k=1}^n \xi_k$ and $\tau = \inf\{n : Y_n \in \{Y_1, \dots, Y_{n-1}\}\}$. If $\tau < \infty$, let $\theta < \tau$ be the smallest integer such that $Y_\tau = Y_\theta$. Denote by θ' the remainder when $n - \theta$ is divided by $\tau - \theta$. Formally, $\theta' = (n - \theta) - (\tau - \theta)\lfloor(n - \theta)/(\tau - \theta)\rfloor$. This remainder will belong to the set $\{0, 1, \dots, \tau - \theta - 1\}$.

We define the walk X_n using the formula:

$$X_n = \begin{cases} Y_n, & n \leq \tau \text{ or } \tau = \infty, \\ Y_{\theta + \theta'}, & \text{otherwise.} \end{cases}$$

Let $l \in \mathbb{R}^d$ be a unit vector. Define $T_m^l = \inf\{n : X_n \cdot l \geq m\}$. For $x \in \mathbb{R}^d$ let us denote by H_x^l the half-space through x determined by the vector l as

$$H_x^l = \mathbb{Z}^d \cap \{x + v : v \in \mathbb{R}^d, l \cdot v \geq 0\}.$$

For $s \in \mathbb{R}$ denote by Z_s^l the hyperplane

$$Z_s^l = \{x \in \mathbb{R}^d : x \cdot l = s\}.$$

Our goal is to prove the large deviations for X_n (see Theorem 5.2). We will be able to use the Ellis-Gärtner theorem to get some further bounds once we establish the following result:

Theorem 2.1. Let X_n be the random walk defined as above. Assume that the random environment satisfies the conditions (i)–(iii). For each unit vector $l \in \mathbb{R}^d$ there exists a concave function $\phi^l : \mathbb{R}_+ \rightarrow \bar{\mathbb{R}}$ such that for all $k \in \mathbb{R}_+$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq nk) = \phi^l(k). \quad (2)$$

Remark. Notice that $\phi^l(k) = \phi^{tl}(tk)$ for all $t \in \mathbb{R}_+$. Therefore $\phi^l(k) = \Phi\left(\frac{1}{k}l\right)$ for a suitable function $\Phi : \mathbb{R}^d \rightarrow \bar{\mathbb{R}}$. Notice that the concavity or even continuity of Φ does not follow from the properties of ϕ^l . One expects that Φ is continuous, but we are not able to prove that, and this is one of the major unsolved problems. In the end we will discuss a bit about difficulties we are facing in trying to address the continuity of Φ .

3 Existence of a Loop

In this section we prove that the previously defined random walk will have a loop in each half-space with a positive probability. This fact will be a consequence of the following elementary lemma. The lemma states that there exists a loop consisting entirely of vectors from a nice set.

Lemma 3.1. Let $\{u_1, \dots, u_m\}$ be a nice set of non-zero vectors. There exist non-negative integers q_1, q_2, \dots, q_m not all equal to 0 such that $q_1 u_1 + \dots + q_m u_m = 0$.

Proof. We will prove the statement using the induction on the dimension d . The statement is easy to prove for $d = 1$ and $d = 2$. We may assume that $\{u_1, \dots, u_m\}$ is a minimal nice set, i.e. there is no proper nice subset of $\{u_1, \dots, u_m\}$. If not, take the proper nice subset and repeat the argument. Let us fix the vector u_m , and let

$$v_i = u_i - \frac{u_i \cdot u_m}{|u_m|^2} u_m \text{ for } i = 1, \dots, m-1.$$

All vectors v_1, \dots, v_{m-1} have rational coordinates. Let r be the common denominator of those fractions and consider the lattice D of size $1/r$ in the vector space determined by the span W of v_1, \dots, v_{m-1} . Let us prove that the set $\{v_1, \dots, v_{m-1}\}$ is nice in W . Let $\tilde{l} \in W$ be a vector with real coordinates. There exists $i \in \{1, 2, \dots, m\}$ such that $u_i \cdot \tilde{l} > 0$. Since $\tilde{l} \in W$ we immediately have that $u_m \cdot \tilde{l} = 0$ and $u_i \cdot \tilde{l} = v_i \cdot \tilde{l}$ hence $v_i \cdot \tilde{l} > 0$. This implies that $\{v_1, \dots, v_{m-1}\}$ is a nice set of vectors in W . According to the induction hypothesis there are non-negative integers q'_1, \dots, q'_{m-1} such that $q'_1 v_1 + \dots + q'_{m-1} v_{m-1} = 0$. We now have

$$|u_m|^2 (q'_1 u_1 + \dots + q'_{m-1} u_{m-1}) = (q'_1 u_1 \cdot u_m + \dots + q'_{m-1} u_{m-1} \cdot u_m) u_m.$$

Let us now prove that $q'_1 u_1 \cdot u_m + \dots + q'_{m-1} u_{m-1} \cdot u_m \leq 0$. Assume the contrary, that this number were greater than 0. Since $\{u_1, \dots, u_{m-1}\}$ is not a nice set (due to our minimality assumption for $\{u_1, \dots, u_m\}$) there exists a vector $l \in \mathbb{R}^d$ such that $l \cdot u_m > 0$ but $l \cdot u_k \leq 0$ for each $k \in \{1, 2, \dots, m-1\}$. This gives that

$$0 \geq l \cdot (q'_1 u_1 + \dots + q'_{m-1} u_{m-1}) |u_m|^2 = (q'_1 u_1 \cdot u_m + \dots + q'_{m-1} u_{m-1} \cdot u_m) u_m \cdot l > 0,$$

a contradiction. Therefore $q'_1 u_1 \cdot u_m + \cdots + q'_{m-1} u_{m-1} \cdot u_m \leq 0$. We can now choose $q_i = |u_m|^2 q'_i$, $i = 1, 2, \dots, m-1$, and $q_m = -(q'_1 u_1 \cdot u_m + \cdots + q'_{m-1} u_{m-1} \cdot u_m)$ to obtain $q_1 u_1 + \cdots + q_m u_m = 0$. This completes the proof. \square

The paper [9] contains a proof that each point of the lattice can be accessed by making only moves governed by the vectors u_1, \dots, u_m .

The following theorem says that in each half-space H_x^l a walk X_n starting from x can have a loop in H_x^l with a probability that is strictly greater than 0. Because of the stationarity it suffices to prove this for half-spaces through the origin.

Theorem 3.1. *Let $U = \{u_1, \dots, u_m\}$ be a nice set of non-zero vectors. There exists a constant $m_0 \in \mathbb{N}$ such that: For each unit vector $l \in \mathbb{R}^d \setminus \{0\}$ there exist an integer $s \leq m_0$ and a sequence $x_0 = 0, x_1, x_2, \dots, x_s \in U$ with the following properties:*

- (i) $\sum_{i=0}^p x_i \cdot l \geq 0$ for all $0 \leq p \leq s$,
- (ii) $\sum_{i=0}^p x_i \neq \sum_{i=0}^q x_i$ for all $0 \leq p < q \leq s-1$, and
- (iii) $\sum_{i=0}^s x_i = 0$.

Proof. Using Lemma 3.1 we have that there exists a sequence $z_1, \dots, z_s \in \{u_1, \dots, u_m\}$ such that $z_1 + \cdots + z_s = 0$. Let us point out that z_i need not be distinct. Assume that this sequence is one of the sequences of minimal length. Let us choose the index π such that that $(z_1 + \cdots + z_\pi) \cdot l$ is minimal (of all $j = 1, 2, \dots, s$). Then $x_1 = z_{\pi+1}, x_2 = z_{\pi+2}, \dots, x_s = z_{\pi+s}$ (indices are modulo s) satisfy $x_1 + x_2 + \cdots + x_s = 0$, and

$$(x_1 + x_2 + \cdots + x_j) \cdot l \geq 0 \quad \text{for all } j = 1, 2, \dots, s.$$

Let us prove the last inequality. Assume the contrary, that $(x_1 + \cdots + x_j) \cdot l < 0$ for some j . Assume first that $\pi + j \leq s$. Then $(z_1 + \cdots + z_{\pi+j}) \cdot l = (z_1 + \cdots + z_\pi) \cdot l + (x_1 + \cdots + x_j) \cdot l < (z_1 + \cdots + z_\pi) \cdot l$ which contradicts the choice of π . If $\pi + j > s$ and $\pi + j \equiv t \pmod{s}$ then $0 = l \cdot (z_1 + \cdots + z_\pi + x_1 + \cdots + x_{s-\pi}) < l \cdot (z_1 + \cdots + z_t + x_1 + \cdots + x_{s-\pi}) = l \cdot (x_1 + \cdots + x_j) < 0$, a contradiction. This proves (i). The condition (ii) follows from the requirement that the sequence z_1, \dots, z_s is the shortest. \square

Let us recall the definition of ξ_i from (1). The following result establishes a useful property of the sequence x_0, \dots, x_s that we constructed in the previous theorem.

Theorem 3.2. *Let $y_i = x_0 + \cdots + x_{i-1}$ for $i \in \{1, 2, \dots, s\}$ and let us denote by $\mathcal{F}_{y_1, \dots, y_s}$ the σ -algebra generated by all random variables η_z for $z \in \mathbb{Z}^d \setminus \{y_1, \dots, y_s\}$ such that $\min_{1 \leq i \leq s} |z - y_i| \leq M$. Then:*

$$\mathbb{P}(\xi_1 = x_1, \xi_2 = x_2, \dots, \xi_s = x_s | \mathcal{F}_{y_1, \dots, y_s}) \geq c_1.$$

Proof. Denote by $\mathcal{G}_{y_1, \dots, y_{s-1}}$ the sigma algebra generated by all random variables η_z for $z \in \mathbb{Z}^d \setminus \{y_s\}$

such that $\min_{1 \leq i \leq s} |z - y_i| \leq M$.

$$\begin{aligned}
& \mathbb{P}(\xi_1 = x_1, \xi_2 = x_2, \dots, \xi_s = x_s | \mathcal{F}_{y_1, \dots, y_s}) \\
&= \mathbb{E} \left(\mathbb{E} \left(1(\xi_1 = x_1) \cdots 1(\xi_s = x_s) | \mathcal{G}_{y_s}^{y_1, \dots, y_{s-1}} \right) \middle| \mathcal{F}_{y_1, \dots, y_s} \right) \\
&= \mathbb{E} \left(1(\xi_1 = x_1) \cdots 1(\xi_{s-1} = x_{s-1}) \cdot \mathbb{E} \left(1(\xi_s = x_s) | \mathcal{G}_{y_s}^{y_1, \dots, y_{s-1}} \right) \middle| \mathcal{F}_{y_1, \dots, y_s} \right) \\
&> c \cdot \mathbb{P}(\xi_1 = x_1, \dots, \xi_{s-1} = x_{s-1} | \mathcal{F}_{y_1, \dots, y_s}) \\
&= c \cdot \mathbb{P}(\mathbb{P}(\xi_1 = x_1, \dots, \xi_{s-1} = x_{s-1} | \mathcal{F}_{y_1, \dots, y_{s-1}}) | \mathcal{F}_{y_1, \dots, y_s}).
\end{aligned}$$

Now we can continue by induction to obtain that

$$\mathbb{P}(\xi_1 = x_1, \dots, \xi_s = x_s | \mathcal{F}_{y_1, \dots, y_s}) > c^s.$$

It remains to take $c_1 = c^s$. □

Using the previous two lemmas we can establish the equality analogous to the one from Theorem 2.1 in which $k = 0$.

Theorem 3.3. For each vector $l \in \mathbb{R}^d$ and X_n defined as before the following equality holds:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq 0) = 0.$$

Proof. The inequality $\mathbb{P}(X_n \cdot l \geq 0) \leq 1$ implies that $\limsup \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq 0) \leq 0$. For the other inequality we use Theorem 3.1. Let x_1, \dots, x_s be the sequence whose existence is claimed by that theorem. Let $y_i = x_0 + \dots + x_{i-1}$ as before. Notice that

$$\begin{aligned}
\mathbb{P}(X_n \cdot l \geq 0) &\geq \mathbb{P}(\xi_1 = x_1, \xi_2 = x_2, \dots, \xi_s = x_s) \\
&= \mathbb{E} \left(\mathbb{P} \left(\xi_1 = x_1, \xi_2 = x_2, \dots, \xi_s = x_s \middle| \mathcal{F}_{y_1, \dots, y_s} \right) \right) \\
&\geq c_1.
\end{aligned}$$

Therefore $\liminf \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq 0) \geq \liminf \frac{1}{n} \log c_1 = 0$. □

We will also need the following deterministic lemma.

Lemma 3.2. Assume that $\{u_1, \dots, u_m\} \subseteq \mathbb{Z}^d$ is a nice set of vectors. Let $\rho : \mathbb{R}^d \rightarrow \mathbb{R}_+ \cup \{0\}$ be the function defined as $\rho(l) = \max_i \{u_i \cdot l\}$. Then

$$\inf_{l: |l|=1} \rho(l) > 0.$$

Proof. First notice that $\rho(l) > 0$ for each $l \in \mathbb{R}^d \setminus \{0\}$. Otherwise the set $\{u_1, \dots, u_m\}$ would not be nice. Notice also that ρ is a continuous function (because it is a maximum of m continuous functions) and the unit sphere is a compact set. Thus the infimum of ρ over the unit sphere must be attained at some point, and we have just proved that value of ρ at any single point is not 0. □

4 Hitting Times of Hyperplanes

The main idea for the proof of Theorem 2.1 is to establish the asymptotic equivalence of the sequence $\frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq nk)$ and a sequence to which we can apply the deterministic superadditive lemmas. First we will prove that the previous sequence behaves like $\frac{1}{n} \log \mathbb{P}(T_{nk}^l \leq n)$. Then we will see that the asymptotic behavior of the latter sequence satisfies

$$\frac{1}{n} \log \mathbb{P}(T_{nk}^l \leq n) \sim \frac{1}{n} \log \mathbb{P}(T_{nk}^l \leq n, T_{nk}^l \leq D_1^l)$$

where D_1^l is the first time the walk returns over the hyperplane Z_0^l . We define D_1^l precisely in the following way. First, let $B_1^l = \inf\{n > 0 : X_n \cdot l < 0\}$. Then $D_1^l = \inf\{n > B_1^l : X_n \cdot l \geq 0\}$. We allow D_1^l to be infinite. The event $\{T_{nk}^l \leq n, T_{nk}^l \leq D_1^l\}$ describes those walks that reach the hyperplane Z_{nk}^l without backtracking over the hyperplane Z_0^l .

We will be able to prove the existence of the limit of the last sequence using a modification of the standard subadditive result that states that $\lim_{n \rightarrow \infty} \frac{a_n}{n} = \inf_{n \rightarrow \infty} \frac{a_n}{n}$ if $a_{n+m} \leq a_n + a_m$ for all $m, n \in \mathbb{N}$.

From now on we fix the unit vector $l \in \mathbb{R}^d$ and omit the superscript l in the variables. Also, some of the new variables that will be defined would need to have a superscript l but we will omit it as well. Our first result in carrying out the formerly described plan is the following lemma:

Lemma 4.1. *The following inequality holds:*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq nk) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(T_{nk} \leq n)$$

In addition, for each $\varepsilon > 0$ we have:

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(T_{n(k+\varepsilon)} \leq n) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq nk).$$

Proof. Clearly, $\{X_n \cdot l \geq kn\} \subseteq \{T_{kn} \leq n\}$. Therefore $\mathbb{P}(X_n \cdot l \geq kn) \leq \mathbb{P}(T_{kn} \leq n)$. This establishes the first inequality. Let x_0, \dots, x_s be the sequence whose existence follows from Theorem 3.1. We now have

$$\begin{aligned} \mathbb{P}(X_n \cdot l \geq kn) &= \mathbb{P}(X_n \cdot l \geq kn, T_{kn} \leq n) \\ &\geq \mathbb{P}(X_n \cdot l \geq kn, T_{kn} \leq n-s, X_{T_{kn}+1} - X_{T_{kn}} = x_1, \dots, X_{T_{kn}+s} - X_{T_{kn}+s-1} = x_s) \\ &= \mathbb{P}(T_{kn} \leq n-s, X_{T_{kn}+1} - X_{T_{kn}} = x_1, \dots, X_{T_{kn}+s} - X_{T_{kn}+s-1} = x_s) \\ &= \mathbb{E} \left[\mathbb{1}(T_{kn} \leq n-s) \mathbb{E} \left[\mathbb{1}(\xi_{T_{kn}} = x_1) \cdots \mathbb{1}(\xi_{T_{kn}+s-1} = x_s) \mid \mathcal{F}_{T_{kn}} \right] \right] \\ &\geq c_1 \cdot \mathbb{P}(T_{kn} \leq n-s). \end{aligned} \tag{3}$$

Here $\mathcal{F}_{T_{kn}}$ denotes the σ -algebra defined by η_z for $z \in \mathbb{Z}^d$ such that $|z - X_i| \leq M$ for $i = 1, 2, \dots, T_{kn}$. The equality in (3) holds because if $T_{kn} \leq n-s$ and $X_{T_{kn}+1} - X_{T_{kn}} = x_1, \dots, X_{T_{kn}+s} - X_{T_{kn}+s-1} = x_s$, then the walk will enter in a loop. This loop will be in the half-space H_{kn} which would guarantee that $X_n \cdot l > kn$.

For each $\varepsilon > 0$, if $n > s + sk/\varepsilon$ we have $\{T_{(k+\varepsilon)(n-s)} \leq n-s\} \subseteq \{T_{kn} \leq n-s\}$, and consequently the first set has the smaller probability than the second. This completes the proof of the lemma. \square

Remark. In the same way we could obtain the analogous inequalities with the walk X_n replaced by $X_{n \wedge \tau}$.

For each integer $i \geq 1$ denote by D_i the time of the i th jump over the hyperplane Z_0 in the direction of the vector l . Define $D_0 = 0$, $B_i = \inf\{n > D_{i-1} : X_n \cdot l < 0\}$ and $D_i = \inf\{n > B_i : X_n \cdot l \geq 0\}$. We allow for D_i to be ∞ .

Lemma 4.2. *Let k and k' be two real numbers such that $0 < k' < k$. Then the following two inequalities hold:*

$$\begin{aligned} \limsup \frac{1}{n} \log \mathbb{P}(T_{nk} \leq n) &\leq \limsup \frac{1}{n} \log \mathbb{P}(T_{nk'} \leq n, T_{nk'} \leq D_1) \\ \liminf \frac{1}{n} \log \mathbb{P}(T_{nk} \leq n, T_{nk} \leq D_1) &\leq \liminf \frac{1}{n} \log \mathbb{P}(T_{nk} \leq n). \end{aligned}$$

Proof. The second inequality is easy and it follows from

$$\{T_{nk} \leq n\} \supseteq \{T_{nk} \leq n, T_{nk} \leq D_1\}.$$

Our goal is to prove the first inequality. We start by noticing that

$$\mathbb{P}(T_u \leq n) = \sum_{i=1}^n \mathbb{P}(T_u \leq n, D_i < T_u \leq D_{i+1}).$$

We will prove that each term from the right-hand side of the previous equality is bounded above by $L^d n^{d-1} \cdot \mathbb{P}(T_{u-L} \leq n, T_{u-L} \leq D_1)$. Let Z'_0 be the set of all points $z \in \mathbb{Z}^d$ such that $z \cdot l > 0$ and the distance between z and Z_0 is at most L . We have:

$$\begin{aligned} \mathbb{P}(T_u \leq n, D_i < T_u \leq D_{i+1}) &= \sum_{z \in Z'_0, |z| \leq nL} \mathbb{E} \left[\mathbf{1}(T_u \leq n) \cdot \mathbf{1}(X_{D_i} = z) \cdot \mathbf{1}(T_u > D_i) \right. \\ &\quad \left. \cdot \mathbf{1}(T_u \leq D_{i+1}) \right] \\ &= \sum_{z \in Z'_0, |z| \leq nL} \mathbb{E} \left[\mathbb{E} \left[\mathbf{1}(T_u \leq n) \cdot \mathbf{1}(X_{D_i} = z) \cdot \mathbf{1}(T_u > D_i) \right. \right. \\ &\quad \left. \left. \cdot \mathbf{1}(T_u \leq D_{i+1}) \middle| \mathcal{F}_{D_i, T_u} \right] \right]. \end{aligned}$$

Here \mathcal{F}_{D_i, T_u} denotes the σ -algebra generated by the random environment that is contained in the M -neighborhood of the walk from D_i to T_u . When conditioning on this σ -algebra we essentially understand our environment in the following way: It consists of two walks: One deterministic that goes from z to Z_u without crossing the hyperplane Z_0 , and another walk that starts at 0 and ends in z by making exactly i crossings over Z_0 , not intersecting the other deterministic walk, and not crossing over Z_u . Therefore:

$$\begin{aligned} &\sum_{z \in Z'_0, |z| \leq nL} \mathbb{E} \left[\mathbb{E} \left[\mathbf{1}(T_u \leq n) \cdot \mathbf{1}(X_{D_i} = z) \cdot \mathbf{1}(T_u > D_i) \cdot \mathbf{1}(T_u \leq D_{i+1}) \middle| \mathcal{F}_{D_i, T_u} \right] \right] \\ &\leq \sum_{z \in Z'_0, |z| \leq nL} \mathbb{E} \left[\mathbf{1}(\tilde{T}_{u-L} \leq n - D_i) \cdot \mathbf{1}(\tilde{T}_{u-L} \leq \tilde{D}_1) \cdot \mathbb{E} \left[\mathbf{1}(X_{D_i} = z) \cdot \mathbf{1}(T_u > D_i) \middle| \mathcal{F}_{D_i, T_u} \right] \right] \\ &\leq \sum_{z \in Z_0, |z| \leq nL} \mathbb{E} \left[\mathbf{1}(\tilde{T}_{u-L} \leq n) \cdot \mathbf{1}(\tilde{T}_{u-L} \leq \tilde{D}_1) \right], \end{aligned}$$

where \tilde{T}_u is defined in analogous way as T_u to correspond to the new walk $\tilde{X}_j = X_{D_i+j}$. In the last equation \tilde{D}_1 is defined as the first time of crossing over the hyperplane Γ parallel to Z_0 that is shifted by the vector $-Ll$. Let us now prove that $\mathbb{P}(\tilde{T}_{u-L} \leq n, \tilde{T}_{u-L} \leq \tilde{D}_1) \leq L^d n^{d-1} \cdot \mathbb{P}(T_{u-2L} \leq n, T_{u-2L} \leq D_1)$. Denote by J the time when the walk comes closest to the hyperplane Γ . For the precise definition of J we first introduce $\mu = \inf\{\text{dist}(X_n, \Gamma) : D_i < n \leq D_{i+1}\}$. Here $\text{dist}(X, \pi)$ denotes the distance between the point X and the hyperplane π . We now define $J = \inf\{n : \text{dist}(X_n, \Gamma) = \mu\}$. The number of possibilities for X_j is at most $L^d n^{d-1}$ and similarly as above, conditioning on the σ -algebra between J and T_{u-L} we get $\mathbb{P}(T_{u-L} \leq n, T_{u-L} \leq \tilde{D}_1) \leq L^d n^{d-1} \cdot \mathbb{P}(T_{u-2L} \leq n, T_{u-2L} \leq D_1)$. We now have

$$\mathbb{P}(T_u \leq n, D_i \leq T_u \leq D_{i+1}) \leq L^{2d} n^{2d-2} \mathbb{P}(T_{u-2L} \leq n, T_{u-2L} \leq D_1).$$

This implies that $\mathbb{P}(T_u \leq n) \leq L^{2d} n^{2d-1} \mathbb{P}(T_{u-2L} \leq n, T_{u-2L} \leq D_1)$. Therefore

$$\limsup \frac{1}{n} \log \mathbb{P}(T_{nk} \leq n) \leq \limsup \frac{1}{n} \log \mathbb{P}(T_{nk-2L} \leq n, T_{nk-2L} \leq D_1).$$

For each $k' < k$ we have

$$\limsup \frac{1}{n} \log \mathbb{P}(T_{nk} \leq n) \leq \limsup \frac{1}{n} \log \mathbb{P}(T_{nk'} \leq n, T_{nk'} \leq D_1).$$

This completes the proof of the lemma. \square

5 Large Deviations Estimates

Now we are able to prove the main theorem:

Proof of Theorem 2.1. We will prove that for each unit vector l there exists a concave function $\psi : \mathbb{R}_+ \rightarrow \bar{\mathbb{R}}$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(T_{nk} \leq n) = \psi(k). \quad (4)$$

Because of Lemma 4.2 it suffices to prove that there exists a concave function $\gamma : \mathbb{R}_+ \rightarrow \bar{\mathbb{R}}$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(T_{nk} \leq n, T_{nk} \leq D_1) = \gamma(k). \quad (5)$$

Let $w \in \mathbb{Z}^d$ be a vector such that $w \cdot l > 0$ and $\mathbb{P}(\eta_z = w | \mathcal{F}_z) \geq c > 0$ for some constant c . Assume that r is an integer such that the distance between the hyperplanes $Z_{r \cdot w}$ and Z_0 is at least M . If u, v, p, q are any four positive real numbers such that $q > r$, then the following inequality holds:

$$\mathbb{P}(T_{u+v} \leq p+q, T_{u+v} \leq D_1) \geq c \cdot \mathbb{P}(T_u \leq p, T_u \leq D_1) \cdot \mathbb{P}(T_v \leq q-r, T_v \leq D_1). \quad (6)$$

If the environment were iid this could have been done by conditioning on \mathcal{F}_u . In our situation the idea is the same, we just need some more work to compensate for the lack of independence.

$$\begin{aligned} \mathbb{P}(T_{u+v} \leq p+q, T_{u+v} \leq D_1) &\geq \mathbb{P}(T_{u+v} \leq p+q, T_{u+v} \leq D_1, T_u \leq p) \\ &\geq \mathbb{E} \left[1(T_u \leq p) \cdot 1(T_u \leq D_1) \cdot 1(\xi_{T_u} = w) \cdots \right. \\ &\quad \left. \cdot 1(\xi_{T_u+r-1} = w) \cdot 1(T_{u+v} \leq p+q) \cdot 1(T_{u+v} \leq D_1) \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[1(T_u \leq p) \cdot 1(T_u \leq D_1) \cdot 1(\xi_{T_u} = w) \cdots \cdots 1(\xi_{T_u+r-1} = w) \right. \right. \\ &\quad \left. \left. \cdot 1(T_{u+v} \leq p+q) \cdot 1(T_{u+v} \leq D_1) \middle| \mathcal{F}_{[X_{T_u}, X_{T_u+r \cdot l}]} \right] \right], \end{aligned}$$

where $\mathcal{F}_{[a,b]}$ is the σ -algebra determined by the environment outside of the strip $[Z_a, Z_b]$. Let us introduce the following notation: $\hat{X}_i = X_{T_u+r-1+i}$, \hat{D}_1 the first time \hat{X}_i jumps over $Z_{X_{T_u}+rw}$, and $\hat{T}_v = \inf\{i : (\hat{X}_i - \hat{X}_0) \cdot l \geq v\}$. We now have that

$$\begin{aligned} \mathbb{P}(T_{u+v} \leq p+q, T_{u+v} \leq D_1) &\geq \mathbb{E} \left[\mathbb{E} \left[1(T_u \leq p) \cdot 1(T_u \leq D_1) \cdot 1(\xi_{T_u} = w) \cdots 1(\xi_{T_u+r-1} = w) \right. \right. \\ &\quad \left. \left. \cdot 1(\hat{T}_v \leq q-r) \cdot 1(\hat{T}_v \leq \hat{D}_1) \middle| \mathcal{F}_{[X_{T_u}, X_{T_u+rw}]} \right] \right] \\ &= \mathbb{E} \left[1(T_u \leq p) \cdot 1(T_u \leq D_1) \cdot 1(\hat{T}_v \leq q-r) \cdot 1(\hat{T}_v \leq \hat{D}_1) \cdot \right. \\ &\quad \left. \mathbb{E} \left[1(\xi_{T_u} = w) \cdots 1(\xi_{T_u+r-1} = w) \middle| \mathcal{F}_{[X_{T_u}, X_{T_u+rw}]} \right] \right] \\ &\geq c \cdot \mathbb{P}(T_u \leq p, T_u \leq D_1) \cdot \mathbb{P}(T_v \leq q-r, T_v \leq D_1). \end{aligned}$$

Let us explain why one can bound $\mathbb{E}[1(\xi_{T_u} = w) \cdots 1(\xi_{T_u+r-1} = w) | \mathcal{F}_{[X_{T_u}, X_{T_u+rw}]}]$ below by c . The quantity in question is a random variable, and we are only considering that random variable on the set $\{T_u \leq D_1\} \cap \{\hat{T}_v \leq \hat{D}_1\}$. On that set the random walk doesn't visit the sites $X_{T_u}, \dots, X_{T_u+r-1}$ hence we can use our ellipticity assumption. This establishes (6). Let $\delta(m, x) = \log \mathbb{P}(T_{mx} \leq m, T_{mx} \leq D_1)$. Applying the inequality (6) to the numbers $u = nk$, $v = mk$, $p = n$, $q = m$ yields the following relation

$$\begin{aligned} \delta(m+n, k) &\geq \log c + \delta(n, k) + \log \mathbb{P}(T_{mk} \leq m-r, T_{mk} \leq D_1) \\ &\geq \log c + \delta(n, k) + \log \mathbb{P}(T_{mk+Lr} \leq m, T_{mk+Lr} \leq D_1) \\ &\geq \log c + \delta(n, k) + \delta(m, k'), \end{aligned}$$

where k' is any real number satisfying $mk' \geq mk + Lr$. From now on we will write c instead of $\log c$. In other words, for each k , and each $k' > k$ we have

$$\delta(m+n, k) \geq c + \delta(n, k) + \delta(m, k'), \quad (7)$$

for all m, n such that $m \geq \frac{Lr}{k'-k}$. Let

$$\underline{\delta}(k) = \liminf_{n \rightarrow \infty} \frac{\delta(n, k)}{n}, \quad \bar{\delta}(k) = \limsup_{n \rightarrow \infty} \frac{\delta(n, k)}{n}.$$

If $k < k'$ then for each $\alpha < \bar{\delta}(k')$ there exists a sequence m_t that goes to infinity as $t \rightarrow \infty$ and that satisfies $\frac{\delta(m_t, k')}{m_t} \geq \alpha$. Let us fix one such m_t . Each $n \geq m_t$ can be represented as $n = am_t + b$ for some $a \geq 0$ and $b \in \{0, 1, 2, \dots, m_t - 1\}$. Then we have

$$\begin{aligned} \frac{\delta(n, k)}{n} &= \frac{\delta(am_t + b, k)}{am_t + b} \geq \frac{a\delta(m_t, k') + \delta(b, k) + ac}{am_t + b} \\ &\geq \frac{\delta(m_t, k')}{m_t} \cdot \frac{1}{1 + \frac{m_t-1}{am_t}} + \frac{\delta(b, k)}{am_t + b} + \frac{c}{m_t + \frac{b}{a}} \\ &\geq \frac{\alpha}{1 + \frac{m_t-1}{am_t}} + \frac{b, k}{am_t + b} + \frac{c}{m_t + \frac{b}{a}}. \end{aligned}$$

Keeping m_t fixed and letting $a \rightarrow \infty$ we get that $\underline{\delta}(k) \geq \alpha$ for all $\alpha < \overline{\delta}(k')$. This immediately yields $\underline{\delta}(k) \geq \overline{\delta}(k')$ for all $k < k'$. Obviously $\overline{\delta}(k) \geq \underline{\delta}(k)$ hence we have

$$\underline{\delta}(k') \leq \overline{\delta}(k') \leq \underline{\delta}(k) \leq \overline{\delta}(k), \text{ for all } k < k'.$$

A consequence of the previous inequality is the monotonicity of the functions $\overline{\delta}$ and $\underline{\delta}$. They are both non-increasing. Let α and β be two positive rational numbers such that $\alpha + \beta = 1$. Let k_1 and k_2 be any two positive real numbers. According to (6) we know that for each $n \in \mathbb{N}$ we have:

$$\begin{aligned} \delta(n, \alpha k_1 + \beta k_2) &= \log \mathbb{P}(T_{n(\alpha k_1 + \beta k_2)} \leq \alpha n + \beta n, T_{n(\alpha k_1 + \beta k_2)} \leq D_1) \\ &\geq \log \mathbb{P}(T_{n\alpha k_1} \leq \alpha n, T_{n\alpha k_1} \leq D_1) + \\ &\quad \log \mathbb{P}(T_{n\beta k_2} \leq \beta n - r, T_{n\beta k_2} \leq D_1) + c \\ &\geq \delta(\alpha n, k_1) + \delta(\beta n, k_2) + c, \end{aligned}$$

for sufficiently large n , where k'_2 is any number larger than k_2 . This implies that

$$\begin{aligned} \overline{\delta}(\alpha k_1 + \beta k_2) &\geq \limsup \frac{1}{n} (\delta(\alpha n, k_1) + \delta(\beta n, k'_2)) \\ &\geq \alpha \underline{\delta}(k_1) + \beta \underline{\delta}(k'_2). \end{aligned} \tag{8}$$

Let us justify the second inequality. The previous lim sup is definitely larger than the lim inf over the sequence of those integers n that are divisible by the denominators of both α and β .

Consider now two positive real numbers $k < k'$ for which $\underline{\delta}(k')$ is a real number (i.e. not $-\infty$). Let α_n and β_n be two sequences of positive rational numbers such that $\alpha_n \rightarrow 1$, $\beta_n \rightarrow 0$, $\alpha_n + \beta_n = 1$. Let k'' be a real number such that $k < k'' < k'$. Then the inequality (8) implies:

$$\overline{\delta}(\alpha_n k + \beta_n k'') \geq \alpha_n \underline{\delta}(k) + \beta_n \underline{\delta}(k').$$

Taking the limit of both sides as $n \rightarrow \infty$ and using the monotonicity of $\overline{\delta}$ we get:

$$\lim_{\varepsilon > 0, \varepsilon \rightarrow 0} \overline{\delta}(k + \varepsilon) \geq \underline{\delta}(k).$$

This inequality together with $\underline{\delta}(k + \varepsilon/2) \geq \overline{\delta}(k + \varepsilon)$ implies that $\underline{\delta}$ is right-continuous.

Let us now choose the sequences $\alpha_n, \beta_n, k_n^1, k_n^2, k_n^3$ that satisfy:

$$\begin{aligned} \alpha_n, \beta_n &\in \mathbb{Q}_+, \quad k_n^1, k_n^2, k_n^3 \in \mathbb{R}_+, \\ \alpha_n + \beta_n &= 1, \quad \alpha_n \rightarrow 0, \quad \beta_n \rightarrow 1, \\ k_n^2 &< k_n^3 < k < k_n^1, \quad k_n^2, k_n^3 \nearrow k, \quad k_n^1 \searrow k, \\ \alpha_n k_n^1 &+ \beta_n k_n^2 = k. \end{aligned}$$

Then the inequality (8) implies

$$\overline{\delta}(k) \geq \alpha_n \underline{\delta}(k_n^1) + \beta_n \underline{\delta}(k_n^3).$$

Letting $n \rightarrow \infty$ and using the monotonicity of $\underline{\delta}$ gives us:

$$\overline{\delta}(k) \geq \lim_{\varepsilon > 0, \varepsilon \rightarrow 0} \underline{\delta}(k - \varepsilon).$$

Using the fact $\underline{\delta}(k - \varepsilon) \geq \overline{\delta}(k - \varepsilon/2)$ gives that $\overline{\delta}(k) \geq \overline{\delta}(k - 0)$. Here $\overline{\delta}(k - 0)$ is defined as

$$\overline{\delta}(k - 0) = \lim_{\varepsilon \rightarrow 0^+} \overline{\delta}(k - \varepsilon).$$

In other words, $\overline{\delta}$ is left-continuous.

Let us now choose the sequences:

$$\begin{aligned} \alpha_n, \beta_n &\in \mathbb{Q}_+, \quad k_n^1, k_n^2, k_n^3 \in \mathbb{R}_+, \\ \alpha_n + \beta_n &= 1, \quad \alpha_n \rightarrow 1, \quad \beta_n \rightarrow 0, \\ k_n^3 &> k_n^2 > k > k_n^1, \quad k_n^2, k_n^3 \searrow k, \quad k_n^1 \nearrow k, \\ \alpha_n k_n^1 + \beta_n k_n^2 &> k. \end{aligned}$$

Placing these sequences in the inequality (8) gives us the following relation:

$$\underline{\delta}(k) \geq \overline{\delta}(\alpha_n k_n^1 + \beta_n k_n^2) \geq \alpha_n \underline{\delta}(k_n^1) + \beta_n \underline{\delta}(k_n^3) \geq \alpha_n \overline{\delta}(k) + \beta_n \underline{\delta}(k_n^3).$$

Letting $n \rightarrow \infty$ implies $\underline{\delta}(k) \geq \overline{\delta}(k)$. Summing up all the facts we got for $\underline{\delta}$ and $\overline{\delta}$ we have that $\underline{\delta}(k) = \overline{\delta}(k) = \gamma(k)$ for some function γ . Moreover, γ is continuous and $\gamma(\alpha k_1 + \beta k_2) \geq \alpha \gamma(k_1) + \beta \gamma(k_2)$ for $\alpha, \beta \in \mathbb{Q}_+$ such that $\alpha + \beta = 1$. Because of the continuity, the last inequality holds for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha + \beta = 1$. This means that γ is concave and the equality (5) is established. Now we have the relation (4) and the concavity of the function ψ . Namely, Lemma 4.2 implies that $\psi = \gamma$.

Using Lemma 4.1 we get that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq nk) &\leq \psi(k), \quad \text{and} \\ \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq nk) &\geq \psi(k + \varepsilon), \end{aligned}$$

for all $\varepsilon > 0$. If k belongs to the interior of $\psi^{-1}(\mathbb{R})$, we can take $\varepsilon \rightarrow 0$ in the previous inequality and use the continuity of ψ to obtain

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq nk) \geq \psi(k).$$

This in turn implies (2) and the concavity of ϕ . □

The Ellis-Gärtner theorem will enable us to get some more information on lower and upper bound large deviations for general sets.

Definition 5.1. Assume that Λ^* is a convex conjugate of the function Λ . We say that $y \in \mathbb{R}^d$ is an exposed point of Λ^* if for some $\lambda \in \mathbb{R}^d$ and all $x \neq y$,

$$\lambda \cdot y - \Lambda^*(y) > \lambda \cdot x - \Lambda^*(x). \tag{9}$$

λ in (9) is called an exposing hyperplane. (see [4])

We are now ready to prove the theorem stated in the introduction.

Theorem 5.1. *There exists a convex function $\Lambda : \mathbb{R}^d \rightarrow \bar{\mathbb{R}}$ such that $\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} \left[e^{X_n \cdot \lambda} \right] = \Lambda(\lambda)$.*

Proof. As noted in the remark after Theorem 2.1 there exists a function $\Phi : \mathbb{R}^d \rightarrow \bar{\mathbb{R}}$ such that for all $l \in \mathbb{R}^d$ and $k \in \mathbb{R}_+$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq kn) = \Phi \left(\frac{1}{k} l \right).$$

For each $\lambda \in \mathbb{R}^d$ and each $k > 0$ we have that

$$\begin{aligned} \liminf \frac{1}{n} \log \mathbb{E} \left(e^{X_n \cdot \lambda} \right) &\geq \liminf \frac{1}{n} \log \mathbb{E} \left(e^{X_n \cdot \lambda} \cdot \mathbf{1}(X_n \cdot \lambda > kn) \right) \\ &\geq \liminf \frac{1}{n} \log \mathbb{E} \left(e^{kn} \cdot \mathbf{1}(X_n \cdot \lambda > kn) \right) = k + \Phi \left(\frac{1}{k} \lambda \right). \end{aligned}$$

Moreover, from Theorem 3.3 we get

$$\begin{aligned} \liminf \frac{1}{n} \log \mathbb{E} \left(e^{X_n \cdot \lambda} \right) &\geq \liminf \frac{1}{n} \log \mathbb{E} \left(e^{X_n \cdot \lambda} \right) \\ &\geq \liminf \frac{1}{n} \log \mathbb{E} \left(e^0 \cdot \mathbf{1}(X_n \cdot \lambda \geq 0) \right) = 0. \end{aligned}$$

Therefore

$$\liminf \frac{1}{n} \log \mathbb{E} \left(e^{X_n \cdot \lambda} \right) \geq \max \left\{ 0, \sup_{k>0} \left\{ k + \Phi \left(\frac{\lambda}{k} \right) \right\} \right\}.$$

From the boundedness of jumps of the random walk X_n we have that $|X_n \cdot \lambda| < L|\lambda|n$. Let $r \in \mathbb{N}$ and $0 = k_0 < k_1 < k_2 < \dots < k_r = L|\lambda|$. Then we have the equality:

$$\begin{aligned} \overline{\lim} \frac{1}{n} \log \mathbb{E} \left(e^{X_n \cdot \lambda} \right) &= \overline{\lim} \frac{1}{n} \log \left(\mathbb{E} \left(e^{X_n \cdot \lambda} \cdot \mathbf{1}(X_n \cdot \lambda \leq 0) \right) + \right. \\ &\quad \left. \sum_{i=0}^{r-1} \mathbb{E} \left(e^{X_n \cdot \lambda} \cdot \mathbf{1}(nk_i < X_n \cdot \lambda \leq nk_{i+1}) \right) \right). \end{aligned}$$

We now use the standard argument to bound each term of the sum by the maximal term.

$$\begin{aligned} \overline{\lim} \frac{1}{n} \log \mathbb{E} \left(e^{X_n \cdot \lambda} \right) &\leq \overline{\lim} \frac{1}{n} \log \left(\mathbb{P}(X_n \cdot \lambda \leq 0) + \sum_{i=0}^{r-1} e^{nk_{i+1}} \cdot \mathbb{P}(X_n \cdot \lambda > nk_i) \right) \\ &\leq \overline{\lim} \frac{1}{n} \log \left(\mathbb{P}(X_n \cdot \lambda \leq 0) + r \max_{0 \leq i \leq r-1} e^{nk_{i+1}} \cdot \mathbb{P}(X_n \cdot \lambda > nk_i) \right) \\ &= \max \left\{ 0, \max_{0 \leq i \leq r-1} \left\{ k_{i+1} + \Phi \left(\frac{\lambda}{k_i} \right) \right\} \right\}. \end{aligned}$$

The last equality is true because $r \in \mathbb{N}$ is a fixed number as $n \rightarrow \infty$ and Theorem 3.3 implies that $\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(X_n \cdot (-\lambda) \geq 0) = 0$. Theorem 2.1 implies that the function $\Phi \left(\frac{1}{k} \lambda \right)$ is continuous in k hence taking $r \rightarrow \infty$ and $k_{i+1} - k_i$ constant we get:

$$\limsup \frac{1}{n} \log \mathbb{E} \left(e^{X_n \cdot \lambda} \right) \leq \max \left\{ 0, \sup_{k>0} \left\{ k + \Phi \left(\frac{\lambda}{k} \right) \right\} \right\}.$$

This proves the existence of the limit from the statement of the theorem with

$$\Lambda(\lambda) = \max \left\{ 0, \sup_{k>0} \left\{ k + \Phi \left(\frac{\lambda}{k} \right) \right\} \right\}.$$

We will not use this representation for Λ to prove its convexity. Notice that all functions $\Lambda_n(\lambda) = \log \mathbb{E} \left(e^{X_n \cdot \lambda} \right)$ are convex when n is fixed. Indeed, for all $\alpha, \beta \in \mathbb{R}_+$ with $\alpha + \beta = 1$ and all $\lambda, \mu \in \mathbb{R}^d$ according to the Holder's inequality we have:

$$\begin{aligned} e^{\Lambda_n(\alpha\lambda + \beta\mu)} &= \mathbb{E} \left[\left(e^{X_n \cdot \lambda} \right)^\alpha \cdot \left(e^{X_n \cdot \mu} \right)^\beta \right] \\ &\leq \left(\mathbb{E} \left[e^{X_n \cdot \lambda} \right] \right)^\alpha \cdot \left(\mathbb{E} \left[e^{X_n \cdot \mu} \right] \right)^\beta = e^{\alpha\Lambda_n(\lambda) + \beta\Lambda_n(\mu)}. \end{aligned}$$

Since the limit of convex functions is convex, as well as the maximum of two convex functions, we are able to conclude that Λ is convex. \square

Since $\Lambda(\lambda) \leq L|\lambda|$, we have that the origin belongs to the interior of the set $\{\lambda \in \mathbb{R}^d : \Lambda(\lambda) < +\infty\} = \mathbb{R}^d$.

Theorem 5.2. *Let X_n be the previously defined deterministic walk in a random environment that satisfies the conditions (i)–(iii). Let Λ be the function from Theorem 5.1 and let Λ^* be its convex conjugate. Let \mathcal{F} be the set of exposed points of Λ^* . For any closed set $F \subseteq \mathbb{R}^d$,*

$$\limsup \frac{1}{n} \log \mathbb{P} \left(\frac{1}{n} X_n \in F \right) \leq - \inf_{x \in F} \Lambda^*(x),$$

and for any open set $G \subseteq \mathbb{R}^d$

$$\liminf \frac{1}{n} \log \mathbb{P} \left(\frac{1}{n} X_n \in G \right) \geq - \inf_{x \in G \cap \mathcal{F}} \Lambda^*(x).$$

Moreover, there exists $\delta > 0$ such that $\Lambda^*(x) < +\infty$ for $|x| < \delta$.

Proof. The conditions for Ellis-Gärtner theorem are now satisfied because of Theorem 5.1. Direct application of that result proves the first part of the statement.

For the second part, we will use Lemma 3.2. There exists $\kappa > 0$ such that for each $\lambda \in \mathbb{R}^d$ there exists $i \in \{1, 2, \dots, m\}$ such that $\lambda \cdot u_i > \kappa|\lambda|$. Then we have $\Lambda(\lambda) \geq \kappa|\lambda| + \Phi \left(\frac{\lambda}{\kappa|\lambda|} \right)$ and:

$$\Phi \left(\frac{\lambda}{\kappa|\lambda|} \right) \geq \liminf \frac{1}{n} \log \mathbb{P} (\xi_1 = \xi_2 = \dots = \xi_n = u_i) \geq c,$$

for some constant c . Therefore $\Lambda(\lambda) \geq \kappa|\lambda| + c$, and

$$\Lambda^*(x) \leq \sup_{\lambda} \{ \lambda \cdot x - \kappa|\lambda| - c \} \leq -c + \sup_{\lambda} \{ |\lambda| \cdot |x| - \kappa|\lambda| \}.$$

Hence if $|x| \leq \kappa$ then $\Lambda^*(x) \leq -c < +\infty$. \square

6 Law of Large Numbers

Let us conclude this study of deterministic walks with some notes about the law of large numbers. We will prove that our walk will have 0 as the limiting velocity. We will also prove that the walk will almost surely end in a loop.

Theorem 6.1. *If X_n is defined as before then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(X_n) = 0.$$

Proof. It suffices to prove that $\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(X_n \cdot l) = 0$ for each $l \in \mathbb{R}^d$, because the zero vector is the only one orthogonal to the entire \mathbb{R}^d . Furthermore, the problem can be reduced to proving that $\frac{1}{n} \mathbb{E}[X_n \cdot l]^+$ converges to 0 because $X_n \cdot l = (X_n \cdot l)^+ + (X_n \cdot (-l))^+$. By Fubini's theorem we have

$$\mathbb{E} \left[\left(\frac{1}{n} X_n \cdot l \right)^+ \right] = \int_0^{+\infty} \mathbb{P}(X_n \cdot l > nt) dt.$$

Since $\{X_n \cdot l > nt\} = \emptyset$ for $t > L$ the previous integration could be performed on the interval $(0, L)$ only. Let x_1, \dots, x_s be a sequence from Theorem 3.1, and let $y_k = \sum_{i=1}^k x_i$. Define the random walk Y_i as $Y_i = X_{s+i}$. The probability that the walk will reach the half-space H_{nt}^l before time n is smaller than the probability of the following event: The walk does not make a loop in first s steps, and after that it reaches the half-space H_{nt-sL}^l . Therefore we deduce that for each $t \in (0, L)$ the following inequality holds:

$$\begin{aligned} \mathbb{P}(X_n \cdot l > nt) &\leq \mathbb{E} \left[\mathbf{1}((X_1, \dots, X_s) \neq (y_1, \dots, y_s)) \cdot \mathbf{1}(Y_{n-s} \cdot l \geq nt - sL) \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\mathbf{1}((X_1, \dots, X_s) \neq (y_1, \dots, y_s)) \cdot \mathbf{1}(Y_{n-s} \cdot l \geq nt - sL) \mid \mathcal{F}_{Y_1, \dots, Y_{n-s}} \right] \right], \end{aligned}$$

where $\mathcal{F}_{Y_1, \dots, Y_s}$ is a σ -field determined by η_z for all $z \in \mathbb{Z}^d \setminus \{X_1, \dots, X_s\}$ such that $\min_{i=1}^n |z - X_i| \leq M$. The previous inequality now implies that

$$\begin{aligned} \mathbb{P}(X_n \cdot l \geq nt) &\leq \mathbb{E} \left[\mathbf{1}(Y_{n-s} \geq nt - sL) \cdot \right. \\ &\quad \left. \mathbb{E} \left[\mathbf{1}((X_1, \dots, X_s) \neq (y_1, \dots, y_s)) \mid \mathcal{F}_{Y_1, \dots, Y_{n-s}} \right] \right]. \end{aligned}$$

From Theorem 3.1 we have that $\mathbb{E} \left[\mathbf{1}((X_1, \dots, X_s) \neq (y_1, \dots, y_s)) \mid \mathcal{F}_{Y_1, \dots, Y_{n-s}} \right] \leq 1 - c$ for some constant $c > 0$. Let us denote $g = 1 - c$. We know that $g \in (0, 1)$. Using mathematical induction, we can repeat the previous sequence of inequalities $\lceil nt/sL \rceil$ times to obtain that $\mathbb{P}(X_n \cdot l \geq nt) \leq g^{\lceil nt/sL \rceil}$. Now we have that for all $t_0 > 0$ the following inequality holds:

$$\begin{aligned} \frac{1}{n} \mathbb{E}(X_n \cdot l)^+ &= \int_0^L \mathbb{P}(X_n \cdot l > nt) dt \\ &= \int_0^{t_0} \mathbb{P}(X_n \cdot l > nt) dt + \int_{t_0}^L \mathbb{P}(X_n \cdot l > nt) dt \\ &\leq t_0 + (L - t_0) \cdot g^{\lceil nt_0/sL \rceil} \\ &\leq t_0 + L \cdot g^{\lceil nt_0/sL \rceil}. \end{aligned}$$

If we keep t_0 fixed and let $n \rightarrow \infty$ it is easy to see that the last quantity converges to 0. Therefore $\limsup \frac{1}{n} \mathbb{E}(X_n \cdot l)^+ \leq t_0$. However, this holds for every $t_0 > 0$ hence $\limsup \frac{1}{n} \mathbb{E}(X_n \cdot l)^+ \leq 0$. This finishes the proof of the theorem. \square

Theorem 6.2. *Under the stated assumptions the walk makes a loop with probability 1.*

Proof. Let x_1, \dots, x_s be the sequence obtained in Theorem 3.1. Define the sequence y_1, \dots, y_s as: $y_i = x_1 + \dots + x_i$ for $i \in \{1, 2, \dots, s\}$. The sequence $y_1, y_2, \dots, y_s = 0$, is a loop. Theorem 3.1 required us to fix a vector l . We are guaranteed that the sequence y_1, \dots, y_s will be in the half-space orthogonal to l . It really does not matter which l was chosen, since any loop will be sufficient for the argument that follows.

Denote by B_n the event that $X_{ns+j} \in \{X_1, \dots, X_{ns+j-1}\}$ for some $j \in \{1, 2, \dots, s\}$. Let N denote the event that the walk doesn't have loops. Then for each $n \in \mathbb{N}$ we have $\mathbb{P}(N) \leq \mathbb{P}(B_0^C \cap B_1^C \cap \dots \cap B_n^C)$. We will prove that the latter probability converges to 0 as $n \rightarrow \infty$. We start by writing

$$\mathbb{P}(B_0^C \cap B_1^C \cap \dots \cap B_n^C) = \mathbb{P}(B_0^C \cap B_1^C \cap \dots \cap B_{n-1}^C) - \mathbb{P}(B_0^C \cap B_1^C \cap \dots \cap B_{n-1}^C \cap B_n).$$

For $i \in \{1, 2, \dots, s\}$ consider the following events:

$$K_i = \left(\bigcap_{j=0}^{i-1} \{X_{ns+j} \notin \{X_1, \dots, X_{ns+j-1}\}\} \right) \cap \{X_{ns+i} \in \{X_1, \dots, X_{ns+i-1}\}\}.$$

The events K_i are all disjoint and $B_n = \bigcup_{i=1}^s K_i$ hence

$$\begin{aligned} \mathbb{P}(B_0^C \cap B_1^C \cap \dots \cap B_{n-1}^C \cap B_n) &= \mathbb{E} \left[1(B_0^C) \cdot 1(B_1^C) \cdot \dots \cdot 1(B_{n-1}^C) \cdot 1(B_n) \right] \\ &= \sum_{i=1}^s \mathbb{E} \left[1(B_0^C) \cdot 1(B_1^C) \cdot \dots \cdot 1(B_{n-1}^C) \cdot 1(K_i) \right]. \end{aligned}$$

Consider the events:

$$\begin{aligned} K'_i &= \{ \{X_{ns} + y_1, \dots, X_{ns} + y_{i-1}\} \cap \{X_1, \dots, X_{ns}\} = \emptyset \} \cap \{X_{ns} + y_i \in \{X_1, \dots, X_{ns}\}\}, \text{ and} \\ K''_i &= \{X_{ns+1} = X_{ns} + y_1, X_{ns+2} = X_{ns} + y_2, \dots, X_{ns+i} = X_{ns} + y_i\}. \end{aligned}$$

We first notice that $K'_i \cap K''_i \subseteq K_i$. The events K'_i are all disjoint and $K'_s = \left(\bigcup_{i=1}^{s-1} K'_i \right)^C$ which implies that $1(K'_1) + \dots + 1(K'_s) = 1$.

Denote by \mathcal{G}_n the σ -algebra generated by the environment at the sites $\{X_1, \dots, X_{ns-1}\}$. The sets B_0, B_1, \dots, B_n are all measurable with respect to this σ -algebra. This is obvious for B_0, \dots, B_{n-1} . The measurability of B_n follows from the following observation: in order to determine whether B_n has happened or not, we actually do not need to know what is the environment at the site X_{ns} , because that environment can affect only the next jump, i.e. $X_{ns+1} - X_{ns}$. Similarly, for each i , the event K'_i is measurable with respect to \mathcal{G}_n because the sequence y_1, \dots, y_s is deterministic. Therefore

$$\begin{aligned} \mathbb{E} \left[1(B_0^C) \cdot 1(B_1^C) \cdot \dots \cdot 1(B_{n-1}^C) \cdot 1(K_i) \right] &\geq \mathbb{E} \left[1(B_0^C) \cdot 1(B_1^C) \cdot \dots \cdot 1(B_{n-1}^C) \cdot 1(K'_i) \cdot 1(K''_i) \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[1(B_0^C) \cdot \dots \cdot 1(B_{n-1}^C) \cdot 1(K'_i) \cdot 1(K''_i) \mid \mathcal{G}_n \right] \right] \\ &= \mathbb{E} \left[1(B_0^C) \cdot \dots \cdot 1(B_{n-1}^C) \cdot 1(K'_i) \cdot \mathbb{E} \left[1(K''_i) \mid \mathcal{G}_n \right] \right]. \end{aligned}$$

For each $i \in \{1, 2, \dots, s\}$ denote by \mathcal{H}_i the σ -algebra generated by the environment consisting of the M -neighborhood of the set $\{X_1, \dots, X_{ns}\} \cup \{X_{ns} + y_1, \dots, X_{ns} + y_{i-1}\}$ excluding the points $X_{ns}, X_{ns} + y_1, \dots, X_{ns} + y_{i-1}$. On the event $B_0^C \cap B_1^C \cap \dots \cap B_{n-1}^C \cap K_i'$ we have

$$\mathbb{E} \left[\mathbf{1}(K_i'') | \mathcal{G}_n \right] = \mathbb{E} \left[\mathbb{E} \left[\mathbf{1}(K_i'') | \mathcal{H}_i \right] | \mathcal{G}_n \right] \geq c^i \geq c^s. \quad (10)$$

The inequality (10) is proved using induction on i . The proof is exactly the same as the proof of Theorem 3.2.

Summing up inequalities (10) for $i = 1, 2, \dots, s$ gives us

$$\mathbb{P} \left(B_0^C \cap \dots \cap B_{n-1}^C \cap B_n \right) \geq c^s \mathbb{E} \left[B_0^C \cap \dots \cap B_{n-1}^C \cdot \sum_{i=1}^s \mathbf{1}(K_i') \right] = c^s \mathbb{P} \left(B_0^C \cap \dots \cap B_{n-1}^C \right).$$

Subtracting both sides of the previous inequality from $\mathbb{P} \left(B_0^C \cap \dots \cap B_{n-1}^C \right)$ yields

$$\mathbb{P} \left(B_0^C \cap \dots \cap B_{n-1}^C \cap B_n^C \right) \leq (1 - c^s) \cdot \mathbb{P} \left(B_0^C \cap \dots \cap B_{n-1}^C \right).$$

Using induction on n we now prove that

$$\mathbb{P} \left(B_0^C \cap \dots \cap B_{n-1}^C \cap B_n^C \right) \leq (1 - c^s)^n.$$

Since $1 - c^s < 1$, the sequence $(1 - c^s)^n$ converges to 0 and we conclude that $\mathbb{P}(N) = 0$. \square

7 Processes in Random Environments with Teleports

7.1 Introduction

Our aim is to study the continuous time process X_t on (Ω, \mathbb{P}) that solves the following ODE:

$$\frac{dX_t}{dt} = b \left(\tau_{X_t} \omega \right). \quad (11)$$

In order to prove Theorem 1.1 we will consider the probabilities $\mathbb{P}(X_t \cdot l > kt)$ for fixed $l \in \mathbb{R}^d$ and fixed $k > 0$.

Let us first outline the main difficulties we have in trying to implement the proof of Theorem 2.1 to continuous setting. The proof of Lemma 4.2 used the fact that by time n , the walk could jump only n times over the hyperplane through 0. The continuous process could jump infinitely many times over that hyperplane and those jumps could happen in relatively short time. Instead, we will look at a strip around 0 of positive width. By strip we mean the set $\{x \in \mathbb{R}^d : x \cdot l \in (\alpha, \beta)\}$ for $\alpha, \beta \in \mathbb{R}$, and a fixed vector $l \in \mathbb{R}^d$. The process can't travel over the strip infinitely many times, because the speed is finite. Thus we will consider the events that the process does not backtrack over the hyperplane Z_{-w} for suitable $w > 0$. We will prove that the probability of the event that the process doesn't backtrack over the hyperplane Z_{-w} is comparable to the probability that the process doesn't backtrack over Z_0 . The difficulty here comes from the fact that the process can approach Z_0 with low speed and it can in some way introduce a lot of dependence in the environment. This can happen especially if the vector field b is continuous. Similar difficulties are known in the study of a class of processes called Lorentz lattice gases (see, for example, [3], [5]). We introduce additional assumptions to have more randomness in the definition of X_t and that randomness helps the process to escape from such environments. So, we will have some assumptions that continuous processes can't satisfy. It turns out that the bounded process won't satisfy some of the requirements, either.

7.2 Description of the environment

We will assume the existence of teleports and discontinuities. They will help us build tunnels that can take the process in the directions we wish it to travel. Also we will be able to build traps that can hold the process for long time.

Before giving formal definition of the process and the environment, let us explain the introduction of Poisson teleports and the discontinuities in the environment. This subsection will not be completely rigorous but we believe that the explanations will help in understanding the motivation behind the formal definitions that will follow in Subsection 7.3.

We will consider a Poisson point process independent of the environment. We will assume that the balls of radius r centered at the points of the process serve as “teleports,” and our process evolves according to (11) until it spends a fixed positive time in some of the teleports. Once that happens, the process will reappear at another location. In some sense, these teleports correspond to the locations where b is infinite.

In order to use the arguments based on subadditivity we need the stationarity of the underlying random field. Our assumptions are going to make the grid \mathbb{Z}^d special, so we can't hope for general stationarity. We will assume only that $\mathbb{P}(b \in A)$ and $\mathbb{P}(b \in \tau_z A)$ have the same distributions for $z \in \mathbb{Z}^d$ (here A is a subset of the set of bounded piecewise continuous vector fields). This is going to be sufficient for our purposes, because we will assume that after each teleportation the process will appear at the point that experiences the same distribution as the initial point. The initial position of the process is assumed to be chosen uniformly inside the unit cube $[0, 1]^d$. The random choice of the point is assumed to be independent from the rest of the environment.

Because of the special role of the grid \mathbb{Z}^d the vector field b can not be continuous. This will require us to modify the notion of solution to (11).

The first idea is to associate a random vector field b_Q to each unit cube Q of \mathbb{Z}^d . However, then (11) does not have a unique solution because of the lack of continuity. Thus, for each cube Q of \mathbb{Z}^d we define a vector field b_Q on an open set that contains Q . While being in this larger set, the process follows the ODE, and once it leaves it, it is in the interior of the domain of another vector field. Since these larger sets overlap, some care is needed in defining the solution to (11).

We now want to define those larger sets that contain the lattice cubes. This can be done in many ways but here we pick the one that will turn out to be convenient for our later study.

7.3 Definition of the environment

Let $\varepsilon > 0$. Consider a unit cube Q whose vertices have all integral coordinates. For each vertex V of Q consider the points that are outside of Q , belong to the lines containing edges of Q , and are at a distance ε from Q . There are d such points corresponding to each V . Taking all 2^d vertices of Q we get a total of $d \cdot 2^d$ points. Denote by $c_\varepsilon(Q)$ the convex hull of all these points. The set $c_\varepsilon(Q)$ contains the cube Q in its interior. Figure 1 shows $c_\varepsilon(Q)$ in two dimensions.

Fix $r \in \left(0, \frac{1}{4\sqrt{d}}\right)$ and $\varepsilon \in \left(0, \frac{r}{2}\right)$. Our environment is given in the following way: For each lattice cube Q we have a vector field $b_Q : c_\varepsilon(Q) \times \Omega \rightarrow \mathbb{R}^d$ that is measurable on $c_\varepsilon(Q) \times \Omega$. For each unit cube Q of the lattice, denote by \mathcal{F}_Q the σ -algebra generated by the values of $b_{Q'}$ for all cubes $Q' \neq Q$. We make the following assumptions on the vector fields b_Q :

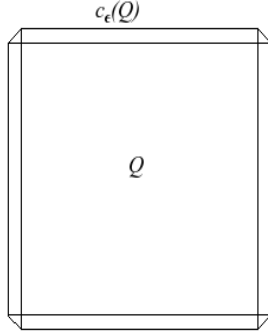


Figure 1: The domain $c_\varepsilon(Q)$ on which b_Q is defined.

- (i) There are positive real numbers c and δ_0 such that for each lattice cube Q of edge length 1 and each vector $l \in \{-1, 0, 1\}^d \setminus \{0\}$ we have:

$$\mathbb{P} \left(b_Q(x, \omega) \cdot l > \delta_0 |l| \text{ and } \frac{b_Q(x, \omega) \cdot l}{|l| \cdot |b_Q(x, \omega)|} > \frac{\sqrt{d-1}}{\sqrt{d}} \text{ for all } x \in c_\varepsilon(Q) \setminus \mathcal{F}_Q \right) > c.$$

- (ii) The vector fields b_Q are bounded, continuous, and have finite range dependence, i.e., there are positive constants L and M such that

$$\begin{aligned} |b_Q(x, \omega)| &\leq L, \quad \text{for all } (x, \omega) \in c_\varepsilon(Q) \times \Omega, \\ |b_Q(x, \omega) - b_Q(y, \omega)| &\leq L|x - y|, \quad \text{for all } x, y \in c_\varepsilon(Q) \text{ and } \omega \in \Omega, \end{aligned}$$

and b_Q is independent of the σ -field generated by the environment in the cells Q' for which $\text{dist}(Q', Q) > M$. In addition, we make an assumption that the sequence of vector fields b_Q is stationary.

- (iii) Fix $t_0 > 0$ such that $t_0 < \frac{r}{4L}$. Fix also $c_3 \in (0, 1)$, $\lambda_0 > 0$, and a sequence $(Y_n)_{n=0}^\infty$ of iid random variables with values in $\{\phi\} \cup [0, 1]^d$ that are independent on the environment. Each Y_n has a probability c_3 of being ϕ , or (with probability $1 - c_3$) it is uniformly distributed in $[0, 1]^d$. There exist a set of vectors $\{u_1, \dots, u_m\} \in \mathbb{R}^d$ such that for each $l \in \mathbb{R}^d \setminus \{0\}$ there is $i \in \{1, 2, \dots, m\}$ that satisfies

$$l \cdot u_i > 3\sqrt{d}.$$

For each i , we consider a Poisson point process \mathcal{P}_i of intensity λ_0 . These processes are assumed to be independent among themselves, independent on the environment, and on the sequence (Y_n) .

For each point x of the process \mathcal{P}_i , the ball $B_r(x)$ will be called a (u_i, t_0) -teleport. In the definitions of the process below, we will see that if the particle spent the time t_0 in a (u_i, t_0) -teleport, it will

have a chance of making a jump in the direction of the vector u_i . When there is no ambiguity, a (u_i, t_0) -teleport will be called simply u_i -teleport. We will often use word teleport for points of Poisson processes (not just for their r -neighborhoods).

7.4 Description of the process X_t

Before the precise definition of X_t let us describe the process informally. The initial position X_0 is uniformly distributed inside $[0, 1]^d$.

Together with the process we will define the sequence of lattice cubes $Q_1 = [0, 1]^d, Q_2, \dots$ and the sequence of times $\tau_0 = 0, \tau_1, \tau_2, \dots$ that tell us which b_Q is used in the defining ODE. The time τ_n can be thought to be the “expiration time” of the cube Q_n , and this is precisely the exit time from the interior of the set $c_\varepsilon(Q_n)$. The sequence τ will be increasing, while the cubes in the sequence Q may repeat.

The vector field b_{Q_j} is governing the process between times τ_{j-1} and τ_j . Since b_{Q_j} is Lipschitz in $c_\varepsilon(Q_j)$ the process X_t is properly defined by the equation (11). Even though these large sets $c_\varepsilon(Q)$ intersect, the process does not switch to the new vector field until it exits the old $c_\varepsilon(Q)$. Once it exits $c_\varepsilon(Q_j)$, it is going to be in the interior of another cube and then it starts following the ODE corresponding to the vector field of that cube and the initial point that is now in the interior of the domain where b is defined. We may think as of having many vector fields that overlap, but the process follows one vector field at any given time.

In addition to switching the environments the process may enter a teleport. Teleports are balls of radius r around Poisson points of processes \mathcal{P}_i . Once the process spends a lot of time inside a teleport i , it makes a jump. The vector for the jump is u_i .

7.5 Definition of the process X_t

We will define simultaneously the sequence $(Q_n)_{n=0}^\infty$ of cubes, the sequence $(\tau_n)_{n=0}^\infty$ of stopping times, and the process X_t .

We start by defining $\tau_0 = 0$, and $Q_1 = [0, 1]^d$. The position X_0 of the process is chosen uniformly at random inside the cube $[0, 1]^d$. Assume that $\tau_1, \dots, \tau_{k-1}$, and Q_1, \dots, Q_k are defined and that the process is defined for $t \in [0, \tau_{k-1}]$. We will now define Q_{k+1} , τ_k , and X_t for $t \in (\tau_{k-1}, \tau_k]$.

The process X_t satisfies

$$\frac{dX_t}{dt} = b_{Q_k}(X_t, \omega) \quad (12)$$

until one of the following two events happen:

- 1° There is an integer $i \in \{1, 2, \dots, m\}$ for which the process spent the entire time $(t - t_0, t)$ in the union of (u_i, t_0) -teleports (notice that teleports may overlap). Then at time t the process X_t does one of the following:
 - 1.1° It stays at the same place if $Y_{[t/t_0]} = \phi$ or there are two different i and j such that the process spent time $(t - t_0, t)$ in both (u_i, t_0) and (u_j, t_0) teleports.

1.2° If $Y_{\lfloor t/t_0 \rfloor} \neq \phi$, the process X_t will make a jump and reappear in the lattice cube containing $X_t + u_i$ at the relative position $Y_{\lfloor t/t_0 \rfloor}$ within the cube.

In each of the cases 1.1° or 1.2° we declare the moment t as a moment in which the process is not in any of the teleports. (This way the time restarts if the process spent t_0 time in a teleport, and we won't have infinitely many successive teleportations in a short time.)

If the jump occurred then we define $\tau_k = t$, and Q_{k+1} is the cube where the process landed after teleportation. If the jump did not occur the process keeps following (12).

2° The process has hit the boundary of $c_e(Q_k)$. Then we define $\tau_k = t$ and Q_{k+1} is the cube that contains X_t . There could be two or more candidates for Q_{k+1} but this is a zero-probability event, and in this case we could choose in any way.

Notice that by fixing the time t_0 we make sure that the average speed of X_t remains bounded. Although there could be a teleportation involved, the particle has to wait to be teleported and the distance it can go is bounded.

There could be regions belonging to more than one teleport. If the particle is a subject to two or more different teleportations, then the jump will be suppressed. A particle could be subject to more than one teleportation if it enters a teleport by a jump. Entering two teleports simultaneously by the means of diffusion is a zero-probability event.

The requirement $t_0 < \frac{r}{4L}$ guarantees that if the process comes to within $r/2$ of a Poisson point, it is going to be teleported for sure, because it won't have enough time to escape.

In the rest of the paper we will write just b instead of b_Q whenever it is clear from the context which Q we are dealing with.

7.6 Main Results

We consider the moment-generating function $\mathbb{E}(\exp(\lambda \cdot X_t))$ for $\lambda \in \mathbb{R}^d$. Our goal is to prove that there is a convex function Λ such that

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{E}(\exp(\lambda \cdot X_t)) = \Lambda(\lambda).$$

We now fix a vector $l \in \mathbb{R}^d \setminus \{0\}$ and recall the definition of the hitting times T_p (for $p \in \mathbb{R}_+$) of hyperplanes from Section 4.

We have an analogous result to Lemma 4.1.

Lemma 7.1. *The following inequality holds:*

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{P}(X_t \cdot l \geq tk) \leq \limsup_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{P}(T_{tk} \leq t)$$

In addition, for each $\varepsilon > 0$ we have:

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{P}(T_{t(k+\varepsilon)} \leq t) \leq \liminf_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{P}(X_t \cdot l \geq tk).$$

Proof. The first inequality follows immediately as in the discrete case. For the second one we have to modify the argument a bit. We are not able to construct a loop as we did in the discrete case. The reason is that the curve X_t may not have a self intersection. It is known that in the case of continuous gradient vector fields the process will never have a loop.

Let us define

$$S = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_i \in \{-1, 0, 1\}, i = 1, 2, \dots, d\}.$$

Denote the points of S by $P_0, P_1, P_2, \dots, P_{3^d-1}$. Assume that P_0 coincides with the origin O . Let C_i ($0 \leq i \leq 3^d - 1$) be the translation of the cube $[0, 1]^d$ for the vector $\overrightarrow{OP_i}$. For each integer $i \in \{1, 2, \dots, 3^d - 1\}$, consider the event

$$D_i = \left\{ \omega \in \Omega : b_{C_i}(z, \omega) \cdot \overrightarrow{P_i O} > \frac{\delta_0}{2} |\overrightarrow{P_i O}| \text{ and } \frac{b_{C_i}(z, \omega) \cdot \overrightarrow{P_i O}}{|b_{C_i}(z, \omega)| \cdot |\overrightarrow{P_i O}|} > \frac{\sqrt{d-1}}{\sqrt{d}} \text{ for all } z \in c_\varepsilon(C_i) \right\}. \quad (13)$$

The constant $\delta_0 > 0$ is chosen according to the assumption (i) on the vector fields b_Q . Let $D = \bigcap_{i=1}^{3^d-1} D_i \cap D'$, where D' is the event that there are no Poisson points in the r -neighborhood of $c_\varepsilon(C_0)$. Denote by $\mathcal{F}_{C_1, \dots, C_{3^d-1}}$ the sigma algebra generated by the environment influencing $b_{Q'}$ for $Q' \notin \{C_1, \dots, C_{3^d-1}\}$. According to our assumptions, we know that there is a constant $c' > 0$ such that

$$\mathbb{P}(D \mid \mathcal{F}_{C_1, \dots, C_{3^d-1}}) > c'.$$

We now prove that if the process ever enters the cube C_0 and gets under the influence of b_{C_0} , it will stay forever in the closure of $c_\varepsilon(C_0)$. We will show that if $Q_k = C_0$, then for each $t \geq t_{k-1}$ we have $X_t \in c_\varepsilon(C_0)$. Assume the contrary, that $X_t \in C_i \setminus \overline{c_\varepsilon(C_0)}$ for some $i \in \{1, \dots, 3^d - 1\}$. We may also assume that C_i is the first cube to which X_t has escaped. Let $Y_t = \overrightarrow{OX_t} \cdot \overrightarrow{P_i O}$. Set

$$t_1 = \inf\{s : X_u \in C_i \setminus c_\varepsilon(C_0) \text{ for all } u \in (s, t)\}.$$

Then X_{t_1} belongs to the boundary of $c_\varepsilon(C_0)$ and because of our choice of c_ε we can guarantee that $Y_t < Y_{t_1}$. Using the fundamental theorem of calculus we obtain

$$0 > Y_t - Y_{t_1} = \int_{t_1}^t \overrightarrow{P_i O} \cdot b(X_s, \omega) ds > 0,$$

since $\overrightarrow{P_i O} \cdot b(X_s, \omega) > 0$ for all $s \in (t_1, t)$ on the event D_i . This is a contradiction.

Now we can finish the proof in the similar way as in the discrete case. Let S_{kt} denote the “shard”-like surface consisting of faces of the grid of size 3 that is in front of the plane Z_{tk} (here “in front of” means with respect to the direction l). Here is the precise definition of S_{tk} . For each point x of the hyperplane Z_{tk} , consider all points z of the integral lattice such that $z \cdot l > x \cdot l$ and denote by z_x the one that minimizes the distance from the point x . Consider the hyperplanes of the integral lattice that pass through z_x . They divide the space in 2^d subsets (d -dimensional dihedral angles) only one of which does not contain points of Z_{tk} in its interior. For every x we form one such subset (notice that there will be only countably many of them). Denote by Σ_{tk} their union, and S_{tk} is the boundary of Σ_{tk} . The Figure 2 shows how S_{tk} looks in 2 dimensions. Let

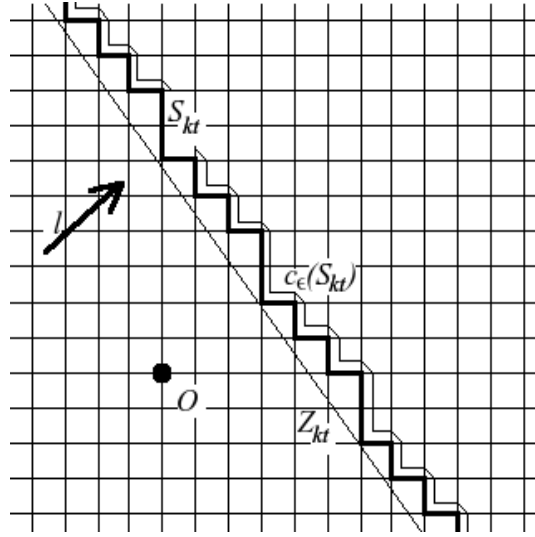


Figure 2: Surfaces $c_{\epsilon}(S_{kt})$, S_{kt} , hyperplane Z_{kt} , and vector l

$$\hat{T}_{tk} = T_{c_{\epsilon}(S_{tk})},$$

where $c_{\epsilon}(S_{tk})$ is defined in the following way: Denote by \mathcal{Q} the set of all lattice cubes Q that share points with S_{tk} but are outside of Σ_{tk} . Denote

$$U_{tk} = \bigcup_{Q \in \mathcal{Q}} c_{\epsilon}(Q),$$

and take $c_{\epsilon}(S_{tk})$ to be the portion of the boundary of U_{tk} that is inside Σ_{tk} .

Denote by C the translation of cube $\bigcup_{i=0}^{3^d-1} C_i$ with the following properties: C contains the point $X_{\hat{T}_{tk}}$ on one of its faces, and is on different side of S_{tk} than the origin. Denote by \hat{D} the event that the environment in C is the translation for of the environment defined by (13) and the paragraph below it. Denote by $\mathcal{F}_{\hat{T}_{tk}}$ the σ -algebra of the environment and teleports determined by the cubes that the walk was influenced by the time \hat{T}_{tk} . Using conditional expectations we get

$$\begin{aligned} \mathbb{P}(X_t \cdot l \geq kt) &= \mathbb{P}(X_t \cdot l \geq kt, T_{kt} \leq t) \\ &\geq \mathbb{P}(X_t \cdot l \geq kt, \hat{T}_{kt} \leq t, \hat{D}) \\ &= \mathbb{E} \left[\mathbb{E} \left[1(\hat{T}_{kt} \leq t) \cdot 1(X_t \cdot l \geq kt) \cdot 1(\hat{D}) \mid \mathcal{F}_{\hat{T}_{kt}} \right] \right] \\ &= \mathbb{E} \left[1(\hat{T}_{kt} \leq t) \cdot \mathbb{E} \left[1(X_t \cdot l \geq kt) \cdot 1(\hat{D}) \mid \mathcal{F}_{\hat{T}_{kt}} \right] \right]. \end{aligned}$$

Denote by R the right-hand side of the last inequality. Let \hat{F} be the event that there exists $i \in \{1, \dots, m\}$ and $t_1 \in \left(t_0 \lfloor \frac{\hat{T}_{tk}}{t_0} \rfloor, \hat{T}_{tk}\right)$ such that X_t spent all the time $(t_1 - t_0, t_1)$ in a u_i -teleport. Then we can write the quantity R in the following way:

$$\begin{aligned} R &= \mathbb{E} \left[1(\hat{T}_{kt} \leq t) \cdot 1(\hat{F}) \cdot \mathbb{E} \left[1(X_t \cdot l \geq kt) \cdot 1(\hat{D}) \mid \mathcal{F}_{\hat{T}_{kt}} \right] \right] \\ &\quad + \mathbb{E} \left[1(\hat{T}_{kt} \leq t) \cdot 1(\hat{F}^C) \cdot \mathbb{E} \left[1(X_t \cdot l \geq kt) \cdot 1(\hat{D}) \mid \mathcal{F}_{\hat{T}_{kt}} \right] \right]. \end{aligned} \tag{14}$$

Consider the first summand on the right hand side of the previous inequality. On the event \hat{F} we know that the random variable $Y_{\lfloor \hat{T}_{tk}/t_0 \rfloor}$ will not be responsible for any further jumps. The process already spent a lot of time in a teleport, so the value of $Y_{\lfloor \hat{T}_{tk}/t_0 \rfloor}$ was already used in making the decision whether there will be a jump or not. We don't care about the outcome, because the process reached the level S_{tk} , but we know for sure that a single value of Y can't be responsible for two decisions about jumps. Hence the event

$$\hat{G} = \left\{ Y_{1+\lfloor \hat{T}_{tk}/t_0 \rfloor} = \dots = Y_{\lfloor \hat{T}_{tk}/t_0 \rfloor + \lceil \frac{2\sqrt{d}}{s_0} \rceil} = \phi \right\}$$

is sufficient to assure that the path X_t will enter the trap before having any possible jumps. There are only finitely many terms listed in this sequence, so the probability of \hat{G} is strictly positive. We also have that $1(X_t \cdot l \geq kt) = 1$ on the intersection $\hat{D} \cap \hat{G} \cap \hat{F} \cap \{\hat{T}_{tk} \leq t\}$. Hence there is a constant c' such that on the event $\{\hat{T}_{kt} \leq t\} \cap \hat{F}$ we have

$$\mathbb{E} \left[1(X_t \cdot l \geq kt) \cdot 1(\hat{D}) \mid \mathcal{F}_{\hat{T}_{kt}} \right] \geq \mathbb{E} \left[1(X_t \cdot l \geq kt) \cdot 1(\hat{D}) \cdot 1(\hat{G}) \mid \mathcal{F}_{\hat{T}_{kt}} \right] = \mathbb{E} \left[1(\hat{D}) \cdot 1(\hat{G}) \mid \mathcal{F}_{\hat{T}_{kt}} \right] \geq c'.$$

Thus we can bound from below the first summand on the right of (14) by $c' \cdot \mathbb{P}(\hat{T}_{kt} \leq t, \hat{F})$.

Let us now consider the second summand. Denote by \hat{E} the event that $Y_{\lfloor \hat{T}_{tk}/t_0 \rfloor} = \phi$. Notice that

$$\begin{aligned} & \mathbb{E} \left[1(\hat{T}_{kt} \leq t) 1(\hat{F}^C) \cdot \mathbb{E} \left[1(X_t \cdot l \geq kt) \cdot 1(\hat{D}) \mid \mathcal{F}_{\hat{T}_{kt}} \right] \right] \\ & \geq \mathbb{E} \left[1(\hat{T}_{kt} \leq t) \cdot 1(\hat{E}) \cdot 1(\hat{F}^C) \cdot \mathbb{E} \left[1(X_t \cdot l \geq kt) \cdot 1(\hat{D}) \cdot 1(\hat{G}) \mid \mathcal{F}_{\hat{T}_{kt}} \right] \right]. \end{aligned} \quad (15)$$

Here \hat{G} is the same event as above. The identity $1(X_t \cdot l \geq kt) = 1$ holds as before, because on the set $\hat{E} \cap \hat{G}$ the process will enter the trap. In the same way as in the case of the first summand of (14) we can bound from below the second expression from (15) by $c' \mathbb{P}(\hat{T}_{kt} \leq t, \hat{E}, \hat{F}^C)$. The unpleasant thing is that by introducing the event \hat{E} we are imposing conditions on the past. However, on the event $\{T_{kt} \leq t, \hat{F}^C\}$ we know that the process reached the level $c_e(S_{kt})$ by time t and it didn't spend sufficient time in a teleport to be considered for a jump in which $Y_{\lfloor \hat{T}_{tk}/t_0 \rfloor}$ would play a deciding role. Therefore \hat{E} is independent of $\{\hat{T}_{kt} \leq t, \hat{F}^C\}$ hence

$$\mathbb{P}(\hat{T}_{kt} \leq t, \hat{E}, \hat{F}^C) = \mathbb{P}(\hat{E}) \cdot \mathbb{P}(\hat{T}_{kt} \leq t, \hat{F}^C) = c_3 \cdot \mathbb{P}(\hat{T}_{kt} \leq t, \hat{F}^C).$$

This allows us to conclude that there is a constant $c'' > 0$ such that

$$\begin{aligned} \mathbb{P}(X_t \cdot l \geq kt) & \geq c'' \left(\mathbb{P}(\hat{T}_{kt} \leq t, \hat{F}) + \mathbb{P}(\hat{T}_{kt} \leq t, \hat{F}^C) \right) = c'' \mathbb{P}(\hat{T}_{kt} \leq t) \\ & \geq c'' \mathbb{P}(T_{kt+5\sqrt{d}} \leq t). \end{aligned}$$

Taking the logarithm of both sides of the last inequality, dividing by t , and taking the \liminf as $t \rightarrow +\infty$ one obtains the following inequality

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{P}(X_t \cdot l \geq kt) \geq \liminf_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{P}(T_{kt+5\sqrt{d}} \leq t).$$

For each ε , there exists \tilde{t} such that every $t > \tilde{t}$ satisfies $\{T_{(k+\varepsilon)t} \leq t-s\} \subseteq \{T_{kt+5\sqrt{d}} \leq t-s\}$. Thus

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{P}(X_t \cdot l \geq kt) \geq \liminf_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{P}(T_{(k+\varepsilon)t} \leq t).$$

This completes the proof of the lemma. □

Following the approach from the discrete case our goal is to establish a statement similar to Lemma 4.2. Let us define $D_1 = \inf\{t > 0 : X_t \cdot l \leq 0\}$. This is analogous to a stopping time we used in the discrete setting: time of first return to 0.

Lemma 7.2. *Let k and k' be two real numbers such that $0 < k' < k$. Then the following two inequalities hold:*

$$\begin{aligned} \limsup \frac{1}{t} \log \mathbb{P}(T_{tk} \leq t) &\leq \limsup \frac{1}{t} \log \mathbb{P}(T_{tk'} \leq t, T_{tk'} \leq D_1) \\ \liminf \frac{1}{t} \log \mathbb{P}(T_{tk} \leq t, T_{tk} \leq D_1) &\leq \liminf \frac{1}{t} \log \mathbb{P}(T_{tk} \leq t). \end{aligned}$$

Proof. The second inequality is obvious since $\{T_{kt} \leq t, T_{kt} \leq D_1\} \subseteq \{T_{kt} \leq t\}$. Our proof of the first inequality from Lemma 4.2 used the fact that in finite time the walk can make only finitely many crossings over the fixed hyperplane. Obviously, we can't use that fact in the continuous setting. The idea is to break the process between its crossings of a strip between hyperplanes Z_{-w} and Z_0 , where w is some fixed real number from the interval $(\frac{1}{2}, 1)$. Define the following stopping times: $G_0 = 0$, $F_0 = \inf\{t : X_t \cdot l \leq -w\}$. Having defined G_i and F_i , for $i \geq 0$, we inductively define

$$\begin{aligned} G_{i+1} &= \inf\{t \geq F_i : X_t \cdot l \geq 0\} \\ F_{i+1} &= \inf\{t \geq G_{i+1} : X_t \cdot l \leq -w\}. \end{aligned}$$

We will need the following lemma:

Lemma 7.3. *For any two real numbers k and k' satisfying $0 < k' < k$ we have*

$$\limsup \frac{1}{t} \log \mathbb{P}(T_{tk} \leq t, T_{tk} \leq F_0) \leq \limsup \frac{1}{t} \log \mathbb{P}(T_{tk'} \leq t, T_{tk'} \leq D_1).$$

Proof. There is at least one of the vectors from the assumption (iii), say u_1 , such that $l \cdot u_1 > 3\sqrt{d}$. Let Γ be the event that every point in the cube $[0, 1]^d$ is at a distance smaller than $\frac{1}{2}r$ of a (u_1, t_0) teleport, every point in the cube $[0, 1]^d$ is at a distance at least r from any other teleport, and that $Y_0 \neq \phi$. The conditional probabilities of that event are bounded below by a constant c . By the time t_0 the process will be away from the origin. Denote by Q the cube to which it gets teleported. The cube Q is the translation of $[0, 1]^d$ for some vector $q \in \mathbb{Z}^d$. Denote by \tilde{O} , \tilde{Z}_0 , and $\tilde{Z}_{-u_1 \cdot l}$ the translations of the origin O , hyperplane Z_0 , and hyperplane $Z_{-u_1 \cdot l}$ by the vector q . Denote by \tilde{X} the process defined by $\tilde{X}_t = X_{t+t_0}$. Let \tilde{G}_i and \tilde{F}_i denote the stopping times corresponding to the process \tilde{X} which are analogous to the stopping times G_i and F_i . Consider the projection \tilde{O}^l of the point \tilde{O} to the hyperplane $\tilde{Z}_{-u_1 \cdot l}$. Consider the $d - 1$ -dimensional ball $B_{\max_i \{|u_i|\}}(\tilde{O}^l)$ in $\tilde{Z}_{-u_1 \cdot l}$, and let B' be the 1-neighborhood of this ball in the space \mathbb{R}^d . Let $\tilde{\Gamma}$ denote the event that each point in B' is at a distance smaller than $\frac{1}{2}r$ to a (u_1, t_0) teleport and larger than r from any other teleport. The probability of that event is strictly positive and the event is independent on the σ -algebra generated by the environment in the positive l -direction of the hyperplane \tilde{Z}_0 . Therefore we have

$$\begin{aligned} \mathbb{P}(T_{tk'} \leq t, T_{tk'} \leq D_1) &\geq \mathbb{P}(T_{tk'} \leq t, T_{tk'} \leq D_1, \Gamma) \\ &\geq \mathbb{P}(\tilde{T}_{tk'} \leq t - t_0, \tilde{T}_{tk'} \leq \tilde{F}_0, \tilde{\Gamma}) \\ &\geq c \cdot \mathbb{P}(T_{tk'} \leq t - t_0, T_{tk'} \leq F_0). \end{aligned}$$

It remains to notice that for sufficiently large t we have that $tk' \leq (t - t_0)k$, hence

$$\mathbb{P}(T_{tk'} \leq t - t_0) \geq \mathbb{P}(T_{(t-t_0)k} \leq t - t_0).$$

This completes the proof of Lemma 7.3. \square

We now return to the proof of Lemma 7.2. For any real number $u > 0$ we write the event $\{T_u \leq t\}$ as the following union:

$$\{T_u \leq t\} = \{T_u \leq t, T_u \leq F_0\} \cup \bigcup_{i=0}^{\infty} \{T_u \leq t, F_i \leq T_u \leq F_{i+1}\}.$$

The last union turns out to be finite, because we can prove that if $T_u \leq t$, then $T_u \leq F_{\lceil tL \rceil}$. We first find a lower bound on $F_{\lceil u \rceil}$. The fundamental theorem of calculus implies

$$-w = X_{F_i} \cdot l - X_{G_i} \cdot l = \int_{G_i}^{F_i} b(\tau_{X_s} \omega) \cdot l \, ds \geq -L \cdot |l| \cdot (F_i - G_i)$$

which together with $|l| = 1$ yields $F_i - G_i \geq \frac{w}{L}$. Using this inequality we obtain

$$F_m = F_0 + \sum_{i=1}^m (F_i - F_{i-1}) \geq \sum_{i=1}^m (F_i - G_i) \geq \frac{wm}{L}.$$

Therefore $F_{\lceil tL/w \rceil} \geq t$ and on $\{T_u \leq t\}$ we immediately get $F_{\lceil tL/w \rceil} \geq T_u$. This implies that

$$\mathbb{P}(T_u \leq t) \leq \sum_{i=0}^{\lceil \frac{tL}{w} \rceil} \mathbb{P}(T_u \leq t, F_i \leq T_u \leq F_{i+1}).$$

Let us prove that each term on the right hand side of the last inequality can be bounded by the quantity $\mathbb{P}(T_u \leq t, T_u \leq F_0)$. Denote by $\mathcal{F}_{G_{i+1}, T_u}$ the sigma algebra generated by the environment contained in the M -neighborhood of the process from G_{i+1} to T_u . Notice that if $T_u \leq t$ and $T_u \geq F_i$, then the process has made at least i trips over the region between the hyperplanes Z_0 and Z_{-w} . Since $T_u \leq F_{i+1}$ and $T_u \leq t$ we conclude that the process has crossed the hyperplane Z_u by time t which means that it had to cross the hyperplane Z_0 again. Therefore $G_{i+1} \leq t$. Let \tilde{X} be the process starting at time G_{i+1} . More precisely, we define $\tilde{X}_t = X_{G_{i+1}+t}$. We use \tilde{F}_i and \tilde{G}_i to denote the stopping times for \tilde{X}_t analogous to F_i and G_i .

$$\begin{aligned} \mathbb{P}(T_u \leq t, F_i \leq T_u \leq F_{i+1}) &= \mathbb{E} \left[\mathbb{E} \left[1(T_u \leq t) \cdot 1(T_u \geq F_i) \cdot 1(T_u \leq F_{i+1}) \mid \mathcal{F}_{G_{i+1}, T_u} \right] \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[1(\tilde{T}_u \leq t - G_{i+1}) \cdot 1(T_u \geq F_i) \cdot 1(\tilde{T}_u \leq \tilde{F}_0) \mid \mathcal{F}_{G_{i+1}, T_u} \right] \right] \\ &\leq \mathbb{E} \left[1(\tilde{T}_u \leq t) \cdot 1(\tilde{T}_u \leq \tilde{F}_0) \cdot \mathbb{E} \left[1(T_u \geq F_i) \mid \mathcal{F}_{G_{i+1}, T_u} \right] \right]. \end{aligned}$$

The last conditional expectation can be bounded above by 1. Although our process does not possess Markov property, we can use the trivial bounds on the indicator functions, namely $1(T_u \geq F_i) \leq 1$. Therefore

$$\mathbb{P}(T_u \leq t) \leq \left\lceil \frac{tL}{w} \right\rceil \cdot \mathbb{P}(T_u \leq t, T_u \leq F_0).$$

Placing $u = tk$ and using Lemma 7.3 we obtain

$$\begin{aligned} \limsup \frac{1}{t} \log \mathbb{P}(T_{tk} \leq t) &\leq \lim_{t \rightarrow +\infty} \frac{1}{t} \log t + \limsup \frac{1}{t} \log \mathbb{P}(T_{tk} \leq t, T_{tk} \leq F_0) \\ &\leq \limsup \frac{1}{t} \log \mathbb{P}(T_{tk'} \leq t, T_{tk'} \leq D_1), \end{aligned}$$

for any two real numbers k' and k such that $0 < k' < k$. The proof of Lemma 7.2 is complete. \square

Proof of Theorem 1.1. The proof will proceed in the same way as in the discrete case. We will first prove the existence of the concave function $\phi^l : \mathbb{R}_+ \rightarrow \bar{\mathbb{R}}$ such that for all $k \in \mathbb{R}_+$:

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{P}(X_t \cdot l \geq tk) = \phi^l(k). \quad (16)$$

As in the discrete case we see that there is a function $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\phi^l(k) = \Phi\left(\frac{l}{k}\right)$. In order to prove (16) we will establish the inequality that has precisely the same form as (6), with a difference that our stopping times are defined in the continuous setting. The proof will be adapted to the continuous in the similar way as the proof of Lemma 7.1. In the proof of the mentioned lemma we needed to construct a trap that will hold the process. Here we need a tunnel that will take the process to the environment sufficiently far away from the previously visited sites.

We start by considering the parallelepiped $Q = [0, \lceil nL \rceil] \times [-3, 3]^{d-1}$. Here n is an integer that is going to be specified in a moment. It will depend on d and l only. Similarly to the proof of Lemma 7.1, denote by S_u the surface consisting only of faces of the grid of size $\lceil nL \rceil$ that is in front of Z_u when looking from the origin in the direction l . Denote by Q'_u the appropriate rotation and translation of Q so it ends up on the side of S_u opposite to the origin. Moreover, Q'_u can be placed in such a way that the hitting point of $c_\varepsilon(S_u)$ by the process X_t is inside Q'_u and located within ε of the base of Q'_u (here by base we mean a $d - 1$ -dimensional face congruent to $[-3, 3]^{d-1}$). We choose the integer n so that there is a hyperplane Z_y orthogonal to l that cuts Q'_u in two prisms each of which has the shortest altitude of at least $2L$. Denote by Q_u the one of the prisms that contains the hitting point of the process X_t .

Denote by \hat{T}_u the hitting time of $c_\varepsilon(S_u)$. Denote by B_1 the event that after reaching the surface $c_\varepsilon(S_u)$, the process encounters the environment that is going to take it through the polytope Q_u in time less than $4L/\delta_0$. Such an environment can be constructed in the following way. We require that the vector field b satisfies $b \cdot l > \frac{\delta_0}{2}$ in the central cells of Q_u . The outer cells serve as traps that bring the process towards the central cells. More precisely if C is one of the outer cells, we choose a vector v_C that points from the center of C towards the simplex that is shared with one of the interior cubes and that satisfies $v_C \cdot l > 0$. We require b to satisfy

$$b \cdot v_C \geq \frac{\sqrt{d-1}}{\sqrt{d}} \cdot |b| \cdot |v_C| \text{ and } b \cdot v_C > \frac{\delta_0}{2}.$$

The environment now acts as a tunnel through which the process must go. This reduces the portion of the environment that gets exposed to the process, and plays the role of the sequence of steps of size w that we used in proving (6). In the end of the tunnel Q_u we require that the environment has a set of teleports in the direction u_1 , while before the end there are no teleports overlapping with the tube. More precisely, every point of the boundary of Q_u that belongs to Z_y is within $r/2$ of a

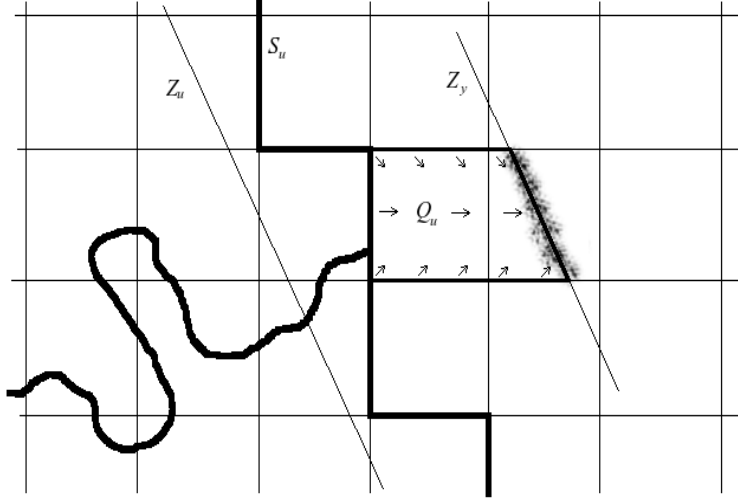


Figure 3: The polytope Q_u and teleports near the boundary that is a part of Z_y

u_1 teleport and at a distance of at least r from all other teleports. Let \hat{T}'_u be the time by which the process has spent time t_0 in the union of u_1 teleports. We assume that the value of the corresponding term $Y_{[\hat{T}'_u/t_0]}$ is not ϕ . This means that \hat{T}'_u is the time of jump. The conditional probabilities of such environments on Q_u are positive, and we are sure that after the passage through the tunnel the process will appear at a uniform location within a cube that is sufficiently far away from the old environment.

Let B_2 be the event that there exists $i \in \{1, 2, \dots, m\}$ and $t_1 \in (t_0 \lfloor \frac{\hat{T}_u}{t_0} \rfloor, \hat{T}_u)$ such that X_t spent all the time $(t_1 - t_0, t_1)$ in a u_i teleport. The definition of B_2 is the same as the definition of \hat{F} from the proof of Lemma 7.1.

We have the following sequence of inequalities

$$\begin{aligned} \mathbb{P}(T_{u+v} \leq p + q, T_{u+v} \leq D_1) &\geq \mathbb{P}(T_{u+v} \leq p + q, T_{u+v} \leq D_1, T_u \leq p) \\ &\geq \mathbb{E} [1(\hat{T}_u \leq p) \cdot 1(B_1) \cdot 1(T_{u+v} \leq p + q) \cdot 1(T_{u+v} \leq D_1)] \\ &= \mathbb{E} \left[\mathbb{E} \left[1(\hat{T}_u \leq p) \cdot 1(B_1) \cdot 1(T_{u+v} \leq p + q) \cdot 1(T_{u+v} \leq D_1) \middle| \mathcal{G}_{[\hat{T}_u, \hat{T}'_u]} \right] \right], \end{aligned}$$

where $\mathcal{G}_{[\hat{T}_u, \hat{T}'_u]}$ is the σ -algebra determined by the environment in the M -neighborhood of the path of the process from 0 to T_{u+v} except the the portion between times \hat{T}_u and \hat{T}'_u . Let us introduce the following notation $\tilde{X}_t = X_{\hat{T}'_u+t}$. Let $\vec{\sigma}$ be the vector that determines the translation which maps $[0, 1]^d$ to the cube that contains \tilde{X}_0 . Let \tilde{O} be the translation of the origin for the vector $\vec{\sigma}$. This way we can understand \tilde{X}_t as a new process, and \tilde{O} plays the role of the origin for that process. Let \tilde{Z}_0 be the hyperplane orthogonal to l that passes through \tilde{O} and define \tilde{D}_1 as the first time \tilde{X}_t backtracks over \tilde{Z}_0 . Define $\tilde{T}_v = \inf\{t : \tilde{X}_t \cdot l \geq v\}$. Let $\mu = \frac{2L}{\delta_0}$. We know that $\hat{T}'_u - \hat{T}_u \leq \mu$ hence

$$\begin{aligned} \mathbb{P}(T_{u+v} \leq p + q, T_{u+v} \leq D_1) &\geq \mathbb{E} \left[\mathbb{E} \left[1(\hat{T}_u \leq p) \cdot 1(B_1) \cdot 1(\tilde{T}_v \leq q - \mu) \cdot 1(\tilde{T}_v \leq \tilde{D}_1) \middle| \mathcal{G}_{[\hat{T}_u, \hat{T}'_u]} \right] \right]. \end{aligned}$$

Since $\{\hat{T}_u \leq p\}$, $\{\tilde{T}_v \leq q - \mu\}$, $\{\tilde{T}_v \leq \tilde{D}_1\}$, $B_2 \in \mathcal{G}_{[\hat{T}_u, \hat{T}'_u]}$ we have

$$\begin{aligned} & \mathbb{E} \left[\mathbb{E} \left[\mathbf{1}(\hat{T}_u \leq p) \cdot \mathbf{1}(B_1) \cdot \mathbf{1}(\tilde{T}_v \leq q - \mu) \cdot \mathbf{1}(\tilde{T}_v \leq \tilde{D}_1) \middle| \mathcal{G}_{[\hat{T}_u, \hat{T}'_u]} \right] \right] \\ &= \mathbb{E} \left[\mathbf{1}(\hat{T}_u \leq p) \cdot \mathbf{1}(\tilde{T}_v \leq q - \mu) \cdot \mathbf{1}(\hat{T}_u \leq D_1) \cdot \mathbf{1}(\tilde{T}_v \leq \tilde{D}_1) \cdot \mathbf{1}(B_2) \cdot \mathbb{E} \left[\mathbf{1}(B_1) \middle| \mathcal{G}_{[\hat{T}_u, \hat{T}'_u]} \right] \right] \\ &+ \mathbb{E} \left[\mathbf{1}(\hat{T}_u \leq p) \cdot \mathbf{1}(\tilde{T}_v \leq q - \mu) \cdot \mathbf{1}(\hat{T}_u \leq D_1) \cdot \mathbf{1}(\tilde{T}_v \leq \tilde{D}_1) \cdot \mathbf{1}(B_2^c) \cdot \mathbb{E} \left[\mathbf{1}(B_1) \middle| \mathcal{G}_{[\hat{T}_u, \hat{T}'_u]} \right] \right]. \end{aligned}$$

Now we bound each of the two terms on the right hand side in the same way as we did in the proof of Lemma 7.1 to get

$$\mathbb{P}(T_{u+v} \leq p + q, T_{u+v} \leq D_1) \geq c \cdot \mathbb{E}(\mathbf{1}(\hat{T}_u \leq p) \cdot \mathbf{1}(\hat{T}_u \leq D_1) \cdot \mathbf{1}(\tilde{T}_v \leq q) \cdot \mathbf{1}(\tilde{T}_v \leq \tilde{D}_1)).$$

We can now replace \tilde{X} with a process on the independent environment because it is sufficiently far away from X and use the independence to obtain

$$\begin{aligned} \mathbb{P}(T_{u+v} \leq p + q, T_{u+v} \leq D_1) &\geq c \cdot \mathbb{P}(\hat{T}_u \leq p, \hat{T}_u \leq D_1) \cdot \mathbb{P}(T_v \leq q - \mu, T_v \leq D_1) \\ &\geq c' \cdot \mathbb{P}(T_u \leq p, T_u \leq D_1) \cdot \mathbb{P}(T_v \leq q - \mu, T_v \leq D_1). \end{aligned}$$

We can now proceed in the same way as in the case of the deterministic walk and establish the existence of the limit in (16). The proof of the concavity of $\phi^l(k)$ is the same as in the discrete case.

We will now prove that

$$\liminf \frac{1}{t} \log \mathbb{E} \left[e^{\lambda \cdot X_t} \right] \geq \max \left\{ 0, \sup_{k>0} \left\{ k + \Phi \left(\frac{\lambda}{k} \right) \right\} \right\}. \quad (17)$$

Denote by $\Lambda(\lambda)$ the right-hand side of (17). For each $k > 0$ we have $\mathbb{E}[\exp\{\lambda \cdot X_t\}] \geq e^{tk} \mathbb{P}(X_t \cdot \lambda \geq kt)$ hence we only need to show that

$$\liminf \frac{1}{t} \log \mathbb{E} \left[e^{\lambda \cdot X_t} \right] \geq 0.$$

For this it suffices to show that $\frac{1}{t} \log \mathbb{P}(\lambda \cdot X_t \geq 0) \geq 0$ and for this we need to find a uniform lower bound on $\mathbb{P}(\lambda \cdot X_t \geq 0)$. In the beginning of the proof of Lemma 7.1 we showed that there is a positive probability that a given cube of size 3 is a trap. Once the process enters there, it can exit only by teleportation. Hence there is a positive probability that the cube $[0, 3]^d$ is a trap. Let D be that event, and D' the event that there are no teleports in proximity of $[0, 3]^d$. Then we have $\mathbb{P}(D \cap D') \geq c$ for some positive constant c . The proof of (17) is complete once we observe that $\{\lambda \cdot X_t \geq 0\} \supseteq D \cap D'$.

Proving inequality

$$\limsup \frac{1}{t} \log \mathbb{E} \left[e^{\lambda \cdot X_t} \right] \leq \max \left\{ 0, \sup_{k>0} \left\{ k + \Phi \left(\frac{\lambda}{k} \right) \right\} \right\}$$

is exactly the same as in the discrete case. The proof only uses the concavity of Φ in the variable k . The convexity of Λ is established also in the same way as in the discrete case, and this completes the proof of Theorem 1.1. \square

We can use similar arguments as in the discrete case to prove the following theorem:

Theorem 7.1. Let X_t be the previously defined process. Let Λ be the function defined after (17) and let Λ^* be its convex conjugate. Let \mathcal{F} be the set of exposed points of Λ^* . For any closed set $F \subseteq \mathbb{R}^d$,

$$\limsup \frac{1}{n} \log \mathbb{P} \left(\frac{1}{n} X_n \in F \right) \leq - \inf_{x \in F} \Lambda^*(x),$$

and for any open set $G \subseteq \mathbb{R}^d$

$$\liminf \frac{1}{n} \log \mathbb{P} \left(\frac{1}{n} X_n \in G \right) \geq - \inf_{x \in G \cap \mathcal{F}} \Lambda^*(x).$$

Moreover, there exists $\delta > 0$ such that $\Lambda^*(x) < +\infty$ for $|x| < \delta$.

Proof. The only difference from the proof in the discrete setting is proving that $\Lambda^*(x) < +\infty$ in some neighborhood of 0. We follow the same idea, though. Denote by e_i the base vectors of \mathbb{R}^d . Define

$$\rho(l) = \max \{ \max_i \{ e_i \cdot l \}, \max_i \{ -e_i \cdot l \} \}.$$

Then $\inf_{l: |l|=1} \rho(l) > \frac{1}{\sqrt{d}}$. Hence, given a vector λ , we may assume that $e_1 \cdot \lambda > \frac{1}{\sqrt{d}} |\lambda|$. Consider the environment such that:

(i) The vector fields b_Q satisfy

$$b_Q \cdot e_1 > \delta_0 \text{ and } b_Q \cdot e_1 > \frac{\sqrt{d-1}}{\sqrt{d}} |b_Q| \text{ in the set } c_\varepsilon(Q),$$

for all unit cubes Q whose lower-left corner has the form $(n, 0, \dots, 0)$ where $n \in \{0, 1, \dots, \lfloor t\delta_0 \rfloor\}$.

(ii) In each unit cube Q that shares a boundary to any of the cubes from (i) we consider the vector $v \in \{-1, 0, 1\}^d \setminus \{0\}$ such that $v \cdot \lambda$ is maximal. Then we require b_Q to satisfy

$$b_Q \cdot v \geq \delta_0 |v| \text{ and } b_Q \cdot v \geq \frac{\sqrt{d-1}}{\sqrt{d}} |b_Q| |v|.$$

(iii) Denote by Q_t the cube whose lower-left corner is $(\lfloor t\delta_0 \rfloor + 1, 0, 0, \dots, 0)$. We require that cells adjacent to Q_t serve as a trap towards Q_t . This environment is precisely described in the beginning of the proof of Lemma 7.1.

(iv) There are no teleports in the $5\sqrt{d} + r$ neighborhood of the segment from O to the lower-left corner of Q_t .

The conditional probabilities of the previously described environment are bounded below by c^t for some $c > 0$. On the realization of such environment we can guarantee that

$$X_t \cdot \lambda > \frac{t\delta_0 |\lambda|}{2\sqrt{d}}.$$

Hence

$$\Phi\left(\frac{\lambda}{\delta_0|\lambda|\cdot\frac{1}{2\sqrt{d}}}\right) = \lim_{t\rightarrow+\infty}\frac{1}{t}\log\mathbb{P}\left(X_t\cdot\lambda > \frac{t\delta_0|\lambda|}{2\sqrt{d}}\right) \geq c.$$

Therefore $\Lambda(\lambda) \geq \frac{\delta_0|\lambda|}{2\sqrt{d}} + c$ and

$$\Lambda^*(x) \leq \sup_x \left\{ \lambda \cdot x - \frac{\delta_0|\lambda|}{2\sqrt{d}} - c \right\} \leq \sup_x \left\{ |\lambda| \cdot |x| - \frac{\delta_0|\lambda|}{2\sqrt{d}} \right\} - c.$$

If $|x| < \frac{\delta_0}{2\sqrt{d}}$ we see that $\Lambda^*(x) \leq -c$. □

7.7 Open Problems

One of the main unsolved questions is the continuity of Φ . We are only able to prove that $k \mapsto \Phi\left(\frac{l}{k}\right)$ is concave for fixed $l \in \mathbb{R}^d$, which gives the continuity in k . However, it would be very interesting to see the continuity of the Φ as a function defined on a domain from \mathbb{R}^d . Ideas developed in this paper can't be directly applied because the hitting times of hyperplanes can't be easily replaced by the hitting times of dihedral angles (or some other interfaces) in the proofs of the previous theorems. The said difficulty appears even in the discrete case.

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