

## Pfaffian Stochastic Dynamics of Strict Partitions

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### Abstract

We study a family of continuous time Markov jump processes on strict partitions (partitions with distinct parts) preserving the distributions introduced by Borodin [Bor99] in connection with projective representations of the infinite symmetric group. The one-dimensional distributions of the processes (i.e., the Borodin's measures) have determinantal structure. We express the dynamical correlation functions of the processes in terms of certain Pfaffians and give explicit formulas for both the static and dynamical correlation kernels using the Gauss hypergeometric function. Moreover, we are able to express our correlation kernels (both static and dynamical) through those of the  $z$ -measures on partitions obtained previously by Borodin and Olshanski in a series of papers.

The results about the fixed time case were announced in the note [Pet10b]. A part of the present paper contains proofs of those results .

**Key words:** random strict partitions; determinantal point process; Pfaffian dynamics.

**AMS 2010 Subject Classification:** Primary 60G55; Secondary: 60J75; 05E18.

Submitted to EJP on April 19, 2011, final version accepted Octobr 15, 2011.

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\*The author was partially supported by the RFBR-CNRS grant 10-01-93114, by A. Kuznetsov's graduate student scholarship and by the Dynasty foundation fellowship for young scientists

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## 1 Introduction

We introduce and study a family<sup>1</sup> of continuous time Markov jump processes on the set of all strict partitions (that is, partitions in which nonzero parts are distinct). Our Markov processes preserve the family of probability measures introduced by Borodin [Bor99] in connection with the harmonic analysis of projective representations of the infinite symmetric group. The construction of our dynamics is similar to that of Borodin and Olshanski [BO06a] and is based on a special coherency property<sup>2</sup> of the measures on strict partitions introduced in [Bor99]. Regarding each strict partition  $\lambda = (\lambda_1 > \dots > \lambda_\ell > 0)$ ,  $\lambda_j \in \mathbb{Z}$ , as a point configuration  $\{\lambda_1, \dots, \lambda_\ell\}$  on the half-lattice  $\mathbb{Z}_{>0} := \{1, 2, \dots\}$ , one can say that the state space of our Markov processes is the space of all finite point configurations on  $\mathbb{Z}_{>0}$ . The fixed time distributions of our dynamics are probability measures on this configuration space. In other words, in the static (fixed time) picture one sees a random point process on  $\mathbb{Z}_{>0}$ . The detailed description of the model and formulation of the main results are given in §2.

The main result of the paper is the computation of the dynamical (or space-time) correlation functions for our family of Markov processes. We show that these correlation functions have certain

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<sup>1</sup>The whole picture depends on two continuous parameters  $\alpha > 0$  and  $0 < \xi < 1$ .

<sup>2</sup>which has a representation-theoretic meaning.

Pfaffian form, and compute the corresponding kernel. Here the kernel is a function  $\Phi_{\alpha,\xi}(s, x; t, y)$  of two space-time variables, where  $x, y \in \mathbb{Z}$  and  $s, t \in \mathbb{R}$ , which is explicitly expressed through the Gauss hypergeometric function. Following the common terminology (e.g., see [NF98], [Joh05], [BO06a]), we call  $\Phi_{\alpha,\xi}$  the *extended (Pfaffian) hypergeometric-type kernel*. Precise formulation of this result and an explicit expression for the Pfaffian kernel are given in Theorem 2 in §2.

In the static case the Pfaffian formula for the correlation functions of our Markov processes can be reduced to a determinantal one. Thus, in the fixed time picture we have a determinantal point process on  $\mathbb{Z}_{>0}$ . Its kernel  $\mathbf{K}_{\alpha,\xi}$  has integrable form and is also expressed through the Gauss hypergeometric function. We call this kernel the *hypergeometric-type kernel*. See Theorem 1 in §2 for a detailed statement and a formula for the kernel  $\mathbf{K}_{\alpha,\xi}$ . The results about the static case were announced in the note [Pet10b]. A part of the present paper (namely, §4–§8) contains complete proofs of those results.

Models with correlation functions of Pfaffian form first appeared in theory of random matrices, e.g., see [Dys70], [MP83a], [MP83b], [NW91a], [NW91b], [TW96], [Nag07], and the book by Mehta [Meh04]. An essentially time-inhomogeneous Pfaffian dynamical model of random-matrix type was considered by Nagao, Katori and Tanemura [KNT04], [Kat05]. Static Pfaffian random point processes of various origins have also been studied, e.g., see [Rai00], [Fer04], [Mat05], [BR05], [Vul07], and §10 of the survey [Bor09]. Borodin and Strahov [BS06], [BS09], [Str10a] considered static models which are discrete analogues of Pfaffian models of random-matrix type, they involve random ordinary (i.e., not necessary strict) partitions and have a representation-theoretic interpretation (see [Str10b]). The dynamical model that we study in the present paper seems to be a first example of a *stationary* (in contrast to the model of [KNT04], [Kat05]) Pfaffian dynamics.

## Comparison with results for the $z$ -measures

Our model of random strict partitions and associated stochastic dynamics is very similar to the one of the  $z$ -measures on ordinary partitions.<sup>3</sup> The structure of static and dynamical correlation functions in that case was investigated in [BO00], [Oko01b], [BO06a], [BO06b]. Let us discuss the relationship of our results with the ones from those papers.

- The main feature of our model is that its dynamical correlation functions are expressed in terms of Pfaffians and not determinants, as it is for the  $z$ -measures.
- Determinantal (static) correlation kernels of random point processes often appear to be projection operators. In particular, this holds for the  $z$ -measures. In our situation the kernel  $\mathbf{K}_{\alpha,\xi}(x, y)$  ( $x, y \in \mathbb{Z}_{>0}$ ) is symmetric, but it is not a projection operator in the corresponding coordinate Hilbert space  $\ell^2(\mathbb{Z}_{>0})$ .
- On the other hand, the static Pfaffian kernel  $\Phi_{\alpha,\xi}(x, y) := \Phi_{\alpha,\xi}(s, x; s, y)$  (where  $x, y \in \mathbb{Z}$ ) in our model has a structure which is very similar to that of the determinantal kernel of the  $z$ -measures on semi-infinite point configurations on the lattice. Viewed as an operator in the Hilbert space  $\ell^2(\mathbb{Z})$ ,  $\Phi_{\alpha,\xi}(\cdot, \cdot)$  is a rank one perturbation of an orthogonal projection operator.

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<sup>3</sup>The  $z$ -measures originated from the problem of harmonic analysis for the infinite symmetric group  $\mathfrak{S}_\infty$  [KOV93], [KOV04] and were studied in detail by Borodin, Okounkov, Olshanski, and other authors, e.g., see the bibliography in [BO09].

- Furthermore, our extended Pfaffian kernel  $\Phi_{\alpha,\xi}(s, x; t, y)$  is obtained from the static kernel  $\Phi_{\alpha,\xi}(x, y)$  in a way which is common for Markov processes on configuration spaces with determinantal dynamical correlation functions. In particular, the same extension happens in the case of the  $z$ -measures.
- Markov processes of [BO06a], as well as many dynamical determinantal models that arise in the theory of random matrices and random tilings (e.g., see [NF98], [War07], [JN06], [ANvM10], [Joh02], [Joh05], [BGR10]), are closely related to orthogonal polynomials. Moreover, connections with orthogonal polynomials also arise in static Pfaffian models of random-matrix and representation-theoretic origin [NW91a], [NW91b], [TW96], [KNT04], [Kat05], [Nag07], [BS06], [BS09], [Str10a]. For our model there also exists a connection with orthogonal polynomials (namely, the Krawtchouk polynomials), but this connection does not help us to compute the correlation kernels as it was for the  $z$ -measures [BO06a], [BO06b].
- The expressions for our correlation kernels involve the same special functions (expressed through the Gauss hypergeometric function) which arise for the kernels in the case of the  $z$ -measures. These functions first appeared in the works of Vilenkin and Klimyk [VK88], [VK95]. In particular, certain degenerations of them lead to the classical Meixner and Krawtchouk orthogonal polynomials.

Using this fact, we are able to express our kernels directly through the corresponding kernels for the  $z$ -measures. These expressions seem to have no direct probabilistic meaning at the level of random point processes, but in particular they allow to study asymptotics of our kernels with the help of results of [BO00], [BO06a], see [Pet10a, §11].

## Method

Our technique of obtaining both static and dynamical correlation kernels in an explicit form is different from those of [BO00], [BO06b], [BO06a], and is based on computations in the fermionic Fock space involving so-called Kerov's operators which span a certain  $\mathfrak{sl}(2, \mathbb{C})$ -module. Both the static and dynamical correlation kernels in our model are expressed through matrix elements related to this module. This approach is similar to the one invented by Okounkov [Oko01b] to calculate the (static) correlation kernel of the  $z$ -measures on ordinary partitions.<sup>4</sup> In computations in this paper we use the ordinary fermionic Fock space instead of the (closely related, but different) infinite wedge space of [Oko01b]. Moreover, our situation also requires to deal with a Clifford algebra (acting in the fermionic Fock space) of a different type. One can say that our Clifford algebra is an infinite-dimensional generalization of the Clifford algebra over an odd-dimensional quadratic space. Similar Clifford algebras were used in [DJKM82], [Mat05], [Vul07]. In the latter two papers the fermionic Fock space is also used for computations of certain correlation functions. That approach is analogous to the formalism of Schur measures and Schur processes [Oko01a], [OR03] and differs from the one used in the present paper.

## Organization of the paper

The present paper is a shortened version of the arXiv preprint [Pet10a].

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<sup>4</sup>A possibility of use of this method in studying the dynamical model related to the  $z$ -measures was pointed out in [BO06a], and later this approach was carried out by Olshanski [Ols08].

In §2 we give main definitions and state main results about our model. In §3 we discuss combinatorial constructions from which our model arises. We also give an argument why the corresponding fixed time random point processes on  $\mathbb{Z}_{>0}$  are determinantal. In §4 we study Kerov's operators on strict partitions. These operators provide us with a convenient way of writing expectations with respect to our point processes. Such formulas are used in the computation of both static and dynamical correlation functions. In §5 we recall the formalism of the fermionic Fock space and define an action of a Clifford algebra in it. These structures are extensively used in our computations.

In §6 we discuss functions (matrix elements of a certain  $\mathfrak{sl}(2, \mathbb{C})$ -module) which are used in explicit expressions for our correlation kernels. These functions are eigenfunctions of a certain second order difference operator on the lattice  $\mathbb{Z}$ . This fact allows later to interpret our kernels through orthogonal spectral projections related to that operator on the lattice.

In §7 we prove that the static correlation functions of our Markov dynamics can be written as certain Pfaffians. We express the Pfaffian kernel through matrix elements related to Kerov's operators, and through the functions discussed in §6. In §8 we write the static correlation functions as determinants and express the determinantal correlation kernel in various forms (including a so-called integrable form).

The Markov processes on strict partitions are defined in §9 in terms of their jump rates. In §10 we show that the dynamical correlation functions of our Markov processes have Pfaffian form, and give an explicit expression for the dynamical (Pfaffian) correlation kernel in terms of the functions discussed in §6.

## Acknowledgements

The author is very grateful to Grigori Olshanski for permanent attention to the work, fruitful discussions, and access to his unpublished manuscript [Ols08]. I would also like to thank Alexei Borodin and Vadim Gorin for useful comments on my work.

## 2 Model and results

### 2.1 Point processes on the half-lattice

Let us first describe the fixed time picture, that is, the random point processes on the half-lattice  $\mathbb{Z}_{>0}$  that we study. They arise from a model of random strict partitions introduced in [Bor99].

By a *strict partition* we mean a partition in which nonzero parts are distinct, that is,  $\lambda = (\lambda_1 > \dots > \lambda_{\ell(\lambda)} > 0)$ , where  $\lambda_j \in \mathbb{Z}_{>0}$ . The number  $|\lambda| := \lambda_1 + \dots + \lambda_{\ell(\lambda)}$  is called the *weight* of the partition, and the number of nonzero parts  $\ell(\lambda)$  is the *length* of the partition. By  $\mathbb{S}_n$  denote the set of all strict partitions of weight  $n = 0, 1, \dots$ <sup>5</sup> Throughout the paper we identify strict partitions and corresponding shifted Young diagrams as in [Mac95, Ch. I, §1, Ex. 9].

The description of the model of [Bor99] starts with the *Plancherel measures* on strict partitions of a fixed weight:

$$Pl_n(\lambda) := \frac{2^{n-\ell(\lambda)} \cdot n!}{(\lambda_1! \dots \lambda_{\ell(\lambda)}!)^2} \prod_{1 \leq k < j \leq \ell(\lambda)} \left( \frac{\lambda_k - \lambda_j}{\lambda_k + \lambda_j} \right)^2, \quad \lambda \in \mathbb{S}_n \quad (2.1)$$

<sup>5</sup>By agreement, the set  $\mathbb{S}_0$  consists of the empty partition  $\emptyset$ .

(by  $\text{Pl}_n(\lambda)$  we denote the measure of a singleton  $\{\lambda\}$ , and the same agreement for other measures on strict partitions is used throughout the paper). The measure  $\text{Pl}_n$  is a probability measure on  $\mathbb{S}_n$ . The set  $\mathbb{S}_n$  parametrizes irreducible truly projective representations of the symmetric group  $\mathfrak{S}_n$  [Sch11], [HH92], and the measures  $\text{Pl}_n$  on  $\mathbb{S}_n$  are analogues (in the theory of projective representations of  $\mathfrak{S}_n$ ) of the well-known Plancherel measures on ordinary partitions. The system of measures  $\{\text{Pl}_n\}$  possesses a coherency property (3.3) which has a representation-theoretic meaning. The Plancherel measures on strict partitions were studied in, e.g., [Bor99], [Iva99], [Iva06], [Pet10c].

Borodin [Bor99] introduced a deformation of the measures  $\text{Pl}_n$  (2.1) depending on one real parameter  $\alpha > 0$  (in [Bor99] this parameter is denoted by  $x$ ):

$$M_{\alpha,n}(\lambda) := \text{Pl}_n(\lambda) \cdot \frac{1}{\alpha(\alpha+2)\dots(\alpha+2n-2)} \cdot \prod_{k=1}^{\ell(\lambda)} \prod_{j=0}^{\lambda_k-1} (j(j+1) + \alpha). \quad (2.2)$$

The deformations  $M_{\alpha,n}$  of the Plancherel measures  $\text{Pl}_n$  preserve the coherency property (3.3). As  $\alpha \rightarrow +\infty$ , the measure  $M_{\alpha,n}$  on  $\mathbb{S}_n$  converges to  $\text{Pl}_n$ . In this paper we do not focus on the Plancherel case, but all our results are translated to that case in a straightforward way, see [Pet10a, §8.3 and §11.2].

**Definition 2.1.** To simplify certain formulas, instead of the parameter  $\alpha$  we will sometimes use another parameter  $\nu(\alpha) := \frac{1}{2}\sqrt{1-4\alpha}$ . If  $0 < \alpha \leq \frac{1}{4}$ , then  $\nu(\alpha)$  is real,  $0 \leq \nu(\alpha) < \frac{1}{2}$ . If  $\alpha > \frac{1}{4}$ , then  $\nu(\alpha)$  can take arbitrary purely imaginary values. The whole picture is symmetric with respect to the replacement of  $\nu(\alpha)$  by  $-\nu(\alpha)$ . Sometimes the argument  $\alpha$  in  $\nu(\alpha)$  is omitted.

Our main object in the present paper is a mixing<sup>6</sup> of the measures  $M_{\alpha,n}$  in terms of the *negative binomial distribution* on the set of indices  $n \in \mathbb{Z}_{\geq 0}$ :

$$\pi_{\alpha,\xi}(n) := (1-\xi)^{\alpha/2} \frac{(\alpha/2)_n}{n!} \xi^n, \quad n = 0, 1, 2, \dots, \quad (2.3)$$

where  $\xi \in (0, 1)$  is an additional parameter. Here

$$(a)_k := a(a+1)\dots(a+k-1) = \Gamma(a+k)/\Gamma(a) \quad (2.4)$$

is the *Pochhammer symbol*, and  $\Gamma(\cdot)$  is the Euler gamma function. This mixing is defined as

$$M_{\alpha,\xi}(\lambda) := \pi_{\alpha,\xi}(|\lambda|) \cdot M_{\alpha,|\lambda|}(\lambda), \quad \lambda \in \mathbb{S}.$$

Regarding each strict partition  $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)})$  as a point configuration  $\{\lambda_1, \dots, \lambda_{\ell(\lambda)}\}$  on  $\mathbb{Z}_{>0}$  (to the empty partition  $\emptyset$  corresponds the empty configuration), we view the resulting mixed distribution  $M_{\alpha,\xi}$  as a random point process on the half-lattice  $\mathbb{Z}_{>0}$ .<sup>7</sup> The process  $M_{\alpha,\xi}$  is supported by finite configurations. The probability of each configuration  $\lambda = \{\lambda_1, \dots, \lambda_\ell\} \subset \mathbb{Z}_{>0}$  has the form

$$M_{\alpha,\xi}(\lambda) = (1-\xi)^{\alpha/2} \cdot \prod_{k=1}^{\ell} w_{\alpha,\xi}(\lambda_k) \cdot \prod_{1 \leq k < j \leq \ell} \left( \frac{\lambda_k - \lambda_j}{\lambda_k + \lambda_j} \right)^2, \quad (2.5)$$

<sup>6</sup>Which also can be viewed as a passage to the *grand canonical ensemble*, cf. [Ver96].

<sup>7</sup>Throughout the paper we use this identification of strict partitions with point configurations on  $\mathbb{Z}_{>0}$  whenever we speak about random point processes and their correlation functions.

where

$$w_{\alpha,\xi}(x) := \frac{\xi^x \cos(\pi\nu(\alpha)) \Gamma(\frac{1}{2} - \nu(\alpha) + x) \Gamma(\frac{1}{2} + \nu(\alpha) + x)}{2\pi (x!)^2}, \quad x \in \mathbb{Z}_{>0}, \quad (2.6)$$

and  $(1 - \xi)^{\alpha/2}$  is a normalizing constant (observe that  $w_{\alpha,\xi}(x) > 0$ ,  $x \in \mathbb{Z}_{>0}$ ).

Our first result is the computation of the correlation functions of the point processes  $M_{\alpha,\xi}$ . Recall that the correlation functions of a random point process on  $\mathbb{Z}_{>0}$  are defined as

$$\rho^{(n)}(x_1, \dots, x_n) := \text{Prob} \{ \text{the random configuration contains } x_1, \dots, x_n \}, \quad (2.7)$$

where  $n = 1, 2, \dots$ , and  $x_1, \dots, x_n$  are pairwise distinct points of  $\mathbb{Z}_{>0}$ . Under mild assumptions, the correlation functions determine the point process uniquely. A point process on  $\mathbb{Z}_{>0}$  is called *determinantal*, if there exists a function  $K$  on  $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$  (called the (*determinantal*) *correlation kernel*) such that the correlation functions  $\rho^{(n)}$ ,  $n = 1, 2, \dots$ , have the following form:

$$\rho^{(n)}(x_1, \dots, x_n) = \det [K(x_k, x_j)]_{k,j=1}^n.$$

About determinantal point processes see, e.g., the surveys [Sos00], [HKPV06], [Bor09].

To formulate the result, we need the following functions in the variable  $x \in \mathbb{Z}$  indexed by  $m \in \mathbb{Z}$  and depending on our parameters  $\alpha > 0$  and  $0 < \xi < 1$ :

$$\begin{aligned} \varphi_m(x; \alpha, \xi) := & \left( \frac{\Gamma(\frac{1}{2} + \nu(\alpha) + x) \Gamma(\frac{1}{2} - \nu(\alpha) + x)}{\Gamma(\frac{1}{2} + \nu(\alpha) - m) \Gamma(\frac{1}{2} - \nu(\alpha) - m)} \right)^{\frac{1}{2}} \xi^{\frac{1}{2}(x+m)} (1 - \xi)^{-m} \times \\ & \times \frac{{}_2F_1(\frac{1}{2} + \nu(\alpha) + m, \frac{1}{2} - \nu(\alpha) + m; x + m + 1; \frac{\xi}{\xi-1})}{\Gamma(x + m + 1)}. \end{aligned}$$

Here  ${}_2F_1(A, B; C; w) := \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{(C)_n n!} w^n$  is the Gauss hypergeometric function.

**Theorem 1.** *For any values of the parameters  $\alpha > 0$  and  $0 < \xi < 1$ , the point process  $M_{\alpha,\xi}$  on the half-lattice  $\mathbb{Z}_{>0}$  is determinantal. The correlation kernel  $\mathbf{K}_{\alpha,\xi}$  of  $M_{\alpha,\xi}$  is expressed through the Gauss hypergeometric function:*

$$\mathbf{K}_{\alpha,\xi}(x, y) = \frac{\sqrt{\alpha\xi xy}}{1 - \xi} \cdot \frac{P(x)Q(y) - Q(x)P(y)}{x^2 - y^2},$$

where  $P(x) := \varphi_0(x; \alpha, \xi)$  and  $Q(x) := \varphi_1(x; \alpha, \xi) - \varphi_{-1}(x; \alpha, \xi)$ .

We call the kernel  $\mathbf{K}_{\alpha,\xi}$  the *hypergeometric-type kernel*. In (8.3) we are able to express the kernel  $\mathbf{K}_{\alpha,\xi}$  through the discrete hypergeometric kernel introduced in [BO00], [BO06b].

## 2.2 Dynamical model

Let us now describe a family of continuous time Markov jump processes  $(\lambda_{\alpha,\xi}(t))_{t \in [0, +\infty)}$  on the space of all strict partitions  $\mathcal{S}$  (which is the same as the set of all finite configurations on  $\mathbb{Z}_{>0}$ ). These processes preserve the measures  $M_{\alpha,\xi}$ . The construction of the processes  $\lambda_{\alpha,\xi}$  uses the same

ideas as in [BO06a]. A closely related Markov dynamical model on strict partitions (but in discrete time) was considered in [Pet10c].

Our first key ingredient is the continuous time birth and death process on  $\mathbb{Z}_{>0}$  denoted by  $(\mathbf{n}_{\alpha,\xi}(t))_{t \in [0,+\infty)}$ . It depends on our parameters  $\alpha$  and  $\xi$  and has the following jump rates:

$$\begin{aligned} \text{Prob}\{\mathbf{n}_{\alpha,\xi}(t+dt) = n+1 \mid \mathbf{n}_{\alpha,\xi}(t) = n\} &= (1-\xi)^{-1} \xi(n+\alpha/2)dt, \\ \text{Prob}\{\mathbf{n}_{\alpha,\xi}(t+dt) = n-1 \mid \mathbf{n}_{\alpha,\xi}(t) = n\} &= (1-\xi)^{-1} ndt. \end{aligned}$$

The process  $\mathbf{n}_{\alpha,\xi}$  preserves the negative binomial distribution  $\pi_{\alpha,\xi}$  (2.3) on  $\mathbb{Z}_{>0}$  and is reversible with respect to it. About birth and death processes in general see, e.g., [KM57], [KM58].

The second key ingredient is the collection of Markov transition kernels  $p_\alpha^\uparrow(n, n+1)$  from  $\mathbb{S}_n$  to  $\mathbb{S}_{n+1}$  and  $p^\downarrow(n+1, n)$  from  $\mathbb{S}_{n+1}$  to  $\mathbb{S}_n$ ,  $n = 0, 1, 2, \dots$ , such that

$$M_{\alpha,n} \circ p_\alpha^\uparrow(n, n+1) = M_{\alpha,n+1} \quad \text{and} \quad M_{\alpha,n+1} \circ p^\downarrow(n+1, n) = M_{\alpha,n}. \quad (2.8)$$

These kernels are canonically associated with the system of measures  $\{M_{\alpha,n}\}_{n=0}^\infty$  (see §3.2 below, and also [Bor99], [Pet10c]), this construction follows the general formalism of Vershik and Kerov [VK87]. Note that the kernels  $p_\alpha^\uparrow(n, n+1)$  depend on the parameter  $\alpha$ , and the kernels  $p^\downarrow(n+1, n)$  do not depend on any parameter. The values  $p_\alpha^\uparrow(n, n+1)_{\mu,\chi}$  and  $p^\downarrow(n+1, n)_{\chi,\mu}$ , where  $\mu \in \mathbb{S}_n$  and  $\chi \in \mathbb{S}_{n+1}$  (these are the individual transition probabilities), *vanish* unless the shifted Young diagram  $\chi$  is obtained from  $\mu$  by adding a box. In other words, the transition kernels  $p_\alpha^\uparrow(n, n+1)$  and  $p^\downarrow(n+1, n)$  describe random procedures of adding and deleting one box, respectively.

We describe the dynamics  $\lambda_{\alpha,\xi}$  on strict partitions in terms of jump rates. The jumps are of two types: one can either add a box to the random shifted Young diagram, or remove a box from it (of course, the result must still be a shifted Young diagram). The events of adding and removing a box are governed by the birth and death process  $\mathbf{n}_{\alpha,\xi} = |\lambda_{\alpha,\xi}|$ . Conditioned on  $\lambda_{\alpha,\xi}(t) = \lambda$  and the jump  $n \rightarrow n+1$  (where  $n = |\lambda|$ ) of the process  $\mathbf{n}_{\alpha,\xi}$  during the time interval  $(t, t+dt)$ , the choice of the box to be added to the diagram  $\lambda$  is made according to the probabilities  $p_\alpha^\uparrow(n, n+1)_{\lambda,\chi}$ , where  $\chi \in \mathbb{S}_{n+1}$ . Similarly, conditioned on  $\lambda_{\alpha,\xi}(t) = \lambda$  and the jump  $n \rightarrow n-1$  of  $\mathbf{n}_{\alpha,\xi}$  during  $(t, t+dt)$ , the choice of the box to be removed from  $\lambda$  is made according to the probabilities  $p^\downarrow(n, n-1)_{\lambda,\mu}$ , where  $\mu \in \mathbb{S}_{n-1}$ .

The fact that the process  $\mathbf{n}_{\alpha,\xi}$  preserves the mixing distribution  $\pi_{\alpha,\xi}$  together with (2.8) implies that the measure  $M_{\alpha,\xi}$  on  $\mathbb{S}$  is invariant for the process  $\lambda_{\alpha,\xi}$ . Moreover, the process is reversible with respect to  $M_{\alpha,\xi}$ . In this paper by  $(\lambda_{\alpha,\xi}(t))_{t \geq 0}$  we mean the equilibrium process (that is, the process starting from the invariant distribution  $M_{\alpha,\xi}$ ).

Let  $(t_1, x_1), \dots, (t_n, x_n) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{>0}$  be pairwise distinct space-time points. The dynamical (or space-time) correlation functions of the Markov process  $\lambda_{\alpha,\xi}$  are defined as

$$\begin{aligned} \rho_{\alpha,\xi}^{(n)}(t_1, x_1; \dots; t_n, x_n) \\ := \text{Prob}\{\text{the configuration } \lambda_{\alpha,\xi}(t) \text{ at time } t = t_j \text{ contains } x_j, j = 1, \dots, n\}. \end{aligned} \quad (2.9)$$

The notion of dynamical correlation functions is a combination of finite-dimensional distributions of a stochastic dynamics and correlation functions of a random point process. Indeed, the finite-dimensional distribution of the process  $\lambda_{\alpha,\xi}$  at times  $t_1, \dots, t_n$  (let these times be distinct for simplicity) is a probability measure on configurations on the space  $\mathbb{Z}_{>0} \sqcup \dots \sqcup \mathbb{Z}_{>0}$  ( $n$  copies), and

$\rho_{\alpha,\xi}^{(n)}(t_1, x_1; \dots; t_n, x_n)$  ( $t_j$ 's fixed) are just the correlation functions of this measure on configurations. The dynamical correlation functions uniquely determine the dynamics  $(\lambda_{\alpha,\xi}(t))_{t \in [0, +\infty)}$ .

The main result of the present paper is the computation of the dynamical correlation functions of  $\lambda_{\alpha,\xi}$ .

To formulate the result, we need a notation. By  $\mathbb{Z}_{\neq 0}$  denote the set of all nonzero integers, and for  $x_1, \dots, x_n \in \mathbb{Z}_{>0}$  put, by definition,  $x_{-k} := -x_k$ ,  $k = 1, \dots, n$ .

**Theorem 2.** *The equilibrium continuous time dynamics  $(\lambda_{\alpha,\xi}(t))_{t \geq 0}$  is Pfaffian, that is, there exists a function  $\Phi_{\alpha,\xi}(s, x; t, y)$ ,  $x, y \in \mathbb{Z}$ ,  $s \leq t$ , such that the dynamical correlation functions of  $\lambda_{\alpha,\xi}$  have the form*

$$\rho_{\alpha,\xi}^{(n)}(t_1, x_1; \dots; t_n, x_n) = \text{Pf} \left( \Phi_{\alpha,\xi} \llbracket T, X \rrbracket \right), \quad 0 \leq t_1 \leq \dots \leq t_n, \quad (2.10)$$

where  $\Phi_{\alpha,\xi} \llbracket T, X \rrbracket$  is the  $2n \times 2n$  skew-symmetric matrix with rows and columns indexed by  $1, -1, \dots, n, -n$ , and the  $kj$ -th entry in  $\Phi_{\alpha,\xi} \llbracket T, X \rrbracket$  above the main diagonal is  $\Phi_{\alpha,\xi}(t_{|k|}, x_k; t_{|j|}, x_j)$ , where  $k, j = 1, -1, \dots, n, -n$  (thus,  $|k| \leq |j|$ ). The kernel  $\Phi_{\alpha,\xi}$  is expressed through the Gauss hypergeometric function:

$$\Phi_{\alpha,\xi}(s, x; t, y) := (-1)^{\min(x,0) + \max(y,0)} \sum_{m=0}^{\infty} 2^{-\delta(m)} e^{-m(t-s)} \varphi_m(x; \alpha, \xi) \varphi_m(-y; \alpha, \xi).$$

We express our dynamical Pfaffian kernel  $\Phi_{\alpha,\xi}$  through the extended discrete hypergeometric kernel of [BO06a], see §10.5. Such an expression helps to study the asymptotic behavior of the dynamical kernel in various limit regimes. This is carried out in [Pet10a, §11].

**Remark 2.2. 1.** Observe that it is enough for  $\Phi_{\alpha,\xi}(s, x; t, y)$  to be defined only for  $x, y \in \mathbb{Z}_{\neq 0}$  because only such values of  $\Phi_{\alpha,\xi}(s, x; t, y)$  are used in the theorem. However, our kernel  $\Phi_{\alpha,\xi}(s, x; t, y)$  extends to  $x, y \in \mathbb{Z}$  in a very natural way, so we always let  $\Phi_{\alpha,\xi}(s, x; t, y)$  to be defined for all  $x, y \in \mathbb{Z}$ . The same is applicable to the static Pfaffian kernel  $\Phi_{\alpha,\xi}(x, y)$ .

**2.** In (2.10) we require that the time moments  $t_j$  are ordered. However, Theorem 2 allows to compute the correlation functions  $\rho_{\alpha,\xi}^{(n)}(t_1, x_1; \dots; t_n, x_n)$  with arbitrary order of time moments: one should simply permute the space-time points  $(t_1, x_1), \dots, (t_n, x_n)$  (this does not change the value of  $\rho_{\alpha,\xi}^{(n)}(t_1, x_1; \dots; t_n, x_n)$ ) in such a way that the time moments become nondecreasing, and then apply (2.10).

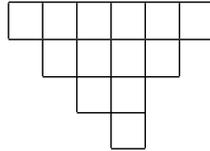
**Remark 2.3** (Hidden determinantal structure in Pfaffian processes). If in Theorem 2 we set  $t_1 = \dots = t_n$ , then the dynamical correlation functions turn into the (static) correlation functions of the point process  $M_{\alpha,\xi}$  on  $\mathbb{Z}_{>0}$ . Thus, Theorem 2 implies that the point process  $M_{\alpha,\xi}$  on  $\mathbb{Z}_{>0}$  is Pfaffian. To show that it is in fact determinantal requires some work (see Theorem 8.1 and Proposition A.2 from Appendix). Thus, one can say that in the static case the determinantal structure of correlation functions is *hidden* under the Pfaffian one.

On the other hand, numerical computations suggest that the dynamical correlation functions of the Markov process  $\lambda_{\alpha,\xi}$  cannot be written as determinants. We plan to give a rigorous proof of this fact in a subsequent work.

### 3 Schur graph and multiplicative measures

#### 3.1 Schur graph

We identify *strict partitions*  $\lambda = (\lambda_1 > \dots > \lambda_{\ell(\lambda)} > 0)$ ,  $\lambda_j \in \mathbb{Z}_{>0}$ , and corresponding *shifted Young diagrams* as in [Mac95, Ch. I, §1, Example 9]. The shifted Young diagram of the form  $\lambda$  consists of  $\ell(\lambda)$  rows. Each  $k$ th row ( $k = 1, \dots, \ell(\lambda)$ ) has  $\lambda_k$  boxes, and for  $j = 1, \dots, \ell(\lambda) - 1$  the first box of the  $(j + 1)$ th row is right under the second box of the  $j$ th row. For example, the shifted Young diagram corresponding to the strict partition  $\lambda = (6, 4, 2, 1)$  looks as follows:



Let  $\mu$  and  $\lambda$  be strict partitions. If  $|\lambda| = |\mu| + 1$  and the shifted diagram  $\lambda$  is obtained from the shifted diagram  $\mu$  by adding a box, then we write  $\mu \nearrow \lambda$ , or, equivalently,  $\lambda \searrow \mu$ . The box that is added is denoted by  $\lambda/\mu$ .

The set  $\mathbb{S} = \bigsqcup_{n=0}^{\infty} \mathbb{S}_n$  of all strict partitions is equipped with a structure of a graded graph: for  $\mu \in \mathbb{S}_{n-1}$  and  $\lambda \in \mathbb{S}_n$  we draw an edge between  $\mu$  and  $\lambda$  iff  $\mu \nearrow \lambda$ . Thus, the edges in  $\mathbb{S}$  are drawn only between consecutive floors. We assume the edges to be oriented from  $\mathbb{S}_{n-1}$  to  $\mathbb{S}_n$ . In this way  $\mathbb{S}$  becomes a graded graph. It is called the *Schur graph*.<sup>8</sup> This graph describes the branching of (suitably normalized) irreducible truly projective characters of symmetric groups, e.g., see [Iva99].

Let  $\dim_{\mathbb{S}} \lambda$  be the total number of oriented paths in the Schur graph from the initial vertex  $\emptyset$  to the vertex  $\lambda$ . This number is given by [Mac95, Ch. III, §8, Example 12]

$$\dim_{\mathbb{S}} \lambda = \frac{|\lambda|!}{\lambda_1! \dots \lambda_{\ell(\lambda)}!} \prod_{1 \leq k < j \leq \ell(\lambda)} \frac{\lambda_k - \lambda_j}{\lambda_k + \lambda_j}, \quad \lambda \in \mathbb{S}. \quad (3.1)$$

Observe that if the components of  $\lambda$  are not distinct, then  $\dim_{\mathbb{S}} \lambda$  vanishes. The numbers  $\dim_{\mathbb{S}} \lambda$  satisfy the recurrence relations

$$\dim_{\mathbb{S}} \lambda = \sum_{\mu: \mu \nearrow \lambda} \dim_{\mathbb{S}} \mu \quad \text{for all } \lambda \in \mathbb{S}, \quad \dim_{\mathbb{S}} \emptyset = 1. \quad (3.2)$$

The number  $\dim_{\mathbb{S}} \lambda$  can also be interpreted as the number of shifted standard tableaux of the form  $\lambda$  [Sag87], [Wor84], and as the (suitably normalized) dimension of the irreducible truly projective representation of the symmetric group  $\mathfrak{S}_{|\lambda|}$  corresponding to the shifted diagram  $\lambda$  [HH92], [Iva99].

Similarly, by  $\dim_{\mathbb{S}}(\mu, \lambda)$  denote the total number of paths from  $\mu$  to  $\lambda$  in the graph  $\mathbb{S}$ . Clearly,  $\dim_{\mathbb{S}}(\mu, \lambda)$  vanishes unless  $\mu \subseteq \lambda$ , that is, unless  $\mu_k \leq \lambda_k$  for all  $k$ . If  $\mu \subseteq \lambda$ , by  $\lambda/\mu$  denote the corresponding skew shifted Young diagram, that is, the set difference of  $\lambda$  and  $\mu$ . We have  $\dim_{\mathbb{S}} \lambda = \dim_{\mathbb{S}}(\emptyset, \lambda)$ .

<sup>8</sup>In [Pet10c] the Schur graph had multiple edges, but now it is more convenient for us to consider simple edges as in, e.g., [Bor99]. The difference between these two choices is inessential because the down transition probabilities (§3.2) are the same.

### 3.2 Coherent systems of measures on the Schur graph

Following the general formalism (e.g., see [KOO98]), one can define coherent systems of measures on the Schur graph. This definition starts from the notion of the *down transition probabilities*. For  $\lambda, \mu \in \mathbb{S}$ , set

$$p^\downarrow(\lambda, \mu) := \begin{cases} \dim_{\mathbb{S}} \mu / \dim_{\mathbb{S}} \lambda, & \text{if } \mu \nearrow \lambda; \\ 0, & \text{otherwise.} \end{cases}$$

By (3.2), the restriction of  $p^\downarrow$  to  $\mathbb{S}_{n+1} \times \mathbb{S}_n$  for all  $n = 0, 1, \dots$  is a Markov transition kernel. We denote it by  $p^\downarrow(n+1, n) = \{p^\downarrow(n+1, n)_{\lambda, \mu}\}_{\lambda \in \mathbb{S}_{n+1}, \mu \in \mathbb{S}_n}$ , and call it the *down transition kernel*.

**Definition 3.1.** Let  $M_n$  be a probability measure on  $\mathbb{S}_n$ ,  $n = 0, 1, \dots$ . We call  $\{M_n\}$  a *coherent system of measures* iff

$$M_n(\lambda) = \sum_{\mathcal{X}: \mathcal{X} \searrow \lambda} M_{n+1}(\mathcal{X}) p^\downarrow(\mathcal{X}, \lambda) \quad \text{for all } n \text{ and } \lambda \in \mathbb{S}_n. \quad (3.3)$$

In other words,  $M_{n+1} \circ p^\downarrow(n+1, n) = M_n$  for all  $n$  (cf. (2.8)).

Having a *nondegenerate* coherent system  $\{M_n\}$  (that is,  $M_n(\lambda) > 0$  for all  $n$  and  $\lambda \in \mathbb{S}_n$ ), we can define the corresponding *up transition probabilities*. They depend on a choice of a coherent system. For  $\lambda, \mathcal{X} \in \mathbb{S}$ , set

$$p^\uparrow(\lambda, \mathcal{X}) := \begin{cases} M_{n+1}(\mathcal{X}) p^\downarrow(\mathcal{X}, \lambda) / M_n(\lambda), & \text{if } \lambda \in \mathbb{S}_n, \mathcal{X} \in \mathbb{S}_{n+1} \text{ and } \lambda \nearrow \mathcal{X}, \\ 0, & \text{otherwise.} \end{cases}$$

By (3.3), the restriction of  $p^\uparrow$  to  $\mathbb{S}_n \times \mathbb{S}_{n+1}$  for all  $n = 0, 1, \dots$  is a Markov transition kernel. We denote it by  $p^\uparrow(n, n+1) = \{p^\uparrow(n, n+1)_{\lambda, \mathcal{X}}\}_{\lambda \in \mathbb{S}_n, \mathcal{X} \in \mathbb{S}_{n+1}}$  and call it the *up transition kernel*. We have  $M_n \circ p^\uparrow(n, n+1) = M_{n+1}$  (cf. (2.8)).

A representation-theoretic meaning of coherent systems of measures on the Schur graph is discussed in, e.g., [Pet10a, §3.2] (see also [Naz92], [Iva99]). A general formalism is explained in [KOO98].

### 3.3 Multiplicative measures

There is a distinguished coherent system on the Schur graph, namely, the Plancherel measures  $\{\text{Pl}_n\}_{n=0}^\infty$  (2.1). Using the function  $\dim_{\mathbb{S}} \lambda$  defined by (3.1), one can write

$$\text{Pl}_n(\lambda) = 2^{n-\ell(\lambda)} (\dim_{\mathbb{S}} \lambda)^2 / n!, \quad n \in \mathbb{Z}_{>0}, \quad \lambda \in \mathbb{S}_n.$$

The Plancherel measures on strict partitions are analogues (in the theory of projective representations of symmetric groups) of the well-known Plancherel measures on ordinary partitions.

Borodin [Bor99] has introduced a deformation  $M_{\alpha, n}$  (2.2) of the measures  $\text{Pl}_n$  on  $\mathbb{S}_n$  depending on one real parameter  $\alpha > 0$ . Here let us recall the characterization of the measures  $\{M_{\alpha, n}\}$  from [Bor99].

**Definition 3.2.** A system of probability measures  $M_n$  on  $\mathbb{S}_n$  is called *multiplicative* if there exists a function  $f : \{(i, j) : j \geq i \geq 1\} \rightarrow \mathbb{C}$  such that

$$M_n(\lambda) = c_n \cdot \text{Pl}_n(\lambda) \cdot \prod_{\square=(i,j) \in \lambda} f(i, j) \quad \text{for all } n \text{ and all } \lambda \in \mathbb{S}_n.$$

Here  $c_n$ ,  $n = 0, 1, \dots$ , are normalizing constants. The product above is taken over all boxes  $\square = (i, j)$  of the shifted Young diagram  $\lambda$ , where  $i$  and  $j$  are the row and column numbers of the box  $\square$ , respectively. (Note that for shifted Young diagrams we always have  $j \geq i$ .)

**Theorem 3.3** ([Bor99]). *Let  $\{M_n\}$  be a nondegenerate coherent system of measures on the Schur graph. It is multiplicative iff the function  $f$  has the form*

$$f(i, j) = (j - i)(j - i + 1) + \alpha \tag{3.4}$$

for some parameter  $\alpha \in (0, +\infty]$ .<sup>9</sup>

If  $f(i, j)$  is given by (3.4), then  $c_n = \alpha(\alpha + 2) \dots (\alpha + 2n - 2)$ . The case  $\alpha = +\infty$  is understood in the limit sense:  $\lim_{\alpha \rightarrow +\infty} \frac{1}{c_n} \prod_{\square=(i,j) \in \lambda} f(i, j) = 1$  for all  $n$  (this gives the Plancherel measures).

We denote by  $\{M_{\alpha, n}\}_{n=0}^{\infty}$  the multiplicative coherent system corresponding to the parameter  $\alpha \in (0, +\infty)$ , these measures are given by (2.2). The up transition kernel on  $\mathbb{S}_n \times \mathbb{S}_{n+1}$  (§3.2) for the coherent system  $\{M_{\alpha, n}\}$  is denoted by  $p_{\alpha}^{\uparrow}(n, n + 1)$ .

### 3.4 Mixing of measures. Point configurations on the half-lattice

For a set  $\mathfrak{X}$ , by  $\text{Conf}(\mathfrak{X})$  denote the space of all (locally finite) point configurations on  $\mathfrak{X}$ , and by  $\text{Conf}_{\text{fin}}(\mathfrak{X}) \subset \text{Conf}(\mathfrak{X})$  denote the subset consisting only of finite configurations. A Borel probability measure (with respect to a certain natural topology) on  $\text{Conf}(\mathfrak{X})$  is called a random point process on  $\mathfrak{X}$ . If  $\mathfrak{X}$  is discrete, then  $\text{Conf}(\mathfrak{X}) \cong \{0, 1\}^{\mathfrak{X}}$ , and we take the standard coordinatewise topology on  $\{0, 1\}^{\mathfrak{X}}$  which turns it into a compact space. More details about random point processes can be found in [Sos00].

As explained in §2.1, we mix the measures  $M_{\alpha, n}$  (2.2) using the negative binomial distribution  $\pi_{\alpha, \xi}$  (2.3) on the set  $\{0, 1, \dots\}$  of indices  $n$ . As a result we get a probability measure  $M_{\alpha, \xi}$  (2.5) on the set  $\mathbb{S}$  of all strict partitions. Identifying strict partitions with point configurations in a natural way (§2.1), we see that the set  $\mathbb{S}$  is the same as  $\text{Conf}_{\text{fin}}(\mathbb{Z}_{>0})$ . Thus,  $M_{\alpha, \xi}$  can be viewed as a random point process on  $\mathbb{Z}_{>0}$  supported by finite configurations.

Let us now prove that the point processes  $M_{\alpha, \xi}$  on  $\mathbb{Z}_{>0}$  are determinantal. Observe that both these processes have a general structure described as follows:

**Definition 3.4.** Let  $w$  be a nonnegative function on  $\mathbb{Z}_{>0}$  such that

$$\sum_{x=1}^{\infty} w(x) < \infty. \tag{3.5}$$

By  $\mathbf{P}^{(w)}$  denote the point process on  $\mathbb{Z}_{>0}$  that lives on finite configurations and assigns the probability

$$\mathbf{P}^{(w)}(\lambda) := \text{const} \cdot \prod_{k=1}^{\ell} w(\lambda_k) \prod_{1 \leq k < j \leq \ell} \left( \frac{\lambda_k - \lambda_j}{\lambda_k + \lambda_j} \right)^2 \tag{3.6}$$

to every configuration  $\lambda = \{\lambda_1, \dots, \lambda_{\ell}\} \subset \mathbb{Z}_{>0}$ , where  $\text{const}$  is a normalizing constant.

<sup>9</sup>For certain negative values of  $\alpha$  one can also define the measures  $M_{\alpha, n}$  by Definition 3.2 with  $f$  given by (3.4) (see §6.5), but such measures are degenerate.

The process  $M_{\alpha,\xi}$  has the form  $\mathbf{P}^{(w)}$  with  $w(x) = w_{\alpha,\xi}(x)$  given by (2.6).

Let  $\mathbf{L}^{(w)}$  be the following  $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$  matrix:

$$\mathbf{L}^{(w)}(x, y) := \frac{2\sqrt{xy \cdot w(x)w(y)}}{x + y}, \quad x, y \in \mathbb{Z}_{>0}. \quad (3.7)$$

Condition (3.5) implies that the operator in  $\ell^2(\mathbb{Z}_{>0})$  corresponding to  $\mathbf{L}^{(w)}$  is of trace class, and, in particular, the Fredholm determinant  $\det(1 + \mathbf{L}^{(w)})$  is well defined.

**Lemma 3.5.** (1) Let  $\lambda = \{\lambda_1, \dots, \lambda_\ell\} \subset \mathbb{Z}_{>0}$  be a point configuration. We have

$$\mathbf{P}^{(w)}(\lambda) = \frac{\det \mathbf{L}^{(w)}(\lambda)}{\det(1 + \mathbf{L}^{(w)})},$$

where  $\mathbf{L}^{(w)}(\lambda)$  denotes the submatrix  $[\mathbf{L}^{(w)}(\lambda_k, \lambda_j)]_{k,j=1}^\ell$  of  $\mathbf{L}^{(w)}$ .

(2) The point process  $\mathbf{P}^{(w)}$  is determinantal with the correlation kernel  $\mathbf{K}^{(w)} = \mathbf{L}^{(w)}(1 + \mathbf{L}^{(w)})^{-1}$ .

*Proof.* The first claim directly follows from the Cauchy determinant identity [Mac95, Ch. I, §4, Ex. 6].

This means that the point process  $\mathbf{P}^{(w)}$  is a so-called L-ensemble corresponding to the matrix  $\mathbf{L}^{(w)}$  defined above (e.g., see [BO00, Prop. 2.1] or [Bor09, §5]). This implies the second claim about the correlation kernel.  $\square$

Note that the normalizing constant in (3.6) is equal to  $(\det(1 + \mathbf{L}^{(w)}))^{-1}$ , so condition (3.5) is necessary for the point process  $\mathbf{P}^{(w)}$  to be well defined.

**Remark 3.6.** The correlation kernel  $\mathbf{K}^{(w)}$  of the process  $\mathbf{P}^{(w)}$  is symmetric, because it has the form  $\mathbf{K}^{(w)} = \mathbf{L}^{(w)}(1 + \mathbf{L}^{(w)})^{-1}$ , where  $\mathbf{L}^{(w)}$  is symmetric. However, the operator of the form  $\mathbf{L}^{(w)}(1 + \mathbf{L}^{(w)})^{-1}$  cannot be a projection operator in  $\ell^2(\mathbb{Z}_{>0})$ . This aspect discriminates our processes from many other determinantal processes appearing in, e.g., random matrix models (see the references given in Introduction).

On the other hand, the static Pfaffian kernel in our model resembles the structure of a spectral projection operator, see Proposition 7.5.

Lemma 3.5 implies, in particular, that our point processes  $M_{\alpha,\xi}$  on  $\mathbb{Z}_{>0}$  are determinantal (for all values of  $\alpha > 0$  and  $0 < \xi < 1$ ). Denote the correlation kernel of  $M_{\alpha,\xi}$  by  $\mathbf{K}_{\alpha,\xi}$  (it is symmetric). However, Lemma 3.5 does not give any suggestions on how to calculate this kernel. Below we compute  $\mathbf{K}_{\alpha,\xi}$  using a fermionic Fock space technique.

## 4 Kerov's operators

### 4.1 Definition

The main tool that we use in the present paper to compute the correlation functions of the point processes  $M_{\alpha,\xi}$  (and also of the associated dynamical models, see §9–§10) is a representation of the

Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$  in the pre-Hilbert space  $\ell_{\text{fin}}^2(\mathbb{S})$  given by the so-called Kerov's operators. This approach was introduced by Okounkov [Oko01b] for the  $z$ -measures on ordinary partitions.

By  $\ell_{\text{fin}}^2(\mathbb{S})$  we denote the space of all finitely supported functions on  $\mathbb{S}$  with the inner product

$$(f, g) := \sum_{\lambda \in \mathbb{S}} f(\lambda)g(\lambda).$$

This is a pre-Hilbert space whose Hilbert completion is the usual space  $\ell^2(\mathbb{S})$  of all functions on  $\mathbb{S}$  which are square integrable with respect to the counting measure on  $\mathbb{S}$ . The standard orthonormal basis in  $\ell^2(\mathbb{S})$  is denoted by  $\{\underline{\lambda}\}_{\lambda \in \mathbb{S}}$ , that is,

$$\underline{\lambda}(\mu) := \begin{cases} 1, & \text{if } \mu = \lambda; \\ 0, & \text{otherwise.} \end{cases} \quad (4.1)$$

**Definition 4.1.** The Kerov's operators in  $\ell_{\text{fin}}^2(\mathbb{S})$  depend on our parameter  $\alpha > 0$  and are defined as

$$\begin{aligned} U\underline{\lambda} &:= \sum_{x: x \searrow \lambda} 2^{-\delta(j-i)/2} \sqrt{(j-i)(j-i+1) + \alpha} \cdot \underline{x}, & (i, j) = x/\lambda; \\ D\underline{\lambda} &:= \sum_{\mu: \mu \nearrow \lambda} 2^{-\delta(j-i)/2} \sqrt{(j-i)(j-i+1) + \alpha} \cdot \underline{\mu}, & (i, j) = \lambda/\mu; \\ H\underline{\lambda} &:= \left(2|\lambda| + \frac{\alpha}{2}\right) \underline{\lambda}. \end{aligned} \quad (4.2)$$

We denote a box by  $(i, j)$  iff its row number is  $i$  and its column number is  $j$ .

The Kerov's operators are closely related to the measures  $M_{\alpha, n}$  (2.2) on  $\mathbb{S}_n$ : since

$$(U^n \underline{\emptyset}, \underline{\lambda}) = (D^n \underline{\lambda}, \underline{\emptyset}) = \dim_{\mathbb{S}} \lambda \cdot 2^{-\ell(\lambda)/2} \prod_{\square=(i,j) \in \lambda} \sqrt{(j-i)(j-i+1) + \alpha}$$

for all  $n$  and  $\lambda \in \mathbb{S}_n$ , we have

$$M_{\alpha, n}(\lambda) = Z_n^{-1} (U^n \underline{\emptyset}, \underline{\lambda}) (D^n \underline{\lambda}, \underline{\emptyset}), \quad (4.3)$$

where  $Z_n = n!(\alpha/2)^n$  is the normalizing constant.

**Lemma 4.2.** *The map*

$$U := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow U, \quad D := \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \rightarrow D, \quad H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow H \quad (4.4)$$

defines a representation of the Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$  in  $\ell_{\text{fin}}^2(\mathbb{S})$ . That is, the operators  $U$ ,  $D$ , and  $H$  satisfy the commutation relations

$$[H, U] = 2U, \quad [H, D] = -2D, \quad [D, U] = H. \quad (4.5)$$

*Proof.* Denote

$$q_{\alpha}(\square) = q_{\alpha}(i, j) := 2^{-\delta(j-i)/2} \sqrt{(j-i)(j-i+1) + \alpha}, \quad (4.6)$$

where  $\square = (i, j)$ . The relation  $[H, U] = 2U$  is straightforward:

$$\begin{aligned} [H, U] \underline{\lambda} &= H \sum_{x \searrow \lambda} q_{\alpha}(x/\lambda) \underline{x} - \left(2|\lambda| + \frac{\alpha}{2}\right) \sum_{x \searrow \lambda} q_{\alpha}(x/\lambda) \underline{x} \\ &= 2(|\lambda| + 2 - |\lambda|) U \underline{\lambda} = 2U \underline{\lambda}, \end{aligned}$$

and the same for the relation  $[H, D] = -2D$ .

It remains to prove that  $[D, U] = H$ . The vector  $[D, U] \underline{\lambda}$  has the form

$$\sum_{x \searrow \lambda} \sum_{\rho \nearrow x} q_\alpha(x/\lambda) q_\alpha(x/\rho) \underline{\rho} - \sum_{\mu \nearrow \lambda} \sum_{\rho \searrow \mu} q_\alpha(\lambda/\mu) q_\alpha(\rho/\mu) \underline{\rho}. \quad (4.7)$$

This is a linear combination of vectors  $\underline{\rho}$ , where  $\rho \in \mathbb{S}_n$  and either  $\rho = \lambda$ , or  $\rho = \lambda + \square_1 - \square_2$  for some boxes  $\square_1 \neq \square_2$ . In the second case the coefficient by the vector  $\underline{\rho}$  with  $\rho = \lambda + \square_1 - \square_2$  is

$$q_\alpha(\square_1) q_\alpha(\square_2) - q_\alpha(\square_2) q_\alpha(\square_1) = 0.$$

Thus, in (4.7) it remains to consider only the terms with  $\rho = \lambda$ . Therefore, one must establish the combinatorial identity

$$\sum_{x: x \searrow \lambda} q_\alpha(x/\lambda)^2 - \sum_{\mu: \mu \nearrow \lambda} q_\alpha(\lambda/\mu)^2 = 2|\lambda| + \frac{\alpha}{2} \quad \text{for all } \lambda \in \mathbb{S}.$$

The proof of this identity (using Kerov's interlacing coordinates of shifted Young diagrams) is essentially contained in §3.1 of the paper [Pet10c] (the arXiv version).  $\square$

For a more detailed discussion about Kerov's operators and their connection with the measures  $M_{\alpha, n}$  see [Pet10a, §4.1].

## 4.2 Kerov's operators and averages with respect to our point processes

The probability assigned to a strict partition  $\lambda$  by the measure  $M_{\alpha, \xi}$  (2.5) (which is a mixture of the measures  $M_{\alpha, n}$ ) can be written for small enough  $\xi$  as follows:

$$M_{\alpha, \xi}(\lambda) = (1 - \xi)^{\alpha/2} (e^{\sqrt{\xi}U} \underline{\emptyset}, \underline{\lambda}) (e^{\sqrt{\xi}D} \underline{\lambda}, \underline{\emptyset}).$$

Here  $e^{\sqrt{\xi}D} \underline{\lambda}$  is clearly an element of  $\ell_{\text{fin}}^2(\mathbb{S})$ . The fact that the vector  $e^{\sqrt{\xi}U} \underline{\lambda}$  belongs to  $\ell^2(\mathbb{S})$  (for small enough  $\xi$ ) requires a justification (see the proof of Proposition 4.3), because the operator  $U$  in  $\ell^2(\mathbb{S})$  is unbounded. This makes the above formula for  $M_{\alpha, \xi}(\lambda)$  not very convenient for taking averages with respect to the measure  $M_{\alpha, \xi}$ .<sup>10</sup> In this subsection we overcome this difficulty and give a convenient way of writing expectations with respect to  $M_{\alpha, \xi}$ . Our approach here is similar to that of Olshanski [Ols08] and is also based on the ideas of [Oko01b].

Recall that the Kerov's operators  $U$ ,  $D$ , and  $H$  (4.2) define (via the map (4.4)) a representation of the complex Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$  in the (complex) pre-Hilbert space  $\ell_{\text{fin}}^2(\mathbb{S})$ . Consider the real form  $\mathfrak{su}(1, 1) \subset \mathfrak{sl}(2, \mathbb{C})$  spanned by the matrices  $U - D$ ,  $i(U + D)$ , and  $iH$  (here  $i = \sqrt{-1}$ ). The corresponding operators  $U - D$ ,  $i(U + D)$ , and  $iH$  act skew-symmetrically in  $\ell_{\text{fin}}^2(\mathbb{S})$ . Now we prove that the representation of the Lie algebra  $\mathfrak{su}(1, 1)$  can be lifted to a representation of a corresponding Lie group:

**Proposition 4.3.** *All vectors of the space  $\ell_{\text{fin}}^2(\mathbb{S})$  are analytic for the described above action of the Lie algebra  $\mathfrak{su}(1, 1)$  in  $\ell_{\text{fin}}^2(\mathbb{S})$ . Consequently, this action of  $\mathfrak{su}(1, 1)$  gives rise to a unitary representation of the universal covering group  $SU(1, 1)^\sim$  in the Hilbert space  $\ell^2(\mathbb{S})$ .*

<sup>10</sup>Static correlation functions are readily expressed as averages with respect to  $M_{\alpha, \xi}$  (see (7.4) below), so we need good tools for computing such averages.

*Proof.* Recall [Nel59] that a vector  $h$  is analytic for an operator  $A$  if the power series  $\sum_{n=0}^{\infty} \frac{\|A^n h\|}{n!} s^n$  in  $s$  has a positive radius of convergence.

We can use Lemma 9.1 in [Nel59] that guarantees the existence of the desired unitary representation of  $SU(1, 1)^\sim$  in  $\ell^2(\mathbb{S})$  if we first prove that for some constant  $s_0 > 0$  we have

$$\|A_{i_1} \dots A_{i_n} h\| \leq \frac{n!}{s_0^n} \quad (4.8)$$

for any  $h \in \ell_{\text{fin}}^2(\mathbb{S})$ , all sufficiently large  $n$  (the bound on  $n$  depends on  $h$ ), and any indices  $i_1, \dots, i_n$  taking values 1, 2, 3, where  $A_1 = U - D$ ,  $A_2 = i(U + D)$ , and  $A_3 = iH$ . Note that this in fact implies that any vector in  $\ell_{\text{fin}}^2(\mathbb{S})$  is analytic for the action of  $\mathfrak{su}(1, 1)$ .

It suffices to prove the estimate (4.8) for  $\hat{A}_1 := U$ ,  $\hat{A}_2 := D$ , and  $\hat{A}_3 := H$ , this can only affect the value of the constant  $s_0$ . Moreover, we can consider only the cases when  $h = \underline{x}$  for an arbitrary  $x \in \mathbb{S}$ . Because all the matrix elements of the operators  $U$ ,  $D$ , and  $H$  are nonnegative in the standard basis  $\{\underline{\lambda}\}_{\lambda \in \mathbb{S}}$ , we have

$$\|\hat{A}_{i_1} \dots \hat{A}_{i_n} \underline{x}\| \leq \|(U + D + H)^n \underline{x}\|.$$

Now the desired estimate would follow if we show that the power series expansion of  $\exp(s(U + D + H)) \underline{x}$  converges for small enough  $s > 0$ . For matrices in  $SL(2, \mathbb{C})$  (see (4.4)) we have

$$\exp(s(U + D + H)) = \exp\left(\frac{s}{1-s}U\right) \exp\left(\log\left(\frac{1}{1-s}\right)H\right) \exp\left(\frac{s}{1-s}D\right).$$

Thus, the power series expansion of  $\exp(s(U + D + H)) \underline{x}$  is the same as that of

$$\exp\left(\frac{s}{1-s}U\right) \exp\left(\log\left(\frac{1}{1-s}\right)H\right) \exp\left(\frac{s}{1-s}D\right) \underline{x}.$$

Since the operator  $D$  is locally nilpotent and the operator  $H$  acts on each  $\underline{\lambda}$  as multiplication by  $(2|\lambda| + \alpha/2)$ , to obtain the desired estimate (4.8) it remains to show that the series

$$\sum_{n=0}^{\infty} \|U^n \underline{\mu}\| s^n / n!$$

converges for all  $\mu \in \mathbb{S}$  for sufficiently small  $s > 0$  (the bound on  $s$  must not depend on  $\mu$ ). Let us fix  $\mu$  with  $|\mu| = k$ . We can write by definition of  $U$ :

$$\|U^n \underline{\mu}\|^2 = \sum_{\lambda \in \mathbb{S}_{k+n}} (U^n \underline{\mu}, \underline{\lambda})^2 = \sum_{\lambda \in \mathbb{S}_{k+n}} \dim_{\mathbb{S}}(\mu, \lambda)^2 \prod_{\square \in \lambda/\mu} q_{\alpha}(\square)^2,$$

where  $q_{\alpha}$  is defined by (4.6). Here the product is taken over all boxes of the skew shifted diagram  $\lambda/\mu$  (see the end of §3.1). Since  $\dim_{\mathbb{S}}(\mu, \lambda) \leq \dim_{\mathbb{S}} \lambda$ , we can estimate

$$\begin{aligned} \|U^n \underline{\mu}\|^2 &\leq \left(\prod_{\square \in \mu} q_{\alpha}(\square)^{-2}\right) \cdot \sum_{\lambda \in \mathbb{S}_{k+n}} (\dim_{\mathbb{S}} \lambda)^2 \prod_{\square \in \lambda} q_{\alpha}(\square)^2 \\ &= \left(\prod_{\square \in \mu} q_{\alpha}(\square)^{-2}\right) \cdot \sum_{\lambda \in \mathbb{S}_{n+k}} (U^n \underline{\emptyset}, \underline{\lambda})^2 = Z_{n+k} \cdot \left(\prod_{\square \in \mu} q_{\alpha}(\square)^{-2}\right). \end{aligned}$$

The factor  $\prod_{\square \in \mu} q_{\alpha}(\square)^{-2}$  has no impact on convergence, and  $Z_n = n!(\alpha/2)_n$ . Putting all together, we get

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} \|U^n \underline{\mu}\| \leq \left(\prod_{\square \in \mu} q_{\alpha}(\square)^{-2}\right)^{\frac{1}{2}} \cdot \sum_{n=0}^{\infty} \frac{s^n}{n!} \sqrt{(n+k)!(\alpha/2)_{n+k}}. \quad (4.9)$$

Using [Erd53, 1.18.(5)], we see that

$$\frac{\sqrt{(n+k)!(\alpha/2)_{n+k}}}{n!} \sim \sqrt{\frac{n^{2k+\alpha/2-1}}{\Gamma(\alpha/2)}},$$

so the series (4.9) converges for small enough  $s > 0$ . This concludes the proof of the proposition.  $\square$

To formulate the central statement of this section, we need some preparation. By  $G_\xi$  denote the matrix

$$G_\xi := \begin{bmatrix} \frac{1}{\sqrt{1-\xi}} & \frac{\sqrt{\xi}}{\sqrt{1-\xi}} \\ \frac{\sqrt{\xi}}{\sqrt{1-\xi}} & \frac{1}{\sqrt{1-\xi}} \end{bmatrix} = \left( \frac{1 + \sqrt{\xi}}{1 - \sqrt{\xi}} \right)^{\frac{U-D}{2}} \in SU(1, 1), \quad 0 \leq \xi < 1. \quad (4.10)$$

Clearly,  $(G_\xi)_{0 \leq \xi < 1}$  is a continuous curve in  $SU(1, 1)$  starting at the unity. By  $(\tilde{G}_\xi)_{0 \leq \xi < 1}$  denote the lifting of this curve to  $SU(1, 1)^\sim$ , again starting at the unity. The unitary operators in  $\ell^2(\mathbb{S})$  corresponding (by Proposition 4.3) to  $\tilde{G}_\xi$  are denoted by  $\tilde{G}_\xi$ .

The next thing we need is the weighted  $\ell^2$  space  $\ell^2(\mathbb{S}, M_{\alpha, \xi})$  — the space of functions on  $\mathbb{S}$  that are square summable with the weight  $M_{\alpha, \xi}$ . This is a Hilbert space with the inner product  $(f, g)_{M_{\alpha, \xi}} := \sum_{\lambda \in \mathbb{S}} f(\lambda)g(\lambda)M_{\alpha, \xi}(\lambda)$ . There is an isometry map  $I_{\alpha, \xi}$  from  $\ell^2(\mathbb{S}, M_{\alpha, \xi})$  to  $\ell^2(\mathbb{S})$ :

$$I_{\alpha, \xi} := \text{multiplication of } f \in \ell^2(\mathbb{S}, M_{\alpha, \xi}) \text{ by the function } \lambda \mapsto \sqrt{M_{\alpha, \xi}(\lambda)}. \quad (4.11)$$

The standard orthonormal basis  $\{\underline{\lambda}\}_{\lambda \in \mathbb{S}}$  (4.1) of the space  $\ell^2(\mathbb{S})$  corresponds to the orthonormal basis  $\{(M_{\alpha, \xi}(\lambda))^{-\frac{1}{2}}\underline{\lambda}\}_{\lambda \in \mathbb{S}}$  of  $\ell^2(\mathbb{S}, M_{\alpha, \xi})$ . To any operator  $A$  in  $\ell^2(\mathbb{S}, M_{\alpha, \xi})$  corresponds the operator  $I_{\alpha, \xi} A I_{\alpha, \xi}^{-1}$  acting in  $\ell^2(\mathbb{S})$ .

Now we can formulate and prove the main statement of this section:

**Proposition 4.4.** *Let  $A$  be a bounded operator in  $\ell^2(\mathbb{S}, M_{\alpha, \xi})$ . Then*

$$(A\mathbf{1}, \mathbf{1})_{M_{\alpha, \xi}} = (\tilde{G}_\xi^{-1}(I_{\alpha, \xi} A I_{\alpha, \xi}^{-1})\tilde{G}_\xi \underline{\varnothing}, \underline{\varnothing}). \quad (4.12)$$

Here  $\mathbf{1} \in \ell^2(\mathbb{S}, M_{\alpha, \xi})$  is the constant identity function. On the left the inner product is in  $\ell^2(\mathbb{S}, M_{\alpha, \xi})$ , while on the right it is taken in  $\ell^2(\mathbb{S})$ .

*Proof.* Let us first show that

$$\tilde{G}_\xi \underline{\varnothing} = \sum_{\lambda \in \mathbb{S}} (M_{\alpha, \xi}(\lambda))^{\frac{1}{2}} \underline{\lambda}. \quad (4.13)$$

In the matrix group  $SL(2, \mathbb{C})$  we have

$$G_\xi = \exp\left(\sqrt{\xi}U\right) \exp\left(\frac{1}{2}\log(1-\xi)H\right) \exp\left(-\sqrt{\xi}D\right).$$

The vector  $\underline{\varnothing} \in \ell_{\text{fin}}^2(\mathbb{S})$  is analytic for the action of  $\mathfrak{su}(1, 1)$  (Proposition 4.3), so on this vector the representation of  $SU(1, 1)^\sim$  can be extended to a representation of the local complexification of the group  $SU(1, 1)^\sim$  (see, e.g., the beginning of §7 in [Nel59]). This means that for small enough  $\xi$  (when  $\tilde{G}_\xi$  is close to the unity of the group  $SU(1, 1)^\sim$ ) we have

$$\tilde{G}_\xi \underline{\varnothing} = \exp\left(\sqrt{\xi}U\right) \exp\left(\frac{1}{2}\log(1-\xi)H\right) \exp\left(-\sqrt{\xi}D\right) \underline{\varnothing}.$$

The operator  $e^{-\sqrt{\xi}D}$  preserves  $\underline{\emptyset}$ , and thus

$$\tilde{G}_\xi \underline{\emptyset} = (1 - \xi)^{a/4} \sum_{\lambda \in \mathbb{S}} \frac{\xi^{|\lambda|}}{|\lambda|!} \dim_{\mathbb{S}} \lambda \cdot \left( \prod_{\square \in \lambda} q_\alpha(\lambda) \right) \underline{\lambda} = \sum_{\lambda \in \mathbb{S}} \left( M_{\alpha, \xi}(\lambda) \right)^{\frac{1}{2}} \underline{\lambda}.$$

We have established (4.13) for small  $\xi$ . The left-hand side of (4.13) is analytic in  $\xi^{11}$  because  $\underline{\emptyset}$  is an analytic vector for the operator  $\tilde{G}_\xi$  by Proposition 4.3. The right-hand side of (4.13) is also analytic in  $\xi$  by definition of  $M_{\alpha, \xi}$ , see §2.1. Thus, (4.13) holds for all  $\xi \in (0, 1)$ .

It follows that  $I_{\alpha, \xi}^{-1} \tilde{G}_\xi \underline{\emptyset} = \mathbf{1} \in \ell^2(\mathbb{S}, M_{\alpha, \xi})$ , see (4.11). Therefore,

$$\left( \tilde{G}_\xi^{-1} I_{\alpha, \xi} A I_{\alpha, \xi}^{-1} \tilde{G}_\xi \underline{\emptyset}, \underline{\emptyset} \right) = \left( \tilde{G}_\xi^{-1} I_{\alpha, \xi} (A\mathbf{1}), \underline{\emptyset} \right) = \left( I_{\alpha, \xi} (A\mathbf{1}), \tilde{G}_\xi \underline{\emptyset} \right),$$

because the operator  $\tilde{G}_\xi$  is unitary and has real matrix elements. We have

$$\begin{aligned} \left( I_{\alpha, \xi} (A\mathbf{1}), \tilde{G}_\xi \underline{\emptyset} \right) &= \left( I_{\alpha, \xi} (A\mathbf{1}), \sum_{\lambda \in \mathbb{S}} \left( M_{\alpha, \xi}(\lambda) \right)^{\frac{1}{2}} \underline{\lambda} \right) = \sum_{\lambda \in \mathbb{S}} \left( I_{\alpha, \xi} (A\mathbf{1}), \underline{\lambda} \right) \left( M_{\alpha, \xi}(\lambda) \right)^{\frac{1}{2}} \\ &= \sum_{\lambda \in \mathbb{S}} \left( I_{\alpha, \xi} (A\mathbf{1}), I_{\alpha, \xi}(\underline{\lambda}) \right) = \left( A\mathbf{1}, \sum_{\lambda \in \mathbb{S}} \underline{\lambda} \right)_{M_{\alpha, \xi}} = (A\mathbf{1}, \mathbf{1})_{M_{\alpha, \xi}}. \end{aligned}$$

This concludes the proof.  $\square$

**Remark 4.5.** The left-hand side of (4.12) can be regarded as an expectation with respect to the measure  $M_{\alpha, \xi}$  of the function  $(A\mathbf{1})(\cdot)$  on  $\mathbb{S}$ . In the special case when the operator  $A$  is diagonal, say,  $A = A_f$  is the multiplication by a (bounded) function  $f(\cdot)$  on  $\mathbb{S}$ , (4.12) is rewritten as the following formula for the expectation:

$$\mathbb{E}_{\alpha, \xi} f := \sum_{\lambda \in \mathbb{S}} f(\lambda) M_{\alpha, \xi}(\lambda) = \left( \tilde{G}_\xi^{-1} A_f \tilde{G}_\xi \underline{\emptyset}, \underline{\emptyset} \right). \quad (4.14)$$

This case is used in the computation of the static correlation functions, and for the dynamical correlation functions we need to use the more general statement of Proposition 4.4.

## 5 Fermionic Fock space

### 5.1 Wick's theorem

We begin with the definition of a certain Clifford algebra over the Hilbert space  $V := \ell^2(\mathbb{Z})$ . Denote the standard orthonormal basis of the space  $V$  by  $\{v_x\}_{x \in \mathbb{Z}}$ . Define a symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on  $V$  by

$$\langle v_x, v_y \rangle := \begin{cases} 1, & \text{if } x = -y \neq 0; \\ 2, & \text{if } x = y = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Let  $V^+$  and  $V^-$  be the spans of  $\{v_x\}_{x \in \mathbb{Z}_{>0}}$  and  $\{v_x\}_{x \in \mathbb{Z}_{<0}}$ , respectively, and let  $V^0$  denote the space  $\mathbb{C}v_0$ . Note that the spaces  $V^+$  and  $V^-$  are maximal isotropic subspaces for the form  $\langle \cdot, \cdot \rangle$ , and

$$V = V^- \oplus V^0 \oplus V^+.$$

<sup>11</sup>Throughout the paper, when speaking about analytic functions in  $\xi$ , we assume that  $\xi$  lies in the unit open disc  $\{z \in \mathbb{C} : |z| < 1\}$ .

By  $Cl(V)$  denote the Clifford algebra over the quadratic space  $(V, \langle \cdot, \cdot \rangle)$ , that is,  $Cl(V)$  is the quotient of the tensor algebra  $\bigoplus_{n=0}^{\infty} V^{\otimes n}$  of the space  $V$  by the two-sided ideal generated by the elements

$$\{v \otimes v' + v' \otimes v - \langle v, v' \rangle : v, v' \in V\}.$$

The tensor product of  $v$  and  $v'$  in  $Cl(V)$  is denoted simply by  $vv'$ . Thus,

$$vv' + v'v = \langle v, v' \rangle \quad \text{for all } v, v' \in V. \quad (5.1)$$

Now let us prove a version of Wick's theorem that allows to write certain functionals on  $Cl(V)$  as Pfaffians. (In §5.3 below we define a functional on  $Cl(V)$  called the vacuum average to which this version of Wick's theorem is applicable.)

**Theorem 5.1.** *Let  $\mathbf{F}$  be a linear functional on  $Cl(V)$  such that  $\mathbf{F}(1) = 1$  and for any  $p, q, r \in \mathbb{Z}_{\geq 0}$ ,  $f_1^+, \dots, f_p^+ \in V^+$ , and  $f_1^-, \dots, f_q^- \in V^-$ , we have*

$$\mathbf{F}(f_1^+ \dots f_p^+ v_0^r f_1^- \dots f_q^-) = 0 \quad (5.2)$$

if at least one of the numbers  $p, q$  is nonzero.

Then for any  $n \geq 1$  and any  $2n$  elements  $f_1, \dots, f_{2n} \in V$  we have

$$\mathbf{F}(f_1 \dots f_{2n}) = \text{Pf}(\mathbf{F}[\![f_1, \dots, f_{2n}]\!] ),$$

where  $\mathbf{F}[\![f_1, \dots, f_{2n}]\!]$  is the skew-symmetric  $2n \times 2n$  matrix in which the  $kj$ -th entry above the main diagonal is  $\mathbf{F}(f_k f_j)$ ,  $1 \leq k < j \leq 2n$ .

*Proof. Step 1.* Consider decompositions

$$f_j = f_j^- + f_j^0 + f_j^+, \quad j = 1, \dots, 2n,$$

where  $f_j^{\pm} \in V^{\pm}$  and  $f_j^0 \in V^0 = \mathbb{C}v_0$ . Thus,

$$\mathbf{F}(f_1 \dots f_{2n}) = \sum_{s_1, \dots, s_{2n}} \mathbf{F}(f_1^{s_1} \dots f_{2n}^{s_{2n}}),$$

where each  $s_j$  is a sign,  $s_j \in \{-, 0, +\}$ , and the sum is taken over all  $3^{2n}$  possible sequences of signs.

**Step 2.** Fix any particular sequence of signs  $(s_1, \dots, s_{2n})$ . Consider first the case when all of the  $s_j$ 's are nonzero. We aim to prove that

$$\mathbf{F}(f_1^{s_1} \dots f_{2n}^{s_{2n}}) = \text{Pf}(\mathbf{F}[\![f_1^{s_1}, \dots, f_{2n}^{s_{2n}}]\!] ), \quad (5.3)$$

where  $\mathbf{F}[\![f_1^{s_1}, \dots, f_{2n}^{s_{2n}}]\!]$  is the  $2n \times 2n$  skew-symmetric matrix in which the  $kj$ th entry above the main diagonal is  $\mathbf{F}(f_k^{s_k} f_j^{s_j})$ .

First, note that if in the sequence  $(s_1, \dots, s_{2n})$  all the “+” signs are on the left and all the “-” signs are on the right,<sup>12</sup> then by (5.2) we get (5.3), because in the Pfaffian in the right-hand side of (5.3) each entry is zero.

<sup>12</sup>Including the case when there are only “+” or only “-” signs.

Next, observe that (5.3) is equivalent to

$$\mathbf{F}(f_1^{s_1} \dots f_{2n}^{s_{2n}}) = \sum_{k=1}^{2n-1} (-1)^{k+1} \mathbf{F}(f_1^{s_1} \dots \widehat{f_k^{s_k}} \dots f_{2n-1}^{s_{2n-1}}) \mathbf{F}(f_k^{s_k} f_{2n}^{s_{2n}}), \quad (5.4)$$

this is just the standard Pfaffian expansion (here  $\widehat{f_k^{s_k}}$  means the absence of  $f_k^{s_k}$ ). It can be readily verified that the right-hand side and the left-hand side of (5.4) vary in the same way under the interchange  $f_r^{s_r} \leftrightarrow f_{r+1}^{s_{r+1}}$  for any  $r = 1, \dots, 2n - 1$ . This implies that (5.4) holds because one can always move the “+” signs to the left and the “-” signs to the right. This argument is similar to the proof of Lemma 2.3 in [Vul07].

**Step 3.** Now assume that among the sequence of signs  $(s_1, \dots, s_{2n})$  there can be zeroes. It is not hard to see that both sides of (5.3) vanish unless the number of zeroes is even. Let the positions of zeroes be  $j_1 < \dots < j_{2k}$ . Thus, moving all  $f_{j_1}^0, \dots, f_{j_{2k}}^0$  to the left, we have

$$\mathbf{F}(f_1^{s_1} \dots f_{2n}^{s_{2n}}) = (-1)^{\sum_{m=1}^{2k} (j_m - m)} \mathbf{F}(f_{j_1}^0 \dots f_{j_{2k}}^0) \mathbf{F}(f_1^{s_1} \dots \widehat{f_{j_1}^0} \dots \widehat{f_{j_{2k}}^0} \dots f_{2n}^{s_{2n}}). \quad (5.5)$$

By (5.3), the factor  $\mathbf{F}(f_1^{s_1} \dots \widehat{f_{j_1}^0} \dots \widehat{f_{j_{2k}}^0} \dots f_{2n}^{s_{2n}})$  is written as the corresponding Pfaffian of order  $(2n - 2k)$ . Assume that  $f_{j_m}^0 = c_m v_0$  (where  $m = 1, \dots, 2k$ ), then

$$\mathbf{F}(f_{j_1}^0 \dots f_{j_{2k}}^0) = c_1 \dots c_{2k}.$$

Since for any  $f \in V^+ \oplus V^-$  we have (using (5.2))  $\mathbf{F}(v_0 f) = \mathbf{F}(f v_0) = 0$ , the right-hand side of (5.5) can be interpreted as the Pfaffian of the block  $2n \times 2n$  matrix with blocks formed by rows and columns with numbers  $j_1, \dots, j_{2k}$  and  $\{1, \dots, 2n\} \setminus \{j_1, \dots, j_{2k}\}$ , respectively. This skew-symmetric  $2n \times 2n$  matrix is exactly  $\mathbf{F}[\![f_1^{s_1}, \dots, f_{2n}^{s_{2n}}]\!]$  for our sequence  $(s_1, \dots, s_{2n})$ .

This implies that (5.3) holds for any choice of signs  $(s_1, \dots, s_{2n})$ ,  $s_j \in \{-, 0, +\}$ .

**Step 4.** Let us now deduce the claim of the theorem from (5.3). We must prove that

$$\sum_{s_1, \dots, s_{2n}} \text{Pf}(\mathbf{F}[\![f_1^{s_1}, \dots, f_{2n}^{s_{2n}}]\!]) = \text{Pf}(\mathbf{F}[\![f_1, \dots, f_{2n}]\!]).$$

This is done by induction on  $n$ . The base is  $n = 1$ :

$$\mathbf{F}(f_1^- f_2^+) + \mathbf{F}(f_1^0 f_2^0) = \mathbf{F}(f_1 f_2)$$

(all other combinations of signs in the left-hand side give zero contribution). The induction step is readily verified using the Pfaffian expansion (5.4). This concludes the proof of the theorem.  $\square$

## 5.2 Fermionic Fock space

Consider the space  $\ell^2(\mathbb{Z}_{>0})$  with the standard orthonormal basis  $\{\varepsilon_k\}_{k \in \mathbb{Z}_{>0}}$ . The exterior algebra  $\wedge \ell^2(\mathbb{Z}_{>0})$  is the vector space with the basis

$$\{\text{vac}\} \cup \{\varepsilon_{i_1} \wedge \dots \wedge \varepsilon_{i_\ell} : \infty > i_1 > \dots > i_\ell \geq 1, \ell = 1, 2, \dots\}, \quad (5.6)$$

where  $\text{vac} \equiv 1$  is called the *vacuum vector*. Define an inner product  $(\cdot, \cdot)$  in the exterior algebra  $\wedge \ell^2(\mathbb{Z}_{>0})$  with respect to which the basis (5.6) is orthonormal. This inner product turns  $\wedge \ell^2(\mathbb{Z}_{>0})$

into a pre-Hilbert space. Its Hilbert completion is called the (*fermionic*) *Fock space* and is denoted by  $\text{Fock}(\mathbb{Z}_{>0})$ . The space  $(\wedge \ell^2(\mathbb{Z}_{>0}), (\cdot, \cdot))$  consisting of finite linear combinations of the basis vectors (5.6) is denoted by  $\text{Fock}_{\text{fin}}(\mathbb{Z}_{>0})$ .

Clearly, the map

$$\underline{\lambda} \mapsto \varepsilon_{\lambda_1} \wedge \cdots \wedge \varepsilon_{\lambda_{\ell(\lambda)}}, \quad \lambda \in \mathbb{S}$$

(in particular,  $\emptyset$  maps to  $\text{vac}$ ) defines an isometry between the pre-Hilbert spaces  $\ell^2_{\text{fin}}(\mathbb{S})$  and  $\text{Fock}_{\text{fin}}(\mathbb{Z}_{>0})$ , and also between their Hilbert completions  $\ell^2(\mathbb{S})$  and  $\text{Fock}(\mathbb{Z}_{>0})$ . Below we identify  $\ell^2(\mathbb{S})$  and  $\text{Fock}(\mathbb{Z}_{>0})$ , and by  $\underline{\lambda}$  we mean the vector  $\varepsilon_{\lambda_1} \wedge \cdots \wedge \varepsilon_{\lambda_{\ell(\lambda)}}$ .

In the next subsection we describe the structure of  $\text{Fock}(\mathbb{Z}_{>0})$  in more detail.

### 5.3 Creation and annihilation operators. Vacuum average

Let  $\phi_k$ ,  $k = 1, 2, \dots$ , be the *creation operators* in  $\text{Fock}(\mathbb{Z}_{>0})$ , that is,

$$\phi_k \underline{\lambda} := \varepsilon_k \wedge \underline{\lambda}, \quad \lambda \in \mathbb{S}.$$

Let  $\phi_k^*$ ,  $k = 1, 2, \dots$ , be the operators that are adjoint to  $\phi_k$  with respect to the inner product in  $\text{Fock}(\mathbb{Z}_{>0})$ . They are called the *annihilation operators* and act as follows:

$$\phi_k^* \underline{\lambda} = \sum_{j=1}^{\ell(\lambda)} (-1)^{j+1} \delta_{k, \lambda_j} \cdot \varepsilon_{\lambda_1} \wedge \cdots \wedge \widehat{\varepsilon_{\lambda_j}} \wedge \cdots \wedge \varepsilon_{\lambda_{\ell(\lambda)}}.$$

We also need the operator  $\phi_0 = \phi_0^*$  acting as

$$\phi_0 \underline{\lambda} := (-1)^{\ell(\lambda)} \underline{\lambda}.$$

To simplify certain formulas below, we organize the operators  $\phi_k$ ,  $\phi_0$  and  $\phi_k^*$  into a single family:

$$\phi_m := \begin{cases} \phi_m, & \text{if } m \geq 0; \\ (-1)^m \phi_{-m}^*, & \text{otherwise,} \end{cases} \quad \text{where } m \in \mathbb{Z}.$$

It can be readily checked that the operators  $\phi_m$  satisfy the following anti-commutation relations:

$$\phi_k \phi_l + \phi_l \phi_k = \begin{cases} 2, & \text{if } k = l = 0; \\ (-1)^l \delta_{k, -l}, & \text{otherwise.} \end{cases} \quad (5.7)$$

In agreement to these definitions, let  $\{\mathbf{v}_x\}_{x \in \mathbb{Z}}$  be another orthonormal basis in the space  $V = \ell^2(\mathbb{Z})$  defined as

$$\mathbf{v}_x := \begin{cases} \mathbf{v}_x, & \text{if } x \geq 0; \\ (-1)^x \mathbf{v}_x, & \text{if } x < 0, \end{cases} \quad \text{where } x \in \mathbb{Z}. \quad (5.8)$$

In other words,  $\mathbf{v}_x = (-1)^{x \wedge 0} \mathbf{v}_x$ . In the Clifford algebra  $Cl(V)$  we have

$$\mathbf{v}_x \mathbf{v}_y + \mathbf{v}_y \mathbf{v}_x = \langle \mathbf{v}_x, \mathbf{v}_y \rangle = \begin{cases} 2, & x = y = 0; \\ (-1)^x \delta_{x, -y}, & \text{otherwise.} \end{cases} \quad (5.9)$$

**Definition 5.2.** Let  $\mathcal{T}$  be a representation of the Clifford algebra  $Cl(V)$  in  $\text{Fock}(\mathbb{Z}_{>0})$  defined on  $V$  by

$$\mathcal{T}(\mathbf{v}_x) := \boldsymbol{\phi}_x, \quad x \in \mathbb{Z},$$

and extended to the whole  $Cl(V)$  by (5.1) and by linearity. The fact that  $\mathcal{T}$  is indeed a representation follows from (5.7) and (5.9).

**Definition 5.3.** The representation  $\mathcal{T}$  allows to consider the following functional on the Clifford algebra  $Cl(V)$ :

$$\mathbf{F}_{\text{vac}}(w) := (\mathcal{T}(w)\text{vac}, \text{vac}), \quad w \in Cl(V)$$

called the *vacuum average*. Here the inner product on the right is taken in  $\text{Fock}(\mathbb{Z}_{>0})$ .

It can be readily verified that the functional  $\mathbf{F}_{\text{vac}}$  on  $Cl(V)$  satisfies the hypotheses of Wick's Theorem 5.1.

## 5.4 The representation $R$

The space  $\ell^2_{\text{fin}}(\mathbb{S})$  is isometric to  $\text{Fock}_{\text{fin}}(\mathbb{Z}_{>0})$ , and thus the Kerov's operators  $U$ ,  $D$ , and  $H$  (4.2) in  $\ell^2_{\text{fin}}(\mathbb{S})$  give rise to certain operators in  $\text{Fock}_{\text{fin}}(\mathbb{Z}_{>0})$ . We obtain a representation of the Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$  in  $\text{Fock}(\mathbb{Z}_{>0})$ , denote this representation by  $R$ .

It can be readily verified that the action of the operators  $R(U)$ ,  $R(D)$ , and  $R(H)$  in  $\text{Fock}_{\text{fin}}(\mathbb{Z}_{>0})$  (this subspace of  $\text{Fock}(\mathbb{Z}_{>0})$  is invariant for the representation  $R$  of  $\mathfrak{sl}(2, \mathbb{C})$ ) can be expressed in terms of the creation and annihilation operators as follows:

$$\begin{aligned} R(U) &= \sum_{k=0}^{\infty} 2^{-\delta(k)/2} (-1)^k \sqrt{k(k+1) + \alpha} \cdot \boldsymbol{\phi}_{k+1} \boldsymbol{\phi}_{-k}, \\ R(D) &= \sum_{k=0}^{\infty} 2^{-\delta(k)/2} (-1)^{k+1} \sqrt{k(k+1) + \alpha} \cdot \boldsymbol{\phi}_k \boldsymbol{\phi}_{-k-1}, \\ R(H) &= \frac{\alpha}{2} + 2 \sum_{k=1}^{\infty} (-1)^k k \boldsymbol{\phi}_k \boldsymbol{\phi}_{-k}. \end{aligned} \tag{5.10}$$

Proposition 4.3 can be reformulated for the representation  $R$ . Namely, the representation  $R$  of  $\mathfrak{sl}(2, \mathbb{C})$  restricted to the real form  $\mathfrak{su}(1, 1) \subset \mathfrak{sl}(2, \mathbb{C})$  gives rise to a unitary representation of the universal covering group  $SU(1, 1)^\sim$  in the Hilbert space  $\text{Fock}(\mathbb{Z}_{>0})$ . Denote this representation also by  $R$ .

Under the identification of  $\ell^2(\mathbb{S})$  with  $\text{Fock}(\mathbb{Z}_{>0})$ , we say that the map  $I_{\alpha, \xi}$  (4.11) is an isometry between  $\ell^2(\mathbb{S}, M_{\alpha, \xi})$  and  $\text{Fock}(\mathbb{Z}_{>0})$ . By Proposition 4.4, for any bounded operator  $A$  in  $\ell^2(\mathbb{S}, M_{\alpha, \xi})$  we have

$$(A\mathbf{1}, \mathbf{1})_{M_{\alpha, \xi}} = (R(\tilde{G}_\xi)^{-1} (I_{\alpha, \xi} A I_{\alpha, \xi}^{-1}) R(\tilde{G}_\xi) \text{vac}, \text{vac}). \tag{5.11}$$

Here  $\tilde{G}_\xi \in SU(1, 1)^\sim$ ,  $0 \leq \xi < 1$  is defined in §4.2, and  $\mathbf{1} \in \ell^2(\mathbb{S}, M_{\alpha, \xi})$  is the constant identity function. The inner products on the left and on the right are taken in the spaces  $\ell^2(\mathbb{S}, M_{\alpha, \xi})$  and  $\text{Fock}(\mathbb{Z}_{>0})$ , respectively.

Formula (4.14) for the expectation of a bounded function  $f(\cdot)$  on  $\mathbb{S}$  with respect to the measure  $M_{\alpha, \xi}$  is rewritten as

$$\mathbb{E}_{\alpha, \xi} f = (R(\tilde{G}_\xi)^{-1} A_f R(\tilde{G}_\xi) \text{vac}, \text{vac}), \tag{5.12}$$

where  $A_f$  is the operator of multiplication by  $f$ .

As we will see below, for averages expressing the correlation functions, the right-hand side of (5.11) (and (5.12)) can be written as a vacuum average. That is, the operator  $R(\tilde{G}_\xi)^{-1}(I_{\alpha,\xi}A I_{\alpha,\xi}^{-1})R(\tilde{G}_\xi)$  (respectively,  $R(\tilde{G}_\xi)^{-1}A_f R(\tilde{G}_\xi)$ ) has the form  $\mathcal{T}(w)$  for a certain  $w \in Cl(V)$ .

## 6 Z-measures and an orthonormal basis in $\ell^2(\mathbb{Z})$

In this section we examine functions  $\varphi_m$  on the lattice which are used in our expressions for correlation kernels (both static and dynamical). They form an orthonormal basis in the Hilbert space  $\ell^2(\mathbb{Z})$  and are eigenfunctions of a certain second order difference operator on the lattice. These functions arise as a particular case of the functions  $\psi_{\alpha'}$  used to describe correlation kernels in the model of the  $z$ -measures on ordinary partitions, and we begin this section by recalling some definitions from [BO06b], [BO06a].

### 6.1 Discrete hypergeometric kernel

For an ordinary (i.e., not necessary strict) partition  $\sigma = (\sigma_1, \dots, \sigma_{\ell(\sigma)})$ , let  $\dim \sigma$  denote the number of standard Young tableaux of shape  $\sigma$  (we identify partitions with ordinary Young diagrams as usual, e.g., see [Mac95, Ch. I, §1]), and  $|\sigma|$  be the number of boxes in the Young diagram  $\sigma$ . Consider the following 3-parameter family of measures on the set of all ordinary partitions:

$$M_{z,z',\xi}(\sigma) := (1 - \xi)^{zz'} \xi^{|\sigma|} (z)_\sigma (z')_\sigma \left( \frac{\dim \sigma}{|\sigma|!} \right)^2, \quad (6.1)$$

where  $(a)_\sigma := \prod_{i=1}^{\ell(\sigma)} (a)_{\sigma_i}$  is a generalization of the Pochhammer symbol. Here the parameter  $\xi \in (0, 1)$  is the same as our parameter  $\xi$  (e.g., in §2.1), and the parameters  $z, z'$  are in one of the following two families (we call such parameters *admissible*):

- (*principal series*) The numbers  $z, z'$  are not real and are conjugate to each other.
- (*complementary series*) Both  $z, z'$  are real and are contained in the same open interval of the form  $(m, m + 1)$ , where  $m \in \mathbb{Z}$ .

To any ordinary partition  $\sigma = (\sigma_1, \dots, \sigma_{\ell(\sigma)}, 0, 0, \dots)$  is associated an infinite point configuration (sometimes called the *Maya diagram*) on the lattice  $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$ :

$$\sigma \mapsto \underline{X}(\sigma) := \{\sigma_i - i + \frac{1}{2}\}_{i=1}^\infty \subset \mathbb{Z}'. \quad (6.2)$$

One can see that the correspondence  $\sigma \mapsto \underline{X}(\sigma)$  is a bijection between ordinary partitions and those (infinite) configurations  $\underline{X} \subset \mathbb{Z}'$  for which the symmetric difference  $\underline{X} \Delta \mathbb{Z}'_-$  is a finite subset containing equally many points in  $\mathbb{Z}'_+$  and  $\mathbb{Z}'_-$  (Here  $\mathbb{Z}'_+$  and  $\mathbb{Z}'_-$  denote the sets of all positive resp. negative half-integers.)

Using the above identification of ordinary partitions with point configurations on the lattice  $\mathbb{Z}'$ , it is possible to speak about the correlation functions of the measures  $M_{z,z',\xi}$  (6.1) in the same way as in (2.7). The resulting random point processes are determinantal with a correlation kernel  $\underline{K}_{z,z',\xi}(x', y')$  (where  $x', y' \in \mathbb{Z}'$ ) which is called the *discrete hypergeometric kernel* [BO00], [BO06b].

**Remark 6.1.** Whenever speaking about points in the shifted lattice  $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$ , we denote them by  $x', y', \dots$ , because we want to reserve the letters  $x, y, \dots$  for the non-shifted integers:  $x, y, \dots \in \mathbb{Z}$ .

**Theorem 6.2** ([BO00], [BO06b]). *Under the correspondence  $\sigma \mapsto \underline{X}(\sigma)$  (6.2), the  $z$ -measures become a determinantal point process on  $\mathbb{Z}'$  with the correlation kernel given by*

$$\underline{K}_{z, z', \xi}(x', y') = \sum_{a' \in \mathbb{Z}'_+} \psi_{a'}(x'; z, z', \xi) \psi_{a'}(y'; z, z', \xi), \quad x', y' \in \mathbb{Z}', \quad (6.3)$$

where the functions  $\psi_{a'}$  are defined in [BO06b, (2.1)].

From [BO06b, §2] it follows that the discrete hypergeometric kernel  $\underline{K}_{z, z', \xi}$  (viewed as an operator in  $\ell^2(\mathbb{Z}')$ ) is an *orthogonal spectral projection* corresponding to the positive part of the spectrum of a certain difference operator  $D(z, z', \xi)$ .

## 6.2 An orthonormal basis $\{\varphi_m\}$ in the Hilbert space $\ell^2(\mathbb{Z})$

For the study of our model, we need the following family of functions:

$$\begin{aligned} \varphi_m(x; \alpha, \xi) &= \left( \frac{\Gamma(\frac{1}{2} + \nu(\alpha) + x) \Gamma(\frac{1}{2} - \nu(\alpha) + x)}{\Gamma(\frac{1}{2} + \nu(\alpha) - m) \Gamma(\frac{1}{2} - \nu(\alpha) - m)} \right)^{\frac{1}{2}} \xi^{\frac{1}{2}(x+m)} (1 - \xi)^{-m} \times \\ &\times \frac{{}_2F_1(\frac{1}{2} + \nu(\alpha) + m, \frac{1}{2} - \nu(\alpha) + m; x + m + 1; \frac{\xi}{\xi - 1})}{\Gamma(x + m + 1)}, \end{aligned} \quad (6.4)$$

where  $\nu(\alpha)$  is given in Definition 2.1. Here the argument  $x$  and the index  $m$  range over the lattice  $\mathbb{Z}$ . Because  $\alpha > 0$ , we have  $\Gamma(\frac{1}{2} + \nu(\alpha) + k) \Gamma(\frac{1}{2} - \nu(\alpha) + k) > 0$  for any  $k \in \mathbb{Z}$ . Thus, the expression in (6.4) which is taken to the power  $\frac{1}{2}$  is positive. Note also that while the Gauss hypergeometric function  ${}_2F_1(A, B; C; w)$  is not defined if  $C$  is a negative integer, the ratio  $\frac{{}_2F_1(A, B; C; w)}{\Gamma(C)}$  (occurring in (6.4)) is well-defined for all  $C \in \mathbb{C}$ . Thus, we see that the functions  $\varphi_m(x; \alpha, \xi)$  are well-defined.

It can be readily verified that  $\varphi_m$ 's arise as a particular case of the functions  $\psi_{a'}$  defined in [BO06b, (2.1)]:

$$\varphi_m(x; \alpha, \xi) = \psi_{m + \frac{1}{2} + d}(x - \frac{1}{2} - d; \nu(\alpha) + \frac{1}{2} + d, -\nu(\alpha) + \frac{1}{2} + d; \xi) \quad (6.5)$$

for any  $d \in \mathbb{Z}$ . For  $x, m, d \in \mathbb{Z}$ , the numbers  $m + \frac{1}{2} + d$  and  $x - \frac{1}{2} - d$  belong to  $\mathbb{Z}'$ , as it should be. Observe that the parameters

$$z = z(\alpha) := \nu(\alpha) + \frac{1}{2} + d, \quad z' = z'(\alpha) := -\nu(\alpha) + \frac{1}{2} + d$$

for  $\alpha > 0$  and any  $d \in \mathbb{Z}$  are admissible (§6.1). For  $0 < \alpha \leq \frac{1}{4}$  these parameters belong to the complementary series, and for  $\alpha > \frac{1}{4}$  they are of principal series.

**Remark 6.3.** The fact that (6.5) holds for any  $d$  is a reflection of a certain translation invariance property of the  $z$ -measures, see [BO98, §10, 11].

From [BO06b, §2] one can readily deduce the corresponding properties of our functions  $\varphi_m$ :

**Proposition 6.4. 1)** The functions  $\varphi_m(x; \alpha, \xi)$ , as the index  $m$  ranges over  $\mathbb{Z}$ , form an orthonormal basis in the Hilbert space  $\ell^2(\mathbb{Z})$ :

$$\sum_{x \in \mathbb{Z}} \varphi_m(x; \alpha, \xi) \varphi_l(x; \alpha, \xi) = \delta_{ml}, \quad m, l \in \mathbb{Z}.$$

2) The functions  $\varphi_m$  are eigenfunctions of the following second order difference operator in  $\ell^2(\mathbb{Z})$  (acting on functions  $f(x)$ , where  $x$  ranges over  $\mathbb{Z}$ ):

$$\begin{aligned} \mathfrak{D}_{\alpha, \xi} f(x) := & \sqrt{\xi(\alpha + x(x+1))} f(x+1) \\ & + \sqrt{\xi(\alpha + x(x-1))} f(x-1) - x(1 + \xi) f(x). \end{aligned} \quad (6.6)$$

This operator is symmetric in  $\ell^2(\mathbb{Z})$ . We have

$$\mathfrak{D}_{\alpha, \xi} \varphi_m(x; \alpha, \xi) = m(1 - \xi) \varphi_m(x; \alpha, \xi), \quad m, x \in \mathbb{Z}.$$

3) The functions  $\varphi_m$  satisfy the following symmetry relations:<sup>13</sup>

$$\varphi_m(x; \alpha, \xi) = \varphi_x(m; \alpha, \xi); \quad (6.7)$$

$$\varphi_m(x; \alpha, \xi) = (-1)^{x+m} \varphi_{-m}(-x; \alpha, \xi), \quad x, m \in \mathbb{Z}. \quad (6.8)$$

4) The functions  $\varphi_m$  satisfy the three-term relation:

$$\begin{aligned} (1 - \xi)x \varphi_m = & \sqrt{\xi(m(m+1) + \alpha)} \varphi_{m-1} \\ & + \sqrt{\xi(m(m-1) + \alpha)} \varphi_{m+1} - m(1 + \xi) \varphi_m, \quad m \in \mathbb{Z}. \end{aligned} \quad (6.9)$$

### 6.3 “Twisting”

To simplify certain formulas in the paper (in particular, the ones involving spectral projections), we will also need certain versions of our functions  $\varphi_m(x; \alpha, \xi)$  which differ from the original ones by multiplying by  $(-1)^{x \wedge 0}$ :

$$\tilde{\varphi}_m(x; \alpha, \xi) := (-1)^{x \wedge 0} \varphi_m(x; \alpha, \xi), \quad x, m \in \mathbb{Z}. \quad (6.10)$$

These functions also form an orthonormal basis in  $\ell^2(\mathbb{Z})$ . They are eigenfunctions of a difference operator  $\tilde{\mathfrak{D}}_{\alpha, \xi}$  in  $\ell^2(\mathbb{Z})$  which is conjugate to  $\mathfrak{D}_{\alpha, \xi}$ :

$$\begin{aligned} (\tilde{\mathfrak{D}}_{\alpha, \xi} f)(x) := & (-1)^{\mathbb{1}_{x < 0}} \sqrt{\xi(\alpha + x(x+1))} f(x+1) \\ & + (-1)^{\mathbb{1}_{x \leq 0}} \sqrt{\xi(\alpha + x(x-1))} f(x-1) - x(1 + \xi) f(x) \end{aligned} \quad (6.11)$$

(here  $\mathbb{1}$  means the indicator),

$$\tilde{\mathfrak{D}}_{\alpha, \xi} \tilde{\varphi}_m(x; \alpha, \xi) = m(1 - \xi) \tilde{\varphi}_m(x; \alpha, \xi), \quad m, x \in \mathbb{Z}.$$

<sup>13</sup>Property (6.7) here means that the functions  $\varphi_m(x; \alpha, \xi)$  are *self-dual* (in contrast to the more general functions  $\psi_{a'}(x; z, z', \xi)$ , cf. [BO06b, Prop. 2.5]).

## 6.4 Matrix elements of $\mathfrak{sl}(2, \mathbb{C})$ -modules

Here we interpret the functions  $\{\varphi_m\}$  (6.4) through certain matrix elements of irreducible unitary representations of the Lie group  $PSU(1, 1) = SU(1, 1)/\{\pm I\}$  ( $I$  is the identity matrix) in the Hilbert space  $\ell^2(\mathbb{Z})$ .

**Remark 6.5.** The more general functions  $\psi_{\alpha}$  [BO06b, (2.1)] first appeared in the works of Vilenkin and Klimyk [VK88], [VK95] as matrix elements of unitary representations of the universal covering group  $SU(1, 1)^\sim$ . In a context similar to ours they were obtained by Okounkov [Oko01b] in a computation of the discrete hypergeometric kernel  $\underline{K}_{z, z', \xi}$  (6.3) using the fermionic Fock space.

Let  $S$  be the representation of the Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$  (spanned by the operators  $U, D$ , and  $H$  (4.4)) in the Hilbert space  $\ell^2(\mathbb{Z})$  with the canonical orthonormal basis  $\{\underline{k}\}_{k \in \mathbb{Z}}$  (that is,  $\underline{k}(x) = \delta_{k, x}$ ) defined as follows:

$$\begin{aligned} S(U)\underline{k} &= \sqrt{k(k+1) + \alpha} \cdot \underline{k+1}; \\ S(D)\underline{k} &= \sqrt{k(k-1) + \alpha} \cdot \underline{k-1}; \\ S(H)\underline{k} &= 2k \cdot \underline{k}. \end{aligned} \tag{6.12}$$

This representation depends on our parameter  $\alpha > 0$ . One can establish an analogue of Proposition 4.3:

**Proposition 6.6.** *All vectors of the space  $\ell^2_{\text{fin}}(\mathbb{Z})$  (consisting of finite linear combinations of the basis vectors  $\{\underline{k}\}$ ) are analytic for the action  $S$  of  $\mathfrak{sl}(2, \mathbb{C})$  (6.12). The representation  $S$  of the Lie algebra  $\mathfrak{su}(1, 1) \subset \mathfrak{sl}(2, \mathbb{C})$  in  $\ell^2_{\text{fin}}(\mathbb{Z})$  lifts to a unitary representation of the Lie group  $PSU(1, 1)$  in the Hilbert space  $\ell^2(\mathbb{Z})$ .*

*Proof.* It is known (e.g., see [Puk64] or [Lan85, Ch. VI, §6]) that for any  $\alpha > 0$  the above representation  $S$  of  $\mathfrak{su}(1, 1)$  in  $\ell^2_{\text{fin}}(\mathbb{Z})$  is irreducible (this is an irreducible Harish–Chandra module) and lifts to a unitary representation of the Lie group  $SU(1, 1)$  in  $\ell^2(\mathbb{Z})$ . Moreover, since  $S(H)\underline{k} = 2k \cdot \underline{k}$ , this is in fact a representation of the group  $PSU(1, 1)$ . The claim about analytic vectors follows from, e.g., [Lan85, Ch. X, §3, Thm. 7].  $\square$

If  $0 < \alpha \leq \frac{1}{4}$ , the above irreducible representation of  $PSU(1, 1)$  in  $\ell^2(\mathbb{Z})$  is of complementary series, and for  $\alpha > \frac{1}{4}$  it is of principal series [Puk64] (cf. the series of the parameters  $(z, z')$  in (6.5)). Denote this representation of  $PSU(1, 1)$  again by  $S$ . For notational reasons (e.g., see Proposition 6.7 below), also by  $S$  let us denote the corresponding representations of  $SU(1, 1)$  and  $SU(1, 1)^\sim$  in  $\ell^2(\mathbb{Z})$  that are obtained from the representation of  $PSU(1, 1)$  by a trivial lifting procedure.

Now let us compute the matrix elements of the operator  $S(G_\xi)^{-1}$  (where  $G_\xi \in SU(1, 1)$  is defined in (4.10)) in the basis  $\{\underline{k}\}_{k \in \mathbb{Z}}$ . These matrix elements will be used below in formulas for our correlation kernels.

**Proposition 6.7.** *For all  $x, k \in \mathbb{Z}$  we have*

$$\left( S(G_\xi)^{-1} \underline{x}, \underline{k} \right)_{\ell^2(\mathbb{Z})} = \varphi_{-k}(x; \alpha, \xi). \tag{6.13}$$

*Proof.* Fix  $x, k \in \mathbb{Z}$ . By Proposition 6.6, the function  $\xi \mapsto (S(G_\xi)^{-1}\underline{x}, \underline{k})_{\ell^2(\mathbb{Z})}$  is analytic. The right-hand side of (6.13) is also analytic in  $\xi$ , see (6.4). Thus, it suffices to prove (6.13) for small  $\xi$ . Also by Proposition 6.6, on  $\underline{x} \in \ell_{\text{fin}}^2(\mathbb{Z})$  the representation  $S$  can be extended to a representation of the local complexification of  $PSU(1, 1)$ . This means that for small  $\xi$  (when  $G_\xi$  is close to the unity of the group) we can write:

$$S(G_\xi)^{-1}\underline{x} = \exp\left(-\sqrt{\xi}S(U)\right) \exp\left(\frac{\sqrt{\xi}}{1-\xi}S(D)\right) \exp\left(\frac{1}{2}\log(1-\xi)S(H)\right)\underline{x}$$

(this follows from the corresponding identity for matrices in  $SL(2, \mathbb{C})$ , see also the proof of Proposition 4.4).

Denote  $c_y := \sqrt{y(y+1)+\alpha}$ , so that  $S(U)\underline{y} = c_y \cdot \underline{y} + 1$  and  $S(D)\underline{y} = c_{y-1} \cdot \underline{y} - 1$ . Note that  $c_y^2 = y(y+1)+\alpha = (y+\nu(\alpha)+\frac{1}{2})(y-\nu(\alpha)+\frac{1}{2})$ . Also set  $a := -\sqrt{\xi}$  and  $b := \sqrt{\xi}/(1-\xi)$ . We have

$$\begin{aligned} (S(G_\xi)^{-1}\underline{x}, \underline{k})_{\ell^2(\mathbb{Z})} &= (1-\xi)^x \left( e^{aS(U)} e^{bS(D)} \underline{x}, \underline{k} \right)_{\ell^2(\mathbb{Z})} \\ &= (1-\xi)^x \sum_{r=0}^{\infty} \sum_{l=0}^{\infty} \frac{a^r b^l}{r!l!} c_{x-l+r-1} \cdots c_{x-l} c_{x-l} \cdots c_{x-1} (\underline{x-l+r}, \underline{k})_{\ell^2(\mathbb{Z})}. \end{aligned}$$

Clearly,  $(\underline{x-l+r}, \underline{k})_{\ell^2(\mathbb{Z})} = \delta_{x-l+r, k}$ . There are two cases:  $x \geq k$ , and  $x \leq k$ . For  $x \geq k$  we perform the above summation over  $r \geq 0$  and set  $l = r + x - k$ . For  $x \leq k$  we sum over  $l$  and set  $r = l + k - x$ . After direct calculations we obtain (we omit the argument in  $\nu(\alpha)$ ):

$$\begin{aligned} (S(G_\xi)^{-1}\underline{x}, \underline{k})_{\ell^2(\mathbb{Z})} &= (1-\xi)^x b^{x-k} \left( \frac{\Gamma(x+\nu+\frac{1}{2})\Gamma(x-\nu+\frac{1}{2})}{\Gamma(k+\nu+\frac{1}{2})\Gamma(k-\nu+\frac{1}{2})} \right)^{\frac{1}{2}} \times \\ &\quad \times \frac{{}_2F_1(\frac{1}{2}-k-\nu, \frac{1}{2}-k+\nu; x-k+1; ab)}{(x-k)!}, \quad \text{if } x \geq k; \\ (S(G_\xi)^{-1}\underline{x}, \underline{k})_{\ell^2(\mathbb{Z})} &= (1-\xi)^x a^{k-x} \left( \frac{\Gamma(k+\nu+\frac{1}{2})\Gamma(k-\nu+\frac{1}{2})}{\Gamma(x+\nu+\frac{1}{2})\Gamma(x-\nu+\frac{1}{2})} \right)^{\frac{1}{2}} \times \\ &\quad \times \frac{{}_2F_1(\frac{1}{2}-x-\nu, \frac{1}{2}-x+\nu; k-x+1; ab)}{(k-x)!}, \quad \text{if } x \leq k. \end{aligned}$$

It is known that the expression  $\frac{{}_2F_1(A, B; C; w)}{\Gamma(C)}$  is well-defined for all  $C \in \mathbb{C}$ , and by [Erd53, 2.1.(3)] we see that

$$\frac{{}_2F_1(A, B; n+1; w)}{\Gamma(n+1)} = \frac{{}_2F_1(A-n, B-n; -n+1; w)}{(A-n)_n (B-n)_n \Gamma(-n+1)} w^{-n}, \quad n = 1, 2, \dots$$

Let us apply this identity in the case  $x \leq k$  above:

$$\begin{aligned} &\frac{{}_2F_1(\frac{1}{2}-x-\nu, \frac{1}{2}-x+\nu; k-x+1; ab)}{(k-x)!} \\ &= \frac{{}_2F_1(\frac{1}{2}-k-\nu, \frac{1}{2}-k+\nu; x-k+1; ab)}{\Gamma(x-k+1)} \cdot \frac{\Gamma(\frac{1}{2}+x-\nu)\Gamma(\frac{1}{2}+x+\nu)}{\Gamma(\frac{1}{2}+k-\nu)\Gamma(\frac{1}{2}+k+\nu)} (ab)^{x-k} \end{aligned}$$

(we have also used the fact that  $(-1)^m \Gamma(\frac{1}{2} + m \pm \nu) = \frac{\Gamma(\frac{1}{2} + \nu) \Gamma(\frac{1}{2} - \nu)}{\Gamma(\frac{1}{2} \mp \nu - m)}$  for  $m \in \mathbb{Z}$ , see (2.4)). Thus, we get the desired result (6.13) for small  $\xi$ , and hence for all  $\xi \in (0, 1)$  by analyticity. This concludes the proof.  $\square$

**Remark 6.8.** The operator  $\mathfrak{D}_{\alpha, \xi}$  (6.6) acting in  $\ell^2(\mathbb{Z})$  can be expressed through the operators of the representation  $S$  of  $\mathfrak{sl}(2, \mathbb{C})$  as follows:

$$\mathfrak{D}_{\alpha, \xi} = \sqrt{\xi}(S(U) + S(D)) - \frac{1}{2}(1 + \xi)S(H) = -(1 - \xi)S(H_\xi),$$

where

$$H_\xi := \frac{1}{2}G_\xi H G_\xi^{-1} \in \mathfrak{sl}(2, \mathbb{C}).$$

Indeed, this is verified by a simple matrix computation (see (4.10)):

$$\frac{1}{2}G_\xi H G_\xi^{-1} = \frac{1}{2(1 - \xi)} \begin{bmatrix} 1 & \sqrt{\xi} \\ \sqrt{\xi} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -\sqrt{\xi} \\ -\sqrt{\xi} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \frac{1 + \xi}{1 - \xi} & -\frac{\sqrt{\xi}}{1 - \xi} \\ \frac{\sqrt{\xi}}{1 - \xi} & -\frac{1}{2} \frac{1 + \xi}{1 - \xi} \end{bmatrix}.$$

An operator corresponding to the matrix  $H_\xi$  under the representation  $R$  (5.10) appears below in Corollary 9.4, where it plays the role of the generator of our Markov dynamics on strict partitions.

## 6.5 Connection with Meixner and Krawtchouk polynomials

The  $z$ -measures  $M_{z, z', \xi}$  (§6.1) for  $\xi \in (0, 1)$  and  $(z, z')$  of principal or complementary series are supported by the set of all ordinary partitions. As is known (e.g., see [BO06b]), the  $z$ -measures admit two degenerate series of parameters:

- (*first degenerate series*)  $\xi \in (0, 1)$ , and one of the numbers  $z$  and  $z'$  (say,  $z$ ) is a nonzero integer while  $z'$  has the same sign and, moreover,  $|z'| > |z| - 1$ .

Here if  $z = N = 1, 2, \dots$ , then the measure  $M_{z, z', \xi}(\sigma)$  vanishes unless  $\ell(\sigma) \leq N$ . Likewise, if  $z = -N$ ,  $M_{z, z', \xi}(\sigma) = 0$  if  $\ell(\sigma') = \sigma_1$  exceeds  $N$  ( $\sigma'$  denotes the transposed Young diagram).

- (*second degenerate series*)  $\xi < 0$ , and  $z = N$  and  $z' = -N'$ , where  $N$  and  $N'$  are positive integers.

In this case, the measure  $M_{z, z', \xi}$  is supported by the (finite) set of all ordinary Young diagrams which are contained in the rectangle  $N \times N'$  (that is,  $\ell(\sigma) \leq N$  and  $\ell(\sigma') \leq N'$ ).

As is explained in [BO06b], in the first case the functions  $\psi_{a'}(x'; z, z', \xi)$  are expressed through the classical Meixner orthogonal polynomials (about their definition, e.g., see [KS96, §1.9]). In the second degenerate series these functions are related to the Krawtchouk orthogonal polynomials [KS96, §1.10].

For our measures  $M_{\alpha, \xi}$  on strict partitions there exists only one degenerate series of parameters:  $\alpha = -N(N + 1)$  for some  $N = 1, 2, \dots$ , and  $\xi < 0$ . In this case, the measure  $M_{\alpha, \xi}$  is supported by the set of all shifted Young diagrams which are contained inside the staircase shifted shape  $(N, N - 1, \dots, 1)$ . This case corresponds to the second degenerate series of the  $z$ -measures, and our functions  $\varphi_m$  are expressed through the Krawtchouk orthogonal polynomials.

The measures  $M_{\alpha, \xi}$  in this case are interpreted as random point processes on the finite lattice  $\{1, \dots, N\}$ , and one could also define a suitable dynamics for them similarly to §2.2. The results

of the present paper about the structure of the static and dynamical correlation functions also hold for the degenerate model, and correlation kernels are expressed through the Krawtchouk polynomials.

## 7 Static correlation functions

In this section we obtain a Pfaffian formula for the correlation functions of the point process  $M_{\alpha,\xi}$ , and discuss the resulting Pfaffian kernel.

### 7.1 Pfaffian formula

Recall that by  $\mathbb{Z}_{\neq 0}$  we denote the set of all nonzero integers. For  $x_1, \dots, x_n \in \mathbb{Z}_{>0}$  we put, by definition,

$$x_{-k} := -x_k, \quad k = 1, \dots, n. \quad (7.1)$$

We use this convention in the formulation of the next theorem. Let the function  $\Phi_{\alpha,\xi}$  on  $\mathbb{Z}_{\neq 0} \times \mathbb{Z}_{\neq 0}$  be defined by (see §5 for definitions of objects below)

$$\Phi_{\alpha,\xi}(x, y) := (-1)^{x \wedge 0 + y \wedge 0} \left( R(\tilde{G}_\xi)^{-1} \phi_x \phi_y R(\tilde{G}_\xi) \text{vac}, \text{vac} \right), \quad (7.2)$$

where the inner product is taken in  $\text{Fock}(\mathbb{Z}_{>0})$ . (For now  $\Phi_{\alpha,\xi}(x, y)$  is defined for  $x, y \in \mathbb{Z}_{\neq 0}$ , but in §7.2 we extend the definition of  $\Phi_{\alpha,\xi}(x, y)$  to zero values of  $x, y$  in a natural way. See also Remark 2.2.1.) In this subsection we prove the following:

**Theorem 7.1.** *The correlation functions  $\rho_{\alpha,\xi}^{(n)}$  (2.7) of the measures  $M_{\alpha,\xi}$  (2.5) are given by the following Pfaffian formula:*

$$\rho_{\alpha,\xi}^{(n)}(x_1, \dots, x_n) = \text{Pf}(\hat{\Phi}_{\alpha,\xi} \llbracket X \rrbracket), \quad (7.3)$$

where  $X = \{x_1, \dots, x_n\} \subset \mathbb{Z}_{>0}$  (here  $x_j$ 's are distinct), and  $\hat{\Phi}_{\alpha,\xi} \llbracket X \rrbracket$  is the skew-symmetric  $2n \times 2n$  matrix with rows and columns indexed by the numbers  $1, 2, \dots, n, -n, \dots, -2, -1$ , and the  $kj$ -th entry in  $\hat{\Phi}_{\alpha,\xi} \llbracket X \rrbracket$  above the main diagonal is  $\Phi_{\alpha,\xi}(x_k, x_j)$ , where  $k, j = 1, \dots, n, -n, \dots, -1$ .<sup>14</sup>

Below in (7.16) we write  $\Phi_{\alpha,\xi}(x, y)$  in terms of the Gauss hypergeometric function. For this reason, we call  $\Phi_{\alpha,\xi}$  the *Pfaffian hypergeometric-type kernel*. Another form of a  $2n \times 2n$  skew-symmetric matrix (constructed using the kernel  $\Phi_{\alpha,\xi}(x, y)$ ) which can be put in the right-hand side of (7.3) is discussed below in §10.4. The above form  $\hat{\Phi}_{\alpha,\xi} \llbracket X \rrbracket$  is most useful when rewriting the Pfaffian in (7.3) as a determinant, see Theorem 8.1 and Proposition A.2 from Appendix.

The rest of this subsection is devoted to proving Theorem 7.1. Consider the following operators in  $\ell^2(\mathbb{S}, M_{\alpha,\xi})$ :

$$\Delta_x \underline{\lambda} := \begin{cases} \lambda, & \text{if } x \in \lambda; \\ 0, & \text{otherwise,} \end{cases} \quad x \in \mathbb{Z}_{>0}.$$

Fix a finite subset  $X = \{x_1, \dots, x_n\} \subset \mathbb{Z}_{>0}$  and set  $\Delta_{\llbracket X \rrbracket} := \Delta_{x_1} \dots \Delta_{x_n}$ . This is a diagonal operator of multiplication by a function which is the indicator of the event  $\{\lambda : \lambda \supseteq X\}$ . We view  $\Delta_{\llbracket X \rrbracket}$  as an

<sup>14</sup>Theorem 7.1 is the same as Proposition 2 in [Pet10b], the only difference is that in [Pet10b] the factor  $(-1)^{\sum_{k=1}^n x_k}$  is put in front of the Pfaffian, and thus in the definition of the Pfaffian kernel in [Pet10b] there is no factor of the form  $(-1)^{x \wedge 0 + y \wedge 0}$ .

operator acting in  $\ell^2(\mathbb{S}, M_{\alpha, \xi})$ . Since this operator is diagonal, it does not change under the isometry  $I_{\alpha, \xi}: \ell^2(\mathbb{S}, M_{\alpha, \xi}) \rightarrow \text{Fock}(\mathbb{Z}_{>0})$  (4.11). Thus,  $\Delta_{\llbracket X \rrbracket}$  also acts in  $\text{Fock}(\mathbb{Z}_{>0})$  (in the same way).

The correlation functions  $\rho_{\alpha, \xi}^{(n)}$  (2.7) of the measures  $M_{\alpha, \xi}$  (2.5) are written as

$$\rho_{\alpha, \xi}^{(n)}(x_1, \dots, x_n) = M_{\alpha, \xi}(\lambda: \lambda \supseteq \{x_1, \dots, x_n\}) = (\Delta_{\llbracket X \rrbracket} \mathbf{1}, \mathbf{1})_{M_{\alpha, \xi}}, \quad (7.4)$$

where  $\mathbf{1} \in \ell^2(\mathbb{S}, M_{\alpha, \xi})$  is the constant identity function. Using (5.11) (or (5.12)), we can rewrite the correlation functions as

$$\rho_{\alpha, \xi}^{(n)}(x_1, \dots, x_n) = \left( R(\tilde{G}_\xi)^{-1} \Delta_{\llbracket X \rrbracket} R(\tilde{G}_\xi) \text{vac}, \text{vac} \right). \quad (7.5)$$

In this formula the operator  $\Delta_{\llbracket X \rrbracket}$  acts in  $\text{Fock}(\mathbb{Z}_{>0})$ . Clearly,  $\Delta_{\llbracket X \rrbracket}$  is expressed through the creation and annihilation operators in the Fock space  $\text{Fock}(\mathbb{Z}_{>0})$  as

$$\Delta_{\llbracket X \rrbracket} = \prod_{k=1}^n \phi_{x_k} \phi_{x_k}^*.$$

It is more convenient for us to rewrite  $\Delta_{\llbracket X \rrbracket}$  using the anti-commutation relations for  $\phi_x$  and  $\phi_x^*$  (see (5.7)) as follows:

$$\Delta_{\llbracket X \rrbracket} = \phi_{x_1} \dots \phi_{x_n} \phi_{x_n}^* \dots \phi_{x_1}^* \quad (7.6)$$

(after moving all the  $\phi_k$ 's to the left and  $\phi_k^*$ 's to the right there is no change of sign).

Our next step is to write (7.5) with  $\Delta_{\llbracket X \rrbracket}$  given by (7.6) as the vacuum average functional  $\mathbf{F}_{\text{vac}}$  applied to a certain element of the Clifford algebra  $Cl(V)$  (§5.1).

Recall that in §5.3 we have defined a representation  $\mathcal{T}$  of  $Cl(V)$  in  $\text{Fock}(\mathbb{Z}_{>0})$  such that  $\mathcal{T}(\mathbf{v}_x) = \phi_x$ ,  $x \in \mathbb{Z}$ , where  $\{\mathbf{v}_x\}_{x \in \mathbb{Z}}$  is the basis of  $V = \ell^2(\mathbb{Z})$  defined by (5.8). Using the anti-commutation relations (5.7), one can readily compute the following commutators between the operators  $\mathcal{T}(\mathbf{v}_x)$  and the operators of the representation  $R$  (5.10):

$$\begin{aligned} [R(U), \mathcal{T}(\mathbf{v}_x)] &= 2^{(\delta(x)-\delta(x+1))/2} \sqrt{x(x+1) + \alpha} \cdot \mathcal{T}(\mathbf{v}_{x+1}); \\ [R(D), \mathcal{T}(\mathbf{v}_x)] &= 2^{(\delta(x)-\delta(x-1))/2} \sqrt{x(x-1) + \alpha} \cdot \mathcal{T}(\mathbf{v}_{x-1}); \\ [R(H), \mathcal{T}(\mathbf{v}_x)] &= 2x \cdot \mathcal{T}(\mathbf{v}_x), \end{aligned} \quad x \in \mathbb{Z}.$$

These formulas motivate the following definition:

**Definition 7.2.** Let  $\check{S}$  be the representation of the Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$  in the (pre-Hilbert) space  $V_{\text{fin}}$  (consisting of all finite linear combinations of the basis vectors  $\{\mathbf{v}_x\}_{x \in \mathbb{Z}}$ ) defined as:

$$\begin{aligned} \check{S}(U)\mathbf{v}_x &:= 2^{(\delta(x)-\delta(x+1))/2} \sqrt{x(x+1) + \alpha} \cdot \mathbf{v}_{x+1}; \\ \check{S}(D)\mathbf{v}_x &:= 2^{(\delta(x)-\delta(x-1))/2} \sqrt{x(x-1) + \alpha} \cdot \mathbf{v}_{x-1}; \\ \check{S}(H)\mathbf{v}_x &:= 2x \cdot \mathbf{v}_x, \end{aligned} \quad x \in \mathbb{Z}. \quad (7.7)$$

The representation  $\check{S}$  is chosen in such a way that for all matrices  $M \in \mathfrak{sl}(2, \mathbb{C})$  and vectors  $v \in V_{\text{fin}}$  we have

$$[R(M), \mathcal{T}(v)] = \mathcal{T}(\check{S}(M)v) \quad (7.8)$$

(the equality of operators in  $\text{Fock}_{\text{fin}}(\mathbb{Z}_{>0})$ ). This follows from definitions of  $R$ ,  $\mathcal{T}$ , and  $\check{S}$ .

Comparing (7.7) and (6.12), we see that the representation  $\check{S}$  is conjugate to the representation  $S$  discussed in §6.4 above. Namely,  $\check{S} = Z^{-1}SZ$ , where  $Z$  is an operator in  $V = \ell^2(\mathbb{Z})$  defined by  $Z\mathbf{v}_x := 2^{\delta(x)/2}\underline{x}$ ,  $x \in \mathbb{Z}$ . This means that Proposition 6.6 also holds for the representation  $\check{S}$ . In particular,  $\check{S}$  lifts to a representation of the group  $PSU(1, 1)$  in the Hilbert space  $V$ .<sup>15</sup> Note that  $\check{S}$  is not unitary (but we do not need this property).

The next proposition (due to Olshanski [Ols08]) is a “group level” version of the identity (7.8).

**Proposition 7.3.** *For all  $g \in SU(1, 1)^\sim$  and all  $v \in V$  we have*

$$R(g)\mathcal{T}(v)R(g)^{-1} = \mathcal{T}(\check{S}(g)v) \quad (7.9)$$

(the equality of operators in  $\text{Fock}(\mathbb{Z}_{>0})$ ).

*Proof. Step 1.* Since the representation  $\mathcal{T}$  is norm preserving, it suffices to take  $v \in V$  from the dense subspace  $V_{\text{fin}}$ . Without loss of generality, we can assume that  $v = \mathbf{v}_x$  for some  $x \in \mathbb{Z}$ .

**Step 2.** Rewrite the claim (7.9) as

$$R(g)\mathcal{T}(\mathbf{v}_x) = \mathcal{T}(\check{S}(g)\mathbf{v}_x)R(g). \quad (7.10)$$

This is an equality of operators in the Hilbert space  $\text{Fock}(\mathbb{Z}_{>0})$ . It is enough to show that these operators agree on  $\text{Fock}_{\text{fin}}(\mathbb{Z}_{>0})$ , which is true if

$$R(g)\mathcal{T}(\mathbf{v}_x)\underline{\lambda} = \mathcal{T}(\check{S}(g)\mathbf{v}_x)R(g)\underline{\lambda} \quad \text{for all } g \in SU(1, 1)^\sim, x \in \mathbb{Z}, \text{ and } \lambda \in \mathbb{S}. \quad (7.11)$$

**Step 3.** Now let us prove that both sides of (7.11) are analytic functions in  $g \in SU(1, 1)^\sim$  with values in  $\text{Fock}(\mathbb{Z}_{>0})$ :

- (left-hand side) The vector  $\mathcal{T}(\mathbf{v}_x)\underline{\lambda}$  belongs to  $\text{Fock}_{\text{fin}}(\mathbb{Z}_{>0})$ , and hence is analytic for the representation  $R$ , see Proposition 4.3. This means that the function  $g \mapsto R(g)\mathcal{T}(\mathbf{v}_x)\underline{\lambda}$  is analytic.
- (right-hand side) By Proposition 6.6 (and remarks before the present proposition), the function  $g \mapsto \check{S}(g)\mathbf{v}_x$  is an analytic function with values in the Hilbert space  $V$ . Since  $\mathcal{T}$  is continuous in the norm topology,  $\mathcal{T}(\check{S}(g)\mathbf{v}_x)$  is an analytic function with values in the Banach space  $\text{End}(\text{Fock}(\mathbb{Z}_{>0}))$  of bounded operators in the space  $\text{Fock}(\mathbb{Z}_{>0})$ . On the other hand, the function  $R(g)\underline{\lambda}$  is also analytic (with values in  $\text{Fock}(\mathbb{Z}_{>0})$ ). Therefore, the function  $g \mapsto \mathcal{T}(\check{S}(g)\mathbf{v}_x)R(g)\underline{\lambda}$  is analytic, too.

**Step 4.** Now it remains to compare the Taylor series expansions of both sides of (7.10) at  $g = e$ , the unity element of  $SU(1, 1)^\sim$ . That is, we need to establish that for any  $M \in \mathfrak{sl}(2, \mathbb{C})$  and any  $x \in \mathbb{Z}$ :

$$\sum_{k=0}^{\infty} \frac{R(M)^k s^k}{k!} \mathcal{T}(\mathbf{v}_x) = \left( \sum_{l=0}^{\infty} \frac{\mathcal{T}(\check{S}(M)^l \mathbf{v}_x) s^l}{l!} \right) \left( \sum_{r=0}^{\infty} \frac{R(M)^r s^r}{r!} \right). \quad (7.12)$$

This should be understood as an equality of formal power series in  $s$  with coefficients being operators in  $\text{Fock}_{\text{fin}}(\mathbb{Z}_{>0})$ . Let us divide both sides by the last formal sum,  $\sum_{r=0}^{\infty} \frac{R(M)^r s^r}{r!}$ . After that it can be readily verified that the identity (7.12) of formal power series is a corollary of the “Lie algebra level” commutation identity (7.8).

This last step concludes the proof of the proposition.  $\square$

<sup>15</sup>Also by  $\check{S}$  we denote the corresponding representations of  $SU(1, 1)$  and  $SU(1, 1)^\sim$  in  $V$  that are obtained from the representation  $\check{S}$  of  $PSU(1, 1)$  by a trivial lifting procedure.

Define (the second equality holds because  $\check{S}$  is a representation of  $PSU(1, 1)$ )

$$v_{x,\xi} := \check{S}(\tilde{G}_\xi)^{-1}v_x = \check{S}(G_\xi)^{-1}v_x \in V, \quad x \in \mathbb{Z}. \quad (7.13)$$

Putting this together with Proposition 7.3, we can rewrite the correlation functions (7.5) as the vacuum average (see Definition 5.3):

$$\rho_{\alpha,\xi}^{(n)}(x_1, \dots, x_n) = \mathbf{F}_{\text{vac}} \left( v_{x_1,\xi} \cdots v_{x_n,\xi} v_{-x_n,\xi} \cdots v_{-x_1,\xi} \right). \quad (7.14)$$

Observe that for  $x, y \in \mathbb{Z}_{\neq 0}$  we have

$$\mathbf{F}_{\text{vac}}(v_{x,\xi}v_{y,\xi}) = (-1)^{x \wedge 0 + y \wedge 0} \left( R(\tilde{G}_\xi)^{-1} \phi_x \phi_y R(\tilde{G}_\xi) \text{vac}, \text{vac} \right) = \Phi_{\alpha,\xi}(x, y),$$

as in (7.2). Therefore, formula (7.14) together with Wick's Theorem 5.1 immediately implies our Theorem 7.1.

## 7.2 Static Pfaffian kernel

Let us express our kernel  $\Phi_{\alpha,\xi}(x, y)$  through the functions  $\varphi_m$  defined by (6.4). This kernel is defined for  $x, y \in \mathbb{Z}_{\neq 0}$  and has the form (see the previous subsection)

$$\Phi_{\alpha,\xi}(x, y) = \mathbf{F}_{\text{vac}}(v_{x,\xi}v_{y,\xi}) = \sum_{k,l \in \mathbb{Z}} (v_{x,\xi}, v_k)_{\ell^2(\mathbb{Z})} (v_{y,\xi}, v_l)_{\ell^2(\mathbb{Z})} \mathbf{F}_{\text{vac}}(v_k v_l)$$

(where the vectors  $v_{x,\xi}, v_{y,\xi}$  are defined by (7.13)). By definitions of §5.3, we have

$$\mathbf{F}_{\text{vac}}(v_k v_l) = \begin{cases} 1, & \text{if } l = -k \geq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (7.15)$$

Therefore,

$$\Phi_{\alpha,\xi}(x, y) = \sum_{m=0}^{\infty} (v_{x,\xi}, v_{-m})_{\ell^2(\mathbb{Z})} (v_{y,\xi}, v_m)_{\ell^2(\mathbb{Z})}.$$

**Proposition 7.4.** *For any  $r, k \in \mathbb{Z}$  we have*

$$(v_{r,\xi}, v_k)_{\ell^2(\mathbb{Z})} = (-1)^{r \wedge 0 + k \wedge 0} 2^{(\delta(r) - \delta(k))/2} \varphi_{-k}(r; \alpha, \xi),$$

where the functions  $\varphi_m$  are defined in §6.2.

*Proof.* By (7.13) and then by (5.8),

$$(v_{r,\xi}, v_k)_{\ell^2(\mathbb{Z})} = (\check{S}(G_\xi)^{-1}v_r, v_k)_{\ell^2(\mathbb{Z})} = (-1)^{r \wedge 0 + k \wedge 0} (\check{S}(G_\xi)^{-1}v_r, v_k)_{\ell^2(\mathbb{Z})}.$$

Using the fact that  $\check{S} = Z^{-1}SZ$  (see the discussion before Proposition 7.3) and Proposition 6.7, we conclude the proof.  $\square$

Therefore, since  $\Phi_{\alpha,\xi}(x, y)$  is defined for  $x, y \in \mathbb{Z}_{\neq 0}$ , we have (in our derivation we have also used (6.8)):

$$\Phi_{\alpha,\xi}(x, y) = (-1)^{x \wedge 0 + y \vee 0} \sum_{m=0}^{\infty} 2^{-\delta(m)} \varphi_m(x; \alpha, \xi) \varphi_m(-y; \alpha, \xi). \quad (7.16)$$

In the rest of the paper we agree that by this formula the kernel  $\Phi_{\alpha,\xi}(x, y)$  is defined for arbitrary  $x, y \in \mathbb{Z}$  (see also Remark 2.2.1). This is needed to view  $\Phi_{\alpha,\xi}$  as an operator in  $\ell^2(\mathbb{Z})$ . One can also write this kernel using the “twisted” functions defined in §6.3:  $\Phi_{\alpha,\xi}(x, y) = \sum_{m=0}^{\infty} 2^{-\delta(m)} \tilde{\varphi}_m(x) \tilde{\varphi}_m(-y)$ .

### 7.3 Interpretation through spectral projections

One can interpret the kernel  $\Phi_{\alpha,\xi}$  through orthogonal spectral projections related to the difference operator  $\tilde{\mathcal{D}}_{\alpha,\xi}$  defined by (6.11). Namely, the projection onto the positive part of the spectrum of  $\tilde{\mathcal{D}}_{\alpha,\xi}$  has the form (see §6.3):

$$\text{Proj}_{>0}(\tilde{\mathcal{D}}_{\alpha,\xi})(x, y) = \sum_{m=1}^{\infty} \tilde{\varphi}_m(x; \alpha, \xi) \tilde{\varphi}_m(y; \alpha, \xi).$$

We also need the projection onto the zero eigenspace, which is simply

$$\text{Proj}_{=0}(\tilde{\mathcal{D}}_{\alpha,\xi})(x, y) = \tilde{\varphi}_0(x; \alpha, \xi) \tilde{\varphi}_0(y; \alpha, \xi).$$

**Proposition 7.5.** *Viewing the static Pfaffian kernel  $\Phi_{\alpha,\xi}$  as an operator in  $\ell^2(\mathbb{Z})$ , we have*

$$\Phi_{\alpha,\xi} = (\text{Proj}_{>0}(\tilde{\mathcal{D}}_{\alpha,\xi}) + \frac{1}{2} \text{Proj}_{=0}(\tilde{\mathcal{D}}_{\alpha,\xi}))R,$$

where  $R: \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$  is the operator corresponding to the reflection of the lattice  $\mathbb{Z}$  with respect to 0:  $(Rf)(x) := f(-x)$ ,  $f \in \ell^2(\mathbb{Z})$ .

Since  $R^2$  is the identity operator, we see that the operator  $\Phi_{\alpha,\xi}R$  is a rank one perturbation of the orthogonal spectral projection operator corresponding to the positive part of the spectrum of  $\tilde{\mathcal{D}}_{\alpha,\xi}$ .

### 7.4 Expression through the discrete hypergeometric kernel

Recall that the discrete hypergeometric kernel  $\underline{K}_{z,z',\xi}(x', y')$  (where  $x', y' \in \mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$ ) serves as a determinantal kernel for the  $z$ -measures on ordinary partitions (§6.1). Under a suitable choice of parameters  $z, z'$ , the functions involved in the formula for  $\underline{K}_{z,z',\xi}(x', y')$  turn into our functions  $\varphi_m$ , see (6.5). This means that one could express our Pfaffian kernel  $\Phi_{\alpha,\xi}$  through the discrete hypergeometric kernel:

**Proposition 7.6.** *For all  $x, y \in \mathbb{Z}$  we have*

$$\begin{aligned} \Phi_{\alpha,\xi}(x, y) = \frac{1}{2}(-1)^{x \wedge 0 + y \vee 0} & \left[ \underline{K}_{\nu(\alpha) - \frac{1}{2}, -\nu(\alpha) - \frac{1}{2}, \xi}(x + \frac{1}{2}, -y + \frac{1}{2}) \right. \\ & \left. + \underline{K}_{\nu(\alpha) + \frac{1}{2}, -\nu(\alpha) + \frac{1}{2}, \xi}(x - \frac{1}{2}, -y - \frac{1}{2}) \right]. \end{aligned} \quad (7.17)$$

*Proof.* Using (6.3) and (6.5) with  $d = -1$  and  $d = 0$ , we observe that for  $x, y \in \mathbb{Z}$ :

$$\sum_{m=1}^{\infty} \varphi_m(x; \alpha, \xi) \varphi_m(y; \alpha, \xi) = \underline{K}_{\nu(\alpha) - \frac{1}{2}, -\nu(\alpha) - \frac{1}{2}, \xi}(x + \frac{1}{2}, y + \frac{1}{2}); \quad (7.18)$$

$$\sum_{m=0}^{\infty} \varphi_m(x; \alpha, \xi) \varphi_m(y; \alpha, \xi) = \underline{K}_{\nu(\alpha) + \frac{1}{2}, -\nu(\alpha) + \frac{1}{2}, \xi}(x - \frac{1}{2}, y - \frac{1}{2}). \quad (7.19)$$

Taking half sum and using (7.16), we conclude the proof.  $\square$

A time-dependent version of (7.17) is (10.12). A similar identity for the (static) determinantal kernel of our model is (8.3) below.

## 7.5 Reduction formulas

It is possible to rewrite the Pfaffian hypergeometric-type kernel  $\Phi_{\alpha,\xi}(x, y)$  in a closed form (without the sum):

**Proposition 7.7.** *For any  $x, y \in \mathbb{Z}$  we have*

$$\Phi_{\alpha,\xi}(x, y) = \frac{(-1)^{x \wedge 0 + y \wedge 0} \sqrt{\alpha\xi}}{2(1-\xi)} \times \frac{\varphi_0(x)(\varphi_1(y) - \varphi_{-1}(y)) - \varphi_0(y)(\varphi_1(x) - \varphi_{-1}(x))}{x+y}.$$

For  $x = -y$  there is a singularity in the numerator (this is seen using (6.8)) as well as in the denominator. In this case the value of  $\Phi_{\alpha,\xi}(x, y)$  is understood according to the L'Hospital's rule using the analytic expression for  $\varphi_m$  (6.4). The same is applicable to all similar formulas below.

*Proof.* There are several ways of establishing this fact. One could use representation-theoretic arguments as in the proof of Theorem 3 in [Oko01b]. Another way is to argue directly using the three-term relations for the functions  $\varphi_m$  (6.9) to simplify the sum (7.16) similarly to the standard derivation of the Christoffel–Darboux formula for orthogonal polynomials.

We use Proposition 7.6 together with the existing closed form expression for  $\underline{K}_{z,z',\xi}$  [BO06b, Proposition 3.10]:<sup>16</sup>

$$\underline{K}_{z,z',\xi}(x', y') = \frac{\sqrt{zz'\xi} \psi_{-\frac{1}{2}}(x')\psi_{\frac{1}{2}}(y') - \psi_{\frac{1}{2}}(x')\psi_{-\frac{1}{2}}(y')}{1-\xi} \frac{1}{x'-y'}, \quad x', y' \in \mathbb{Z}'.$$

(of course, the parameters of the functions  $\psi$  above are  $z, z', \xi$ ). We plug this formula into (7.17), and then express each function  $\psi_{a'}$  through  $\varphi_m$  using (6.5) with  $d = -1$  and  $d = 0$ . Observe that for such  $d$  we have  $z(\alpha)z'(\alpha) = \alpha$ . After that we apply (6.8) to simplify the resulting expression. This concludes the proof.  $\square$

**Corollary 7.8** (Reduction formulas for  $\Phi_{\alpha,\xi}$ ). *For all  $x, y \in \mathbb{Z}$  we have:*

- (1)  $\Phi_{\alpha,\xi}(x, -y) = \Phi_{\alpha,\xi}(y, -x)$ ;
- (2)  $\Phi_{\alpha,\xi}(x, -y) = -\Phi_{\alpha,\xi}(-x, y)$  if  $x \neq y$ ;
- (3)  $(x+y)\Phi_{\alpha,\xi}(x, y) = (x-y)\Phi_{\alpha,\xi}(x, -y)$  (note that  $\Phi_{\alpha,\xi}(x, x) = 0$  for all  $x \neq 0$ ).

*Proof.* Claim (1) is best seen from (7.17), because the kernel  $\underline{K}$  is symmetric. Claims (2) and (3) follow from Proposition 7.7 and (6.8).  $\square$

<sup>16</sup>In fact, [BO06b, Proposition 3.10] itself is established using the three-term relations for the functions  $\psi_{a'}$ .

## 8 Static determinantal kernel

Here we compute and discuss the determinantal correlation kernel  $\mathbf{K}_{\alpha,\xi}$  of the point process  $M_{\alpha,\xi}$  on  $\mathbb{Z}_{>0}$ , thus completing the proof of Theorem 1 from §2.

**Theorem 8.1.** *For all  $\alpha > 0$  and  $0 < \xi < 1$ , the point process  $M_{\alpha,\xi}$  on  $\mathbb{Z}_{>0}$  is determinantal. Its correlation kernel  $\mathbf{K}_{\alpha,\xi}$  can be expressed in several ways (here  $x, y \in \mathbb{Z}_{>0}$ ):*

(1) *As an infinite sum*

$$\mathbf{K}_{\alpha,\xi}(x, y) = \frac{2\sqrt{xy}}{x+y} \sum_{m=0}^{\infty} 2^{-\delta(m)} \varphi_m(x; \alpha, \xi) \varphi_m(y; \alpha, \xi) \quad (8.1)$$

(the functions  $\varphi_m$  are defined in §6.2).

(2) *In an integrable form*

$$\mathbf{K}_{\alpha,\xi}(x, y) = \frac{\sqrt{\alpha\xi xy}}{1-\xi} \cdot \frac{P(x)Q(y) - Q(x)P(y)}{x^2 - y^2}, \quad (8.2)$$

where  $P(x) = \varphi_0(x; \alpha, \xi)$  and  $Q(x) = \varphi_1(x; \alpha, \xi) - \varphi_{-1}(x; \alpha, \xi)$ .

(3) *In terms of the discrete hypergeometric kernel of the  $z$ -measures (§6.1)*

$$\begin{aligned} \tilde{\mathbf{K}}_{\alpha,\xi}(x, y) &= \underline{K}_{\nu(\alpha)+\frac{1}{2}, -\nu(\alpha)+\frac{1}{2}, \xi}(x - \frac{1}{2}, y - \frac{1}{2}) \\ &\quad + (-1)^y \underline{K}_{\nu(\alpha)-\frac{1}{2}, -\nu(\alpha)-\frac{1}{2}, \xi}(x + \frac{1}{2}, -y + \frac{1}{2}), \end{aligned} \quad (8.3)$$

where we have denoted  $\tilde{\mathbf{K}}_{\alpha,\xi}(x, y) := \sqrt{\frac{x}{y}} \cdot \mathbf{K}_{\alpha,\xi}(x, y)$ .

(4) *Viewed as an operator in  $\ell^2(\mathbb{Z}_{>0})$ ,  $\tilde{\mathbf{K}}_{\alpha,\xi}$  can be interpreted in terms of orthogonal spectral projections corresponding to the difference operator  $\tilde{\mathcal{D}}_{\alpha,\xi}$  (6.11) as follows (we restrict the operator below to  $\ell^2(\mathbb{Z}_{>0}) \subset \ell^2(\mathbb{Z})$ ):*

$$\tilde{\mathbf{K}}_{\alpha,\xi} = (\text{Proj}_{>0}(\tilde{\mathcal{D}}_{\alpha,\xi}) + \frac{1}{2} \text{Proj}_{=0}(\tilde{\mathcal{D}}_{\alpha,\xi})) (\mathbf{I} + \mathbf{R}),$$

where  $\mathbf{I}$  is the identity operator and  $\mathbf{R}$  is the reflection, see Proposition 7.5.

*Proof.* The fact that the process  $M_{\alpha,\xi}$  is determinantal is guaranteed by Lemma 3.5. On the other hand, the reduction formulas for the Pfaffian kernel  $\Phi_{\alpha,\xi}$  (Corollary 7.8) allow us to apply Proposition A.2 from Appendix. This implies that

$$\rho_{\alpha,\xi}^{(n)}(X) = \text{Pf}(\hat{\Phi}_{\alpha,\xi} \llbracket X \rrbracket) = \det \left[ \mathbf{K}_{\alpha,\xi}(x_k, x_j) \right]_{k,j=1}^n,$$

where  $X = \{x_1, \dots, x_n\} \subset \mathbb{Z}_{>0}$  (with pairwise distinct  $x_j$ 's),  $\hat{\Phi}_{\alpha,\xi} \llbracket X \rrbracket$  is the skew-symmetric  $2n \times 2n$  matrix introduced in Theorem 7.1, and

$$\mathbf{K}_{\alpha,\xi}(x, y) = \frac{2\sqrt{xy}}{x+y} \Phi_{\alpha,\xi}(x, -y), \quad x, y \in \mathbb{Z}_{>0}.$$

This gives an argument (independently of Lemma 3.5) that the process  $M_{\alpha,\xi}$  is determinantal. Moreover, this also provides us with explicit formulas for the kernel  $K_{\alpha,\xi}$ . Namely, claims 1 and 2 of the present theorem directly follow from the expressions of  $\Phi_{\alpha,\xi}$  as a series (7.16) and in a closed form (Proposition 7.7).

To prove claims 3 and 4, observe that

$$K_{\alpha,\xi}(x, y) = \sqrt{\frac{y}{x}} \left[ \frac{x-y}{x+y} + 1 \right] \Phi_{\alpha,\xi}(x, -y) = \sqrt{\frac{y}{x}} \left[ \Phi_{\alpha,\xi}(x, y) + \Phi_{\alpha,\xi}(x, -y) \right]$$

(the last equality is by Corollary 7.8.(3)), so

$$\tilde{K}_{\alpha,\xi}(x, y) = \Phi_{\alpha,\xi}(x, y) + \Phi_{\alpha,\xi}(x, -y).$$

Now we see that claim 3 follows from (7.16) and (7.18)–(7.19), and claim 4 is due to Proposition 7.5. This concludes the proof.  $\square$

### Comments to Theorem 8.1

1. Formulas (8.1) and (8.2) for the correlation kernel  $K_{\alpha,\xi}$  are the same as the statements of Theorems 2.1 and 2.2 in [Pet10b]. This can be seen from the expression (6.4) for the functions  $\varphi_m$ .
2. It is possible to obtain double contour integral expressions for the kernel  $K_{\alpha,\xi}(x, y)$  (given in [Pet10b, Propositions 3 and 4]). They can be derived from (8.1) in the same way as in the proof of [BO06b, Theorem 3.3].
3. The form (8.2) of the kernel  $K_{\alpha,\xi}$  is called *integrable* because the operator (8.2) in  $\ell^2(\mathbb{Z}_{>0})$  can be viewed as a discrete analogue of an integrable operator (if we take  $x^2$  and  $y^2$  as variables). About integrable operators, e.g., see [IJKS90], [Dei99]. Discrete integrable operators are discussed in [Bor00] and [BO00, §6].
4. The expression  $\tilde{K}_{\alpha,\xi}(x, y)$  is a so-called *gauge transformation* of the original correlation kernel  $K_{\alpha,\xi}$ , that is,  $\tilde{K}_{\alpha,\xi}$  is related to  $K_{\alpha,\xi}$  by a conjugation by a diagonal matrix. This means that the  $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$  matrix  $\tilde{K}_{\alpha,\xi}$  can also serve as a correlation kernel for the point process  $M_{\alpha,\xi}$ .
5. Relation (8.3) seems to be purely formal and have no consequences at the level of random point processes.
6. Consider the  $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$  matrix  $L_{\alpha,\xi}$  which is defined by (3.7), where  $w(x) = w_{\alpha,\xi}(x)$  is given by (2.6). Then one can show similarly to the proof of Theorem 3.3 in [BO00] (and also using the identities from Appendix in that paper) that  $K_{\alpha,\xi} = L_{\alpha,\xi}(1 + L_{\alpha,\xi})^{-1}$ . That is, the symmetric kernel  $K_{\alpha,\xi}$  is precisely the one given by Lemma 3.5.

## 9 Markov processes

In §2.2 we have described a family of continuous time Markov processes on the set  $\mathcal{S}$  of all strict partitions. They depend on our parameters  $\alpha > 0$  and  $0 < \xi < 1$  and are defined in terms of jump rates (here  $\lambda \in \mathcal{S}_n$ ,  $n = 0, 1, \dots$ ):

$$Q_{\lambda,x} := (1 - \xi)^{-1} \xi(n + \alpha/2) p_\alpha^\uparrow(n, n+1)_{\lambda,x}, \quad \text{where } x \searrow \lambda;$$

$$\begin{aligned}
\mathbb{Q}_{\lambda,\mu} &:= (1 - \xi)^{-1} n p^\downarrow(n, n - 1)_{\lambda,\mu}, & \text{where } \mu \nearrow \lambda; \\
\mathbb{Q}_{\lambda,\lambda} &:= - \sum_{x: x \searrow \lambda} \mathbb{Q}_{\lambda,x} - \sum_{\mu: \mu \nearrow \lambda} \mathbb{Q}_{\lambda,\mu} \\
&= -(1 - \xi)^{-1} \{ \xi(\alpha/2 + n) + n \}.
\end{aligned} \tag{9.1}$$

All other jump rates are zero. Here  $p^\downarrow(n, n - 1)$  and  $p^\uparrow_\alpha(n, n + 1)$  are the down and up transition kernels, respectively (see §3).

Under the projection  $\mathbb{S} \rightarrow \mathbb{Z}_{\geq 0}$ ,  $\lambda \mapsto |\lambda|$ , the  $\mathbb{S} \times \mathbb{S}$  matrix  $\mathbb{Q}$  (9.1) turns into the  $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$  matrix of jump rates of the birth and death process  $\mathbf{n}_{\alpha,\xi}$  (see §2.2). This means that the processes on  $\mathbb{S}$  “extend” the birth and death processes  $\mathbf{n}_{\alpha,\xi}$ .

The jump rates  $\mathbb{Q}$  (9.1) uniquely define a continuous time Markov process on  $\mathbb{S}$  that can start from any point and any probability distribution. This fact is proven similarly to the case of the ordinary partitions [BO06a, §4]. In our situation the details are explained in [Pet10a, §9]. The process with jump rates (9.1) preserves the measure  $M_{\alpha,\xi}$  on  $\mathbb{S}$ . By  $(\lambda_{\alpha,\xi}(t))_{t \in [0, +\infty)}$  we denote the equilibrium version of this process. It is reversible with respect to the measure  $M_{\alpha,\xi}$  on strict partitions.

**Remark 9.1.** In contrast to [BO06a], we restrict our attention to the stationary (time homogeneous) case, that is, we assume that the parameter  $\xi$  does not vary in time. The introduction of the non-stationary processes in [BO06a] was motivated by the technique of handling the stationary case (in particular, by the method of the computation of the dynamical correlation functions). The technique that we use in the present paper does not require dealing with non-stationary processes.

Let us discuss the pre-generator of the Markov process  $\lambda_{\alpha,\xi}$ . We regard the  $\mathbb{S} \times \mathbb{S}$  matrices  $(\mathbb{P}_{\lambda,\mu}(t))_{t \geq 0}$  of transition probabilities of the process  $\lambda_{\alpha,\xi}$  as operators acting on functions on  $\mathbb{S}$  (from the left):

$$(\mathbb{P}(t)f)(\lambda) := \sum_{\mu \in \mathbb{S}} \mathbb{P}_{\lambda,\mu}(t) f(\mu).$$

Here  $\mathbb{P}_{\lambda,\mu}(t)$  is the probability that the process starting from  $\lambda$  will be at  $\mu$  after time  $t$ . The family  $(\mathbb{P}(t))_{t \geq 0}$  is a Markov semigroup of self-adjoint contractive operators in the weighted space  $\ell^2(\mathbb{S}, M_{\alpha,\xi})$  (see §4.2 for the definition of  $\ell^2(\mathbb{S}, M_{\alpha,\xi})$ ). This semigroup has a generator which is an unbounded operator. By  $\mathbb{Q}$  let us denote the restriction of this generator to  $\ell^2_{\text{fin}}(\mathbb{S}, M_{\alpha,\xi}) \subset \ell^2(\mathbb{S}, M_{\alpha,\xi})$ , the dense subspace of all finitely supported functions in  $\ell^2(\mathbb{S}, M_{\alpha,\xi})$ . The operator  $\mathbb{Q}$  acts as

$$(\mathbb{Q}f)(\lambda) = \sum_{\mu \in \mathbb{S}} \mathbb{Q}_{\lambda,\mu} f(\mu), \quad f \in \ell^2_{\text{fin}}(\mathbb{S}, M_{\alpha,\xi}), \tag{9.2}$$

where  $\mathbb{Q}_{\lambda,\mu}$  (9.1) are the jump rates of the process  $\lambda_{\alpha,\xi}$ .

The operator  $\mathbb{Q}$  is symmetric with respect to the inner product  $(\cdot, \cdot)_{M_{\alpha,\xi}}$ . Moreover, it is closable in  $\ell^2(\mathbb{S}, M_{\alpha,\xi})$ , and its closure generates the semigroup  $(\mathbb{P}(t))_{t \geq 0}$  (see Remark 9.5 below). That is,  $\mathbb{Q}$  is the *pre-generator* of the process  $\lambda_{\alpha,\xi}$ .

**Remark 9.2.** As a wider domain for the operator  $\mathbb{Q}$  (9.2) one can take the space of all functions  $f$  on  $\mathbb{S}$  such that both  $f$  and  $\mathbb{Q}f$  (defined by (9.2)) belong to  $\ell^2(\mathbb{S}, M_{\alpha,\xi})$ . This space clearly includes finitely supported functions.

Using the isometry  $I_{\alpha,\xi}: \ell^2(\mathbb{S}, M_{\alpha,\xi}) \rightarrow \ell^2(\mathbb{S})$  (4.11), we get a symmetric operator  $\mathbb{B}$  in  $\ell^2_{\text{fin}}(\mathbb{S})$  and a Markov semigroup  $(\mathbb{V}(t))_{t \geq 0}$  of self-adjoint contractive operators in  $\ell^2(\mathbb{S})$  corresponding to  $\mathbb{Q}$  and

$(\mathbb{P}(t))_{t \geq 0}$ , respectively.<sup>17</sup> Let us compute the matrix elements of the operator  $\mathbb{B}$  in the standard orthonormal basis  $\{\underline{\lambda}\}_{\lambda \in \mathbb{S}}$ .

**Proposition 9.3.** *We have*

$$\begin{aligned} \mathbb{B}\underline{\lambda} &= \sum_{\nu \in \mathbb{S}} (\mathbb{B}\underline{\lambda}, \underline{\nu}) \underline{\nu} = -(1 - \xi)^{-1} \{|\lambda| + \xi(|\lambda| + \frac{\alpha}{2})\} \underline{\lambda} \\ &\quad + \frac{\sqrt{\xi}}{1 - \xi} \sum_{\mu: \mu \nearrow \lambda} q_{\alpha}(\lambda/\mu) \underline{\mu} + \frac{\sqrt{\xi}}{1 - \xi} \sum_{x: x \searrow \lambda} q_{\alpha}(x/\lambda) \underline{x}. \end{aligned}$$

Here  $q_{\alpha}$  is the function of a box defined by (4.6).

*Proof.* Fix  $\lambda \in \mathbb{S}$ , and for any  $\nu \in \mathbb{S}$  one has

$$\begin{aligned} (\mathbb{B}\underline{\lambda}, \underline{\nu}) &= ((M_{\alpha, \xi}(\lambda))^{-\frac{1}{2}} \mathbb{Q}\underline{\lambda}, (M_{\alpha, \xi}(\nu))^{-\frac{1}{2}} \underline{\nu})_{M_{\alpha, \xi}} \\ &= (M_{\alpha, \xi}(\lambda) M_{\alpha, \xi}(\nu))^{-\frac{1}{2}} (\mathbb{Q}\underline{\lambda}, \underline{\nu})_{M_{\alpha, \xi}} = M_{\alpha, \xi}(\nu)^{\frac{1}{2}} M_{\alpha, \xi}(\lambda)^{-\frac{1}{2}} \mathbb{Q}_{\nu, \lambda} \end{aligned}$$

( $\mathbb{Q}$  is given in (9.1)), and Proposition follows from a direct computation.  $\square$

**Corollary 9.4.** *The operator  $\mathbb{B}$  in  $\text{Fock}_{\text{fin}}(\mathbb{Z}_{>0})$  has the form*

$$\mathbb{B} = -R(H_{\xi}) + \frac{\alpha}{4} \mathbf{I},$$

where  $\mathbf{I}$  is the identity operator, the unitary representation  $R$  of  $\mathfrak{sl}(2, \mathbb{C})$  in the Hilbert space  $\text{Fock}_{\text{fin}}(\mathbb{Z}_{>0})$  is defined in §5.4, and  $H_{\xi}$  is given in Remark 6.8.

*Proof.* This is a straightforward consequence of Proposition 9.3 (where, of course, we identify  $\ell^2(\mathbb{S})$  and  $\text{Fock}(\mathbb{Z}_{>0})$ ) and the matrix computation in Remark 6.8.  $\square$

**Remark 9.5.** From the above corollary it follows that the operator  $\mathbb{B}$  (with domain  $\text{Fock}_{\text{fin}}(\mathbb{Z}_{>0})$ ) is essentially self-adjoint because, by Proposition 4.3, all vectors of the space  $\text{Fock}_{\text{fin}}(\mathbb{Z}_{>0})$  are analytic for the operator  $R(H_{\xi})$ . The same also holds for the operator  $R(H)$  (corresponding to the case  $\xi = 0$ ). Moreover, the closure of  $\mathbb{B}$  looks as  $\overline{\mathbb{B}} = \frac{\alpha}{4} \mathbf{I} - \overline{R(H_{\xi})} = \frac{\alpha}{4} \mathbf{I} - R(\tilde{G}_{\xi}) \overline{R(H)} R(\tilde{G}_{\xi})^{-1}$ , and this operator generates the semigroup  $(\mathbb{V}(t))_{t \geq 0}$ .

These properties of  $\mathbb{B}$  in fact imply (using the isometry  $I_{\alpha, \xi}$  (4.11)) that the operator  $\mathbb{Q}$  is closable in  $\ell^2(\mathbb{S}, M_{\alpha, \xi})$ , and its closure generates the semigroup  $(\mathbb{P}(t))_{t \geq 0}$  of the Markov process  $\lambda_{\alpha, \xi}$  on strict partitions.

## 10 Dynamical correlation functions

In this section we prove a Pfaffian formula for the dynamical correlation functions  $\rho_{\alpha, \xi}^{(n)}$  (2.9) of the Markov processes  $\lambda_{\alpha, \xi}$  on strict partitions, thus proving Theorem 2 from §2.

<sup>17</sup>We denote, e.g., the operators  $\mathbb{B}$  and  $\mathbb{Q}$  by different symbols only to indicate in what spaces they act. Essentially, these operators are the same.

## 10.1 Dynamical correlation functions and Markov semigroups

Let us fix  $n \geq 1$  and pairwise distinct space-time points  $(t_1, x_1), \dots, (t_n, x_n) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{>0}$ . We assume that the time moments are ordered as  $0 \leq t_1 \leq \dots \leq t_n$ . Recall the operators  $\Delta_x$  (where  $x \in \mathbb{Z}_{>0}$ ) in the Hilbert space  $\ell^2(\mathbb{S}, M_{\alpha, \xi})$  defined in §7.1.

**Lemma 10.1.** *The dynamical correlation functions of  $\lambda_{\alpha, \xi}$  have the form*

$$\rho_{\alpha, \xi}^{(n)}(t_1, x_1; \dots; t_n, x_n) = (\Delta_{x_1} \mathbb{P}(t_2 - t_1) \Delta_{x_2} \dots \Delta_{x_{n-1}} \mathbb{P}(t_n - t_{n-1}) \Delta_{x_n} \mathbf{1}, \mathbf{1})_{M_{\alpha, \xi}},$$

where  $(\mathbb{P}(t))_{t \geq 0}$  is the semigroup of the process  $\lambda_{\alpha, \xi}$  in the space  $\ell^2(\mathbb{S}, M_{\alpha, \xi})$  and  $\mathbf{1} \in \ell^2(\mathbb{S}, M_{\alpha, \xi})$  is the constant identity function.

*Proof.* This is a simple consequence of the Markov property of the process  $\lambda_{\alpha, \xi}$ . Indeed, let us assume (for simplicity) that  $t_j$ 's are distinct. The  $n$ -dimensional distribution of the process  $\lambda_{\alpha, \xi}$  at time moments  $t_1 < \dots < t_n$  is a probability measure on  $\mathbb{S} \times \dots \times \mathbb{S}$  ( $n$  copies) which assigns the probability

$$M_{\alpha, \xi}(\lambda^{(1)}) \mathbb{P}_{\lambda^{(1)}, \lambda^{(2)}}(t_2 - t_1) \dots \mathbb{P}_{\lambda^{(n-1)}, \lambda^{(n)}}(t_n - t_{n-1}) \quad (10.1)$$

to every point  $(\lambda^{(1)}, \dots, \lambda^{(n)})$ ,  $\lambda^{(i)} \in \mathbb{S}$ . By definition,  $\rho_{\alpha, \xi}^{(n)}(t_1, x_1; \dots; t_n, x_n)$  is exactly the mass of the set  $\{\lambda^{(1)} \ni x_1, \dots, \lambda^{(n)} \ni x_n\}$  under the measure (10.1). This proves the claim for distinct  $t_j$ 's. It can be readily verified that the claim also holds if some of  $t_j$ 's coincide. This concludes the proof.  $\square$

Let us consider the following operator in  $\text{Fock}(\mathbb{Z}_{>0})$ :

$$\Delta_{\llbracket T, X \rrbracket} := \Delta_{x_1} \mathbb{V}(t_2 - t_1) \Delta_{x_2} \dots \Delta_{x_{n-1}} \mathbb{V}(t_n - t_{n-1}) \Delta_{x_n}.$$

Here  $(\mathbb{V}(t))_{t \geq 0}$  is the semigroup in  $\ell^2(\mathbb{S})$  defined in §9, and we have identified  $\ell^2(\mathbb{S})$  with  $\text{Fock}(\mathbb{Z}_{>0})$  as in §5.2. The operators  $\Delta_x$ ,  $x \in \mathbb{Z}_{>0}$ , are now acting in  $\text{Fock}(\mathbb{Z}_{>0})$ .

**Proposition 10.2.** *The correlation functions of  $\lambda_{\alpha, \xi}$  have the form*

$$\rho_{\alpha, \xi}^{(n)}(t_1, x_1; \dots; t_n, x_n) = (R(\tilde{G}_\xi)^{-1} \Delta_{\llbracket T, X \rrbracket} R(\tilde{G}_\xi) \text{vac}, \text{vac}). \quad (10.2)$$

*Note that now the expectation is taken in  $\text{Fock}(\mathbb{Z}_{>0})$ .*

*Proof.* Since  $\mathbb{V}(t) = I_{\alpha, \xi} \mathbb{P}(t) I_{\alpha, \xi}^{-1}$ , the claim is a direct consequence of Lemma 10.1 and formula (5.11) with  $A = \Delta_{x_1} \mathbb{P}(t_2 - t_1) \Delta_{x_2} \dots \Delta_{x_{n-1}} \mathbb{P}(t_n - t_{n-1}) \Delta_{x_n}$ .  $\square$

Note that in contrast to the static case (7.5), the operator  $\Delta_{\llbracket T, X \rrbracket}$  is not diagonal (see also Remark 4.5). It is worth noting that formula (10.2) does not hold if  $t_j$ 's are not ordered as  $t_1 \leq \dots \leq t_n$ .

## 10.2 Pre-generator and Kerov's operators

Our next aim is to extend the definition of the semigroup  $(\mathbb{V}(t))_{t \geq 0}$  from real nonnegative values of  $t$  to complex values of  $t$  with  $\Re t \geq 0$ . This will be needed in the next subsection for computation of the dynamical correlation functions.

Observe that the matrix  $iH_\xi$  (where  $H_\xi$  is defined in Remark 6.8 and here and below  $i = \sqrt{-1}$ ) belongs to the real form  $\mathfrak{su}(1, 1) \subset \mathfrak{sl}(2, \mathbb{C})$ . Denote

$$W_\xi(\tau) := e^{-i\tau H_\xi} = G_\xi \begin{bmatrix} e^{-i\tau/2} & 0 \\ 0 & e^{i\tau/2} \end{bmatrix} G_\xi^{-1} \in SU(1, 1), \quad \tau \in \mathbb{R}.$$

The family  $\{W_\xi(\tau)\}_{\tau \in \mathbb{R}}$  for any fixed  $\xi \in [0, 1)$  is a continuous curve in  $SU(1, 1)$  passing through the unity at  $\tau = 0$ . By  $\{\widetilde{W}_\xi(\tau)\}_{\tau \in \mathbb{R}}$  denote the lifting of this curve to the universal covering group  $SU(1, 1)^\sim$ .

For real  $\tau$  one can consider unitary operators

$$R(\widetilde{W}_\xi(\tau)) = R(\widetilde{G}_\xi)R(\widetilde{W}_0(\tau))R(\widetilde{G}_\xi)^{-1}$$

in the Fock space  $\text{Fock}(\mathbb{Z}_{>0})$ . Here the operator  $R(\widetilde{W}_0(\tau))$  (corresponding to  $\xi = 0$ ) acts in  $\text{Fock}_{\text{fin}}(\mathbb{Z}_{>0})$  as

$$R(\widetilde{W}_0(\tau))\underline{\lambda} = e^{-i\tau R(H)/2}\underline{\lambda} = e^{-i\tau(\lambda|\frac{\alpha}{4})}\underline{\lambda}, \quad \lambda \in \mathbb{S}, \quad \tau \in \mathbb{R}. \quad (10.3)$$

Informally speaking, for  $s \in \mathbb{R}_{\geq 0}$ , the operator  $\mathbb{V}(s)$  means  $e^{s\mathbb{B}}$ , and for  $\tau \in \mathbb{R}$ , the operator  $R(\widetilde{W}_\xi(\tau))e^{i\tau\frac{\alpha}{4}\mathbb{I}}$  means  $e^{i\tau\mathbb{B}}$  (here  $\mathbb{B}$  is the generator of the semigroup  $(\mathbb{V}(s))_{s \geq 0}$ ). Thus, it is natural to give the following definition:

**Definition 10.3.** For  $t = s + i\tau \in \mathbb{C}_+ := \{w \in \mathbb{C} : \Re w \geq 0\}$  let  $\mathbb{V}(t)$  be the following operator in  $\text{Fock}(\mathbb{Z}_{>0})$ :

$$\mathbb{V}(t) := \mathbb{V}(s)R(\widetilde{W}_\xi(\tau))e^{i\tau\frac{\alpha}{4}\mathbb{I}}$$

For real nonnegative  $t$  the operator  $\mathbb{V}(t)$  is self-adjoint and bounded, it was defined in §9. For purely imaginary  $t$ , the operator  $\mathbb{V}(t)$  is unitary. Thus, the operators  $\mathbb{V}(t)$  are bounded for all  $t \in \mathbb{C}_+$ . Moreover,  $\mathbb{V}(t_1 + t_2) = \mathbb{V}(t_1)\mathbb{V}(t_2)$  for any  $t_1, t_2 \in \mathbb{C}_+$ , so  $\{\mathbb{V}(t)\}_{t \in \mathbb{C}_+}$  is a semigroup (with complex parameter) that can be viewed as an analytic continuation of the semigroup  $\{\mathbb{V}(s)\}_{s \in \mathbb{R}_{\geq 0}}$ . In particular, the operators  $\mathbb{V}(t)$  commute with each other. Moreover, it is clear that the function  $t \mapsto \mathbb{V}(t)h$  is bounded and continuous in  $\mathbb{C}_+$  and holomorphic in the interior  $\{w \in \mathbb{C}_+ : \Re w > 0\}$  of  $\mathbb{C}_+$  for any vector  $h \in \text{Fock}(\mathbb{Z}_{>0})$  which is analytic for the operator  $\mathbb{B}$ .

## 10.3 Pfaffian formula for dynamical correlation functions

**Theorem 10.4.** *The dynamical correlation functions of the equilibrium Markov process  $(\lambda_{\alpha,\xi}(t))_{t \geq 0}$  have the form*

$$\rho_{\alpha,\xi}^{(n)}(t_1, x_1; \dots; t_n, x_n) = \text{Pf}(\Phi_{\alpha,\xi} \llbracket T, X \rrbracket), \quad (10.4)$$

where the function  $\Phi_{\alpha,\xi}(s, x; t, y)$  ( $x, y \in \mathbb{Z}, s \leq t$ ) is given by

$$\Phi_{\alpha,\xi}(s, x; t, y) := (-1)^{x \wedge 0 + y \vee 0} \sum_{m=0}^{\infty} 2^{-\delta(m)} e^{-m(t-s)} \varphi_m(x; \alpha, \xi) \varphi_m(-y; \alpha, \xi) \quad (10.5)$$

(see (6.4) for definition of  $\varphi_m$ ). In (10.4),  $(t_1, x_1), \dots, (t_n, x_n) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{>0}$  are pairwise distinct space-time points such that  $0 \leq t_1 \leq \dots \leq t_n$ , and  $\Phi_{\alpha,\xi}[[T, X]]$  is the  $2n \times 2n$  skew-symmetric matrix with rows and columns indexed by the numbers  $1, -1, \dots, n, -n$ , such that the  $kj$ -th entry in  $\Phi_{\alpha,\xi}[[T, X]]$  above the main diagonal is  $\Phi_{\alpha,\xi}(t_{|k|}, x_k; t_{|j|}, x_j)$ , where  $k, j = 1, -1, \dots, n, -n$  (thus,  $|k| \leq |j|$ ).<sup>18</sup>

The rest of this subsection is devoted to proving Theorem 10.4.

**Lemma 10.5** ([Ols08]). *Let  $F(z)$  be a function on the right half-plane  $\mathbb{C}_+$  which is bounded and continuous in  $\mathbb{C}_+$  and is holomorphic in the interior of  $\mathbb{C}_+$ . Then  $F$  is uniquely determined by its values on the imaginary axis  $\{w \in \mathbb{C} : \Re w = 0\}$ .*

*Proof.* Conformally transforming  $\mathbb{C}_+$  to the unit disc  $|\zeta| < 1$ , we get a function  $G$  on the disc which is holomorphic in the interior of the disc and bounded and continuous up to the boundary (with possible exception of one point corresponding to  $w = \infty \in \mathbb{C}_+$ ).

For any fixed  $\zeta_0$  with  $|\zeta_0| < 1$ , the value  $G(\zeta_0)$  is represented by Cauchy's integral over the circle  $|\zeta| = r$ , for  $|\zeta_0| < r < 1$ . By our hypotheses, this Cauchy's integral has a limit as  $r \rightarrow 1$ , which gives an expression of  $G(\zeta_0)$  through the boundary values.  $\square$

Let us fix pairwise distinct space-time points  $(t_1, x_1), \dots, (t_n, x_n) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{>0}$  such that  $0 \leq t_1 \leq \dots \leq t_n$ . For convenience, set  $t_{kj} := t_k - t_j$ . Above we have expressed the dynamical correlation functions as (10.2), that is,

$$\begin{aligned} \rho_{\alpha,\xi}^{(n)}(t_1, x_1; \dots; t_n, x_n) \\ = \left( R(\tilde{G}_\xi)^{-1} \Delta_{x_1} \mathbb{V}(t_{2,1}) \Delta_{x_2} \dots \Delta_{x_{n-1}} \mathbb{V}(t_{n,n-1}) \Delta_{x_n} R(\tilde{G}_\xi) \text{vac}, \text{vac} \right). \end{aligned} \quad (10.6)$$

Denote the right-hand side of (10.6) by  $\mathcal{F}(t_{2,1}, \dots, t_{n,n-1}; x_1, \dots, x_n)$ . As a function in  $n-1$  variables  $t_{2,1}, \dots, t_{n,n-1}$ ,  $\mathcal{F}$  is initially defined for  $t_{j,j-1}$  taking real nonnegative values. However, as explained in §10.2, the definition of each operator  $\mathbb{V}(t_{j,j-1})$  can be extended to  $t_{j,j-1} \in \mathbb{C}_+$ , so  $\mathcal{F}$  is defined on  $(\mathbb{C}_+)^{n-1} \subset \mathbb{C}^{n-1}$ . Moreover,  $\mathcal{F}$  is continuous and bounded in  $(t_{2,1}, \dots, t_{n,n-1})$  belonging to the closed domain  $(\mathbb{C}_+)^{n-1}$  and is holomorphic in the interior of this domain. Therefore, by Lemma 10.5,  $\mathcal{F}(t_{2,1}, \dots, t_{n,n-1}; x_1, \dots, x_n)$  is uniquely determined by its values when all the variables  $t_{j,j-1}$  are purely imaginary.

From now on in the computation we will assume that the variables  $t_j = i\tau_j$  (where  $\tau_j \in \mathbb{R}, j = 1, \dots, n$ ) are purely imaginary. This implies that the differences  $t_{k,j} = i(\tau_k - \tau_j)$  are also purely imaginary. For such  $t_j$ , each operator  $\mathbb{V}(t_{j,j-1})$  is unitary, and, moreover,

$$\mathbb{V}(t_{j,j-1}) = \mathbb{V}(t_{j-1,1})^{-1} \mathbb{V}(t_{j,1}), \quad j = 1, \dots, n \quad (10.7)$$

(here by agreement  $t_{1,1} = 0$ , and  $\mathbb{V}(0) = \mathbf{I}$ , the identity operator).

<sup>18</sup>Here and below we use convention (7.1).

Now we want to rewrite  $\mathcal{F}(t_{2,1}, \dots, t_{n,n-1}; x_1, \dots, x_n)$  as a certain Pfaffian. First, we need a notation. Recall that in §7.1 we have defined a representation  $\check{S}$  of  $SU(1, 1)$  in the Hilbert space  $V = \ell^2(\mathbb{Z})$  with the standard orthonormal basis  $\{v_x\}_{x \in \mathbb{Z}}$ . Denote

$$v_{x,\xi}^{(t)} := \check{S}(W_0(\tau))\check{S}(G_\xi)^{-1}v_x \in V, \quad x \in \mathbb{Z}, \quad t = i\tau \in i\mathbb{R}. \quad (10.8)$$

For  $t = 0$  this vector becomes  $v_{x,\xi}$  defined by (7.13).

**Lemma 10.6.** *For  $t_j = i\tau_j \in i\mathbb{R}$  and distinct  $x_j \in \mathbb{Z}_{>0}$  ( $j = 1, \dots, n$ ) we have*

$$\mathcal{F}(t_{2,1}, \dots, t_{n,n-1}; x_1, \dots, x_n) = \text{Pf}(\mathcal{F}[[T, X]]), \quad (10.9)$$

where  $\mathcal{F}[[T, X]]$  is the  $2n \times 2n$  skew-symmetric matrix with rows and columns indexed by the numbers  $1, -1, \dots, n, -n$ , such that the  $kj$ -th entry in  $\mathcal{F}[[T, X]]$  above the main diagonal is  $\mathbf{F}_{\text{vac}}(v_{x_k, \xi}^{(t_{|k|, 1})}, v_{x_j, \xi}^{(t_{|j|, 1})})$ , where  $k, j = 1, -1, \dots, n, -n$  (and thus  $|k| \leq |j|$ ). Here  $\mathbf{F}_{\text{vac}}$  is the vacuum average on the Clifford algebra  $Cl(V)$ , see §5.

*Proof.* The operators  $\Delta_x$  have the form

$$\Delta_x = \mathcal{T}(v_x)\mathcal{T}(v_{-x}) = \mathcal{T}(v_x v_{-x}), \quad x \in \mathbb{Z}_{>0}.$$

By (10.8) and Proposition 7.3, we have

$$R(\widetilde{W}_0(\tau))R(\widetilde{G}_\xi)^{-1}\Delta_x R(\widetilde{G}_\xi)R(\widetilde{W}_0(\tau))^{-1} = \mathcal{T}(v_{x,\xi}^{(t)}v_{-x,\xi}^{(t)}), \quad x \in \mathbb{Z}, \quad t = i\tau \in i\mathbb{R}.$$

A straightforward computation using Definition 10.3 and (10.7) allows us to rewrite the operator in the right-hand side of (10.6) as

$$\begin{aligned} R(\widetilde{G}_\xi)^{-1}\Delta_{x_1}\mathbb{V}(t_{2,1})\Delta_{x_2}\dots\Delta_{x_{n-1}}\mathbb{V}(t_{n,n-1})\Delta_{x_n}R(\widetilde{G}_\xi) \\ = \mathcal{T}(v_{x_1,\xi}^{(t_{1,1})}v_{-x_1,\xi}^{(t_{1,1})}\dots v_{x_n,\xi}^{(t_{n,1})}v_{-x_n,\xi}^{(t_{n,1})})R(\widetilde{W}_0(\tau_{n,1}))e^{i\tau_{n,1}\frac{\alpha}{4}\mathbf{I}}. \end{aligned}$$

Observe that from (10.3) it follows that  $R(\widetilde{W}_0(\tau_{n,1}))e^{i\tau_{n,1}\frac{\alpha}{4}\mathbf{I}}\text{vac} = \text{vac}$ , so

$$\begin{aligned} \mathcal{F}(t_{2,1}, \dots, t_{n,n-1}; x_1, \dots, x_n) &= (\mathcal{T}(v_{x_1,\xi}^{(t_{1,1})}v_{-x_1,\xi}^{(t_{1,1})}\dots v_{x_n,\xi}^{(t_{n,1})}v_{-x_n,\xi}^{(t_{n,1})})\text{vac}, \text{vac}) \\ &= \mathbf{F}_{\text{vac}}\left(v_{x_1,\xi}^{(t_{1,1})}v_{-x_1,\xi}^{(t_{1,1})}\dots v_{x_n,\xi}^{(t_{n,1})}v_{-x_n,\xi}^{(t_{n,1})}\right). \end{aligned}$$

An application of Wick's Theorem 5.1 concludes the proof.  $\square$

Now that we have established a Pfaffian formula for purely imaginary time variables  $t_{j,j-1}$  ( $j = 2, \dots, n$ ), we want to extend it to the case when all  $t_{j,j-1}$ 's are real nonnegative. Let us look closer at the function  $\mathbf{F}_{\text{vac}}(v_{x,\xi}^{(s)}v_{y,\xi}^{(t)})$ , where  $s = i\sigma$  and  $t = i\tau$  are purely imaginary. We have

$$v_{x,\xi}^{(s)} = \check{S}(W_0(\sigma))v_{x,\xi}, \quad v_{y,\xi}^{(t)} = \check{S}(W_0(\tau))v_{y,\xi},$$

where  $v_{x,\xi}$  and  $v_{y,\xi}$  are defined by (7.13). Therefore, we get

$$\mathbf{F}_{\text{vac}}(v_{x,\xi}^{(s)}v_{y,\xi}^{(t)}) = \sum_{k,l \in \mathbb{Z}} (v_{x,\xi}, v_k)_{\ell^2(\mathbb{Z})} (v_{y,\xi}, v_l)_{\ell^2(\mathbb{Z})} \mathbf{F}_{\text{vac}}\left((\check{S}(W_0(\sigma))v_k)(\check{S}(W_0(\tau))v_l)\right).$$

On the space  $V_{\text{fin}} \subset V = \ell^2(\mathbb{Z})$  consisting of finite linear combinations of the basis vectors  $\{v_x\}_{x \in \mathbb{Z}}$ , the operator  $\check{S}(W_0(u))$  acts as  $e^{-iu\check{S}(H)/2}$  (where  $u \in \mathbb{R}$ ). From this fact and (7.15), we see that

$$\begin{aligned} \mathbf{F}_{\text{vac}}(v_{x,\xi}^{(s)} v_{y,\xi}^{(t)}) &= \sum_{m=0}^{\infty} e^{-m(t-s)} (v_{x,\xi}, v_{-m})_{\ell^2(\mathbb{Z})} (v_{y,\xi}, v_m)_{\ell^2(\mathbb{Z})} \\ &= (-1)^{x \wedge 0 + y \vee 0} \sum_{m=0}^{\infty} 2^{-\delta(m)} e^{-m(t-s)} \varphi_m(x) \varphi_m(-y). \end{aligned} \quad (10.10)$$

Note that here  $s$  and  $t$  are still purely imaginary. However, one can view the right-hand side of (10.10) as a function in  $(t-s) \in \mathbb{C}_+$ . This function is bounded and continuous in  $\mathbb{C}_+$  and is holomorphic in the interior of  $\mathbb{C}_+$ . Indeed, this follows from the fact that the series in (10.10) converges rapidly because the functions  $\varphi_m(x)$  for fixed  $x$  and  $m \rightarrow +\infty$  decay as  $\text{Const} \cdot m^{-x - \frac{1}{2}} \xi^{\frac{m}{2}}$  (this can be observed from the analytic expression (6.4)). We are interested in the restriction of the right-hand side of (10.10) to real nonnegative values of  $(t-s)$ . Observe that this is exactly the kernel  $\Phi_{\alpha,\xi}(s, x; t, y)$  (10.5). By application of Lemma 10.5, we see that formula (10.9) holds for real nonnegative  $t_1, \dots, t_{n-1}$ , that is, for  $0 \leq t_1 \leq \dots \leq t_n$ . This fact together with Proposition 10.2 implies Theorem 10.4.

Thus, we have established Theorem 2 from §2.

## 10.4 Skew-symmetric matrices in Pfaffian formulas

In the right-hand sides of our Pfaffian formulas (7.3) and (10.4) for static and dynamical correlation functions we see certain skew-symmetric  $2n \times 2n$  matrices constructed using Pfaffian kernels  $\Phi_{\alpha,\xi}(x, y)$  and  $\Phi_{\alpha,\xi}(s, x; t, y)$ , respectively. It is clear from (7.16) and (10.5) that for  $t = s$ , the dynamical kernel  $\Phi_{\alpha,\xi}(s, x; t, y)$  turns into the static one. (In fact, this is the reason why we use the same notation for these kernels.) However, the matrix of (10.4) for  $t_1 = \dots = t_n$  does not become the one from (7.3). Let us explain how one can transform (10.4) to get the expected behavior in the static picture.

One can readily verify that conjugating the matrix  $\Phi_{\alpha,\xi} \llbracket T, X \rrbracket$  from (10.4) by the matrix  $C$  of the permutation  $(1, 2n, 2, 2n-1, \dots, n, n+1)$ , we get the following  $2n \times 2n$  skew-symmetric matrix:

$$\begin{aligned} &(C \Phi_{\alpha,\xi} \llbracket T, X \rrbracket C^T)_{i,j} \\ &= \begin{cases} \Phi_{\alpha,\xi}(t_i, x_i; t_j, x_j), & \text{if } 1 \leq i < j \leq n; \\ \Phi_{\alpha,\xi}(t_i, x_i; t_{j'}, -x_{j'}), & \text{if } 1 \leq i \leq n < j \leq 2n \text{ and } i \leq j'; \\ -\Phi_{\alpha,\xi}(t_{j'}, -x_{j'}; t_i, x_i), & \text{if } 1 \leq i \leq n < j \leq 2n \text{ and } i > j'; \\ -\Phi_{\alpha,\xi}(t_{j'}, -x_{j'}; t_{i'}, -x_{i'}), & \text{if } n < i < j \leq 2n, \end{cases} \end{aligned} \quad (10.11)$$

where  $1 \leq i < j \leq 2n$ , and  $i' := 2n+1-i$ ,  $j' := 2n+1-j$ . The permutation matrix  $C$  has determinant one (cf. how (7.6) is obtained), so the Pfaffian does not change under such a conjugation. It is worth noting that matrices similar to (10.11) appeared in [Mat05, Thm. 3.1] and [Vul07, Thm. 2.2]). Using Corollary 7.8, it is clear that for  $t_1 = \dots = t_n$ , (10.11) becomes the matrix  $\hat{\Phi}_{\alpha,\xi} \llbracket X \rrbracket$ . Observe that Corollary 7.8.(2) does not hold in the dynamical case, so one cannot put a matrix of the form  $\hat{\Phi}_{\alpha,\xi} \llbracket T, X \rrbracket$  in the right-hand side of (10.4).

## 10.5 Expression through the extended discrete hypergeometric kernel

The extended discrete hypergeometric kernel introduced in [BO06a] serves as a determinantal kernel for a Markov dynamics preserving the  $z$ -measures on ordinary partitions (§6.1). It is given by

[BO06a, Thm. A (Part 2)]:

$$\underline{K}_{z,z',\xi}(t, x'; s, y') = \pm \sum_{a' \in \mathbb{Z}'_+} e^{-a'|t-s|} \psi_{\pm a'}(x'; z, z', \xi) \psi_{\pm a'}(y'; z, z', \xi),$$

where  $x', y' \in \mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$ , the “+” sign is taken for  $t \geq s$ , and the “−” sign is taken for  $t < s$ . Our kernel  $\Phi_{\alpha,\xi}(s, x; t, y)$  can be expressed through  $\underline{K}_{z,z',\xi}(t, x'; s, y')$ :

**Proposition 10.7.** *For all  $x, y \in \mathbb{Z}$  and  $s \leq t$ , we have*

$$\begin{aligned} \Phi_{\alpha,\xi}(s, x; t, y) &= \frac{1}{2}(-1)^{x \wedge 0 + y \vee 0} \left[ e^{-\frac{1}{2}(t-s)} \underline{K}_{v(\alpha)-\frac{1}{2}, -v(\alpha)-\frac{1}{2}, \xi}(t, x + \frac{1}{2}; s, -y + \frac{1}{2}) \right. \\ &\quad \left. + e^{\frac{1}{2}(t-s)} \underline{K}_{v(\alpha)+\frac{1}{2}, -v(\alpha)+\frac{1}{2}, \xi}(t, x - \frac{1}{2}; s, -y - \frac{1}{2}) \right], \end{aligned} \quad (10.12)$$

*Proof.* This is established similarly to (7.17): using the above formula for the kernel  $\underline{K}_{z,z',\xi}(t, x'; s, y')$  and (6.5), one can write two identities analogous to (7.18) and (7.19), then take their half sum and use (10.5).  $\square$

When  $t = s$ , (10.12) reduces to the static version (7.17). Observe that the parameters  $z, z'$  (depending on  $\alpha$ ) in (10.12) are admissible (see §6.1). However, similarly to the static identity, (10.12) seems to imply no probabilistic connections between the dynamics related to the  $z$ -measures [BO06a] and our Markov processes  $\lambda_{\alpha,\xi}$ . On the other hand, this identity helps to study the asymptotics of our dynamical kernel  $\Phi_{\alpha,\xi}$  in various limit regimes similar to the ones discussed in [BO06a, §9–10]. This analysis is carried out in [Pet10a, §11].

## A Reduction of Pfaffians to determinants

Let us first recall basic definitions and properties related to Pfaffians. We use the following notations for matrices. Let  $\mathfrak{X}$  be an abstract finite space of indices and  $\mathbf{a} = (a_1, \dots, a_{2n})$  be a sequence of length  $2n$  of points of  $\mathfrak{X}$ . Let  $F : \mathfrak{X} \times \mathfrak{X} \rightarrow \mathbb{C}$  be some function. Form a  $2n \times 2n$  skew-symmetric matrix

$$\begin{pmatrix} 0 & F(a_1, a_2) & \dots & F(a_1, a_{2n-1}) & F(a_1, a_{2n}) \\ -F(a_1, a_2) & 0 & \dots & F(a_2, a_{2n-1}) & F(a_2, a_{2n}) \\ \dots & \dots & \dots & \dots & \dots \\ -F(a_1, a_{2n-1}) & -F(a_2, a_{2n-1}) & \dots & 0 & F(a_{2n-1}, a_{2n}) \\ -F(a_1, a_{2n}) & -F(a_2, a_{2n}) & \dots & -F(a_{2n-1}, a_{2n}) & 0 \end{pmatrix}.$$

Denote this matrix by  $F[\mathbf{a}]$ . This skew-symmetric matrix has rows and columns indexed by  $a_1, \dots, a_{2n}$ , such that the  $ij$ th element above the main diagonal is equal to  $F(a_i, a_j)$  (here  $1 \leq i < j \leq 2n$ ).

**Definition A.1.** Let  $\mathbf{a} = (a_1, \dots, a_{2n})$  and  $F[\mathbf{a}]$  be as defined above. The determinant  $\det(F[\mathbf{a}])$  is a perfect square as a polynomial in  $F(a_i, a_j)$  (where  $i < j$ ). The Pfaffian of  $F[\mathbf{a}]$ , denoted by  $\text{Pf}(F[\mathbf{a}])$ , is defined to be the square root of  $\det F[\mathbf{a}]$  having the “+” sign by the monomial  $F(a_1, a_2) \dots F(a_{2n-1}, a_{2n})$ .

The following properties of Pfaffians are well known:

- Let  $A$  be a skew-symmetric  $2n \times 2n$  matrix and  $B$  be any  $2n \times 2n$  matrix, then

$$\text{Pf}(BAB^T) = \det B \cdot \text{Pf}(A). \quad (\text{A.1})$$

where  $B^T$  means the transposed matrix;

- If  $M$  is any  $n \times n$  matrix, then

$$\text{Pf} \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix} = (-1)^{n(n-1)/2} \det M. \quad (\text{A.2})$$

Now we give a sufficient condition under which a  $2n \times 2n$  Pfaffian can be reduced to a certain  $n \times n$  determinant. Assume that the set  $\mathfrak{X}$  is divided into two parts  $\mathfrak{X} = \mathfrak{X}_+ \sqcup \mathfrak{X}_-$ , and there exists a bijection between  $\mathfrak{X}_+$  and  $\mathfrak{X}_-$ . By  $a \mapsto \hat{a}$  we denote the corresponding involution of the space  $\mathfrak{X}$  that interchanges  $\mathfrak{X}_+$  and  $\mathfrak{X}_-$ . Let  $\mathbf{a} := (a_1, \dots, a_n, \hat{a}_n, \dots, \hat{a}_1)$ , and  $a_i \in \mathfrak{X}_+$  (so  $\hat{a}_i \in \mathfrak{X}_-$ ),  $i = 1, \dots, n$ .

**Proposition A.2.** *Suppose that the function  $F$  on  $\mathfrak{X} \times \mathfrak{X}$  satisfies the following properties:<sup>19</sup>*

- (1)  $F(a, \hat{b}) = F(b, \hat{a})$  for any  $a, b \in \mathfrak{X}$ .
- (2)  $F(a, b) = -F(b, a)$  for any  $a, b \in \mathfrak{X}$  such that  $a \neq \hat{b}$ .
- (3) There exists a strictly positive function  $f : \mathfrak{X}_+ \rightarrow \mathbb{R}$  with the property  $f(a) \neq f(b)$  if  $a \neq b$ , such that

$$(f(a) - f(b)) F(a, \hat{b}) = (f(a) + f(b)) F(a, b) \quad \text{for any } a, b \in \mathfrak{X}_+.$$

Then

$$\text{Pf}(F \llbracket a_1, \dots, a_n, \hat{a}_n, \dots, \hat{a}_1 \rrbracket) = \det[K(a_r, a_s)]_{r,s=1}^n,$$

where  $K$  has the form

$$K(u, v) = \frac{2F(u, \hat{v})\sqrt{f(u)f(v)}}{f(u) + f(v)}, \quad u, v \in \mathfrak{X}_+. \quad (\text{A.3})$$

Note that the third property above implies that  $F(a, a) = 0$  for all  $a \in \mathfrak{X}_+$ .

*Proof.* In this proof we denote the matrix  $F \llbracket a_1, \dots, a_n, \hat{a}_n, \dots, \hat{a}_1 \rrbracket$  simply by  $F$ .

We act on  $F$  by  $SL(2, \mathbb{C})^n$ : each  $j$ th copy of  $SL(2, \mathbb{C})$  acts as  $F \mapsto C_j F C_j^T$ , where  $C_j$  is the  $2n \times 2n$  identity matrix except for the  $2 \times 2$  submatrix with determinant 1 formed by rows and columns with numbers  $j$  and  $2n + 1 - j$ . By (A.1), this action of  $SL(2, \mathbb{C})^n$  does not change the Pfaffian of  $F$ . We want to choose  $C \in SL(2, \mathbb{C})^n$  such that the matrix  $CFC^T$  becomes a block matrix as in (A.2).

Define  $g(a) := \frac{1}{2} \log f(a)$ ,  $a \in \mathfrak{X}_+$ . As the  $j$ th element in  $C \in SL(2, \mathbb{C})^n$  we take the hyperbolic rotation  $\begin{pmatrix} \cosh g(a_j) & \sinh g(a_j) \\ \sinh g(a_j) & \cosh g(a_j) \end{pmatrix}$ . The whole matrix  $C$  looks as

$$C = \left( \begin{array}{ccc|ccc} \cosh g(a_1) & \dots & 0 & 0 & \dots & \sinh g(a_1) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \cosh g(a_n) & \sinh g(a_n) & \dots & 0 \\ 0 & \dots & \sinh g(a_n) & \cosh g(a_n) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \sinh g(a_1) & \dots & 0 & 0 & \dots & \cosh g(a_1) \end{array} \right).$$

<sup>19</sup>cf. Corollary 7.8.

It can be readily verified using the properties of  $F$  that  $CFC^T = \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix}$ , where the rows of  $M$  are indexed by  $i = 1, 2, \dots, n$ , and columns are indexed by  $j = n + 1, \dots, 2n$ , and

$$M_{ij} = \begin{cases} \frac{2F(a_i, a_{2n+1-j})\sqrt{f(a_i)f(a_{2n+1-j})}}{f(a_i) - f(a_{2n+1-j})}, & \text{if } i + j \neq 2n, \\ F(a_i, \hat{a}_i), & \text{otherwise.} \end{cases}$$

Set, for  $r, s = 1, \dots, n$ ,

$$K(a_r, a_s) := M_{r, 2n+s-1}, \tag{A.4}$$

and note that  $\det [K(a_r, a_s)]_{r,s=1}^n = (-1)^{n(n-1)/2} \det M$ . Thus, from (A.1) and (A.2) we get  $\text{Pf}(F) = \text{Pf}(CFC^T) = (-1)^{n(n-1)/2} \det M = \det [K(a_r, a_s)]_{r,s=1}^n$ . It remains to observe that  $K(\cdot, \cdot)$  (A.4) that now has the form

$$K(u, v) = \begin{cases} \frac{2F(u, v)\sqrt{f(u)f(v)}}{f(u) - f(v)}, & \text{if } u \neq v, \\ F(u, \hat{u}), & \text{otherwise,} \end{cases}$$

where  $u, v \in \mathfrak{X}_+$ , can be rewritten as (A.3) using the properties of  $F$ . □

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