Self-interacting diffusions IV: Rate of convergence*

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Abstract

Self-interacting diffusions are processes living on a compact Riemannian manifold defined by a stochastic differential equation with a drift term depending on the past empirical measure \( \mu_t \) of the process. The asymptotics of \( \mu_t \) is governed by a deterministic dynamical system and under certain conditions \( (\mu_t) \) converges almost surely towards a deterministic measure \( \mu^* \) (see Benaïm, Ledoux, Raimond (2002) and Benaïm, Raimond (2005)). We are interested here in the rate of convergence of \( \mu_t \) towards \( \mu^* \). A central limit theorem is proved. In particular, this shows that greater is the interaction repelling faster is the convergence.

Key words: Self-interacting random processes, reinforced processes.

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1 Introduction

Self-interacting diffusions

Let $M$ be a smooth compact Riemannian manifold and $V : M \times M \to \mathbb{R}$ a sufficiently smooth mapping\footnote{The mapping $V_x : M \to \mathbb{R}$ defined by $V_x(y) = V(x,y)$ is $C^2$ and its derivatives are continuous in $(x,y)$.}. For all finite Borel measure $\mu$, let $V\mu : M \to \mathbb{R}$ be the smooth function defined by

$V\mu(x) = \int_M V(x,y)\mu(dy)$.

Let $(e_\alpha)$ be a finite family of vector fields on $M$ such that $\sum_\alpha e_\alpha(e_\alpha f)(x) = \Delta f(x)$, where $\Delta$ is the Laplace operator on $M$ and $e_\alpha(f)$ stands for the Lie derivative of $f$ along $e_\alpha$. Let $(B^\alpha)$ be a family of independent Brownian motions.

A self-interacting diffusion on $M$ associated to $V$ can be defined as the solution to the stochastic differential equation (SDE)

$$dX_t = \sum_\alpha e_\alpha(X_t) \circ dB^\alpha_t - \nabla(V\mu_t)(X_t) dt$$

where $\mu_t = \frac{1}{t} \int_0^t \delta_{X_s} ds$ is the empirical occupation measure of $(X_t)$.

In absence of drift (i.e $V = 0$), $(X_t)$ is just a Brownian motion on $M$ but in general it defines a non Markovian process whose behavior at time $t$ depends on its past trajectory through $\mu_t$. This type of process was introduced in Benaim, Ledoux and Raimond (2002) (\cite{3}) and further analyzed in a series of papers by Benaim and Raimond (2003, 2005, 2007) (\cite{4, 5} and \cite{6}). We refer the reader to these papers for more details and especially to \cite{3} for a detailed construction of the process and its elementary properties. For a general overview of processes with reinforcement we refer the reader to the recent survey paper by Pemantle (2007) (\cite{16}).

Notation and Background

We let $\mathcal{M}(M)$ denote the space of finite Borel measures on $M$, $\mathcal{P}(M) \subset \mathcal{M}(M)$ the space of probability measures. If $I$ is a metric space (typically, $I = M$, $\mathbb{R}^+ \times M$ or $[0, T] \times M$) we let $C(I)$ denote the space of real valued continuous functions on $I$ equipped with the topology of uniform convergence on compact sets. The normalized Riemann measure on $M$ will be denoted by $\lambda$.

Let $\mu \in \mathcal{P}(M)$ and $f : M \to \mathbb{R}$ a nonnegative or $\mu$–integrable Borel function. We write $\mu f$ for $\int f \, d\mu$, and $f \mu$ for the measure defined as $f \mu(A) = \int_A f \, d\mu$. We let $L^2(\mu)$ denote the space of functions for which $\mu|f|^2 < \infty$, equipped with the inner product $(f, g)_\mu = \mu(f g)$ and the norm $\|f\|_\mu = \sqrt{\mu|f|^2}$. We simply write $L^2$ for $L^2(\lambda)$.

Of fundamental importance in the analysis of the asymptotics of $(\mu_t)$ is the mapping $\Pi : \mathcal{M}(M) \to \mathcal{P}(M)$ defined by

$$\Pi(\mu) = \xi(V\mu)\lambda$$

\text{(1)}
where $\xi : C(M) \to C(M)$ is the function defined by

$$
\xi(f)(x) = \frac{e^{-f(x)}}{\int_M e^{-f(y)} \lambda(dy)}.
$$

In [3], it is shown that the asymptotics of $\mu_t$ can be precisely related to the long term behavior of a certain semiflow on $\mathcal{P}(M)$ induced by the ordinary differential equation (ODE) on $\mathcal{M}(M)$:

$$
\dot{\mu} = -\mu + \Pi(\mu).
$$

Depending on the nature of $V$, the dynamics of (3) can either be convergent or nonconvergent leading to similar behaviors for $\{\mu_t\}$ (see [3]). When $V$ is symmetric, (3) happens to be a quasigradient and the following convergence result holds.

**Theorem 1.1** ([5]). Assume that $V$ is symmetric, i.e. $V(x,y) = V(y,x)$. Then the limit set of $\{\mu_t\}$ (for the topology of weak* convergence) is almost surely a compact connected subset of

$$
\text{Fix}(\Pi) = \{\mu \in \mathcal{P}(M) : \mu = \Pi(\mu)\}.
$$

In particular, if $\text{Fix}(\Pi)$ is finite then $(\mu_t)$ converges almost surely toward a fixed point of $\Pi$. This holds for a generic function $V$ (see [5]). Sufficient conditions ensuring that $\text{Fix}(\Pi)$ has cardinal one are as follows:

**Theorem 1.2** ([5], [6]). Assume that $V$ is symmetric and that one of the two following conditions hold

(i) Up to an additive constant $V$ is a Mercer kernel: For some constant $C$, $V(x,y) = K(x,y) + C$, and for all $f \in L^2$,

$$
\int K(x,y)f(x)f(y)\lambda(dx)\lambda(dy) \geq 0.
$$

(ii) For all $x \in M, y \in M, u \in T_xM, v \in T_yM$

$$
\text{Ric}_x(u,u) + \text{Ric}_y(v,v) + \text{Hess}_{x,y} V((u,v),(u,v)) \geq K(\|u\|^2 + \|v\|^2)
$$

where $K$ is some positive constant. Here $\text{Ric}_x$ stands for the Ricci tensor at $x$ and $\text{Hess}_{x,y}$ is the Hessian of $V$ at $(x,y)$.

Then $\text{Fix}(\Pi)$ reduces to a singleton $\{\mu^*\}$ and $\mu_t \to \mu^*$ with probability one.

As observed in [6] the condition (i) in Theorem 1.2 seems well suited to describe self-repelling diffusions. On the other hand, it is not clearly related to the geometry of $M$. Condition (ii) has a more geometrical flavor and is robust to smooth perturbations (of $M$ and $V$). It can be seen as a Bakry-Emery type condition for self interacting diffusions.

In [5], it is also proved that every stable (for the ODE (3)) fixed point of $\Pi$ has a positive probability to be a limit point for $\mu_t$; and any unstable fixed point cannot be a limit point for $\mu_t$.
Organisation of the paper

Let $\mu^* \in \text{Fix}(\Pi)$. We will assume that

**Hypothesis 1.3.** $\mu_t$ converges a.s. towards $\mu^*$.

In this paper we intend to study the rate of this convergence. Let

$$\Delta_t = e^{t/2}(\mu_t - \mu^*).$$

It will be shown that, under some conditions to be specified later, for all $g = (g_1, \ldots, g_n) \in C(M)^n$ the process $[\Delta_s g_1, \ldots, \Delta_s g_n, V\Delta_s]_{s \geq t}$ converges in law, as $t \to \infty$, toward a certain stationary Ornstein-Uhlenbeck process $(Z^g, Z)$ on $\mathbb{R}^n \times C(M)$. This process is defined in Section 2. The main result is stated in section 3 and some examples are developed. It is in particular observed that a strong repelling interaction gives a faster convergence. The section 4 is a proof section.

In the following $K$ (respectively $C$) denotes a positive constant (respectively a positive random constant). These constants may change from line to line.

## 2 The Ornstein-Uhlenbeck process $(Z^g, Z)$.

For a more precise definition of Ornstein-Uhlenbeck processes on $C(M)$ and their basic properties, we refer the reader to the appendix (section 5). Throughout all this section we let $\mu \in \mathcal{P}(M)$ and $g = (g_1, \ldots, g_n) \in C(M)^n$. For $x \in M$ we set $V_x : M \to \mathbb{R}$ defined by

$$V_x(y) = V(x, y).$$

### 2.1 The operator $G_\mu$

Let $h \in C(M)$ and let $G_{\mu, h} : \mathbb{R} \times C(M) \to \mathbb{R}$ be the linear operator defined by

$$G_{\mu, h}(u, f) = u/2 + \text{Cov}_\mu(h, f),$$

where $\text{Cov}_\mu$ is the covariance on $L^2(\mu)$, that is the bilinear form acting on $L^2 \times L^2$ defined by

$$\text{Cov}_\mu(f, h) = \mu(fh) - \mu(f)\mu(h).$$

We define the linear operator $G_\mu : C(M) \to C(M)$ by

$$G_\mu f(x) = G_{\mu, V_x}(f(x), f) = f(x)/2 + \text{Cov}_\mu(V_x, f).$$

It is easily seen that $\|G_\mu f\|_\infty \leq (2\|V\|_\infty + 1/2)\|f\|_\infty$. In particular, $G_\mu$ is a bounded operator. Let $\{e^{-tG_\mu}\}$ denote the semigroup acting on $C(M)$ with generator $-G_\mu$. From now on we will assume the following:

**Hypothesis 2.1.** There exists $\kappa > 0$ such that $\mu << \lambda$ with $d\mu/d\lambda \|_\infty < \infty$, and such that for all $f \in L^2(\lambda)$, $\langle G_\mu f, f \rangle_\lambda \geq \kappa\|f\|_\lambda^2$.

Let

$$\lambda(-G_\mu) = \lim_{t \to \infty} \frac{\log(\|e^{-tG_\mu}\|_\lambda)}{t}.$$ 

This limit exists by subadditivity. Then
Lemma 2.2. Hypothesis 2.1 implies that $\lambda(-G_\mu) \leq -\kappa < 0$.

Proof: For all $f \in L^2(\lambda)$,  
\[
\frac{d}{dt} \|e^{-tG_\mu}f\|^2_\lambda = -2\langle G_\mu e^{-tG_\mu}f, e^{-tG_\mu}f \rangle_\lambda \leq -2\kappa \|e^{-tG_\mu}f\|_\lambda.
\]
This implies that $\|e^{-tG_\mu}f\|_\lambda \leq e^{-\kappa t} \|f\|_\lambda$. Denote by $g_t$ the solution of the differential equation  
\[
\frac{d}{dt} g_t(x) = \text{Cov}_\mu(V_{\mu}, g_t)
\]
with $g_0 = f \in C(M)$. Note that $e^{-tG_\mu}f = e^{-t/2}g_t$. It is straightforward to check that (using the fact that $\|\frac{d\mu}{d\lambda}\|_\infty < \infty$) $\frac{d}{dt} \|g_t\|_\lambda \leq K \|g_t\|_\lambda$ with $K$ a constant depending only on $V$ and $\mu$. Thus $\sup_{t \in [0,1]} \|g_t\|_\lambda \leq K \|f\|_\lambda$. Now, since for all $x \in M$ and $t \in [0,1]$  
\[
\left| \frac{d}{dt} g_t(x) \right| \leq K \|g_t\|_\lambda \leq K \|f\|_\lambda,
\]
we have $\|g_1\|_\infty \leq K \|f\|_\lambda$. This implies that $\|e^{-G_\mu}f\|_\infty \leq K \|f\|_\lambda$.

Now for all $t > 1$, and $f \in C(M)$,  
\[
\|e^{-tG_\mu}f\|_\infty = \|e^{-tG_\mu}e^{-(t-1)G_\mu}f\|_\infty \leq K \|e^{-(t-1)G_\mu}f\|_\lambda \leq K e^{-\kappa(t-1)} \|f\|_\lambda \leq K e^{-\kappa t} \|f\|_\infty.
\]
This implies that $\|e^{-tG_\mu}\| \leq Ke^{-\kappa t}$, which proves the lemma. QED

The adjoint of $G_\mu$ is the operator on $\mathcal{M}(M)$ defined by the relation  
\[
m(G_\mu f) = (G_\mu^* m)f
\]
for all $m \in \mathcal{M}(M)$ and $f \in C(M)$. It is not hard to verify that  
\[
G_\mu^* m = \frac{1}{2} m + (Vm)\mu - (\mu(Vm))\mu.
\]

2.2 The generator $A_\mu$ and its inverse $Q_\mu$

Let $H^2$ be the Sobolev space of real valued functions on $M$, associated with the norm $\|f\|_H^2 = \|f\|_\lambda^2 + \|\nabla f\|_\lambda^2$. Since $\Pi(\mu)$ and $\lambda$ are equivalent measures with continuous Radon-Nykodim derivative, $L^2(\Pi(\mu)) = L^2(\lambda)$. We denote by $K_\mu$ the projection operator, acting on $L^2(\Pi(\mu))$, defined by  
\[
K_\mu f = f - \Pi(\mu)f.
\]
We denote by $A_\mu$ the operator acting on $H^2$ defined by  
\[
A_\mu f = \frac{1}{2} \Delta f - \langle \nabla V\mu, \nabla f \rangle.
\]
Note that for \( f \) and \( h \) in \( H^2 \) (denoting \( \langle \cdot, \cdot \rangle \) the Riemannian inner product on \( M \))

\[
\langle A_\mu f, h \rangle_{\Pi(\mu)} = -\frac{1}{2} \int \langle \nabla f, \nabla h \rangle(x) \Pi(\mu)(dx).
\]

For all \( f \in C(M) \) there exists \( Q_\mu f \in H^2 \) such that \( \Pi(\mu)(Q_\mu f) = 0 \) and

\[
f - \Pi(\mu)f = K_\mu f = -A_\mu Q_\mu f.
\]

It is shown in [3] that \( Q_\mu f \) is \( C^1 \) and that there exists a constant \( K \) such that for all \( f \in C(M) \) and \( \mu \in \mathcal{P}(M) \),

\[
\|Q_\mu f\|_\infty + \|\nabla Q_\mu f\|_\infty \leq K\|f\|_\infty.
\]

Finally, note that for \( f \) and \( h \) in \( L^2 \),

\[
\int \langle \nabla Q_\mu f, \nabla Q_\mu h \rangle(x) \Pi(\mu)(dx) = -2\langle A_\mu Q_\mu f, Q_\mu h \rangle_{\Pi(\mu)} = 2\langle f, Q_\mu h \rangle_{\Pi(\mu)}.
\]

### 2.3 The covariance \( C^g_\mu \)

We let \( \tilde{C}_\mu \) denote the bilinear continuous form \( \tilde{C}_\mu : C(M) \times C(M) \to \mathbb{R} \) defined by

\[
\tilde{C}_\mu(f, h) = 2\langle f, Q_\mu h \rangle_{\Pi(\mu)}.
\]

This form is symmetric (see its expression given by (9)). Note also that for some constant \( K \) depending on \( \mu \), \( |\tilde{C}_\mu(f, h)| \leq K\|f\|_\infty \times \|h\|_\infty \).

We let \( C_\mu \) denote the mapping \( C_\mu : M \times M \to \mathbb{R} \) defined by \( C_\mu(x, y) = \tilde{C}_\mu(V_x, V_y) \). Let \( \tilde{M} = \{1, \ldots, n\} \cup M \) and \( C^g_\mu : \tilde{M} \times \tilde{M} \to \mathbb{R} \) be the function defined by

\[
C^g_\mu(x, y) = \begin{cases} 
\tilde{C}_\mu(g_x, g_y) & \text{for } x, y \in \{1, \ldots, n\}, \\
C_\mu(x, y) & \text{for } x, y \in M, \\
\tilde{C}_\mu(V_x, g_y) & \text{for } x \in M, y \in \{1, \ldots, n\}.
\end{cases}
\]

Then \( C_\mu \) and \( C^g_\mu \) are covariance functions (as defined in subsection 5.2).

In the following, when \( n = 0 \), \( \tilde{M} = M \) and \( C^g_\mu = C_\mu \). When \( n \geq 1 \), \( C(\tilde{M}) \) can be identified with \( \mathbb{R}^n \times C(M) \).

**Lemma 2.3.** There exists a Brownian motion on \( \mathbb{R}^n \times C(M) \) with covariance \( C^g_\mu \).

**Proof:** Since the argument are the same for \( n \geq 1 \), we just do it for \( n = 0 \). Let

\[
d_{C_\mu}(x, y) := \sqrt{C_\mu(x, x) - 2C_\mu(x, y) + C_\mu(y, y)} = \|\nabla Q_\mu(V_x - V_y)\|_{\Pi(\mu)} \leq K\|V_x - V_y\|_\infty
\]

where the last inequality follows from (9). Then \( d_{C_\mu}(x, y) \leq Kd(x, y) \). Thus \( d_{C_\mu} \) satisfies (30) and we can apply Theorem 5.4 of the appendix (section 5). **QED**
2.4 The process \((Z^g, Z)\)

Let \(G^g_\mu : \mathbb{R}^n \times C(M) \to \mathbb{R}^n \times C(M)\) be the operator defined by

\[
G^g_\mu = \begin{pmatrix}
I_n/2 & A^g_\mu \\
0 & G_\mu
\end{pmatrix}
\]

where \(I_n\) is the identity matrix on \(\mathbb{R}^n\) and \(A^g_\mu : C(M) \to \mathbb{R}^n\) is the linear map defined by \(A^g_\mu(f) = \left(\text{Cov}_\mu(f, g_1), \ldots, \text{Cov}_\mu(f, g_n)\right)\).

Since \(G^g_\mu\) is a bounded operator, for any law \(\nu\) on \(\mathbb{R}^n \times C(M)\), there exists \(\tilde{Z} = (Z^g, Z)\) an Ornstein-Uhlenbeck process of covariance \(C^g_\mu\) and drift \(-G^g_\mu\), with initial distribution given by \(\nu\) (using Theorem 5.6). More precisely, \(\tilde{Z}\) is the unique solution of

\[
\begin{align*}
\frac{dZ_t}{dt} &= W_t - G_\mu Z_t dt \\
\frac{dZ^g_t}{dt} &= W^g_t - \left(\frac{Z^g_t}{2} + \text{Cov}_\mu(Z_t, g_i)\right) dt, \quad i = 1, \ldots, n
\end{align*}
\]

where \(\tilde{Z}_0\) is a \(\mathbb{R}^n \times C(M)\)-valued random variable of law \(\nu\) and \(\tilde{W} = (W^g, W)\) is a \(\mathbb{R}^n \times C(M)\)-valued Brownian motion of covariance \(C^g_\mu\) independent of \(\tilde{Z}\). In particular, \(Z\) is an Ornstein-Uhlenbeck process of covariance \(C_\mu\) and drift \(-G_\mu\). Denote by \(P^g_t\) the semigroup associated to \(\tilde{Z}\).

**Proposition 2.4.** Assume hypothesis [2.1] Then there exists \(\pi^{g, \mu}\) the law of a centered Gaussian variable in \(\mathbb{R}^n \times C(M)\), with variance \(\text{Var}(\pi^{g, \mu})\) where for \(u, m \in \mathbb{R}^n \times M(M)\),

\[
\text{Var}(\pi^{g, \mu})(u, m) := \mathbb{E}\left((mZ_\infty + \langle u, Z^g_\infty \rangle)^2\right) = \int_0^\infty \tilde{C}_\mu(f_t, f_t) dt
\]

with \(f_t = e^{-t/2} \sum_i u_i g_i + V m_t\), and where \(m_t\) is defined by

\[
m_t f = m_0(e^{-G_\mu t}f) + \sum_{i=1}^n u_i \int_0^t e^{-(t-s)G_\mu} \text{Cov}_\mu(g_i, e^{-(t-s)G_\mu} f) ds.
\]

Moreover,

(i) \(\pi^{g, \mu}\) is the unique invariant probability measure of \(P_t\).

(ii) For all bounded continuous function \(\varphi\) on \(\mathbb{R}^n \times C(M)\) and all \((u, f) \in \mathbb{R}^n \times C(M)\),

\[
\lim_{t \to \infty} P^g_t \varphi(u, f) = \pi^{g, \mu} \varphi.
\]

**Proof:** This is a consequence of Theorem [5.7]. To apply it one can remark that \(G^g_\mu\) is an operator like the ones given in example [5.11].

The variance \(\text{Var}(\pi^{g, \mu})\) is given by \(\text{Var}(\pi^{g, \mu})(v) = \int_0^\infty \langle v, e^{-sG_\mu} e^{s(G_\mu)^*} v \rangle ds\) for \(v = (u, m) \in \mathbb{R}^n \times M(M) = C(M)^*\). Thus \(\text{Var}(\pi^{g, \mu})(u, m) = \int_0^\infty \tilde{C}_\mu(f_t, f_t) dt\) with \(f_t = \sum_i u_i(i) g_i + V m_t\) and where \((u_t, m_t) = e^{-t(G_\mu)^*}(u, m)\). Now

\[
(G_\mu)^* = \begin{pmatrix}
I_n/2 & 0 \\
A_\mu^* & (G_\mu)^*
\end{pmatrix}
\]
\[ (A_{i\mu}^g)^* u = \sum_i u_i (g_i - \mu g_i) \mu. \] 

Thus \( u_t = e^{-t/2} u \) and \( m_t \) is the solution with \( m_0 = m \) of

\[
\frac{dm_t}{dt} = -e^{-t/2} \left( \sum_i u_i (g_i - \mu g_i) \right) \mu - (G_{i\mu}^g)^* m_t. \tag{13}
\]

Note that (13) is equivalent to

\[
\frac{d}{dt}(m_t f) = -e^{-t/2} \text{Cov}_{\mu} \left( \sum_i u_i g_i, f \right) - m_t (G_{i\mu}^g)
\]

for all \( f \in C(M) \), and \( m_0 = m \). From which we deduce that

\[
m_t = e^{-t G_{i\mu}^*} m_0 - \int_0^t e^{-s/2} e^{-(t-s) G_{i\mu}^*} \left( \sum_i u_i (g_i - \mu g_i) \right) \mu \ ds
\]

which implies the formula for \( m_t \) given by (12). QED

An Ornstein-Uhlenbeck process of covariance \( C_{i\mu}^g \) and drift \( -G_{i\mu}^g \) will be called stationary when its initial distribution is \( \pi_{g, \mu} \).

3 A central limit theorem for \( \mu_t \)

We state here the main results of this article. We assume \( \mu^* \in \text{Fix}(\Pi) \) satisfies hypotheses 1.3 and 2.1. Set \( \Delta_t = e^{t/2}(\mu_t - \mu^*) \), \( D_t = V \Delta_t \) and \( D_{t+} = (D_{t+})_{t \geq 0} \). Then

**Theorem 3.1.** \( D_{t+} \) converges in law, as \( t \to \infty \), towards a stationary Ornstein-Uhlenbeck process of covariance \( C_{i\mu^*} \) and drift \( -G_{i\mu^*} \).

For \( g \in C(M)^n \), we set \( D_t^g = (\Delta_t g, D_t) \) and \( D_{t+}^g = (D_{t+}^g)_{t \geq 0} \). Then

**Theorem 3.2.** \( D_{t+}^g \) converges in law towards a stationary Ornstein-Uhlenbeck process of covariance \( C_{i\mu^*}^g \) and drift \( -G_{i\mu^*}^g \).

Define \( \tilde{C} : C(M) \times C(M) \to \mathbb{R} \) the symmetric bilinear form defined by

\[
\tilde{C}(f, h) = \int_0^\infty \tilde{C}_{i\mu^*}(f_t, h_t) dt,
\]

with \( (h_t) \) is defined by the same formula, with \( h \) in place of \( f \)

\[
f_t(x) = e^{-t/2} f(x) - \int_0^t e^{-s/2} \text{Cov}_{\mu^*}(f, e^{-(t-s) G_{i\mu^*}} V_x) ds.
\]

**Corollary 3.3.** \( \Delta_t g \) converges in law towards a centered Gaussian variable \( Z_{\infty}^g \) of covariance

\[
E[Z_{\infty}^{g_i} Z_{\infty}^{g_j}] = \tilde{C}(g_i, g_j).
\]

**Proof:** Follows from theorem 3.2 and the calculus of \( \text{Var}(\pi_{g, \mu}) (u_i, 0) \). QED
3.1 Examples

3.1.1 Diffusions

Suppose \( V(x, y) = V(x) \), so that \((X_t)\) is just a standard diffusion on \( M \) with invariant measure 
\[
\mu^* = \frac{\exp(-V)}{\lambda \exp(-V)}.
\]
Let \( f \in C(M) \). Since \( e^{-tG_{\mu^*}}1 = e^{-t/2}1 \), \( f_t \) defined by (15) is equal to \( e^{-t/2}f \). Thus 
\[
\tilde{C}(f, g) = 2\mu^*(f Q_{\mu^*} g).
\]
(16)

Corollary 3.3 says that

Theorem 3.4. For all \( g \in C(M)^n \), \( \Delta^g_t \) converges in law toward a centered Gaussian variable 
\((Z_{\infty}^g, \ldots, Z_{\infty}^g)) \), with covariance given by 
\[
E(Z_{\infty}^g Z_{\infty}^g) = 2\mu^*(g Q_{\mu^*} g).
\]

Remark 3.5. This central limit theorem for Brownian motions on compact manifolds has already been 
considered by Baxter and Brosamler in [1] and [2]; and by Bhattacharya in [7] for ergodic diffusions.

3.1.2 The case \( \mu^* = \lambda \) and \( V \) symmetric.

Suppose here that \( \mu^* = \lambda \) and that \( V \) is symmetric. We assume (without loss of generality since 
\( \Pi(\lambda) = \lambda \) implies that \( V \lambda \) is a constant function) that \( V \lambda = 0 \).

Since \( V \) is compact and symmetric, there exists an orthonormal basis \((e_\alpha)_{\alpha \geq 0}\) in \( L^2(\lambda) \) and a se-
quenue of reals \((\lambda_\alpha)_{\alpha \geq 0}\) such that \( e_0 \) is a constant function and 
\[
V = \sum_{\alpha \geq 1} \lambda_\alpha e_\alpha \otimes e_\alpha.
\]

Assume that for all \( \alpha \), \( 1/2 + \lambda_\alpha > 0 \). Then hypothesis [2.1] is satisfied, and the convergence of \( \mu_t \) 
towards \( \lambda \) holds with positive probability (see [3]).

Let \( f \in C(M) \) and \( f_t \) defined by (15), denoting \( f^\alpha = \langle f, e_\alpha \rangle_\lambda \) and \( f^\alpha_t = \langle f_t, e_\alpha \rangle_\lambda \), we have \( f^0_t = e^{-t/2}f^0 \) and for \( \alpha \geq 1 \), 
\[
f^\alpha_t = e^{-t/2}f^\alpha - \lambda_\alpha e^{-(1/2 + \lambda_\alpha)t} \left( \frac{e^{\lambda_\alpha t} - 1}{\lambda_\alpha} \right) f^\alpha
= e^{-(1/2 + \lambda_\alpha)t} f^\alpha.
\]

Using the fact that \( \tilde{C}_\lambda(f, g) = 2\lambda(f Q_\lambda g) \), this implies that 
\[
\tilde{C}(f, g) = 2 \sum_{\alpha \geq 1} \sum_{\beta \geq 1} \frac{1}{1 + \lambda_\alpha + \lambda_\beta} \langle f, e_\alpha \rangle_\lambda \langle g, e_\beta \rangle_\lambda \lambda(e_\alpha Q_\lambda e_\beta).
\]

This, with corollary 3.3 proves
Theorem 3.6. Assume hypothesis 1.3 and that \( 1/2 + \lambda_\alpha > 0 \) for all \( \alpha \). Then for all \( g \in C(M)^n \), \( \Delta^\delta_t \) converges in law toward a centered Gaussian variable \((Z^\delta_{\infty} \ldots , Z^\delta_{\infty})\), with covariance given by \( \text{E}(Z^\delta_{\infty} Z^\delta_{\infty}) = \bar{C}(g_i,g_j) \).

In particular,

\[
\text{E}(Z^\delta_{\infty} Z^\delta_{\infty}) = \frac{2}{1 + \lambda_\alpha + \lambda_\beta} \lambda(e^{\alpha}Q^e\lambda e^{\beta}).
\]

When all \( \lambda_\alpha \) are positive, which corresponds to what is named a self-repelling interaction in [6], the rate of convergence of \( \mu_t \) towards \( \lambda \) is bigger than when there is no interaction, and the bigger is the interaction (that is larger \( \lambda_\alpha \)'s) faster is the convergence.

4 Proof of the main results

We assume hypothesis 1.3 and \( \mu^* \) satisfies hypothesis 2.1. For convenience, we choose for the constant \( \kappa \) in hypothesis 2.1 a constant less than \( 1/2 \). In all this section, we fix \( g = (g_1, \ldots , g_n) \in C(M)^n \).

4.1 A lemma satisfied by \( Q_{\mu} \)

We denote by \( X(M) \) the space of continuous vector fields on \( M \), and equip the spaces \( \mathcal{P}(M) \) and \( X(M) \) respectively with the weak convergence topology and with the uniform convergence topology.

Lemma 4.1. For all \( f \in C(M) \), the mapping \( \mu \mapsto \nabla Q_{\mu} f \) is a continuous mapping from \( \mathcal{P}(M) \) in \( X(M) \).

Proof : Let \( \mu \) and \( \nu \) be in \( \mathcal{M}(M) \), and \( f \in C(M) \). Set \( h = Q_{\mu} f \). Then \( f = -A_{\mu} h + \Pi(\mu) f \) and

\[
\| \nabla Q_{\mu} f - \nabla Q_{\nu} f \|_\infty = \| - \nabla Q_{\mu} A_{\mu} h + \nabla Q_{\nu} A_{\mu} h \|_\infty \\
\leq \| \nabla h + \nabla Q_{\nu} A_{\mu} h \|_\infty \leq \| (h + Q_{\nu} A_{\mu} h) \|_\infty + \| \nabla Q_{\nu} (A_{\mu} - A_{\nu}) h \|_\infty.
\]

Since \( \nabla (h + Q_{\nu} A_{\mu} h) = 0 \) and \( (A_{\mu} - A_{\nu}) h = (\nabla V_{\mu - \nu} , \nabla h) \), we get

\[
\| \nabla Q_{\mu} f - \nabla Q_{\nu} f \|_\infty \leq K \| (\nabla V_{\mu - \nu} , \nabla h) \|_\infty. \tag{17}
\]

Using the fact that \( (x, y) \mapsto \nabla V_x(y) \) is uniformly continuous, the right hand term of (17) converges towards 0, when \( d(\mu, \nu) \) converges towards 0, \( d \) being a distance compatible with the weak convergence. QED

4.2 The process \( \Delta \)

Set \( h_t = V_{\mu_t} \) and \( h^* = V_{\mu^*} \). Recall \( \Delta_t = e^{t/2}(\mu^*_t - \mu^*) \) and \( D_t(x) = V_{\Delta_t} (x) = \Delta_t V_x \). Then \( (D_t) \) is a continuous process taking its values in \( C(M) \) and \( D_t = e^{t/2}(h^*_t - h^*) \).
To simplify the notation, we set $K_s = K_{\mu_s}$, $Q_s = Q_{\mu_s}$ and $A_s = A_{\mu_s}$. Let $(M^f_t)_{t \geq 1}$ be the martingale defined by $M^f_t = \sum_{\alpha} \int_1^t e_a(Q_s f)(X_s) dB^\alpha_t$. The quadratic covariation of $M^f$ and $M^h$ (with $f$ and $h$ in $C(M)$) is given by

$$\langle M^f, M^h \rangle_t = \int_1^t \langle \nabla Q_s f, \nabla Q_s h \rangle(X_s) ds.$$  

Then for all $t \geq 1$ (with $Q_t = \frac{d}{dt} Q_t$),

$$Q_t f(X_t) - Q_1 f(X_1) = M^f_t + \int_1^t \dot{Q}_s f(X_s) ds - \int_1^t K_s f(X_s) ds.$$  

Thus

$$\mu_t f = \frac{1}{t} \int_1^t K_s f(X_s) ds + \frac{1}{t} \int_1^t \Pi(\mu_s) f ds + \frac{1}{t} \int_0^1 f(X_s) ds$$

$$= -\frac{1}{t} \left( Q_t f(X_t) - Q_1 f(X_1) - \int_1^t \dot{Q}_s f(X_s) ds \right)$$

$$+ \frac{M^f_t}{t} + \frac{1}{t} \int_1^t (\xi(h_s), f)_\lambda ds + \frac{1}{t} \int_0^1 f(X_s) ds.$$  

For $f \in C(M)$ (using the fact that $\mu^* f = \langle \xi(h^*), f \rangle_\lambda$), $\Delta_t f = \sum_{i=1}^5 \Delta_i^f$ with

$$\Delta_1^f = e^{-t/2} \left( -Q_{e^{-t}} f(X_{e^{-t}}) + Q_1 f(X_1) + \int_1^{e^{-t}} \dot{Q}_s f(X_s) ds \right)$$

$$\Delta_2^f = e^{-t/2} M^f_{e^{-t}}$$

$$\Delta_3^f = e^{-t/2} \int_1^{e^{-t}} (\xi(h_s) - \xi(h^*) - D\xi(h^*)(h_s - h^*), f)_\lambda ds$$

$$\Delta_4^f = e^{-t/2} \int_1^{e^{-t}} \langle D\xi(h^*)(h_s - h^*), f \rangle_\lambda ds$$

$$\Delta_5^f = e^{-t/2} \left( \int_0^1 f(X_s) ds - \mu^* f \right).$$  

Then $D_t = \sum_{i=1}^5 D^i_t$, where $D^i_t = V \Delta^i_t$. Finally, note that

$$\langle D\xi(h^*)(h - h^*), f \rangle_\lambda = -\text{Cov}_{\mu^*} (h - h^*, f).$$  

(18)

### 4.3 First estimates

We recall the following estimate from [3]: There exists a constant $K$ such that for all $f \in C(M)$ and $t > 0$,

$$\|Q_t f\|_\infty \leq \frac{K}{t} \|f\|_\infty.$$  

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This estimate, combined with (8), implies that for \( f \) and \( h \) in \( C(M) \),
\[
\langle M^f - M^h \rangle_t \leq K\|f - h\|_\infty \times t
\]
and that

**Lemma 4.2.** There exists a constant \( K \) depending on \( \|V\|_\infty \) such that for all \( t \geq 1 \), and all \( f \in C(M) \)
\[
\|\Delta_1^t f\|_\infty + \|\Delta_5^t f\|_\infty \leq K (1 + t) e^{-t/2}\|f\|_\infty,
\]
which implies that \((\Delta^1 + \Delta^5)\) both converge towards 0 (respectively in \( \mathcal{M}(M) \) and in \( C(\mathbb{R}^+ \times M) \)).

We also have

**Lemma 4.3.** There exists a constant \( K \) such that for all \( t \geq 0 \) and all \( f \in C(M) \),
\[
E[(\Delta_2^t f)^2] \leq K \|f\|_\infty^2,
\]
\[
|\Delta_3^t f| \leq K \|f\|_\lambda \times e^{-t/2} \int_0^t \|D_s\|_\lambda^2 ds,
\]
\[
|\Delta_4^t f| \leq K \|f\|_\lambda \times e^{-t/2} \int_0^t e^{s/2} \|D_s\|_\lambda^2 ds.
\]

**Proof:** The first estimate follows from
\[
E[(\Delta_2^t f)^2] = e^{-t} E[(M^f)^2] = e^{-t} E[\langle M^f \rangle_{e^t}] \leq K \|f\|_\infty^2.
\]
The second estimate follows from the fact that
\[
\|\xi(h) - \xi(h^*) - D\xi(h^*)(h - h^*)\|_\lambda = O(\|h - h^*\|_\lambda^2).
\]
The last estimate follows easily after having remarked that
\[
|\langle D\xi(h^*)(h - h^*), f \rangle| = |\text{Cov}_{\mu^t}(h - h^*, f)| \leq K \|f\|_\lambda \times \|h - h^*\|_\lambda.
\]
This proves this lemma. **QED**

### 4.4 The processes \( \Delta' \) and \( D' \)

Set \( \Delta' = \Delta^2 + \Delta^3 + \Delta^4 \) and \( D' = D^2 + D^3 + D^4 \). For \( f \in C(M) \), set
\[
e^t \xi_{e^t} = e^{t/2} \langle \xi(h_{e^t}) - \xi(h^*) - D\xi(h^*)(h_{e^t} - h^*), f \rangle_\lambda.
\]
Then
\[
d\Delta'_f = -\frac{\Delta'_f}{2} dt + dN^f_t + e^t dtd + \langle D\xi(h^*)(D_t), f \rangle_\lambda dt
\]
where for all \( f \in C(M) \), \( N^f_t \) is a martingale. Moreover, for \( f \) and \( h \) in \( C(M) \),
\[
\langle N^f, N^h \rangle_t = \int_0^t \langle \nabla Q_{e^t} f(X_{e^t}), \nabla Q_{e^t} h(X_{e^t}) \rangle ds.
\]
Then, for all \( x \),
\[
dD'(x) = -\frac{D'(x)}{2} dt + dM_t(x) + \epsilon_t(x) dt + (D\xi(h^t)(D_t), V_x)_\lambda dt
\]
where \( M \) is the martingale in \( C(M) \) defined by \( M(x) = N^{V_x} \) and \( \epsilon_t(x) = \epsilon_{V_x}^t \). We also have
\[
G_{\mu^*}(D')_t(x) = \frac{D'(x)}{2} - (D\xi(h^t)(D'_t), V_x)_\lambda.
\]
Denoting \( L_{\mu^*} = L_{-G_{\mu^*}} \) (defined by equation (32) in the appendix (section 5)),
\[
dL_{\mu^*}(D')_t(x) = dD'(x) + G_{\mu^*}(D')_t(x) dt
\]
and we have
\[
L_{\mu^*}(D')_t(x) = M_t(x) + \int_0^t \epsilon'_s(x) ds
\]
with \( \epsilon'_s(x) = \epsilon'_s V_x \) where for all \( f \in C(M) \),
\[
\epsilon'_t f = \epsilon'_s f + (D\xi(h^t)((D^1 + D^5)_t, f)_\lambda.
\]
Using lemma 5.5,
\[
D'_t = L_{\mu^*}^{-1}(M_t) + \int_0^t e^{-(t-s)G_{\mu^*}} \epsilon'_s ds.
\]
Denote \( \Delta_t g = (\Delta_t g_1, \ldots, \Delta_t g_n) \), \( \Delta'_t g = (\Delta'_t g_1, \ldots, \Delta'_t g_n) \), \( N^g_t = (N^{g_1}_t, \ldots, N^{g_n}_t) \) and \( \epsilon'_t g = (\epsilon'_t g_1, \ldots, \epsilon'_t g_n) \). Then, denoting \( L^g_{\mu^*} = L_{-G^g_{\mu^*}} \) (with \( G^g_{\mu^*} \) defined by (10)) we have
\[
L^g_{\mu^*}(\Delta'_t g, D'_t)_t = (N^g_t, M_t) + \int_0^t (\epsilon'_s g, \epsilon'_s)_ds
\]
so that (using lemma 5.5 and integrating by parts)
\[
(\Delta'_t g, D'_t)_t = (L^g_{\mu^*})^{-1}(N^g_t, M_t) + \int_0^t e^{-(t-s)G^g_{\mu^*}} (\epsilon'_s g, \epsilon'_s)_ds.
\]
Moreover
\[
(L^g_{\mu^*})^{-1}(N^g_t, M_t) = \left( \tilde{N}^{g_1}_t, \ldots, \tilde{N}^{g_n}_t, L_{\mu^*}^{-1}(M_t) \right),
\]
where
\[
\tilde{N}^{g_i}_t = N^{g_i}_t - \int_0^t \left( \frac{N^{g_i}_s}{2} + \tilde{G}_{\mu^*}(L_{\mu^*}^{-1}(M_t), g_i)_\lambda \right) ds.
\]
4.5 Estimation of $e'_t$

4.5.1 Estimation of $\|L^{-1}_\mu(M)_t\|_\lambda$

Lemma 4.4. (i) For all $\alpha \geq 2$, there exists a constant $K_\alpha$ such that for all $t \geq 0$,

$$E[\|L^{-1}_\mu(M)_t\|_\lambda^2]^{1/\alpha} \leq K_\alpha.$$ 

(ii) a.s. there exists $C$ with $E[C] < \infty$ such that for all $t \geq 0$,

$$\|L^{-1}_\mu(M)_t\|_\lambda \leq C(1 + t).$$

Proof: We have

$$dL^{-1}_\mu(M)_t = dM_t - G_\mu, L^{-1}_\mu(M)_t, dt.$$ 

Let $N$ be the martingale defined by

$$N_t = \int_t^0 \left( \frac{L^{-1}_\mu(M)_s}{\|L^{-1}_\mu(M)_s\|_\lambda} \right) dM_s.$$

We have $\langle N \rangle_t \leq K t$ for some constant $K$. Then

$$d\|L^{-1}_\mu(M)_t\|_\lambda^2 = 2\|L^{-1}_\mu(M)_t\|_\lambda dN_t - 2\langle L^{-1}_\mu(M)_t, G_\mu, L^{-1}_\mu(M)_t \rangle_\lambda dt + d\left( \int \langle M(x) \rangle_t \lambda(dx) \right).$$

Note that there exists a constant $K$ such that

$$\frac{d}{dt} \left( \int \langle M(x) \rangle_t \lambda(dx) \right) \leq K$$

and that (see hypothesis 2.1)

$$\langle L^{-1}_\mu(M)_t, G_\mu, L^{-1}_\mu(M)_t \rangle_\lambda \geq K\|L^{-1}_\mu(M)_t\|_\lambda^2.$$ 

This implies that

$$\frac{d}{dt} E[\|L^{-1}_\mu(M)_t\|_\lambda^2] \leq -2\kappa E[\|L^{-1}_\mu(M)_t\|_\lambda^2] + K$$

which implies (i) for $\alpha = 2$. For $\alpha > 2$, we find that

$$\frac{d}{dt} E[\|L^{-1}_\mu(M)_t\|_\lambda^{\alpha/2}] \leq -\alpha\kappa E[\|L^{-1}_\mu(M)_t\|_\lambda^{\alpha/2}] + K E[\|L^{-1}_\mu(M)_t\|_\lambda^{\alpha/2 - 2}]$$

$$\leq -\alpha\kappa E[\|L^{-1}_\mu(M)_t\|_\lambda^{\alpha/2}] + K E[\|L^{-1}_\mu(M)_t\|_\lambda^{\alpha/2 - 2}]$$

which implies that $E[\|L^{-1}_\mu(M)_t\|_\lambda^\alpha]$ is bounded.

We now prove (ii). Fix $\alpha > 1$. Then there exists a constant $K$ such that

$$\frac{\|L^{-1}_\mu(M)_t\|_\lambda^2}{(1 + t)^\alpha} \leq \|L^{-1}_\mu(M)_0\|_\lambda^2 + 2 \int_0^t \frac{\|L^{-1}_\mu(M)_s\|_\lambda}{(1 + s)^\alpha} dN_s + K.$$ 

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Then Bürkholder-Davies-Gundy inequality (BDG inequality in the following) inequality implies that
\[
E \left[ \sup_{t \geq 0} \frac{\|L^{-1}_\mu(M)\|_2^2}{(1 + t)^a} \right] \leq K + 2 \sup_{t \geq 0} \left( \int_0^t \frac{Kds}{(1 + s)^{2a}} \right)^{1/2}
\]
which is finite. This implies the lemma by taking \(a = 2\).  \textbf{QED}

### 4.5.2 Estimation of \(\|D_t\|_\lambda\)

Note that for all \(f \in C(M)\), \(|e'_t f| \leq Ke^{-t/2}\|D_t\|_\lambda^2 \times \|f\|_\infty\). Thus
\[
|e'_t f| \leq Ke^{-t/2}(1 + t + \|D_t\|_\lambda^2) \times \|f\|_\infty.
\]
This implies (using lemma 2.2 and the fact that \(0 < \kappa < 1/2\))

**Lemma 4.5.** There exists \(K\) such that
\[
\left| \int_0^t e^{-s}G_e e'_s ds \right| \leq Ke^{-\kappa t} \left( 1 + \int_0^t e^{-(1/2-\kappa)s}\|D_s\|_\lambda^2 ds \right).
\] (22)

This lemma with lemma 4.4(ii) implies the following

**Lemma 4.6.** \(a.s.\) there exists \(C\) with \(E[C] < \infty\) such that
\[
\|D_t\|_\lambda \leq C \left[ 1 + \int_0^t e^{-s/2}\|D_s\|_\lambda^2 ds \right].
\] (23)

**Proof:** First note that
\[
\|D_t\|_\lambda \leq \|D'_t\|_\lambda + K(1 + t)e^{-t/2}.
\]
Using the expression of \(D'_t\) given by (20), we get
\[
\|D'_t\|_\lambda \leq \|L^{-1}_\mu(M)\|_\lambda + \left| \int_0^t e^{-(t-s)G_e e'_s ds} \right|_\infty \leq C(1 + t) + Ke^{-\kappa t} \left( 1 + \int_0^t e^{-(1/2-\kappa)s}\|D_s\|_\lambda^2 ds \right)
\]
(with \(E[C] < \infty\)) which implies the lemma.  \textbf{QED}

**Lemma 4.7.** Let \(x\) and \(f\) be real functions, and \(\alpha\) a real constant. Assume that for all \(t \geq 0\), we have \(x_t \leq \alpha + \int_0^t e_s x_s ds\). Then \(x_t \leq \alpha \exp \left( \int_0^t e_s ds \right) \).

**Proof:** Similarly to the proof of Gronwall’s lemma, we set \(y_t = \int_0^t e_s x_s ds\) and take \(\lambda_t = y_t \exp \left( -\int_0^t e_s ds \right)\). Then \(\dot{\lambda}_t \leq \alpha e_t \exp \left( -\int_0^t e_s ds \right)\) and
\[
y_t \leq \alpha \int_0^t e_s \exp \left( \int_s^t e_u du \right) ds \leq \alpha \exp \left( \int_0^t e_u du \right) - \alpha.
\]
This implies the lemma.  \textbf{QED}
Lemma 4.8. \( \text{a.s., there exists } C \text{ such that for all } t, \|D_t\|_\lambda \leq C(1 + t). \)

Proof: Lemmas 4.6 and 4.7 imply that
\[
\|D_t\|_\lambda \leq C(1 + t) \times \exp \left( C \int_0^t e^{-s/2} \|D_s\|_\lambda \, ds \right).
\]

Since hypothesis 1.3 implies that \( \lim_{s \to \infty} e^{-s/2} \|D_s\|_\lambda = 0 \), then a.s. for all \( \epsilon > 0 \), there exists \( C_\epsilon \) such that
\[
\|D_t\|_\lambda \leq C_\epsilon e^{\epsilon t}.
\]
Taking \( \epsilon < 1/4 \), we get
\[
\int_0^\infty e^{-s/2} \|D_s\|_\lambda^2 \, ds \leq C_\epsilon.
\]
This proves the lemma. QED

4.5.3 Estimation of \( e'_t \)

Lemma 4.9. \( \text{a.s. there exists } C \text{ such that for all } f \in C(M), \)
\[
|e'_t f| \leq C(1 + t)^2 e^{-t/2} \|f\|_\infty
\]

Proof: We have \( |e'_t f| \leq |e'_f| + K(1 + t)e^{-t/2} \|f\|_\infty \) and
\[
|e'_f| \leq K \|f\|_\lambda \times e^{-t/2} \|D_t\|_\lambda^2 \leq C \|f\|_\infty \times (1 + t)^2 e^{-t/2}
\]
by lemma 4.8. QED

4.6 Estimation of \( \|D_t - L^{-1}_\mu(M)_t\|_\infty \)

Lemma 4.10. (i) \( \|D_t - L^{-1}_\mu(M)_t\|_\infty \leq Ce^{-\kappa t}. \)

(ii) \( \|(\Delta_t g, D_t) - (\tau_{\Delta_t g, -1}(N^g, M)_t\|_\infty \leq C(1 + \|g\|_\infty) e^{-\kappa t}. \)

Proof: Note that (i) is implied by (ii). We have \( \|(\Delta_t g, D_t) - (\Delta'_t g, D'_t)\|_\infty \leq K(1 + \|g\|_\infty)(1 + t)e^{-\kappa t}. \) So to prove this lemma, using (21), it suffices to show that
\[
\left\| \int_0^t e^{-(t-s)G^g_{\mu'}}(e_s', e'_s) \, ds \right\|_\infty \leq K(1 + \|g\|_\infty) e^{-\kappa t}. \tag{24}
\]

Using hypothesis 2.1 and the definition of \( G^g_{\mu'} \), we have that for all positive \( t, \|e^{-(t-s)G^g_{\mu'}}\|_\infty \leq Ke^{-\kappa t}. \)
This implies \( \|e^{-(t-s)G^g_{\mu'}}(e'_s, e'_s)\|_\infty \leq Ke^{-\kappa(t-s)}\|e'_s\|_\infty \times (1 + \|g\|_\infty). \) Thus the term (24) is dominated by
\[
K(1 + \|g\|_\infty) \int_0^t e^{-\kappa(t-s)} \|e'_s\|_\infty \, ds,
\]
from which we prove (24) like in the previous lemma. QED
4.7 Tightness results

We refer the reader to section 5.1 in the appendix (section 5), where tightness criteria for families of \(C(M)\)-valued random variables are given. They will be used in this section.

4.7.1 Tightness of \((L^{-1}_\mu(M))_{t \geq 0}\)

In this section we prove the following lemma which in particular implies the tightness of \((D_t)_{t \geq 0}\) and of \((D'_t)_{t \geq 0}\).

**Lemma 4.11.** \((L^{-1}_\mu(M))_{t \geq 0}\) is tight.

**Proof:** We have the relation (that defines \(L^{-1}_\mu(M)\))

\[
 dL^{-1}_\mu(M)_t(x) = -G_\mu L^{-1}_\mu(M)_t(x)dt + dM_t(x).
\]

Thus, using the expression of \(G_\mu\)

\[
 dL^{-1}_\mu(M)_t(x) = -\frac{1}{2}L^{-1}_\mu(M)_t(x)dt + A_t(x)dt + dM_t(x),
\]

with

\[
 A_t(x) = \widehat{C}_\mu(V_x, L^{-1}_\mu(M)_t).
\]

Since \(\mu^*\) is absolutely continuous with respect to \(\lambda\), we have that (with \(\text{Lip}(A_t)\) the Lipschitz constant of \(A_t\), see (36)).

\[
 ||A_t||_\infty + \text{Lip}(A_t) \leq K||L^{-1}_\mu(M)_t||_\lambda.
\]

Therefore (using lemma 4.4 (i) for \(\alpha = 2\), \(\sup_t E[||A_t||^2_\infty] < \infty\).

To prove this tightness result, we first prove that for all \(x\), \((L^{-1}_\mu(M))_{t \geq 0}\) is tight. Setting \(Z^x_t = L^{-1}_\mu(M)_t(x)\) we have

\[
 \frac{d}{dt}E[(Z^x_t)^2] \leq -E[(Z^x_t)^2] + 2E[|Z^x_t| \times |A_t(x)|] + \frac{d}{dt}E[(M(x))_t]
\]

\[
 \leq -E[(Z^x_t)^2] + KE[(Z^x_t)^2]^{1/2} + K
\]

which implies that \((L^{-1}_\mu(M))_{t \geq 0}\) is bounded in \(L^2(\mathbb{P})\) and thus tight.

We now estimate \(E[|Z^x_t - Z^y_t|^\alpha]^{1/\alpha}\) for \(\alpha\) greater than 2 and the dimension of \(M\). Setting \(Z^{x,y}_t =\)
Lemma 4.13.

Thus, if \( x \)

Using this expression it is easy to prove that

Set \( A \)

Remark 4.12. Kolmogorov's theorem (see theorem 1.4.1 and its proof in Kunita (1990)), with the estimates given in the proof of this lemma, implies that

\[
\sup_t E[||L^{-1}_\mu^r(M)_t||_\infty] < \infty.
\]

4.7.2 Tightness of \(((L^g_{\mu^r})^{-1}(N^g, M))_{t\geq 0}\)

Let \( \overline{\Delta} g \) be defined by the relation

\[
(\overline{\Delta} g, L^{-1}_\mu^r(M)) = (L^g_{\mu^r})^{-1}(N^g, M).
\]

Set \( A_t g = (A_t g_1, \ldots, A_t g_n) \) with \( A_t g_i = \widehat{c}_i(g_i, L^{-1}_\mu^r(M)_t) \). Then

\[
d \overline{\Delta} g = dN^g_t - \frac{\overline{\Delta} g}{2} dt + A_t g dt.
\]

Thus,

\[
\overline{\Delta} g = e^{-t/2} \int_0^t e^{s/2} dN^g_s + e^{-t/2} \int_0^t e^{s/2} A_s g ds.
\]

Using this expression it is easy to prove that \( (\overline{\Delta} g)_{t\geq 0} \) is bounded in \( L^2(P) \). This implies, using also lemma 4.11

Lemma 4.13. \(((L^g_{\mu^r})^{-1}(N^g, M))_{t\geq 0}\) is tight.
4.8 Convergence in law of \( (N^{g}, M)_{t+} - (N^{g}, M)_{t} \)

In this section, we denote by \( E_{t} \) the conditional expectation with respect to \( \mathcal{F}_{t} \). We also set \( Q = Q_{\mu^{*}} \) and \( C = \tilde{C}_{\mu^{*}} \).

4.8.1 Preliminary lemmas.

For \( f \in C(M) \) and \( t \geq 0 \), set \( N^{f,t}_{s} = N^{f}_{s+t} - N^{f}_{t} \).

**Lemma 4.14.** For all \( f \) and \( h \) in \( C(M) \), \( \lim_{t \to \infty} \langle N^{f,t}, N^{h,t} \rangle_{s} = s \times C(f, h) \).

**Proof:** For \( z \in M \) and \( u > 0 \), set

\[
\begin{align*}
G(z) &= \langle \nabla Qf, \nabla h \rangle(z) - C(f, h); \\
G_{u}(z) &= \langle \nabla Quf, \nabla Quh \rangle(z) - C(f, h).
\end{align*}
\]

We have

\[
\langle N^{f,t}, N^{h,t} \rangle_{s} - s \times C(f, h) = \int_{t}^{\infty} G_{u}(X_{u}) \frac{du}{u}
\]

\[
= \int_{t}^{\infty} (G_{u} - G)(X_{u}) \frac{du}{u} + \int_{t}^{\infty} G(X_{u}) \frac{du}{u}.
\]

Integrating by parts, we get that

\[
\int_{t}^{\infty} G(X_{u}) \frac{du}{u} = (\mu^{*}_{t}, G - \mu^{*}_{t} G) + \int_{0}^{\infty} (\mu^{*}_{t} G) du.
\]

Since \( \mu^{*} G = 0 \), this converges towards 0 on the event \( \{\mu_{t} \to \mu^{*}\} \). The term \( \int_{t}^{\infty} (G_{u} - G)(X_{u}) \frac{du}{u} \) converges towards 0 because \( (\mu, z) \to \nabla Quf(z) \) is continuous. This proves the lemma. \( \text{QED} \)

Let \( f_{1}, \ldots, f_{n} \) be in \( C(M) \). Let \( \{t_{k}\} \) be an increasing sequence converging to \( \infty \) such that the conditional law of \( M^{n,k} = (N^{f_{1},t_{k}}, \ldots, N^{f_{n},t_{k}}) \) given \( \mathcal{F}_{t_{k}} \) converges in law towards a \( \mathbb{R}^{n} \)-valued process \( W^{n} = (W_{1}, \ldots, W_{n}) \).

**Lemma 4.15.** \( W^{n} \) is a centered Gaussian process such that for all \( i \) and \( j \),

\[
E[W_{i}^{n}(s)W_{j}^{n}(t)] = (s \wedge t)C(f_{i}, f_{j}).
\]

**Proof:** We first prove that \( W^{n} \) is a martingale. For all \( k \), \( M^{n,k} \) is a martingale. For all \( u \leq v \), BDG inequality implies that \( (M^{n,k}(v) - M^{n,k}(u))_{k} \) is bounded in \( L^{2} \).

Let \( l \geq 1, \varphi \in C(\mathbb{R}^{l}) \), \( 0 \leq s_{1} \leq \cdots \leq s_{l} \leq u \) and \( (i_{1}, \ldots, i_{l}) \in \{1, \ldots, n\}^{l} \). Then for all \( k \) and \( i \in \{1, \ldots, n\} \), the martingale property implies that

\[
E_{t_{k}}[(M_{i}^{n,k}(v) - M_{i}^{n,k}(u))Z_{k}] = 0
\]
where \( Z_k \) is of the form

\[
Z_k = \phi(M_{i_1}^{n,k}(s_1), \ldots, M_{i_l}^{n,k}(s_l)).
\]

Using the convergence of the conditional law of \( M_{n,k} \) given \( \mathcal{F}_{\xi_k} \) towards the law of \( W^n \) and since \((M_{n,k}(v) - M_{n,k}(u))_k\) is uniformly integrable (because it is bounded in \( L^2 \)), we prove that \( E[(W^n_t(v) - W^n_t(u))Z] = 0 \) where \( Z \) is of the form

\[
Z = \phi(W^n_{i_1}(s_1), \ldots, W^n_{i_l}(s_l)).
\]

This implies that \( W^n \) is a martingale.

We now prove that for \((i,j) \in \{1, \ldots, n\} \) (with \( C = C_{\mu^n} \)),

\[
\langle W^n_t, W^n_j \rangle_s = s \times C(f_i, f_j).
\]

By definition of \( (M_{i}^{n,k}, M_{j}^{n,k}) \) (in the following \((\cdot, \cdot)\) = \((\cdot, \cdot)_v - (\cdot, \cdot)_u\)

\[
E_{t_k} \left[ \left( (M_{i}^{n,k}(v) - M_{i}^{n,k}(u))(M_{j}^{n,k}(v) - M_{j}^{n,k}(u)) - (M_{i}^{n,k}, M_{j}^{n,k}) u \right) Z_k \right] = 0
\]

where \( Z_k \) is of the form \((25)\). Using the convergence in law and the fact that \((M_{n,k}(v) - M_{n,k}(u))_k^2\) is bounded in \( L^2 \) (still using BDG inequality), we prove that as \( k \to \infty \),

\[
E_{t_k} [(M_{i}^{n,k}(v) - M_{i}^{n,k}(u))(M_{j}^{n,k}(v) - M_{j}^{n,k}(u))Z_k]
\]

converges towards \( E[(W^n_t(v) - W^n_t(u))(W^n_j(v) - W^n_j(u))Z] \) with \( Z \) of the form \((26)\). Now,

\[
E_{t_k} [(M_{i}^{n,k}, M_{j}^{n,k})_v Z] = v \times E[Z] \times C(x_i, x_j)
\]

\[
= E_{t_k} [(M_{i}^{n,k}, M_{j}^{n,k})_v - v \times C(f_i, f_j)] Z_k \] + \( v \times (E_{t_k} [Z_k] - E[Z]) \times C(f_i, f_j)
\]

The convergence in \( L^2 \) of \( (M_{i}^{n,k}, M_{j}^{n,k})_v \) towards \( v \times C(f_i, f_j) \) shows that the first term converges towards 0. The convergence of the conditional law of \( M_{n,k} \) with respect to \( \mathcal{F}_{\xi_k} \) towards \( W^n \) shows that the second term converges towards 0. Thus

\[
E \left[ \left( (W^n_t(v) - W^n_t(u))(W^n_j(v) - W^n_j(u)) - (v - u)C(f_i, f_j) \right) Z \right] = 0.
\]

This shows that \( \langle W^n_t, W^n_j \rangle_s = s \times C(f_i, f_j) \). We conclude using Lévy’s theorem. QED

### 4.8.2 Convergence in law of \( M_{t^+} - M_t \)

In this section, we denote by \( \mathcal{L}_t \) the conditional law of \( M_{t^+} - M_t \) knowing \( \mathcal{F}_t \). Then \( \mathcal{L}_t \) is a probability measure on \( C(\mathbb{R}^+ \times M) \).

**Proposition 4.16.** When \( t \to \infty \), \( \mathcal{L}_t \) converges weakly towards the law of a \( C(M) \)-valued Brownian motion of covariance \( C_{\mu^n} \).

**Proof:** In the following, we will denote \( M_{t^+} - M_t \) by \( M_t^f \). We first prove that

**Lemma 4.17.** \( \{\mathcal{L}_t : t \geq 0\} \) is tight.
Proof: For all \( x \in M, t \) and \( u \) in \( \mathbb{R}^+ \),
\[
E_t[(M_u^t(x))^2] = E_t \left[ \int_t^{t+u} d(M(x))_s \right] \leq Ku.
\]
This implies that for all \( u \in \mathbb{R}^+ \) and \( x \in M, (M_u^t(x))_{t \geq 0} \) is tight.

Let \( \alpha > 0 \). We fix \( T > 0 \). Then for \( (u, x) \) and \( (v, y) \) in \([0, T] \times M\), using BDG inequality,
\[
E_t[|M_u^t(x) - M_v^t(y)|^\alpha]^{\frac{1}{\alpha}} \leq E_t[|M_u^t(x) - M_v^t(x)|^\alpha]^{\frac{1}{\alpha}} + E_t[|M_v^t(y) - M_v^t(y)|^\alpha]^{\frac{1}{\alpha}}
\]
\[
\leq K_\alpha \times (\sqrt{T}d(x, y) + \sqrt{|v - u|})
\]
where \( K_\alpha \) is a positive constant depending only on \( \alpha, \|V\|_\infty \) and \( \text{Lip}(V) \) the Lipschitz constant of \( V \).

We now let \( D_T \) be the distance on \([0, T] \times M\) defined by
\[
D_T((u, x), (v, y)) = K_\alpha \times (\sqrt{T}d(x, y) + \sqrt{|v - u|}).
\]
The covering number \( N\{0, T\} \times M, D_T, \epsilon \) is of order \( \epsilon^{-d-1/2} \) as \( \epsilon \to 0 \). Taking \( \alpha > d + 1/2 \), we conclude using proposition \ref{prop:covering}

\[ \text{QED} \]

Let \( (t_k) \) be an increasing sequence converging to \( \infty \) and \( N \) a \( C(M) \)-valued random process (or a \( C(\mathbb{R}^+ \times M) \) random variable) such that \( \mathcal{L}_{t_k} \) converges in law towards \( N \).

Lemma 4.18. \( N \) is a \( C(M) \)-valued Brownian motion of covariance \( C_{\mu^*} \).

Proof: Let \( W \) be a \( C(M) \)-valued Brownian motion of covariance \( C_{\mu^*} \). Using lemma \ref{lem:bdd}, we prove that for all \( (x_1, \ldots, x_n) \in M^n \), \( (N(x_1), \ldots, N(x_n)) \) has the same distribution as \( (W(x_1), \ldots, W(x_n)) \).

This implies the lemma. \[ \text{QED} \]

Since \( \{\mathcal{L}_t\} \) is tight, this lemma implies that \( \mathcal{L}_t \) converges weakly towards the law of a \( C(M) \)-valued Brownian motion of covariance \( C_{\mu^*} \). \[ \text{QED} \]

4.8.3 Convergence in law of \( (N^g, M)_t \rightarrow (N^g, M)_t \)

Let \( \mathcal{L}_{t_k}^g \) denote the conditional law of \( (N^g, M)_{t_k} \rightarrow (N^g, M)_t \) knowing \( \mathcal{F}_t \). Then \( \mathcal{L}_{t_k}^g \) is a probability measure on \( C(\mathbb{R}^+ \times M \cup \{1, \ldots, n\}) \). Let \( (N^g, M)_t \) denote the process \( (N^g, M)_{t_k} \rightarrow (N^g, M)_t \).

Let \( (W^g_t)_{t \in \mathbb{R}^+ \times C(M)} \) be a \( \mathcal{X}(M) \)-valued Brownian motion of covariance \( C_{\mu^*} \). Denoting \( W_t(x) = W^g_{t_v} \), then \( W = (W_t(x))_{(t, x) \in \mathbb{R}^+ \times M} \) is a \( C(M) \)-valued Brownian motion of covariance \( C_{\mu^*} \). Let \( W^g \) denote \( (W^g_1, \ldots, W^g_n) \), and let \( (W^g, W) \) denote the process \( (W^g_t, (W_t(x))_{x \in M})_{t \geq 0} \).

Proposition 4.19. As \( t \) goes to \( \infty \), \( \mathcal{L}_{t_k}^g \) converges weakly towards the law of \( \mathcal{L}_t \).

Proof: We first prove that \( \{\mathcal{L}_{t_k}^g : t \geq 0\} \) is tight. This is a straightforward consequence of the tightness of \( \{\mathcal{L}_t\} \) and of the fact that for all \( \alpha > 0 \), there exists \( K_\alpha \) such that for all nonnegative \( u \) and \( v, E_t[|N^{g, t}_u - N^{g, t}_v|^\alpha]^{\frac{1}{\alpha}} \leq K_\alpha \sqrt{|v - u|}. \]

Let \( (t_k) \) be an increasing sequence converging to \( \infty \) and \( \tilde{N}^g, \tilde{M} \) a \( \mathbb{R}^n \times C(M) \)-valued random process (or a \( C(\mathbb{R}^+ \times M \cup \{1, \ldots, n\}) \) random variable) such that \( \mathcal{L}_{t_k}^g \) converges in law towards \( (\tilde{N}^g, \tilde{M}) \). Then lemmas \ref{lem:tightness} and \ref{lem:bdd} imply that \( \tilde{N}^g, \tilde{M} \) has the same law as \( (W^g, W) \). Since \( \{\mathcal{L}_{t_k}^g\} \) is tight, \( \mathcal{L}_{t_k}^g \) converges towards the law of \( (W^g, W) \). \[ \text{QED} \]
4.9 Convergence in law of $D$

4.9.1 Convergence in law of $(D_{t+s} - e^{-sG_{\mu^*}}D_t)_{s \geq 0}$

We have

$$D'_{t+s} - e^{-sG_{\mu^*}}D'_t = L_{\mu^*}^{-1}(M')_s + \int_0^s e^{-(s-u)G_{\mu^*}}e'_{t+u}du.$$  

Since (using lemma 4.9), $\left|\int_0^s e^{-(s-u)G_{\mu^*}}e'_{t+u}du\right|_\infty \leq Ke^{-\kappa t}$ and $\|D_t - D'_t\|_\infty \leq K(1 + t)e^{-t/2}$, this proves that $(D_{t+s} - e^{-sG_{\mu^*}}D_t - L_{\mu^*}^{-1}(M_{t+s} - M_t))_{s \geq 0}$ converges towards 0. Since $L_{\mu^*}^{-1}$ is continuous, this proves that the law of $L_{\mu^*}^{-1}(M_{t+s} - M_t)$ converges weakly towards $L_{\mu^*}^{-1}(W)$. Since $L_{\mu^*}^{-1}(W)$ is an Ornstein-Uhlenbeck process of covariance $C_{\mu^*}$ and drift $-G_{\mu^*}$ started from 0, we have

**Theorem 4.20.** The conditional law of $(D_{t+s} - e^{-sG_{\mu^*}}D_t)_{s \geq 0}$ given $\mathcal{F}_t$ converges weakly towards an Ornstein-Uhlenbeck process of covariance $C_{\mu^*}$ and drift $-G_{\mu^*}$ started from 0.

4.9.2 Convergence in law of $D_{t+}$.

We can now prove theorem 3.1. We here denote by $P_t$ the semigroup of an Ornstein-Uhlenbeck process of covariance $C_{\mu^*}$ and drift $-G_{\mu^*}$, and we denote by $\pi$ its invariant probability measure.

Since $(D_t)_{t \geq 0}$ is tight, there exists $\nu \in \mathcal{P}(C(M))$ and an increasing sequence $t_n$ converging towards $\infty$ such that $D_{t_n}$ converges in law towards $\nu$. Then $D_{t_n+}$ converges in law towards $(L_{\mu^*}^{-1}(W) + e^{-sG_{\mu^*}}Z_0)$, with $Z_0$ independent of $W$ and distributed like $\nu$. This proves that $D_{t_n+}$ converges in law towards an Ornstein-Uhlenbeck process of covariance $C_{\mu^*}$ and drift $-G_{\mu^*}$.

We now fix $t > 0$. Let $s_n$ be a subsequence of $t_n$ such that $D_{s_n+t^+}$ converges in law. Then $D_{s_n-t^+}$ converges towards $t$ a law we denote by $\nu_t$ and $D_{s_n-t^+}$. converges in law towards an Ornstein-Uhlenbeck process of covariance $C_{\mu^*}$ and drift $-G_{\mu^*}$. Since $D_{s_n} = D_{s_n+t^+},D_{s_n}$ converges in law towards $\nu_t P_t$. On the other hand $D_{s_n}$ converges in law towards $\nu_t$. Thus $\nu_t P_t = \nu$.

Let $\varphi$ be a Lipschitz bounded function on $C(M)$. Then

$$|\nu_t P_t \varphi - \pi \varphi| = \left|\int (P_t \varphi(f) - \pi \varphi)\nu_t(df)\right| \leq \int |P_t \varphi(f) - P_t \varphi(0)|\nu_t(df) + |P_t \varphi(0) - \pi \varphi| \quad (28)$$

where the second term converges towards 0 (using proposition 2.4 (ii) or theorem 5.7 (ii)) and the first term is dominated by (using lemma 5.8) $Ke^{-\kappa t} \int \|f\|\nu_t(df)$.

It is easy to check that

$$\int \|f\|\nu_t(df) = \lim_{k \to \infty} \int (\|f\|\wedge k)\nu_t(df)$$

$$= \lim_{k \to \infty} \lim_{n \to \infty} \sup_t E[\|D_{s_n-t^+}\|\wedge k] \leq \sup_t E[\|D_t\|\wedge k].$$
Let \((\text{random variable), we let}\)

\[ P \subset C \]

The next proposition gives a useful criterium for a class of random variables to be tight. It follows

\[ P \]

\[ C \]

\[ 5.1 \text{Tightness in} \]

\[ C \text{ Appendix : Ornstein-Uhlenbeck processes on} \]

\[ C \]

\[ \text{From this theorem, like in section 4.9.2, we prove theorem 3.2.} \quad \text{QED} \]

\[ 4.9.3 \text{ Convergence in law of} \]

\[ D^g \]

Set \(D^g_t = (\Delta^g_t, D^g_t)\). Since \(\|D^g_t - D^g_{t+s}\|_\infty \leq K(1 + t)e^{-t/2}\), instead of studying \(D^g\), we can only study \(D^g_t\). Then

\[ D^g_t - e^{-sG^g_{\mu^*}}D^g_t = (L^g_{\mu^*})^{-1}(N^g_t, M^t) + \int_0^s e^{-(s-u)G^g_{\mu^*}}(e^{t+u}g, e^{t+u})du. \]

The norm of the second term of the right hand side (using the proof of lemma 4.10) is dominated by

\[ K(1 + \|g\|_\infty) \int_0^s e^{-\kappa(s-u)}\|e^{t+u}\|_\infty du \leq K \int_0^s e^{-\kappa(s-u)}(1 + t + u)^2e^{-(t+u)/2}du \]

which is less than \(Ke^{-\kappa t}\). Like in section 4.9.1, since \((L^g_{\mu^*})^{-1}(W^g, W)\) is an Ornstein-Uhlenbeck process of covariance \(C^g_{\mu^*}\) and drift \(-G^g_{\mu^*}\) started from 0,

**Theorem 4.21.** The conditional law of \((\Delta^g_t, D)_{t+s} - e^{-sG^g_{\mu^*}(\Delta^g_t, D)_t})_{t \geq 0}\) given \(\mathcal{F}_t\) converges weakly towards an Ornstein-Uhlenbeck process of covariance \(C^g_{\mu^*}\) and drift \(-G^g_{\mu^*}\) started from 0.

From this theorem, like in section 4.9.2 we prove theorem 3.2. \quad \text{QED}

5 \text{ Appendix : Ornstein-Uhlenbeck processes on} \( C(M) \)

5.1 \text{Tightness in} \( \mathcal{P}(C(M)) \)

Let \((M, d)\) be a compact metric space. Denote by \(\mathcal{P}(C(M))\) the space of Borel probability measures on \(C(M)\). Since \(C(M)\) is separable and complete, Prohorov theorem (see \[8\]) asserts that \(\mathcal{X} \subset \mathcal{P}(C(M))\) is tight if and only if it is relatively compact.

The next proposition gives a useful criterium for a class of random variables to be tight. It follows directly from \[15\] (Corollary 11.7 p. 307 and the remark following Theorem 11.2). A function \(\psi : \mathbb{R}_+ \to \mathbb{R}_+\) is a Young function if it is convex, increasing and \(\psi(0) = 0\). If \(Z\) is a real valued random variable, we let

\[ \|Z\|_\psi = \inf\{c > 0 : E(\psi(|Z|/c)) \leq 1\}. \]

For \(e > 0\), we denote by \(N(M, d; e)\) the covering number of \(E\) by balls of radius less than \(e\) (i.e. the minimal number of balls of radius less than \(e\) that cover \(E\)), and by \(D\) the diameter of \(M\).
Proposition 5.1. Let \((F_t)_{t \in I}\) be a family of \(C(M)\)-valued random variables and \(\psi\) a Young function. Assume that

(i) There exists \(x \in E\) such that \((F_t(x))_{t \in I}\) is tight;

(ii) \(\|F_t(x) - F_t(y)\|_\psi \leq K d(x, y)\);

(iii) \(\int_0^\delta \psi^{-1}(N(M, d; \epsilon)) d\epsilon < \infty\).

Then \((F_t)_{t \geq 0}\) is tight.

Proposition 5.2. Suppose \(M\) is a compact finite dimensional manifold of dimension \(r\), \(d\) is the Riemannian distance, and

\[
[E|F_t(x) - F_t(y)|^a]^{1/a} \leq K d(x, y)
\]

for some \(\alpha > r\). Then conditions (ii) and (iii) of Proposition 5.1 hold true.

Proof: One has \(N(E, d; \epsilon)\) of order \(\epsilon^{-r}\); and for \(\psi(x) = x^\alpha, \|\cdot\|_\psi\) is the \(L^\alpha\) norm. Hence the result. QED

5.2 Brownian motions on \(C(M)\).

Let \(C : M \times M \to \mathbb{R}\) be a covariance function, that is a continuous symmetric function such that \(\sum_{ij} a_i a_j C(x_i, x_j) \geq 0\) for every finite sequence \((a, x_i)\) with \(a_i \in \mathbb{R}\) and \(x_i \in M\).

A Brownian motion on \(C(M)\) with covariance \(C\) is a continuous \(C(M)\)-valued stochastic process \(W = \{W_t\}_{t \geq 0}\) such that \(W_0 = 0\) and for every finite subset \(S \subset \mathbb{R}^+ \times \tilde{M}\), \(\{W_t(x)\}_{(t, x) \in S}\) is a centered Gaussian random vector with

\[
E[W_t(x)W_t(y)] = (s \wedge t)C(x, y).
\]

For \(d'\) a pseudo-distance on \(M\) and for \(\epsilon > 0\), let

\[
\omega(\epsilon) = \sup\{\eta > 0 : d(x, y) \leq \eta \Rightarrow d'(x, y) \leq \epsilon\}. \tag{29}
\]

Then \(N(M, d; \omega_C(\epsilon)) \geq N(M, d'; \epsilon)\). We will consider the following hypothesis that \(d'\) may or may not satisfy:

\[
\int_0^1 \log(N(M, d; \omega(\epsilon))) d\epsilon < \infty. \tag{30}
\]

Let \(d_C\) be the pseudo-distance on \(M\) defined by

\[
d_C(x, y) = \sqrt{C(x, x) - 2C(x, y) + C(y, y)}.
\]

When \(d' = d_C\), the function \(\omega\) defined by (29) will be denoted by \(\omega_C\).

Remark 5.3. Assume that \(M\) is a compact finite dimensional manifold and that \(d_C(x, y) \leq K d(x, y)^\alpha\) for some \(\alpha > 0\). Then \(\omega_C(\epsilon) \leq \left(\frac{\epsilon}{K}\right)^{1/\alpha}\) and \(N(M, d; \eta) = O(\eta^{\text{dim}(M)})\); so that \(d_C\) satisfies (30).

Theorem 5.4. Assume \(d_C\) satisfies (30). Then there exists a Brownian motion on \(C(M)\) with covariance \(C\).
Proof: By Mercer Theorem (see e.g. [11]) there exists a countable family of function \( \Psi_i \in C(M) \), \( i \in \mathbb{N} \), such that \( C(x, y) = \sum_i \Psi_i(x) \Psi_i(y) \), and the convergence is uniform. Let \( B^i, i \in \mathbb{N} \), be a family of independent standard Brownian motions. Set \( W^n_t(x) = \sum_{i \leq n} B^i_t \Psi_i(x) \), \( n \geq 0 \). Then, for each \((t, x) \in \mathbb{R}^+ \times M\), the sequence \((W^n_t(x))_{n \geq 1}\) is a martingale. It is furthermore bounded in \( L^2 \) since

\[
E[(W^n_t(x))^2] = t \sum_{i \leq n} \Psi_i(x)^2 \leq tC(x, x).
\]

Hence by Doob’s convergence theorem one may define \( W_t(x) = \sum_{i \geq 0} B^i_t \Psi_i(x) \). Let now \( S \subset \mathbb{R}^+ \times M \) be a countable and dense set. It is easily checked that the family \((W_t(x))_{(t, x) \in S}\) is a centered Gaussian family with covariance given by

\[
E[W_t(x)W_t(y)] = (s \wedge t)C(x, y),
\]

In particular, for \( t \geq s \)

\[
E[(W_t(x) - W_t(y))^2] = sC(x, x) - 2sC(x, y) + tC(y, y) \leq K(t - s) + sdC(x, y)^2
\]

This later bound combined with classical results on Gaussian processes (see e.g. Theorem 11.17 in [15]) implies that \((t, x) \mapsto W_t(x)\) admits a version uniformly continuous over \( S_T = \{(t, x) \in S : t \leq T\} \). By density it can be extended to a continuous (in \((t, x)\)) process \( W = (W_t(x))_{(t, x) \in \mathbb{R}^+ \times M} \).

QED

5.3 Ornstein-Ulhenbeck processes

Let \( A : C(M) \to C(M) \) be a bounded operator and \( C \) a covariance satisfying hypothesis 30. Let \( W \) be \( C(M) \)-valued Brownian motion with covariance \( C \).

An Ornstein-Ulhenbeck process with drift \( A \), covariance \( C \) and initial condition \( F_0 = f \in C(M) \) is defined to be a continuous \( C(M) \)-valued stochastic process such that

\[
F_t - f = \int_0^t A F_s ds + W_t. \tag{31}
\]

We let \((e^{tA})_{t \in \mathbb{R}}\) denote the linear flow induced by \( A \). For each \( t \), \( e^{tA} \) is a bounded operator on \( C(M) \).

Let \( L_A : C(\mathbb{R}^+ \times M) \to C(\mathbb{R}^+ \times M) \) be defined by

\[
L_A(f)_t = f_t - f_0 - \int_0^t A f_s ds, \quad t \geq 0. \tag{32}
\]

Lemma 5.5. The restriction of \( L_A \) to \( C_0(\mathbb{R}^+ \times M) = \{f \in C(\mathbb{R}^+ \times M) : f_0 = 0\} \) is bijective with inverse \((L_A)^{-1}\) defined by

\[
L_A^{-1}(g)_t = g_t + \int_0^t e^{(t-s)A} g_s ds. \tag{33}
\]
Proof: Observe that \( L_A(f) = 0 \) implies that \( f_t = e^{tA}f_0 \). Hence \( L_A \) restricted to \( C_0(\mathbb{R}^+ \times M) \) is injective. Let \( g \in C_0(\mathbb{R}^+ \times M) \) and let \( f_t \) be given by the right hand side of (33). Then

\[
h_t = L_A(f)_t - g_t = \int_0^t e^{(t-s)A}Ag_s ds - \int_0^t Af_s ds.
\]

It is easily seen that \( h \) is differentiable and that \( \frac{d}{dt}h_t = 0 \). This proves that \( h_t = h_0 = 0 \). QED

This lemma implies for all \( f \in C(M) \), \( g \in C_0(\mathbb{R}^+ \times M) \) the solution to \( L_A(f) = g \), with \( f_0 = f \) is given by \( f_t = e^{tA}f + L_A^{-1}(g)_t \). This implies

**Theorem 5.6.** Let \( A \) be a bounded operator acting on \( C(M) \). Let \( C \) be a covariance function satisfying hypothesis [30]. Then there exists a unique solution to (31), given by

\[
F_t = e^{tA}f + L_A^{-1}(W)_t.
\]

Note that \( L_A^{-1}(W)_t \) is Gaussian and its variance \( \text{Var}_{F_t}(\mu) := E[\langle \mu, F_t \rangle^2] \) (with \( \mu \in \mathcal{M}(M) \)) is given by

\[
\text{Var}_{F_t}(\mu) = \int_0^t \langle \mu, e^{sA}Ce^{sA}^\ast \mu \rangle ds.
\]

where \( C : \mathcal{M}(M) \rightarrow C(M) \) is the operator defined by \( C\mu(x) = \int_M C(x, y)\mu(dy) \). *** We refer to [10] for the calculation of \( \text{Var}_{F_t} \). Note that the results given in Theorem 5.6 are not included in [10].

### 5.3.1 Asymptotic Behaviour

Let \( \lambda(A) = \lim_{t \to \infty} \frac{\log(\|e^{tA}\|)}{t} \). Denote by \( P_t \) the semigroup associated to an Ornstein-Uhlenbeck process of covariance \( C \) and drift \( A \). Then for all bounded measurable \( \varphi : C(M) \rightarrow \mathbb{R} \) and \( f \in C(M) \),

\[
P_t \varphi(f) = E[\varphi(F_t)],
\]

where \( F_t \) is the solution to (31), with \( F_0 = f \).

**Theorem 5.7.** Assume that \( \lambda(A) < 0 \). Then there exists a centered Gaussian variable in \( C(M) \), with variance \( \mathbb{V} \) given by

\[
\mathbb{V}(\mu) = \int_0^\infty \langle \mu, e^{tA}Ce^{tA}^\ast \mu \rangle dt.
\]

Let \( \pi \) denote the law of this Gaussian variable. Let \( d_\mathbb{V} \) be the pseudo-distance defined by \( d_\mathbb{V}(x, y) = \sqrt{\delta_x - \delta_y} \). Assume furthermore that \( d_c \) and \( d_\mathbb{V} \) satisfy (30). Then

(i) \( \pi \) is the unique invariant probability measure of \( P_t \).

(ii) For all bounded continuous function \( \varphi \) on \( C(M) \) and all \( f \in C(M) \),

\[
\lim_{t \to \infty} P_t \varphi(f) = \pi \varphi.
\]
Proof: The fact that $\lambda(A) < 0$ implies that $\lim_{t \to \infty} \text{Var}_{F_t}(\mu) = V(\mu) < \infty$. Let $v_t$ denote the law of $F_t$, where $F_t$ is the solution to (31), with $F_0 = f$. Since $F_t$ is Gaussian, every limit point of $\{v_t\}$ (for the weak* topology) is the law of a $C(M)$-valued Gaussian variable with variance $V$. The proof then reduces to show that $(v_t)$ is relatively compact or equivalently that $\{F_t\}$ is tight. We use Proposition 5.1. The first condition is clearly satisfied. Let $\psi(x) = e^{x^2} - 1$. It is easily verified that for any real valued Gaussian random variable $Z$ with variance $\sigma^2$, $||Z||_\psi = \sigma \sqrt{8/3}$. Hence $||F_t(x) - F_t(y)||_\psi \leq 2d_V(x, y)$ so that condition (ii) holds with $d_V$. Denoting $\omega$ (defined by (29)) by $\omega_V$ when $d' = d_V$, $N(M, d; \omega_V(\epsilon)) \geq N(M, d_V; \epsilon)$ and since $\psi^{-1}(u) = \sqrt{\log(u - 1)}$ condition (iii) is verified. QED

Even though we don’t have the speed of convergence in (ii), we have

Lemma 5.8. Assume that $\lambda(A) < 0$. For all bounded Lipschitz continuous $\varphi : C(M) \to \mathbb{R}$, all $f$ and $g$ in $C(M)$,

$$|P_t \varphi(f) - P_t \varphi(g)| \leq Ke^{\lambda(A)t} ||f - g||_\infty.$$  

Proof: We have $P_t \varphi(f) = E[\varphi(L^{-1}_t(W) + e^{tA}f)]$. So, using the fact that $\varphi$ is Lipschitz,

$$|P_t \varphi(f) - P_t \varphi(g)| \leq K||e^{tA}(f - g)||_\infty \leq Ke^{\lambda(A)t} ||f - g||_\infty.$$  

QED

To conclude this section we give a set of simple sufficient conditions ensuring that $d_V$ satisfies (30). For $f \in C(M)$ we let

$$\text{Lip}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} \in \mathbb{R}^+ \cup \{\infty\}. \tag{36}$$

A map $f$ is said to be Lipschitz provided $\text{Lip}(f) < \infty.$

Proposition 5.9. Assume that

(i) $N(d, M; \epsilon) = O(e^{-r})$ for some $r > 0$;

(ii) $C$ is Lipschitz;

(iii) There exists $K > 0$ such that $\text{Lip}(Af) \leq K(\text{Lip}(f) + ||f||_\infty)$;

(iv) $\lambda(A) < 0$.

Then $d_C$ and $d_V$ satisfy (30).

Note that (i) holds when $M$ is a finite dimensional manifold. We first prove

Lemma 5.10. Under hypotheses (iii) and (iv) of proposition 5.9 there exist constants $K$ and $\alpha$ such that

$$\text{Lip}(e^{tA}f) \leq e^{\alpha t}(\text{Lip}(f) + K||f||_\infty).$$

Proof: For all $x, y$

$$|e^{tA}f(x) - e^{tA}f(y)| = \left| \int_0^t [Ae^{sA}f(x) - Ae^{sA}f(y)] ds + f(x) - f(y) \right| \leq K \left( \int_0^t \left[ \text{Lip}(e^{sA}f) + ||e^{sA}f||_\infty \right] ds + \text{Lip}(f) \right) d(x, y).$$
Since $\lambda(A) = -\lambda < 0$, there exists $K' > 0$ such that $\|e^{sA}\| \leq K'e^{-s\lambda}$. Thus

$$Lip(e^{sA}f) \leq K \int_0^T Lip(e^{sA}f) ds + \frac{KK'}{\lambda} \|f\|_\infty + Lip(f)$$

and the result follows from Gronwall's lemma. \textbf{QED}

\textbf{Proof of proposition 5.9:} Set $\mu = \delta_x - \delta_y$ and $f_\delta = Ce^{sA} \mu$ so that

$$\langle \mu, e^{sA} C e^{sA} \mu \rangle = e^{sA} f_\delta(x) - e^{sA} f_\delta(y).$$

It follows from (ii) and (iv) that $Lip(f_\delta) + \|f_\delta\|_\infty \leq Ke^{-s\lambda}$. Therefore, by the preceding lemma, $Lip(e^{sA}f_\delta) \leq Ke^{as}$ and we have

$$d_V(x, y)^2 \leq d(x, y) \int_0^T Lip(e^{sA}f_\delta) ds + \int_0^\infty \|e^{sA}f(x) - e^{sA}f(y)\| ds$$

$$\leq d(x, y) \int_0^T Ke^{as} ds + 2 \int_T^\infty \|e^{sA}f_\delta\|_\infty ds$$

$$\leq K \left( d(x, y)e^{aT} + \int_T^\infty e^{-s\lambda} ds \right)$$

$$\leq K(d(x, y)e^{aT} + e^{-\lambda T}).$$

Let $\gamma = \frac{a}{\lambda}, \epsilon > 0$, and $T = -\ln(\epsilon)/\lambda$. Then $d_V^2(x, y) \leq K(e^{-\gamma d(x, y)} + \epsilon)$. Therefore $d(x, y) \leq e^{\gamma + 1} \Rightarrow d_V^2(x, y) \leq K\epsilon$, so that $N(d, M; \omega_V(\epsilon)) = O(\epsilon^{-2r(\gamma + 1)})$ and $d_V$ satisfies (30). \textbf{QED}

Example 5.11. Let

$$Af(x) = \int f(y) k_0(x, y) \mu(dy) + \sum_{i=1}^n a_i(x)f(b_i(x))$$

where $\mu$ is a bounded measure on $M$, $k_0(x, y)$ is bounded and uniformly Lipschitz in $x$, $a_i : M \to \mathbb{R}$ and $b_i : M \to M$ are Lipschitz. Then hypothesis (iii) of proposition 5.9 is verified.

\textbf{References}


