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Exact asymptotic for distribution densities of Lévy functionals

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Abstract

A version of the saddle point method is developed, which allows one to describe exactly the asymptotic behavior of distribution densities of Lévy driven stochastic integrals with deterministic kernels. Exact asymptotic behavior is established for (a) the transition probability density of a real-valued Lévy process; (b) the transition probability density and the invariant distribution density of a Lévy driven Ornstein-Uhlenbeck process; (c) the distribution density of the fractional Lévy motion.

Key words: Lévy process, Lévy driven Ornstein-Uhlenbeck process, transition distribution density, saddle point method, Laplace method.

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1 Introduction

In this paper, we develop a version of the *saddle point method*, which allows one to describe exactly the asymptotic behavior of distribution densities of Lévy processes and, more generally, Lévy driven stochastic integrals with deterministic kernels. We start the exposition with the outline of the principal idea of the approach.

Let $(Z_t)_{t\geq 0}$ be a real-valued Lévy process with characteristic exponent ψ; that is,

$$Ee^{izZ_t} = e^{t\psi(z)}, \quad t > 0. \tag{1.1}$$

The function $\psi : \mathbb{R} \to \mathbb{C}$ (the *characteristic exponent* of the process Z) admits the Lévy-Khinchin representation

$$\psi(z) = iaz - bz^2 + \int_{\mathbb{R}} \left(e^{iuz} - 1 - izu1_{\{|u|\leq 1\}}\right) \mu(du), \tag{1.2}$$

where $a \in \mathbb{R}$, $b \geq 0$, and $\mu(\cdot)$ is a *Lévy measure*, i.e. $\int_{\mathbb{R}}(1 \wedge u^2)\mu(du) < \infty$. Under some conditions (see Section 2 below), the function $e^{t\psi}$ is integrable, and hence the transition probability density $p_t(x)$ of the process Z_t has the integral representation as the inverse Fourier transform of the characteristic function (1.1):

$$p_t(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-izx + t\psi(z)}\, dz. \tag{1.3}$$

Our intent is to investigate the oscillatory integral (1.3) using the *saddle point method*. According to this method (see [27]), one can, under the assumption that the characteristic exponent ψ admits an analytic extension to the complex plane, apply the Cauchy theorem in order to change the integration path in (1.3):

$$p_t(x) = \frac{1}{2\pi} \int_{C} e^{-izx + t\psi(z)}\, dz. \tag{1.4}$$

Here C is certain properly chosen contour that allows one to apply the *Laplace method* ([27], [29], [30]) for estimating integral (1.4). A perfect choice of the contour C would be the proper branch of the curve $\{z : \mathrm{Im}(-izx + t\psi(z)) = \mathrm{Im}(-iz_0x + t\psi(z_0))\}$, where z_0 is a critical point of the function $-izx + t\psi(z)$ (*a saddle point*). Under such a choice the integrand in (1.4) is real-valued; in this case the *saddle point method* coincides with the *fastest descent method*, see [27]. However the complicated "oscillatory" structure of the Lévy-Khinchin representation of ψ does not give an opportunity to solve the equation $\mathrm{Im}(-izx + t\psi(z)) = \mathrm{Im}(-iz_0x + t\psi(z_0))$ explicitly. Instead, we put in (1.4) $C = \mathbb{R} + i\xi_0$ with $i\xi_0$ being a critical point of the function $-izx + t\psi(z)$. Under such a choice, we develop an appropriate version of the Laplace method and give exact asymptotics for the transition probability density $p_t(x)$.

The saddle point method is a classic tool for estimating a distribution density in various versions of the local limit theorem with the normal domain of attraction (see [33], chapters 8, 10, and the references therein). In the Lévy processes setting, the idea of applying the complex analysis technique was used, for instance, in [40] for getting upper estimates for (1.3) in the case when the characteristic exponent is real valued.

Since we require the characteristic exponent ψ to have an analytic extension to the complex plane, a standing assumption on the Lévy measure within our approach is that it is exponentially integrable;

that is,

$$\int_{|y|\geq 1} e^{Cy}\mu(dy) < \infty \quad \text{for all } C \in \mathbb{R}. \tag{1.5}$$

Equivalently, (1.5) means that the variable Z_1 has exponential moments, i.e. $Ee^{cZ_1} < \infty$ for all $c \in \mathbb{R}$, see [53], §25 − 26. Assumption (1.5) is non-restrictive, and is satisfied, for instance, for a *generalized tempered* Lévy measure of the form $\mu(du) = \psi(u)\tilde{\mu}(du)$, where $\tilde{\mu}$ is another Lévy measure, and ψ has a super-exponential decay, i.e., $e^{Cu}\psi(u) \to 0$, $u \to \infty$, for all $C \in \mathbb{R}$. For various results on generalized tempered Lévy processes and models that lead to processes of such a type, we refer the reader to Rosinski and Singlair [51], Sztonyk [56], [57], Bianchi et. al. [8], [37]. The notion of a generalized tempered Lévy measure is closely related to the notions of a *tempered* and a *layered* Levy measure, with the function $\psi(u)$ in the above definition respectively being completely monotonous or having a polynomial decay rate (for both these classes (1.5) fails). For the results on tempered and layered Lévy processes and related models, see Rosinski [50], Cohen and Rosinski [25], Cont and Tankov [26], Carr et. al. [13], [14], Baeumer and Meerschaert [1], Kim et. al. [36], Houdré and Kawai [32]. Of course, this list of references is far from complete.

The method described above can be extended naturally for Lévy driven stochastic integrals with deterministic kernels. Let

$$Y_t := \int_I f(t,s)dZ_s, \tag{1.6}$$

where $I \subset \mathbb{R}$ is an interval, f is a deterministic function, and Z_t is a Lévy process (in some particularly important cases, one should take $I = \mathbb{R}$, and then Z should be assumed to be two-sided; see details in Section 2 below). The characteristic exponent of Y_t can be written explicitly (see (2.4) below), which makes it possible to apply the method described above to study the asymptotic behaviour of the distribution density of Y_t.

We mention two particular classes of processes, frequently used in applications, and having representation (1.6). The *Lévy driven Ornstein-Uhlenbeck process* is defined as the solution to the linear SDE

$$dX_t = \gamma X_t\, dt + dZ_t, \quad t \geq 0, \tag{1.7}$$

and has the integral representation

$$X_t = e^{\gamma t}X_0 + \int_0^t e^{\gamma(t-s)}\, dZ_s, \quad t \geq 0. \tag{1.8}$$

If the initial value X_0 is non-random, the distributional properties of X_t are determined by the second term in the right hand side of (1.8), which clearly has the form (1.6) with $I = \mathbb{R}^+$ and $f(t,s) = e^{\gamma(t-s)}\mathbb{I}_{s\leq t}$. In what follows, we call such a process a *non-stationary version of the Ornstein-Uhlenbeck process*.

The Ornstein-Uhlenbeck process is Markov one. It is ergodic (i.e. possesses unique invariant distribution), if and only if, $\gamma < 0$ and

$$\int_{|u|\geq 1} \ln|u|\mu(du) < +\infty; \tag{1.9}$$

see [54]. Clearly, our standing assumption (1.5) provides (1.9). Respective *stationary version of the Ornstein-Uhlenbeck process* can be represented as

$$X_t = \int_{-\infty}^{t} e^{\gamma(t-s)}\, dZ_s, \quad t \in \mathbb{R},$$

which is clearly of the form (1.6) with $I = \mathbb{R}$, $f(t,s) = e^{\gamma(t-s)}\mathbb{1}_{s\leq t}$. Conditions on the existence and smoothness of the distribution densities for Lévy driven Ornstein-Uhlenbeck processes were studied in [45], [9], [48], [55]. In some exceptional stationary cases, the density can be represented explicitly, see [5]. However, as far as we know, any references concerning general estimates or a description of the asymptotic behaviour of such a density are not available.

Another example of a process of the type (1.6) is the *fractional Lévy motion*, defined, analogously to the fractional Brownian motion, by the stochastic Weyl integral

$$Z^H(t) = \frac{1}{\Gamma(H+1/2)} \int_{\mathbb{R}} \left[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right] dZ_s, \quad t \in \mathbb{R}, \tag{1.10}$$

where $x_+ = \max(x,0)$, and $H \in (0,1)$ is *the Hurst index*; see [52], [7], [44], [38] and references therein. In what follows, we will study the asymptotic behaviour of the distribution density of $Z^H(t)$ under the assumption that $H > 1/2$, which is the so called *long memory case*, see Definition 1.1 in [44]. Note that in this case Z^H is not a Markov process, in contrast to the Lévy process Z, or the Lévy driven Ornstein-Uhlenbeck process (1.8).

Heat kernel estimates for symmetric jump processes were studied systematically by Barlow, Bass, Chen and Kassman [3], Chen, Kim, Kumagai [24], [18], [15], Barlow, Grigoryan, Kumagai [4], Chen, Kumagai [16], [23], Chen, Kim, Kumagai [17]; see also Bass and Levin [6] for the transition density estimates for a Markov chain on $\mathbb{Z}^d$. The approach used in the papers listed above relies on the paper by Carlen, Kusuoka and Stroock [12]. For heat kernel estimates in domains we refer to the papers by Bogdan and Jakubowski [10], Banuelos and Bogdan [2], Bogdan, Grzywny, and Ryznar [11], Chen, Kim and Song [19] – [22]. Of course, this list of references is far from complete.

In particular, heat kernel estimates for symmetric jump processes on $\mathbb{R}^d$ with jump kernel $J(x,y)$, either bounded both from above and below by $\frac{1}{|x-y|^{d+\alpha}} 1_{|x-y|\leq 1}$, $0 < \alpha < 2$, $d \geq 1$, or decaying as $e^{-\gamma|x-y|^\beta}$, $\beta \in [0,\infty)$, as $|x-y| \to \infty$, are studied in [24] and [18], respectively. Under the particular choice of the jump kernel $J(x,y) = J(x-y)$, the processes studied in [24] and [18] become symmetric Lévy processes. We postpone to Section 3 (Example 3.4) the detailed comparison of the asymptotic results for the distribution densities of such processes obtained in [24] and [18], with the results obtained by our approach. Here we just mention that our approach, based on the the complex analysis technique, can be applied both for non-symmetric Markov jump processes, like the Lévy driven Ornstein-Uhlenbeck process, and for non-Markov processes such as the fractional Lévy motion.

Let us outline the rest of the paper. Our main result on asymptotic behavior of the distribution densities of Lévy driven stochastic integrals Y, including the Lévy process Z itself, is formulated and proved in Section 2. To simplify the exposition, we give one-sided asymptotics; that is, we formulate the main result for the distribution density $p_t(x)$ only for $x \geq 0$. Clearly, one can easily deduce from this result the two-sided asymptotics, assuming additionally that the Lévy measure of the process $(-Z)$ satisfies conditions of Theorem 2.1.

Conditions of the main result, Theorem 2.1, are quite abstract, and require an additional analysis in order to provide a verifiable criteria. For the reader's convenience and to clarify the exposition, we separate such an analysis in two parts. In Section 3 we consider an "individual" asymptotic behavior of the distribution density of Y_t with fixed t. We formulate an "individual" version of Theorem 2.1 with verifiable conditions on the Lévy measure μ and the kernel f. These conditions, in particular, reveal the "smoothifying" effect provided by the kernel f: typically, both to provide existence of the distribution density of the Lévy driven stochastic integral Y_t and to describe its asymptotic behavior, fewer restrictions on the Lévy measure are required than in the case of the Lévy process Z_t itself. An illustrative example of such an effect is provided by the fractional Lévy motion, where the assumptions on the Lévy measure are finally reduced to

$$\mu(\mathbb{R}^+) > 0. \tag{1.11}$$

In Section 4 we establish the asymptotic behavior of the distribution density of Y_t, involving both state space variable x and time variable t. To shorten the exposition, we restrict ourselves to the case of a self-similar kernel f. The class of the Lévy driven stochastic integrals with self-similar kernels, although not being the most general possible, is wide enough to cover the important particular cases of the Lévy process Z itself and the fractional Lévy motion Z^H. As a corollary of the main result of Section 4 (Theorem 4.1), we obtain asymptotic relation

$$p_t(x) \sim \frac{1}{\sqrt{2\pi t \mathscr{K}_Z\left(\frac{x}{t}\right)}} e^{t\mathscr{D}_Z\left(\frac{x}{t}\right)}, \quad t + x \to \infty, \quad (t,x) \in [t_0, +\infty) \times \mathbb{R}^+, \tag{1.12}$$

for the distribution density of the Lévy process Z, and

$$p_t(x) \sim \frac{1}{\sqrt{2\pi t^{2H} \mathscr{K}_{Z^H}\left(\frac{x}{t^{H+1/2}}\right)}} e^{t\mathscr{D}_{Z^H}\left(\frac{x}{t^{H+1/2}}\right)}, \quad t + x \to \infty, \quad (t,x) \in [t_0, +\infty) \times \mathbb{R}^+, \tag{1.13}$$

for the distribution density of the fractional Lévy motion Z^H. Here $t_0 > 0$ is arbitrary, $\mathscr{D}_Y$, $\mathscr{K}_Y$ with $Y = Z, Z^H$ are some functions, defined in terms of the Lévy measure μ and the kernel f; see Section 4 below. Observe that the asymptotic formulae for distribution densities of Z and Z^H possess the self-similarity property in spite of the fact that, in general, the families of these densities are not self-similar.

Formally, Z^H includes Z as a partial case with $H = 1/2$, and (1.13) with $H = 1/2$ transforms to (1.12). However, there is a substantial difference between the conditions under which these asymptotic results are available (see Corollary 4.1). To get (1.12), one should impose some "regularity" conditions (N_1) and (C) together with some "tail" conditions (T_1) and (T_2). To get (1.13) with $H \in (1/2, 1)$, it is sufficient to claim only "tail" conditions and non-degeneracy condition (1.11): there is no need for additional "regularity" conditions. Such a difference is caused by the "smoothifying" effect provided by the kernel in the integral (1.10).

Theorem 2.1 and Theorem 4.1 describe the asymptotic behaviour of the distribution density precisely, but in an implicit form. In Section 5 we use these theorems in order to deduce explicit, although less precise, asymptotic expressions. In the same section we give another application of Theorem 2.1, and study the asymptotic behavior (as $x \to \infty$ for a fixed a) of the ratio

$$r_a(x) = \frac{p(x+a)}{p(x)} \tag{1.14}$$

for the invariant distribution density p of the Ornstein-Uhlenbeck process. Such a study is of particular theoretical interest, since the ratio (1.14) appears in the formula for the generator of the dual (i.e., time-reversed) process corresponding to the solution to SDE (1.7). Therefore, knowledge of the asymptotic properties of (1.14) would be useful when one is interested in studying the stationary version of the solution, respective Dirichlet form etc. For instance, in the forthcoming paper [42] the estimate given in Theorem 5.2 below is used substantially in the proof of the spectral gap property for the Lévy driven Ornstein-Uhlenbeck process.

Formula (1.12) and Theorem 5.1 provide a detailed description of the asymptotic behavior of the distribution densities of the Lévy process and the fractional Lévy motion. This behavior exhibits two different regimes. In the first regime, where the ratio $\frac{x}{t}$ (resp., $\frac{x}{t^{H+1/2}}$) stays bounded, the principal behavior of $p_t(x)$ is determined by the values of the functions $\mathscr{D}_Y$, $\mathscr{K}_Y$ (with $Y = Z$ or Z^H) on a bounded domain. For instance, for any $x \geq 0$

$$p_t(tx) \sim \frac{1}{\sqrt{2\pi t \mathscr{K}_Z(x)}} e^{t\mathscr{D}_Z(x)}, \quad t \to +\infty, \tag{1.15}$$

(for the Lévy process Z) and

$$p_t(t^{H+1/2}x) \sim \frac{1}{\sqrt{2\pi t^{2H} \mathscr{K}_{Z^H}(x)}} e^{t\mathscr{D}_{Z^H}(x)}, \quad t \to +\infty, \tag{1.16}$$

(for the fractional Lévy motion Z^H). In the second regime, where the ratio $\frac{x}{t}$ (resp., $\frac{x}{t^{H+1/2}}$) tends to $+\infty$, the principal behavior of $p_t(x)$ is determined by the asymptotics of $\mathscr{D}_Y$, $\mathscr{K}_Y$ (with $Y = Z$ or Z^H) on $+\infty$. Such asymptotics are described in Theorem 4.1 for two cases: for the Lévy measure μ being either "truncated" (i.e. supported in a bounded set) or "exponentially damped" (i.e. its tail satisfies certain exponential estimate, see (3.23)). This description gives some constant c_*, determined in terms of the Lévy measure μ only (see (5.15) and (5.16)), such that the statements below hold true (see Corollary 5.1 and Corollary 5.2 below).

I. *Case of the Lévy process Z.* For any constants $c_1 > c_*$ and $c_2 < c_*$ there exists $y = y(c_1, c_2)$ such that for $x/t > y$, either

$$\exp\left(-c_1 x \ln\left(\frac{x}{t}\right)\right) \leq p_t(x) \leq \exp\left(-c_2 x \ln\left(\frac{x}{t}\right)\right), \tag{1.17}$$

(if μ is truncated), or

$$\exp\left(-c_1 x \ln^{\frac{\beta-1}{\beta}}\left(\frac{x}{t}\right)\right) \leq p_t(x) \leq \exp\left(-c_2 x \ln^{\frac{\beta-1}{\beta}}\left(\frac{x}{t}\right)\right), \tag{1.18}$$

(if μ is is exponentially damped).

II. *Case of the fractional Lévy motion Z^H.* For any constants $c_1 > c_*$ and $c_2 < c_*$ there exists $y = y(c_1, c_2)$ such that for $x/t^{H+1/2} > y$, either

$$\exp\left(-\frac{c_1 x}{\Gamma(H+1/2)t^{H-1/2}} \ln\left(\frac{x}{t^{H+1/2}}\right)\right) \leq p_t(x) \leq \exp\left(-\frac{c_2 x}{\Gamma(H+1/2)t^{H-1/2}} \ln\left(\frac{x}{t^{H+1/2}}\right)\right), \tag{1.19}$$

(if μ is truncated), or

$$\exp\left(-\frac{c_1 x}{\Gamma(H+1/2)t^{H-1/2}}\ln^{\frac{\beta-1}{\beta}}\left(\frac{x}{t^{H+1/2}}\right)\right)\le p_t(x)\le\exp\left(-\frac{c_2 x}{\Gamma(H+1/2)t^{H-1/2}}\ln^{\frac{\beta-1}{\beta}}\left(\frac{x}{t^{H+1/2}}\right)\right) \tag{1.20}$$

(if μ is exponentially damped).

In this paper we restrict ourselves to the case of one-dimensional processes in order to make the exposition reasonably short, and to give the main results in their most transparent form. These results have straightforward generalizations to the multi-dimensional case; we postpone the discussion of these generalizations to a further publication. We also restrict our considerations of the Lévy process Z and the fractional Lévy motion Z^H to the case where the time variable t is separated from 0. The small time estimates require additional analysis of the local behavior of the Lévy measure of the noise; this analysis is performed in the separate article [39].

2 The main result

2.1 Preliminaries

Everywhere below Z is a Lévy process and ψ is its characteristic exponent; that is, (1.1) and (1.2) hold.

To exclude from consideration the trivial cases, we assume that $b = 0$ and $\mu(\mathbb{R}) > 0$; that is, Z does not contain a diffusion part, and contains a non-trivial jump part. Moreover, we assume that μ satisfies (1.11), which is motivated by our intent to analyze the distribution density on the positive half-line. Finally, we assume Z to be centered, which means that the characteristic exponent is of the form

$$\psi(z)=\int_{\mathbb{R}}\left(e^{iuz}-1-izu\right)\mu(du),\quad z\in\mathbb{R}. \tag{2.1}$$

This assumption does not restrict the generality: under (1.5), the increments of Z have moments of all orders, therefore the difference between the processes with characteristic exponents (1.2) and (2.1) is given by the explicitly calculable constant, which clearly does not effect the distributional properties.

We consider Lévy driven stochastic integrals of the form

$$Y_t=\int_I f(t,s)dZ_s,\quad t\in\mathbb{T}, \tag{2.2}$$

where $\mathbb{T}\subset\mathbb{R}$ is some set, and $I\subset\mathbb{R}$ is an interval. We allow the case where the interval I belongs not only to the half-line, but to whole $\mathbb{R}$. In this case, the process given by (2.2) is assumed to be well defined on the whole line $\mathbb{R}$, and to have independent and stationary distributed increments, with the characteristic exponent of the increments still being of the form (2.1). A standard version of such a process is the so called two-sided Lévy process

$$Z_t=\begin{cases}Z^1_t, & t\ge 0\\ -Z^2_{-t-}, & t<0\end{cases},$$

where Z^1 and Z^2 are two independent copies of a Lévy process, defined on $\mathbb{R}^+$.

We interpret (2.2) as an integral with respect to an infinitely divisible random measure; for the general theory of such integrals we refer to [49]. Under (1.5), the integral (2.2) is well defined if, and only if,

$$\int_I f^2(t,s)\,ds < +\infty, \quad t \in \mathbb{T}, \tag{2.3}$$

and in that case its characteristic function admits the representation

$$Ee^{izY_t} = \exp\left[\int_I \int_{\mathbb{R}} \left(e^{izf(t,s)u} - 1 - izf(t,s)u\right) \mu(du)ds\right], \quad z \in \mathbb{R}, \quad t \in \mathbb{T}, \tag{2.4}$$

see Theorem 2.7 from [49]. In what follows we assume that f satisfies (2.3), and $f(t,\cdot)$ is bounded for every $t \in \mathbb{T}$. To exclude the trivial case $Y_t = 0$ a.s., we assume $\int_I f^2(t,s)\,ds > 0$, $t \in \mathbb{T}$. We also assume

$$\int_I (f(t,s) \vee 0)^2\,ds > 0, \quad t \in \mathbb{T}. \tag{2.5}$$

This does not restrict generality since otherwise one can consider $-Y_t$ instead of Y_t.

For a Borel set $A \subset \mathbb{R}$, denote

$$\Theta(t,z,A) = \iint_{\{(s,u)\in I\times\mathbb{R}:\, f(t,s)u\in A\}} (1 - \cos(f(t,s)zu))\mu(du)ds, \quad t \in \mathbb{R}^+, z \in \mathbb{R}.$$

The functions $\Theta(\cdot,\cdot,A)$, with properly chosen sets A, will be used below as a tool for studying the properties of distribution densities of Lévy driven stochastic integrals Y. One statement of such a type is formulated in the proposition below, which is in fact the classic Hartman-Wintner sufficient condition ([31]), reformulated in the context of Lévy driven stochastic integrals.

As usual, we denote by $C_b^k(\mathbb{R})$ the class of function, continuous and bounded together with their derivatives up to order k.

Proposition 2.1. *For given $t \in \mathbb{T}$, $k \in \mathbb{Z}_+$, and $|z|$ large enough, let*

$$\Theta(t,z,\mathbb{R}) \geq (k+1+\delta)\ln|z| \tag{2.6}$$

with some $\delta > 0$.

Then Y_t has a distribution density p_t, which belongs to the class $C_b^k(\mathbb{R})$.

In particular, if for a given $t \in \mathbb{T}$

$$\Theta(t,z,\mathbb{R}) \gg \ln|z| \quad \text{as} \quad |z| \to \infty, \tag{2.7}$$

then Y_t has a distribution density $p_t \in C_b^\infty$.

Proof. By (2.4), condition (2.6) implies

$$|Ee^{izY_t}| \leq |z|^{-k-1-\delta} \quad \text{for } |z| \text{ large enough.} \tag{2.8}$$

Hence the required statement follows by the inversion formula for the Fourier transform. □

As usual, we write $f(\xi) \sim g(\xi)$, $\xi \to \infty$, or $f(\xi) = o(g(\xi))$, $\xi \to \infty$, if $\lim_{\xi\to\infty} \frac{f(\xi)}{g(\xi)} = 1$ or $\lim_{\xi\to\infty} \frac{f(\xi)}{g(\xi)} = 0$, respectively. We also use the notation $f(\xi) \ll g(\xi)$, $\xi \to \infty$, instead of $f(\xi) = o(g(\xi))$, $\xi \to \infty$, when it is more convenient. The same conventions are used when functions f and g depend on t and/or on x.

2.2 The main result: formulation and discussion

Since f is bounded, the exponential integrability assumption (1.5) implies that for $t \in \mathbb{T}$ the function

$$\Psi(t,z) = \int_I \int_{\mathbb{R}} \left(e^{-izf(t,s)u} - 1 + izf(t,s)u\right) \mu(du)ds, \quad t \in \mathbb{T}, \quad z \in \mathbb{C},$$

is well defined and analytic with respect to z. Denote

$$H(t,x,z) = ixz + \Psi(t,z),$$

and observe that, assuming (2.6), we have

$$p_t(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{H(t,x,z)} dz, \quad x \in \mathbb{R}, \tag{2.9}$$

which is just the inversion formula for for the characteristic function of Y_t, combined with the change of variables $z \mapsto -z$.

Denote

$$\mathscr{M}_k(t,\xi) = \frac{\partial^k}{\partial \xi^k} \Psi(t,i\xi), \quad k \geq 1, \quad \xi \in \mathbb{R}.$$

Clearly,

$$\mathscr{M}_k(t,\xi) = \int_I \int_{\mathbb{R}} u^k f^k(t,s) e^{\xi f(t,s)u} \mu(du)ds = \frac{\partial^k}{\partial \xi^k} H(t,x,i\xi), \quad k \geq 2.$$

Since μ and $f(t,\cdot)$ are assumed to be non-degenerate, we have $\mathscr{M}_2(t,\xi) > 0$. Therefore there exists at most one solution $\xi(t,x)$ to the equation

$$\frac{\partial}{\partial \xi} H(t,x,i\xi) = 0. \tag{2.10}$$

Clearly, for any $t \in \mathbb{T}$ we have $\xi(t,0) = 0$. Note that

$$\mathscr{M}_1(t,\xi) = \frac{\partial}{\partial \xi} \Psi(t,i\xi) = \int_I \int_{\mathbb{R}} uf(t,s)\left(e^{\xi f(t,s)u} - 1\right) \mu(du)ds = \int_{I \times \mathbb{R}} v(e^{\xi v} - 1)\mu_{t,f}(dv),$$

where $\mu_{t,f}$ denotes the image of the measure $\mu(du)ds$ under the mapping

$$I \times \mathbb{R} \ni (s,u) \mapsto f(t,s)u \in \mathbb{R}.$$

Under the assumptions (1.11) and (2.5), which we assume to hold everywhere below, we have $\mu_{t,f}(\mathbb{R}^+) > 0$. Therefore $\mathscr{M}_1(t,\xi) \to +\infty$ as $\xi \to +\infty$, which means that $\xi(t,x)$ is well defined and positive for $x > 0$, and

$$\xi(t,x) \to +\infty, \quad \text{as} \quad x \to +\infty. \tag{2.11}$$

Note that $z = i\xi(t,x)$ is the unique critical point for $H(t,x,\cdot)$ on the line $i\mathbb{R}$.

We put

$$D(t,x) = H(t,x,i\xi(t,x)), \quad K(t,x) = \mathscr{M}_2(t,\xi(t,x)) = \frac{\partial^2}{\partial \xi^2} H(t,x,i\xi)\Big|_{\xi=\xi(t,x)}.$$

In the sequel, we fix $\mathscr{A} \subset \mathbb{T} \times \mathbb{R}^+$ and denote

$$\mathscr{T} = \{t : \exists x \in \mathbb{R}^+, (t,x) \in \mathscr{A}\}, \quad \mathscr{B} = \{(t,\xi) : \exists (t,x) \in \mathscr{A}, (t,\xi) = (t,\xi(t,x))\}.$$

For instance, if $\mathscr{A} = \mathbb{T}' \times \mathbb{R}^+$ with some $\mathbb{T}' \subset \mathbb{T}$, then $\mathscr{T} = \mathbb{T}'$ and $\mathscr{B} = \mathscr{A}$.

In the following theorem, which represents the main result of the paper, the function $\theta : \mathscr{T} \to (0,+\infty)$ is assumed to be bounded away from zero on $\mathbb{T}$, and the function $\chi : \mathscr{T} \to (0,+\infty)$ is assumed to be bounded away from zero on every set $\{t : \theta(t) \le c\}$, $c > 0$. For a particular process Y, the choice of the "scaling" functions θ and χ is determined by the structure of the kernel f, see Section 4 below.

Theorem 2.1. *Assume that the following conditions hold true:*

(H_1) $\mathscr{M}_4(t,\xi) \ll \mathscr{M}_2^2(t,\xi)$, $\theta(t) + \xi \to \infty$, $(t,\xi) \in \mathscr{B}$.

(H_2)

$$\ln\left(\left(\chi^{-2}(t)\frac{\mathscr{M}_4(t,\xi)}{\mathscr{M}_2(t,\xi)}\right) \vee 1\right) + \ln\left(\left(\ln\left((1 \vee \chi^{-1}(t))\mathscr{M}_2(t,\xi)\right)\right) \vee 1\right)$$
$$\ll \ln\theta(t) + \chi(t)\xi, \quad \theta(t) + \xi \to \infty, \quad (t,\xi) \in \mathscr{B}.$$

(H_3) There exist $R > 0$ and $\delta > 0$ such that

$$\Theta(t,z,\mathbb{R}^+) \ge (1+\delta)\ln(\chi(t)|z|), \quad t \in \mathscr{T}, \quad |z| > R. \tag{2.12}$$

(H_4) There exists $r > 0$ such that for every $\varepsilon > 0$,

$$\inf_{|z|>\varepsilon} \Theta(t,z,[r\chi(t),+\infty)) \ge \theta(t)\Big((\varepsilon\chi(t))^2 \wedge 1\Big).$$

Then for every $t \in \mathscr{T}$ the law of Y_t has a continuous bounded distribution density $p_t(x)$, and

$$p_t(x) \sim \frac{1}{\sqrt{2\pi K(t,x)}} e^{D(t,x)}, \quad \theta(t) + x \to \infty, \quad (t,x) \in \mathscr{A}. \tag{2.13}$$

Remark 2.1. *(On conditions)*. The conditions of Theorem 2.1 are rather technical and abstract. In Sections 3 and 4 below we give their more explicit versions, formulated in terms of the Lévy measure μ and the kernel f. Note that (H_1) and (H_2) are, in fact, the assumptions on the growth of the tails of the Lévy measure μ. In addition, (H_2) is balanced with (H_4), which in turn is closely related to the so called *Cramer condition* (see, for example, [43], [33], Chapter 3 §3, and the discussion prior to Lemma 3.5 below). Finally, (H_3) is a proper uniform version of the Hartman-Wintner condition, see Proposition 2.1. Clearly, one can consider the stronger version of condition (2.12) with $k+1+\delta$ instead of $1+\delta$ (cf. (2.6) and (2.7)), and provide the asymptotic relations similar to (2.13) for the derivatives of the distribution density $p_t(x)$ up to order k.

Remark 2.2. *(On relation (2.13))*. 1. Note that the asymptotic relation (2.13) corresponds completely to the standard form of an asymptotic relation obtained by the Laplace method. Typically, within this method one can prove that the integral

$$\int_{(a,b)} e^{-F(\lambda,x)}\,dx$$

is asymptotically equivalent to

$$\sqrt{\frac{2\pi}{F''_{xx}(\lambda, x_\lambda)}}e^{-F(\lambda,x_\lambda)}, \quad x_\lambda := \arg\min_x F(\lambda, x). \tag{2.14}$$

Clearly, (2.13) is exactly of the form (2.14) with appropriate F and additional normalizer $1/(2\pi)$, which comes from the inverse Fourier transform formula.

2. Our approach is in some sense related to the Large Deviations Principle (LDP). Namely, if $P^l_t(dx)$ is the probability measure associated with $Y^l_t := \frac{1}{l}\sum_{i=1}^l Z^i_t$, where $\{Z^i_t\}_{i=1}^l$ are independent copies of $(Z_t)_{t\geq 0}$, then $P^l_t(dx)$ satisfies the LDP with a *good rate function* $\Lambda_t(x) := -D(t,x)$, in the sense that for all measurable subsets $A \subset \mathbb{R}$

$$-\inf_{x\in \text{interior}(A)} \Lambda_t(x) \leq \liminf_{l\to\infty} \frac{1}{l}\ln P^l_t(A) \leq \limsup_{l\to\infty}\frac{1}{l}\ln P^l_t(A) \leq -\inf_{x\in\text{closure}(A)}\Lambda_t(x);$$

see [28]. Moreover, assuming the exponential integrability condition (1.5) and existence of the transition probability density $p_t(x)$ for $t > t_0$, it is shown in [40] that

$$\lim_{l\to\infty}\frac{\ln p_{tl}(lx)}{l} = D(t,x), \tag{2.15}$$

cf. (1.15) and (1.16).

2.3 Proof

Note that $\Theta(t,z,A)$ depends on the set A monotonously. Hence (H_3) yields (2.6), and therefore (2.9) holds. In what follows, we analyze the right hand side of (2.9). We divide this analysis into several steps.

Step 1: changing the integration contour. We prove that

$$p_t(x) = \frac{1}{2\pi}\int_{i\xi(t,x)+\mathbb{R}} e^{H(t,x,z)}\,dz = \frac{1}{2\pi}\int_{\mathbb{R}} e^{H(t,x,\eta+i\xi(t,x))}\,d\eta. \tag{2.16}$$

Recall that we assumed $x \geq 0$, which in turn implies $\xi(t,x)\geq 0$. Consider the domain

$$G_M := \Big\{z\in\mathbb{C} : \operatorname{Im} z \in [0,\xi(t,x)], \quad \operatorname{Re} z\in[-M,M], \quad M>0\Big\}. \tag{2.17}$$

The function $H(t,x,z)$ is analytic in G_M, hence by the Cauchy theorem

$$\int_{\partial G_M} e^{H(t,x,z)}\,dz = 0. \tag{2.18}$$

Consider the integrals

$$\int_0^1 e^{H(t,x,\pm M + iv\xi(t,x))}dv. \tag{2.19}$$

We have

$$\operatorname{Re} H(t,x,\eta+i\xi) = -x\xi - \int_I\int_{\mathbb{R}}\big(1-e^{f(t,s)\xi u}\cos(f(t,s)\eta u) + f(t,s)\xi u\big)\mu(du)ds$$

$$= H(t,x,i\xi) - \int_I \int_{\mathbb{R}} e^{f(t,s)\xi u}(1-\cos(f(t,s)\eta u))\mu(du)ds, \quad \xi,\eta \in \mathbb{R}. \tag{2.20}$$

The function $\xi \mapsto H(t,x,i\xi)$ is real-valued, convex, and attains its minimal value at the point $\xi(t,x)$. Then $H(t,x,i\xi) \le H(t,x,0) = 0$ for $\xi \in [0,\xi(t,x)]$. On the other hand, for every $\xi \ge 0$

$$\int_I \int_{\mathbb{R}} e^{f(t,s)\xi u}(1-\cos(f(t,s)\eta u))\mu(du)ds \ge \iint_{\{(s,u)\in I\times\mathbb{R}:\, f(t,s)u>0\}} (1-\cos(f(t,s)\eta u))\mu(du)ds$$
$$= \Theta(t,\eta,\mathbb{R}^+).$$

Therefore

$$\operatorname{Re} H(t,x,\pm M + i\nu\xi(t,x)) \le -\Theta(t,\pm M,\mathbb{R}^+), \quad \nu \in [0,1].$$

Thus, condition (H_3) implies that the integrals in (2.19) tend to 0 as $M \to +\infty$, which together with (2.17) gives (2.16).

In what follows we denote

$$R(t,x,\eta) = \operatorname{Re} H(t,x,\eta + i\xi(t,x)),$$

$$I(t,x,\eta) = \operatorname{Im} H(t,x,\eta + i\xi(t,x)) = x\eta - \int_I \int_{\mathbb{R}} \big(e^{f(t,s)\xi(t,x)u}\sin(\eta f(t,s)u) - \eta f(t,s)u\big)\mu(du).$$

Since a distribution density is real valued, we derive from (2.16)

$$p_t(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{R(t,x,\eta)}\cos(I(t,x,\eta))\,d\eta. \tag{2.21}$$

Before proceeding further on, let us give a short description of the rest of the proof. We will estimate the integral (2.21) using the appropriate version of the Laplace method (see [27] for its description). In our case the application of the Laplace method meets some difficulties, since the expression under the integral contains two functions R and I. Therefore we introduce two intervals $[-\alpha,\alpha]$ and $[-\beta,\beta]$, on which R and I are controllable in terms of their Taylor's expansions. Then we split the integral into the sum of integrals over $\{|\eta| \le \alpha\}, \{|\eta| \in (\alpha,\beta]\}$, and $\{|\eta| > \beta\}$, and estimate these integrals separately. As in the standard Laplace method, the first two integrals are controlled by using Taylor expansion arguments. For the third integral, any standard considerations, like convexity arguments from [30], cannot be applied. Therefore, we use the specific arguments based on the structure of the functional under consideration.

Step 2: choosing α,β. Following the explanations given above, we split the integral (2.21) into the sum

$$\frac{1}{2\pi}\Big[\int_{|\eta|\le\alpha} + \int_{|\eta|\in(\alpha,\beta]} + \int_{|\eta|>\beta}\Big]\Big(e^{R(t,x,\eta+i\xi(t,x))}\cos(I(t,x,\eta+i\xi(t,x)))\,d\eta\Big)$$
$$= J_1(t,x) + J_2(t,x) + J_3(t,x), \tag{2.22}$$

where $\alpha \equiv \alpha(t,x)$ and $\beta \equiv \beta(t,x)$ are auxiliary functions. The function β is defined by

$$\beta(t,x) = \sqrt{\frac{\mathscr{M}_2(t,\xi(t,x))}{\mathscr{M}_4(t,\xi(t,x))}}. \tag{2.23}$$

Our aim in this step is to construct the function α in such a way that

$$0 < \alpha(t,x) \le \beta(t,x), \quad (t,x) \in \mathscr{A}, \tag{2.24}$$

$$\begin{aligned} \frac{1}{\mathscr{M}_2(t,\xi(t,x))} \ll \alpha^2(t,x) \ll \frac{\mathscr{M}_2(t,\xi(t,x))}{\mathscr{M}_4(t,\xi(t,x))}, \\ \alpha^3(t,x) \ll \frac{1}{\mathscr{M}_3(t,\xi(t,x))}, \quad \theta(t)+x \to \infty, \quad (t,x) \in \mathscr{A}. \end{aligned} \tag{2.25}$$

By the Cauchy inequality and condition (H_1), we have

$$\mathscr{M}_3^2(t,\xi) \le \mathscr{M}_2(t,\xi)\mathscr{M}_4(t,\xi) \ll \mathscr{M}_2^3(t,\xi), \quad \theta(t)+\xi \to \infty, \quad (t,\xi) \in \mathscr{B}.$$

Hence, there exists a function $\kappa = \kappa(t,\xi)$, such that

$$\begin{aligned} 1 \ll \kappa(t,\xi), \quad \kappa(t,\xi) \ll \mathscr{M}_2(t,\xi)\mathscr{M}_4^{-1/2}(t,\xi), \\ \kappa(t,\xi) \ll \mathscr{M}_2^{1/2}(t,\xi)\mathscr{M}_3^{-1/3}(t,\xi), \quad \theta(t)+\xi \to \infty, \quad (t,\xi) \in \mathscr{B}. \end{aligned} \tag{2.26}$$

Without loss of generality, we can assume the function κ to be locally bounded. Then we put

$$\alpha(t,x) = c\kappa(t,\xi(t,x))\mathscr{M}_2^{-1/2}(t,\xi(t,x)) \tag{2.27}$$

with some constant $c > 0$. By (2.26) and (2.11), we have (2.25). Since κ is locally bounded, the constant c can be chosen small enough to provide (2.24).

Step 3: estimating $J_1(t,x)$ in (2.22). A straightforward computation shows that

$$\frac{\partial}{\partial\eta}R(t,x,\eta)\Big|_{\eta=0} = \frac{\partial^3}{\partial\eta^3}R(t,x,\eta)\Big|_{\eta=0} = 0, \quad \frac{\partial^2}{\partial\eta^2}R(t,x,\eta)\Big|_{\eta=0} = -\mathscr{M}_2(t,\xi(t,x)),$$

$$\left|\frac{\partial^4}{\partial\eta^4}R(t,x,\eta)\right| = \left|\int_I\int_{\mathbb{R}} u^4 f^4(t,s) e^{\xi f(t,s)u}\cos\Big(\eta f(t,s)u\Big)\,\mu(du)ds\right| \le \mathscr{M}_4(t,\xi(t,x)), \quad \eta \in \mathbb{R}, \tag{2.28}$$

which gives

$$-\mathscr{M}_2(t,\xi(t,x)) - \frac{\eta^2}{2}\mathscr{M}_4(t,\xi(t,x)) \le \frac{\partial^2}{\partial\eta^2}R(t,x,\eta) \le -\mathscr{M}_2(t,\xi(t,x)) + \frac{\eta^2}{2}\mathscr{M}_4(t,\xi(t,x)) \tag{2.29}$$

for all $\eta \in \mathbb{R}$. Therefore by the estimate for α^2 in (2.25) we get

$$\begin{aligned} \sup_{|\eta|\le\alpha} \frac{\partial^2}{\partial\eta^2}R(t,x,\eta) \sim -\mathscr{M}_2(t,\xi(t,x)), \\ \inf_{|\eta|\le\alpha} \frac{\partial^2}{\partial\eta^2}R(t,x,\eta) \sim -\mathscr{M}_2(t,\xi(t,x)), \quad \theta(t)+x \to \infty, \quad (t,x) \in \mathscr{A}. \end{aligned} \tag{2.30}$$

Next, similarly to (2.3) we get

$$I(t,x,\eta)\Big|_{\eta=0} = \frac{\partial}{\partial\eta}I(t,x,\eta)\Big|_{\eta=0} = \frac{\partial^2}{\partial\eta^2}I(t,x,\eta)\Big|_{\eta=0} = 0, \quad \left|\frac{\partial^3}{\partial\eta^3}I(t,x,\eta)\right| \le \mathscr{M}_3(t,\xi(t,x)).$$

Note that the equality for $\frac{\partial}{\partial \eta} I$ holds true because $z = i\xi(t,x)$ is a critical point for $H(t,x,\cdot)$. Hence the estimate for α^3 in (2.25) implies

$$\sup_{|\eta|\le\alpha} |I(t,x,\eta)| \to 0, \quad \theta(t)+x\to\infty, \quad (t,x)\in\mathscr{A}. \tag{2.31}$$

Recall that $K(t,x) \equiv \mathscr{M}_2(t,\xi(t,x))$ and $D(t,x) \equiv H(t,x,i\xi(t,x)) = R(t,x,0)$. From (2.30) and (2.31) we get

$$\begin{aligned}
\int_{|\eta|\le\alpha} e^{R(t,x,\eta)} \cos I(t,x,\eta) d\eta &\sim e^{R(t,x,0)} \int_{|\eta|\le\alpha} e^{-\frac{K(t,x)\eta^2}{2}} d\eta \\
&= \sqrt{\frac{2\pi}{K(t,x)}} e^{R(t,x,0)} \int_{|\eta|\le\sqrt{K(t,x)}\alpha} \frac{e^{-\frac{|\eta|^2}{2}}}{\sqrt{2\pi}} d\eta \\
&\sim \sqrt{\frac{2\pi}{K(t,x)}} e^{D(t,x)}, \quad \theta(t)+x\to\infty, \quad (t,x)\in\mathscr{A},
\end{aligned} \tag{2.32}$$

where in the last relation we used the lower estimate for α in (2.25). Thus,

$$J_1(t,x) \sim \frac{1}{\sqrt{2\pi K(t,x)}} e^{D(t,x)}, \quad \theta(t)+x\to\infty, \quad (t,x)\in\mathscr{A}. \tag{2.33}$$

Step 4: proving that $J_2(t,x)$ in (2.22) is negligible. On the set $\{|\eta|\le\beta\}$, the function R is controlled by its Taylor expansion. Hence for the integral $J_2(t,x)$ we can apply standard arguments of the Laplace method.

By (2.29) we have for $|\eta|\le\beta$

$$R(t,x,\eta) \le R(t,x,0) - \frac{1}{4}\mathscr{M}_2(t,\xi(t,x))\eta^2,$$

which, together with the lower estimate for α in (2.25), gives

$$\begin{aligned}
|J_2(t,x)| &\le \int_{|\eta|\in(\alpha,\beta]} e^{R(t,x,\eta)} d\eta \le e^{R(t,x,0)} \int_{|\eta|>\alpha} e^{-\frac{\mathscr{M}_2(t,\xi(t,x))\eta^2}{4}} d\eta \\
&= \frac{e^{D(t,x)}}{\sqrt{K(t,x)}} \int_{|y|>\alpha\sqrt{K(t,x)}} e^{-\frac{\eta^2}{4}} d\eta \ll J_1(t,x), \quad \theta(t)+x\to\infty, \quad (t,x)\in\mathscr{A}.
\end{aligned} \tag{2.34}$$

Step 5: proving that $J_3(t,x)$ in (2.22) is negligible. By (2.3),

$$\begin{aligned}
|J_3(t,x)| &\le \frac{1}{2\pi} \int_{|\eta|>\beta} e^{R(t,x,\eta)} d\eta \\
&\le \frac{1}{2\pi} e^{D(t,x)} \int_{|\eta|>\beta} \exp\left\{ -\int_I \int_{\mathbb{R}} e^{f(t,s)\xi u} (1-\cos(f(t,s)\eta u)) \mu(du) ds \right\} d\eta.
\end{aligned}$$

Therefore, by (2.33), to prove $J_3(t,x) \ll J_1(t,x)$ we need to check that

$$\int_{|\eta|>\beta} e^{-\Delta(t,x,\eta)} d\eta \ll K^{-1/2}(t,x) = \mathscr{M}_2^{-1/2}(t,\xi(t,x)), \quad \theta(t)+x\to\infty, \quad (t,x)\in\mathscr{A}, \tag{2.35}$$

where

$$\Delta(t,x,\eta) = \int_I \int_{\mathbb{R}} e^{f(t,s)\xi(t,x)u}(1-\cos(f(t,s)\eta u))\mu(du)ds.$$

Recall that $\xi(t,x) \geq 0$. Then for any $\sigma \in (0,1)$ we have for some $r > 0$

$$\begin{aligned}\Delta(t,x,\eta) &\geq \iint_{\{(s,u)\in I\times\mathbb{R}: f(t,s)u\geq 0\}} e^{f(t,s)\xi(t,x)u}(1-\cos(f(t,s)\eta u))\mu(du)ds \\ &\geq (1-\sigma)\Theta(t,\eta,\mathbb{R}^+) + \sigma e^{r\chi(t)\xi(t,x)}\Theta(t,\eta,[r\chi(t),+\infty)).\end{aligned} \tag{2.36}$$

Condition (H_3), combined with the trivial observation that $\Theta(t,\eta,\mathbb{R}^+)$ is non-negative, yields

$$\int_{\mathbb{R}} e^{-(1-\sigma)\Theta(t,\eta,\mathbb{R}^+)}\, d\eta < c(1 \vee \chi^{-1}(t)), \tag{2.37}$$

provided that σ is chosen such that $(1-\sigma)(1+\delta) > 1$. Applying condition (H_4) with $\varepsilon = \beta(t,x)$ gives for $|\eta| \geq \beta(t,x)$

$$e^{r\chi(t)\xi(t,x)}\Theta(t,\eta,[r\chi(t),+\infty)) \geq e^{r\chi(t)\xi(t,x)}\theta(t)\Big((\beta(t,x)\chi(t))^2 \wedge 1\Big).$$

Thus, in view of (2.37), to show (2.35) it is enough to prove as $\theta(t)+x \to \infty$, $(t,x) \in \mathscr{A}$,

$$(1 \vee \chi^{-1}(t)) \exp\Big[-\sigma e^{r\chi(t)\xi(t,x)}\theta(t)\Big((\beta(t,x)\chi(t))^2 \wedge 1\Big)\Big] \ll \mathscr{M}_2^{-1/2}(t,\xi(t,x)), \tag{2.38}$$

for every $\sigma > 0$. By the definition (2.23) of $\beta(t,x)$, we have

$$\Big((\beta(t,x)\chi(t))^2 \wedge 1\Big) = \left(\left(\chi^{-2}(t)\frac{\mathscr{M}_4(t,\xi)}{\mathscr{M}_2(t,\xi)}\right) \vee 1\right)^{-1}. \tag{2.39}$$

Condition (H_2) and the assumptions on θ and χ, imposed prior to Theorem 2.1, yield for any $\sigma > 0$, as $\theta(t)+\xi \to \infty$, $(t,\xi) \in \mathscr{B}$,

$$\left(\left(\chi^{-2}(t)\frac{\mathscr{M}_4(t,\xi)}{\mathscr{M}_2(t,\xi)}\right) \vee 1\right) \ln\Big((1 \vee \chi^{-1}(t))\mathscr{M}_2(t,\xi)\Big) \ll \sigma\theta(t)e^{r\chi(t)\xi}.$$

This relation, combined with (2.39), (2.11), and the relation $\xi(t,x) \geq 0$, yields

$$\ln\Big((1 \vee \chi^{-1}(t))\mathscr{M}_2(t,\xi)\Big) \ll \sigma\theta(t)e^{r\chi(t)\xi(t,x)}\Big((\beta(t,x)\chi(t))^2 \wedge 1\Big), \quad \theta(t)+x \to \infty, \quad (t,x) \in \mathscr{A},$$

which in turn implies (2.38) and completes the proof of (2.35).

We have proved

$$J_2(t,x) \ll J_1(t,x), \quad J_3(t,x) \ll J_1(t,x).$$

By (2.33) we get the statement of the theorem. □

3 Explicit conditions: fixed time setting

Our further aim is to give explicit and tractable sufficient conditions which provide assumptions (H_1) – (H_4) of Theorem 2.1. In this section we consider the case where the time variable is fixed. Therefore everywhere below in this section we assume

$$\mathbb{T} = \{t\}, \quad \mathscr{A} = \mathscr{B} = \{t\} \times \mathbb{R}, \quad \mathscr{T} = \{t\}.$$

We skip the variable t in the notation and write, for instance, Y, $f(s)$, $\mathscr{M}_k(\xi)$ instead of Y_t, $f(t,s)$, $\mathscr{M}_k(t,\xi)$, respectively.

In the fixed time setting the assumptions (H_1) – (H_4) look more simple: in particular, functions $\theta(t)$ and $\chi(t)$ degenerate to some constants θ and χ. Therefore it is appropriate to introduce the set of conditions which will be useful later on.

$(\hat{H}_1)$ $\mathscr{M}_4(\xi) \ll \mathscr{M}_2^2(\xi)$, $\xi \to +\infty$.

$(\hat{H}_2)$ $\ln\left(\left(\frac{\mathscr{M}_4(\xi)}{\mathscr{M}_2(\xi)}\right) \vee 1\right) + \ln\ln \mathscr{M}_2(\xi) \ll \xi$, $\xi \to +\infty$.

$(\hat{H}_3)$ There exist $R > 0$, $\delta > 0$ such that

$$\Theta(z, \mathbb{R}^+) \geq (1+\delta) \ln|z|, \quad |z| > R. \tag{3.1}$$

$(\hat{H}_4)$ There exist $q > 0$ and $\vartheta > 0$ such that for every $\varepsilon > 0$

$$\inf_{|z|>\varepsilon} \Theta(z, [q, +\infty)) \geq \vartheta\left(\varepsilon^2 \wedge 1\right).$$

One can easily see that in the fixed time setting conditions (H_1) – (H_4) are equivalent to $(\hat{H}_1)$ – $(\hat{H}_4)$. Indeed, the constants $\theta > 0$ and $\chi > 0$, which come, respectively, from the functions $\theta(t)$, $\chi(t)$, are suppressed in (H_2) by the term ξ. In (H_4), the constant χ can be eliminated by a proper change of the constants r and θ; we denote these new constants by q and ϑ.

Clearly, Y is infinitely divisible with the Lévy measure

$$\mu_f(A) = \iint_{I\times\mathbb{R}} \mathbb{1}_{uf(s)\in A}\mu(du)ds.$$

In what follows we demonstrate that conditions $(\hat{H}_1)$ – $(\hat{H}_4)$, which are in fact the assumptions on μ_f, can be verified efficiently in the terms of the kernel f and the initial Lévy measure μ.

3.1 Assumptions $(\hat{H}_1)$ and $(\hat{H}_2)$

Observe that $(\hat{H}_1)$ and $(\hat{H}_2)$ control the growth rate of the "tails" of μ_f. The following two lemmas show that these assumptions can be verified in the terms of similar "tail" conditions imposed on μ. Denote

$$M_1(\xi) = \int_{\mathbb{R}} u(e^{\xi u} - 1)\mu(du), \quad M_k(\xi) = \int_{\mathbb{R}} u^k e^{\xi u}\mu(du), \quad k \geq 2. \tag{3.2}$$

Clearly,

$$\mathscr{M}_k(\xi) = \int_I f^k(s) M_k(f(s)\xi)\, ds, \quad k \geq 1. \tag{3.3}$$

Lemma 3.1. *Assume*

(T_1) there exists $\gamma \in (0,1)$ such that $M_4(\xi) \ll M_2^2(\gamma\xi)$, $\xi \to +\infty$.

Then ($\hat{H}_1$) holds true.

Proof. Under the assumption (1.11) we have $M_k(\xi) \to +\infty$, $\xi \to +\infty$, $k \geq 1$. In addition, by Hölder inequality, for every $a \in (0,1)$ and $k \geq 2$,

$$M_k(a\xi) \leq [M_k(\xi)]^a \left[\int_{\mathbb{R}} u^k \mu(du)\right]^{1-a}$$

implying

$$M_k(a\xi) \ll M_k(\xi), \quad \xi \to +\infty. \tag{3.4}$$

Denote $F = \operatorname{esssup}_{s\in I} f(s)$ and $I_{f,\gamma} = \{s : f(s) \geq \gamma F\}$. Recall that f is assumed to be bounded, which together with (2.5) yields $F \in (0,+\infty)$. Since f^2 is integrable on I and f is bounded, $f^k, k \geq 3$ is integrable as well. Then (3.4) yields

$$\mathscr{M}_k(\xi) = \left(\int_{I_{f,\gamma}} f^k(s) M_k(f(s)\xi)\,ds\right)\Big[1 + o(1)\Big], \quad \xi \to +\infty. \tag{3.5}$$

Since M_2 is convex and $M_2(\xi) \to +\infty$, as $\xi \to +\infty$, there exists ξ_0 such that M_2 is increasing on $[\xi_0, +\infty)$. Then for $\xi > \xi_0\gamma^{-1}F^{-1}$ we have

$$\begin{aligned}\left(\int_{I_{f,\gamma}} f^2(s) M_2(f(s)\xi)\,ds\right)^2 &= \int_{I_{f,\gamma}}\int_{I_{f,\gamma}} f^2(s_1) f^2(s_2) M_2(f(s_1)\xi) M_2(f(s_2)\xi)\,ds_1 ds_2 \\ &\geq M_2^2(\gamma F\xi)\left(\int_{I_{f,\gamma}} f^2(s)\,ds\right)^2.\end{aligned}$$

Similarly, for sufficiently large ξ we have

$$\int_{I_{f,\gamma}} f^4(s) M_4(f(s)\xi)\,ds \leq M_4(F\xi)\int_{I_{f,\gamma}} f^4(s)\,ds.$$

These relations, together with (3.5) and (T_1), imply ($\hat{H}_1$). □

Lemma 3.2. *Assume*

(T_2) $\ln\left(\frac{M_4(\xi)}{M_2(\xi)} \vee 1\right) + \ln\ln M_2(\xi) \ll \xi$, $\xi \to +\infty$.

Then ($\hat{H}_2$) holds true.

Proof. Fix an arbitrary $\gamma \in (0,1)$. For ξ large enough, we have by (3.5)

$$\mathscr{M}_2(\xi) \sim \int_{I_{f,\gamma}} f^2(s) M_2(f(s)\xi)\,ds \leq M_2(F\xi)\left(\int_{I_{f,\gamma}} f^2(s)\,ds\right), \quad \xi \to +\infty,$$

which together with (T_2) gives

$$\ln\ln \mathscr{M}_2(\xi) \ll \xi, \quad \xi \to +\infty. \tag{3.6}$$

On the other hand, (T_2) implies that for every $\varepsilon > 0$ and ξ large enough,

$$M_4(\xi) \le e^{\varepsilon\xi} M_2(\xi).$$

Then for ξ large enough we have by (3.5)

$$\begin{aligned}\mathscr{M}_4(\xi) &\sim \int_{I_{f,\gamma}} f^4(s) M_4(f(s)\xi)\, ds \le e^{\varepsilon F \xi} \int_{I_{f,\gamma}} f^4(s) M_2(f(s)\xi)\, ds \\ &\le F^2 e^{\varepsilon F\xi} \int_{I_{f,\gamma}} f^2(s) M_2(f(s)\xi)\, ds \sim F^2 e^{\varepsilon F \xi} \mathscr{M}_2(\xi).\end{aligned}$$

Consequently,

$$\limsup_{\xi\to+\infty} \xi^{-1} \ln\left(\frac{\mathscr{M}_4(\xi)}{\mathscr{M}_2(\xi)}\right) \le \varepsilon F.$$

Since $\varepsilon > 0$ is arbitrary, this relation combined with (3.6) implies $(\hat{H}_2)$. $\square$

3.2 Assumptions $(\hat{H}_3)$ and $(\hat{H}_4)$

It will be convenient to consider, together with the assumption $(\hat{H}_3)$, its stronger version

$$(\hat{H}_3^s) \qquad \Theta(z, \mathbb{R}^+) \gg \ln|z|, \quad z \to \infty.$$

To proceed with the assumption $(\hat{H}_3^s)$, we introduce several conditions on the kernel f.

(F_1) $\int_I (f(s) \vee 0)^2\, ds > 0$.

(F_2) On some interval $[a, b] \subset I$, the function f is positive and has a continuous non-zero derivative.

(F_3) On some interval $(-\infty, b] \subset I$, the function f is positive, convex, and has at most exponential decay at $-\infty$; that is, there exists $\gamma > 0$ such that

$$\lim_{s\to-\infty} e^{-\gamma s} f(s) = +\infty. \tag{3.7}$$

(F_4) On some interval $(-\infty, b) \subset I$, the function f is positive, convex, and has a subexponential decay at $-\infty$; that is, (3.7) holds true for every $\gamma > 0$.

Note that when i increases from 1 to 4, the respective conditions (F_i) become stronger. Condition (F_1) is just our standing non-degeneracy assumption (2.5), listed here for further reference convenience. Conditions (F_1) – (F_4) are well designed to handle the particularly interesting classes of processes, mentioned in the Introduction. Namely,

1. for the Lévy process Z, one has $f(t,s) = \mathbb{I}_{[0,t]}(s)$, which satisfies (F_1) for every $t > 0$;

2. for the non-stationary version of a Lévy driven Ornstein-Uhlenbeck process, one has $f(t,s) = e^{\gamma(t-s)} \mathbb{I}_{[0,t]}(s)$, which satisfies (F_2) for every $t > 0$;

3. for the stationary version of a Lévy driven Ornstein-Uhlenbeck process, one has $f(s) = e^{\gamma s}\mathbb{I}_{(-\infty,0]}(s)$, which satisfies (F_3) with $b=0$;

4. for the fractional Lévy motion, one has $f(t,s) = \frac{1}{\Gamma(H+1/2)}\left[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\right]$, which for every $t>0$ satisfies (F_4) with $b=0$.

Recall several conditions which appeared in the literature in the context of the problem of studying local properties of infinitely divisible distributions.

A Lévy measure ν on $\mathbb{R}$ is said to satisfy the *Hartman-Wintner condition* ([31]), if

$$\int_{\mathbb{R}} (1-\cos zu)\nu(du) \gg \ln|z|, \quad z\to\infty. \tag{3.8}$$

Clearly, $(\hat{H}_3^s)$ is exactly the assumption on μ_f, restricted to $\mathbb{R}^+$, to satisfy the Hartman-Wintner condition.

An elementary inequality

$$cx^2\mathbb{I}_{|x|\le 1} \le 1-\cos x \le x^2\wedge 1, \quad x\in\mathbb{R}, \tag{3.9}$$

(where $c>0$ is some constant) provides the following pair of conditions, sufficient and necessary for (3.8), respectively:

$$\int_{|u|\le|z|^{-1}} (uz)^2\nu(du) \gg \ln|z|, \quad |z|\to\infty; \tag{3.10}$$

$$\int_{\mathbb{R}} [(uz)^2\wedge 1]\nu(du) \gg \ln|z|, \quad |z|\to\infty. \tag{3.11}$$

Condition (3.10) was introduced in [35], and is called the *Kallenberg condition*. Condition (3.11) was introduced in [41], where it was proved to be necessary for the existence of a *bounded* transition probability density of the solution to a (not necessarily linear) Lévy driven SDE. At the same time, for an Ornstein-Uhlenbeck process (1.7) with non-trivial drift ($\gamma\neq 0$), this condition is sufficient for the existence of C^∞ distribution density ([9]). Thus, for the non-stationary version of the Ornstein-Uhlenbeck process (1.7), condition (3.11) is a criterion.

We denote by μ_+ the restriction of μ to $\mathbb{R}^+$, and formulate the following set of "non-degeneracy" conditions on the measure μ.

(N_1) μ_+ satisfies (3.10).

(N_2) μ_+ satisfies (3.11).

(N_3) $\mu(\mathbb{R}^+) = +\infty$.

(N_4) $\mu(\mathbb{R}^+) > 0$.

Note that when i increases from $i=1$ to $i=4$, the respective conditions (N_i) become more mild; (N_4) is just our fixed non-degeneracy assumption (1.11), listed here for further reference convenience.

Lemma 3.3. *Assume for some $i=1,\dots,4$ conditions (N_i) and (F_i) hold.*

Then $(\hat{H}_3^s)$ holds true.

Proof. *Case* $i = 1$. From the positivity of $1 - \cos x$ and the first inequality in (3.9), it follows that

$$\begin{aligned}\Theta(z, \mathbb{R}^+) &= \iint_{(s,u):uf(s)>0} (1 - \cos(uf(s)z))\mu(du)ds \\ &\geq \iint_{(s,u):u>0,0<uf(s)<1/|z|} (1 - \cos(uf(s)z))\mu(du)ds \\ &\geq c \iint_{(s,u):u>0,0<uf(s)<1/|z|} u^2 f^2(s) z^2 \mu(du)ds \geq c z^2 \left[\iint_I f_+(s)^2\, ds\right] \int_{(0,(F|z|)^{-1})} u^2 \mu(du),\end{aligned}$$

here we keep the notation $F = \operatorname{esssup}_{s\in I} f(s)$. Combined with (3.10) for μ_+, the estimates above provide $(\hat{H}_3^s)$.

Case $i = 2$. Since f is positive on $[a, b]$, we have

$$\Theta(z, \mathbb{R}^+) \geq \iint_{(a,b)\times\mathbb{R}^+} (1 - \cos(uf(s)z))\, \mu(du)ds.$$

Let us show that

$$\int_a^b (1 - \cos(xf(s)))\, ds \geq c(x^2 \wedge 1) \tag{3.12}$$

holds true with some constant $c > 0$, which would imply $(\hat{H}_3^s)$ provided that the assumption (3.11) is satisfied. Consider the function

$$\Upsilon(x) = \int_a^b (1 - \cos(xf(s)))\, ds.$$

Clearly, $\Upsilon(x) \sim c_1 x^2$ as $x \to 0$, with $c_1 = (1/2)\int_a^b f^2(s)\, ds > 0$. Further, one can write

$$\Upsilon(x) = \int_{f(a)}^{f(b)} (1 - \cos(xv)) g'(v)\, dv, \tag{3.13}$$

where $g := f^{-1}$. By our assumptions on f we have $g \in C^1$, which implies $\Upsilon(x) > 0$ for every $x \neq 0$. Finally, by the Riemann-Lebesgue lemma,

$$\int_{f(a)}^{f(b)} \cos(xv) g'(v)\, dv \to 0, \quad x \to \infty,$$

which implies $\lim_{x\to\infty} \Upsilon(x) > 0$ and completes the proof of (3.12).

Cases $i = 3$ *and* $i = 4$. We show that the inequality

$$\int_{-\infty}^b (1 - \cos(xf(s)))\, ds \geq c \ln|x| \tag{3.14}$$

holds true (i) *for some* $c > 0$ and $|x|$ large enough provided that f satisfies (F_3); (ii) *for every* $c > 0$ and $|x|$ large enough provided that f satisfies (F_4). Keeping the notation g for the inverse function for f, we have

$$\Upsilon(x) := \int_{-\infty}^b (1 - \cos(xf(s)))\, ds = \int_0^{f(b)} (1 - \cos(xv)) g'(v)\, dv.$$

Since f is convex, f' is non-decreasing. In addition, f itself is increasing: this follows from the convexity, condition (3.7), and the fact that $f(s) \to 0$ as $s \to -\infty$ (which comes from the integrability of $f^2(s)$). Therefore, $g'(v) = [f'(g(v))]^{-1}$ is positive and non-increasing.

By positivity of g',

$$\Upsilon(x) \geq \int_{\pi/(2|x|)}^{f(b)} (1 - \cos(xv)) g'(v)\, dv \tag{3.15}$$

when $\pi/(2|x|) \leq f(b)$. Denote $I_k := \left[\frac{(2k-1)\pi}{2|x|}, \frac{(2k+1)\pi}{2|x|}\right], k \geq 1$. Then, since g' is positive,

$$(-1)^k \int_{I_k} \cos(xv) g'(v)\, dv > 0$$

for every $k \geq 1$, and since g' is non-increasing, we have

$$\int_{I_{k-1}} \cos(xv) g'(v)\, dv + \int_{I_k} \cos(xv) g'(v)\, dv \leq 0 \tag{3.16}$$

for every even $k \geq 2$. Note that, on the axis $[0, +\infty)$, the "negative" interval I_{k-1} is located to the left from the "positive" interval I_k. Then, for any $A > 0$, inequality (3.16) still holds true with I_{k-1} and I_k replaced, respectively, by $I_{k-1} \cap [0, A]$ and $I_k \cap [0, A]$. Consequently, for any $A \geq \pi/(2|x|)$

$$\begin{aligned}\int_{\pi/(2|x|)}^{A} \cos(xv) g'(v)\, dv &= \sum_{k=1}^{\infty} \int_{I_k \cap [0,A]} \cos(xv) g'(v)\, dv \\ &= \sum_{m=1}^{\infty} \left(\int_{I_{2m-1} \cap [0,A]} \cos(xv) g'(v)\, dv + \int_{I_{2m} \cap [0,A]} \cos(xv) g'(v)\, dv \right) \leq 0.\end{aligned}$$

Therefore we obtain by (3.15)

$$\Upsilon(x) \geq \int_{\pi/(2|x|)}^{f(b)} g'(v)\, dv = g(f(b)) - g(\pi/(2|x|)) = b - g(\pi/(2|x|)) \tag{3.17}$$

for $|x|$ large enough. It follows from (3.7) that

$$\rho := \liminf_{v \to 0} \left(-\frac{g(v)}{\ln(1/v)} \right)$$

is positive when f satisfies (F_3), and equals to $+\infty$ when f satisfies (F_4). Combined with (3.17), this yields (3.14).

Now we can complete the proof. In the case $i = 3$, take $c > 0$ and $Q > 0$ such that (3.14) holds true for $|x| \geq Q$. Since $\mu(\mathbb{R}^+) = +\infty$, there exists $q > 0$ such that $\mu([q, +\infty)) \geq (1 + \delta)c^{-1}$. Then (3.14) with $x = uz$ implies

$$\Theta(z, \mathbb{R}^+) \geq \int_{[q,+\infty)} \left(\int_{-\infty}^{b} (1 - \cos(uf(s)z))\, ds \right) \mu(du) \geq c\mu([q, +\infty)) \ln(|qz|), \quad |z| \geq q^{-1}Q, \tag{3.18}$$

which provides $(\hat{H}_3^s)$ because $\ln(|qz|) \sim \ln|z|$, $|z| \to \infty$.

In the case $i = 4$, the assumption $\mu(\mathbb{R}^+) > 0$ implies the existence of $q > 0$ for which $\mu([q, +\infty)) > 0$. Take c satisfying $c\mu([q, +\infty)) > (1 + \delta)$, and let $Q > 0$ be such that (3.14) holds true with this c and $|x| \geq Q$. Then (3.18) holds true as well, which provides $(\hat{H}_3^s)$. □

Lemma 3.3 shows that the kernel f is "smoothifying" in the following sense: when f satisfies some additional assumption like (F_2) – (F_4), the Hartman-Wintner type condition $(\hat{H}_3^s)$ holds true under milder assumptions on the Lévy measure of the noise. The following lemma shows that such "smoothifying" effect concerns the condition $(\hat{H}_4)$, as well.

Lemma 3.4. *Under the assumption* (1.11) *assume additionally that the function f satisfies (F_2).*
Then $(\hat{H}_4)$ holds true for $q > 0$ small enough.

Proof. Similarly to the proof of Lemma 3.3, case $i = 2$, we assume that f is positive on $[a, b]$. Take $\rho > 0$ such that $\mu([\rho, +\infty)) > 0$. Then, for $0 < q < \rho \min_{s\in(a,b)} f(s)$, we have by (3.12)

$$\begin{aligned}\Theta(z, [q, +\infty)) &\geq \int_{u\geq\rho}\int_a^b (1-\cos(uf(s)z))ds\mu(du)\\ &\geq c\int_{u\geq\rho}\Big((uz)^2\wedge 1\Big)\mu(du) \geq c\mu([\rho, +\infty))\Big((\rho z)^2 \wedge 1\Big),\end{aligned}$$

which implies the required estimate. □

To proceed with the assumption $(\hat{H}_4)$ when f is not "smoothifying", recall that a finite measure $\varkappa$ is said to satisfy the *Cramer's condition* if

$$\sup_{|z|\geq\varepsilon}\left|\int_{\mathbb{R}} e^{iyz}\varkappa(dy)\right| < \varkappa(\mathbb{R}) \quad \text{for all } \varepsilon > 0 \tag{3.19}$$

(see, for example, [43] or [33], chapter 3 §3). Cramer's condition means that $\varkappa$ is in some sense regular. For instance, if $\varkappa$ has a non-trivial absolutely continuous part, then (3.19) follows from the Riemann-Lebesgue lemma, although, in general, a measure satisfying Cramer's condition should not be necessarily absolutely continuous (see Example 3.3 below).

Note that (3.19) leads to

$$\Xi(\varepsilon) := \inf_{|z|\geq\varepsilon}\int_{\mathbb{R}}(1-\cos yz)\varkappa(dy) > 0 \quad \text{for all } \varepsilon > 0.$$

In addition, assuming $\varkappa$ to have finite second moment, we get $\Xi(\varepsilon) \sim c\varepsilon^2$ as $\varepsilon \to 0$ with some positive c, and thus

$$\Xi(\varepsilon) = \inf_{|z|\geq\varepsilon}\int_{\mathbb{R}}(1-\cos yz)\varkappa(dy) \geq c(\varepsilon^2\wedge 1) \quad \text{for all } \varepsilon > 0 \tag{3.20}$$

and some positive c. Note that the function $\Theta(z,A)$ involved in $(\hat{H}_4)$ is just the term under the supremum in (3.20), with $\varkappa$ equal to μ_f restricted to A. By the standing assumptions on μ and f, the measure μ_f restricted to $\mathbb{R}\setminus(-q, q)$ has finite second moment for any $q > 0$. Therefore, $(\hat{H}_4)$ holds true, provided that for some $q > 0$ the restriction of μ_f to $[r, +\infty)$ satisfies the Cramer's condition.

Lemma 3.5. *Assume in addition to standing assumptions on μ and f that*
(C) for some $\rho > 0$ the restriction of μ to $[\rho, +\infty)$ satisfies the Cramer's condition.
Then $(\hat{H}_4)$ holds true for $q > 0$ small enough.

Proof. Take $r < \gamma F \rho$ with $F = \text{esssup}_{s\in I} f(s)$ and some $\gamma \in (0,1)$. Then

$$\Theta(z,[r,+\infty)) = \iint_{(s,u):uf(s)\geq r} (1-\cos(uf(s)z))\mu(du)ds$$
$$\geq \int_{f(s)>\gamma F}\int_{u\geq\rho}(1-\cos(uf(s)z))\mu(du)ds \geq \left(\int_{f(s)>\gamma F} ds\right)\Xi_\rho(\gamma F\varepsilon), \quad |z|\geq\varepsilon,$$

with $\Xi_\rho(\varepsilon) = \inf_{|z|\geq\varepsilon}\int_\rho^\infty (1-\cos uz)\mu(du)$. Since $\Xi_\rho(\varepsilon)$ satisfies (3.20) and the set $\{s : f(s) > \gamma F\}$ has positive Lebesgue measure, we obtain the required estimate for $\Theta(z,[q,+\infty))$. □

To summarise, let us formulate in the fixed time setting the asymptotic results for the distribution densities of particular processes, listed in the Introduction.

Corollary 3.1. *Let Y be a Lévy driven stochastic integral, specified below. Assume that the Lévy measure of the noise satisfies (1.11), (1.5), and "tail" conditions (T_1), (T_2).*

Then for every $t > 0$ the distribution density p_t exists, belongs to C_b^∞, and satisfies

$$p_t(x) \sim \frac{1}{\sqrt{2\pi K(t,x)}} e^{D(t,x)}, \quad x \to \infty, \tag{3.21}$$

with respective functions $K(t,x)$ and $D(t,x)$, in the following cases:

(1) Y is the Lévy process Z, μ satisfies (N_1) and (C);

(2) Y is the non-stationary version of a Lévy driven Ornstein-Uhlenbeck process, μ satisfies (N_2);

(3) Y is the stationary version of a Lévy driven Ornstein-Uhlenbeck process, μ satisfies (N_3) (in that case, $p_t(x)$, $K(t,x)$ and $D(t,x)$ actually don't depend on t);

(4) Y is the fractional Lévy motion.

3.3 Examples

In this section we give several examples that illustrate the conditions on the measure μ, introduced above.

The first two examples illustrate two typical situations where "tail" conditions (T_1) and (T_2) hold.

Example 3.1. Let μ be supported in a bounded subset. Denote by σ_+ the minimal positive constant σ such that $\mu((\sigma,+\infty)) = 0$. One can easily show that for all $\varepsilon > 0$ and $k \geq 1$ one has

$$M_k(\xi) \gg e^{(\sigma_+-\varepsilon)\xi}, \quad M_k(\xi) - \sigma_+^k \mu(\{\sigma_+\})e^{\sigma_+\xi} \ll e^{\sigma_+\xi}, \quad \xi \to +\infty. \tag{3.22}$$

This relation yields both (T_1) and (T_2). Indeed, for $\varepsilon > 0$ small enough one has $\gamma := \frac{\sigma_++\varepsilon}{2(\sigma_+-\varepsilon)} \in (0,1)$ and

$$M_4(\xi) \ll e^{(\sigma_++\varepsilon)\xi} = e^{2(\sigma_+-\varepsilon)\gamma\xi} \ll M_2^2(\gamma\xi),$$

which is (T_1). Similarly, for ξ large enough

$$\ln\left(\frac{M_4(\xi)}{M_2(\xi)}\vee 1\right)+\ln\ln M_2(\xi)\le 2\varepsilon\xi+\ln\xi,$$

which provides (T_2) because $\varepsilon>0$ is arbitrary.

Example 3.2. Assume that for u large enough

$$\frac{1}{Q(u)}e^{-bu^{\beta}}\le\mu([u,+\infty))\le Q(u)e^{-bu^{\beta}}, \tag{3.23}$$

where $b>0$, $\beta>1$ are some constants, and Q is some polynomial.

For $\sigma>0$ denote

$$M_k^{\sigma}(\xi):=\int_{[\sigma,+\infty)}u^k e^{\xi u}\mu(du).$$

Clearly,

$$M_k^{\sigma}(\xi)\gg e^{A\xi},\quad \xi\to+\infty$$

for any $A>0$, and

$$M_k(\xi)-M_k^{\sigma}(\xi)\ll e^{\sigma\xi},\quad \xi\to+\infty.$$

This means that, for any $\sigma>0$

$$\begin{aligned} M_k(\xi)&\sim\int_{[\sigma,+\infty)}u^k e^{\xi u}\mu(du)\\ &=\sigma^k e^{\xi\sigma}\mu([\sigma,+\infty))+\int_{[\sigma,+\infty)}\left[ku^{k-1}+\xi u^k\right]e^{\xi u}\mu([u,+\infty))\,du. \end{aligned} \tag{3.24}$$

For any $\sigma>0$, $m\in\mathbb{Z}$, we have

$$\int_{\sigma}^{\infty}u^m e^{\xi u}e^{-bu^{\beta}}\,du\sim c_1(\beta,b,m)\xi^{\frac{2m+2-\beta}{2(\beta-1)}}e^{c_2(\beta)b^{\frac{1}{\beta-1}}\xi^{\frac{\beta}{\beta-1}}},\quad \xi\to+\infty, \tag{3.25}$$

where $c_2(\beta)=\beta^{-\frac{1}{\beta-1}}-\beta^{-\frac{\beta}{\beta-1}}$ (we have no need to specify the constant $c_1(\beta,b,m)$). One can prove (3.25) applying the Laplace method in a standard way; we omit the detailed calculations.

Take σ large enough; then (3.23) holds true for $u\ge\sigma$. Then (3.24) and (3.25) yield for every $k\ge 1$,

$$\frac{1}{Q_k(\xi)}e^{c_2(\beta)b^{\frac{1}{\beta-1}}\xi^{\frac{\beta}{\beta-1}}}\le M_k(\xi)\le Q_k(\xi)e^{c_2(\beta)b^{\frac{1}{\beta-1}}\xi^{\frac{\beta}{\beta-1}}} \tag{3.26}$$

for ξ large enough, where Q_k is some polynomial.

By (3.26),

$$M_4(\xi)\le Q_4(\xi)e^{c_2(\beta)b^{\frac{1}{\beta-1}}\xi^{\frac{\beta}{\beta-1}}}\ll\left(\frac{1}{Q_2(\gamma\xi)}e^{c_2(\beta)b^{\frac{1}{\beta-1}}(\gamma\xi)^{\frac{\beta}{\beta-1}}}\right)^2\le M_2^2(\gamma\xi),\quad \xi\to+\infty,$$

as soon as $2\gamma^{\frac{\beta}{\beta-1}} > 1$, which implies (T_1). Further, for ξ large enough (3.26) gives

$$\ln\left(\frac{M_4(\xi)}{M_2(\xi)} \vee 1\right) + \ln\ln M_2(\xi)$$
$$\le \ln\left(Q_2(\xi)Q_4(\xi) \vee 1\right) + \ln\left(c_2(\beta)b^{\frac{1}{\beta-1}} \ln Q_2(\xi)\right) + \frac{\beta}{\beta-1}\ln\xi \ll \xi, \quad \xi \to \infty,$$

which provides (T_2).

The following example illustrates condition (C) and the relations between the conditions (N_1) – (N_4). All the measures in this example have bounded supports, therefore "tail" conditions (T_1) and (T_2) are satisfied.

Example 3.3. (a) Let $\mu = \sum_{n=1}^{\infty} n^{\rho}\delta_{n^{-1}}$, $\rho < 1$; the assumption on ρ provides that μ is a Lévy measure. In the case $\rho \in (-1, 1)$, the asymptotic behavior of the integrals $\int_{|u|\le\varepsilon} u^2\mu(du)$ is the same as for the α-stable case with $\alpha = 1 + \rho$, i.e. is of a power type:

$$\int_{|u|\le\varepsilon} u^2\mu(du) \asymp \varepsilon^{2-\alpha}, \quad \varepsilon \to 0 \tag{3.27}$$

(cf. [46], [47], [34]). Therefore, conditions (N_1) – (N_4) holds true. The analogy with the α-stable case is not complete: condition (C) does not hold, because for every $r > 0$ the restriction of μ to $[r, +\infty)$ has finite number of atoms. Therefore, statements (2) – (4) in Corollary 3.1 hold true, but one can not claim (3.21) for the Lévy process Z itself. Statement (1) of Corollary 3.1 becomes applicable when μ is replaced by $\mu + \varkappa$, where $\varkappa$ is a measure with a bounded support, satisfying Cramer's condition. For instance, either $\varkappa$ may be absolutely continuous (and then Cramer's condition is provided by the Riemann-Lebesgue lemma), or $\varkappa$ may be equal to the Cantor measure on $[0, 1]$ (and then Cramer's condition is verified by straightforward calculations).

When $\rho = -1$, (N_1) and (N_2) fail, but (N_3) and (N_4) hold true. When $\rho < -1$, only (N_4) hold true, while (N_1) – (N_3) fail. It is clear that, in the latter case, the laws of the Lévy process Z and of the non-stationary version of a Lévy driven Ornstein-Uhlenbeck process contain non-trivial discrete components. Therefore one definitely can not expect any asymptotic relation like (3.21) to hold for these processes. On the other hand, (3.21) holds true for the fractional Lévy motion Z^H with $H \in (1/2, 1)$. This well illustrates the "smoothifying" role of the kernel f.

(b) $\nu = \sum_{n=1}^{\infty} n\delta_{(n!)^{-1}}$. Then condition (N_1) fails, while (N_2) – (N_4) hold true, see [9], Example 1. In these example it is shown that the law of Z_t is singular for all $t > 0$. Thus, the asymptotic relation (3.21) clearly can not be valid for the Lévy process Z itself. In this case, the Lévy measure provides some "hidden smoothness" in the sense that the law of the Lévy process Z is singular, but the distributions of the respective (both non-stationary and stationary) Lévy driven Ornstein-Uhlenbeck processes and fractional Lévy motion possess C_b^{∞} distribution densities which, moreover, admit asymptotical description (3.21).

In the last example, in the case of a Lévy process, we compare our conditions with those introduced in [24] and [18].

Example 3.4. In the paper [24] the authors give the transition density estimates for a symmetric α-stable–like process whose jump intensity kernel $J(x,y)$ is of the form

$$J(x,y) = \frac{c(x,y)}{|x-y|^{n+\alpha}} 1_{|x-y|\leq 1},$$

where $c(x,y)$ is a symmetric Borel measurable function on $\mathbb{R}^n \times \mathbb{R}^n$, bounded from above and below by two positive constants. When $c(x,y) \equiv c(x-y)$, that is, $J(x,y) = J(x-y)$, this process is a Lévy one with the Lévy measure $\mu(dx) = J(x)dx$. We check in the one-dimensional case that such a Lévy measure satisfies the conditions imposed above.

Since the Lévy measure μ has bounded support, the exponential integrability condition is satisfied and, moreover, conditions (T_1) and (T_2) hold true (see Example 3.1). By the Riemann-Lebesgue lemma, the absolute continuity of μ implies condition (C). Finally, (3.27) holds true, which provides the Kallenberg condition (3.10) for the measure μ. Since μ is assumed to be symmetric, this yields (N_1).

The paper [18] is devoted to the estimates of the transition density of a Markov process whose jump intensity $J(x,y)$ satisfies

$$\frac{c_1}{|x-y|^n \phi(c_2|x-y|)} \leq J(x,y) \leq \frac{c_3}{|x-y|^n \phi(c_4|x-y|)}, \quad x,y \in \mathbb{R}^n \times \mathbb{R}^n, \quad x \neq y, \tag{3.28}$$

for some c_i, $i = 1,2,3,4$, where $\phi : [0,\infty) \to [0,\infty)$ is of the form $\phi(r) = \phi_1(r)\psi(r)$, $r > 0$, and

i) ψ is increasing on $[0,\infty)$, $\psi(r) = 1$ for $0 < r \leq 1$, and for some $0 < \gamma_1 \leq \gamma_2$, $\beta > 0$,

$$c_1 e^{\gamma_1 r^\beta} \leq \psi(r) \leq c_2 e^{\gamma_2 r^\beta}, \quad 1 < r < \infty; \tag{3.29}$$

ii) ϕ_1 is strictly increasing on $[0,\infty)$ with $\phi_1(0) = 0$, $\phi_1(1) = 1$, and, in particular, satisfies for $c_2 > c_1 > 0$, $c_3 > 0$, $0 < \beta_1 \leq \beta_2 < 2$, the inequality

$$c_1 \left(\frac{R}{r}\right)^{\beta_1} \leq \frac{\phi_1(R)}{\phi_1(r)} \leq c_2 \left(\frac{R}{r}\right)^{\beta_2} \quad \text{for every } 0 < r < R < \infty. \tag{3.30}$$

Again, let $n = 1$ and $J(x,y) = J(x-y)$, where J is the density of the Lévy measure μ. To achieve the exponential integrability (1.5) we need to assume $\beta > 1$ in (3.29). Assuming additionally that $\gamma_1 = \gamma_2$, one has (T_1) and (T_2) (see Example 3.2). By (3.30), the Lévy measure μ satisfies the lower bound in (3.27) with $\alpha = \beta_1$; that is,

$$\int_{|u|\leq\varepsilon} u^2 \mu(du) \geq c\varepsilon^{2-\beta_1}$$

with some positive c and $\varepsilon > 0$ small enough, which implies (N_1). Finally, condition (C) holds true by the absolute continuity of μ.

Since (N_1), (C), (T_1), and (T_2) hold true, by statement (1) in Corollary 4.1 and Corollary 5.1 below, the transition probability density of the Levy process Z satisfies (1.12) and either (1.17) (in the "truncated" case [24]) or (1.18) (in the case treated in [18]). Let us compare these relations with the estimates for the transition probability density of a symmetric jump process from [24] and [18].

For $t \geq t_0$, these estimates are given in the form

$$C_1 g_t(C_2|x-y|) \leq p(t,x,y) \leq C_3 g_t(C_4|x-y|), \tag{3.31}$$

where $C_1, \ldots, C_4$ are some positive constants, and

$$g_t(x) = \exp\left(-|x| \ln^{\delta} \frac{|x|}{t}\right) \vee \left(\frac{1}{t^{d/2}} \exp\left(-\frac{|x|^2}{t}\right)\right) \tag{3.32}$$

with $\delta = 1$ in the “truncated" case [24], and $\delta = \frac{\beta}{\beta-1}$ in the case treated in [18] (d is the dimension of the space; in the current paper $d = 1$).

For a Levy process, (3.31) with $p(t,x,y) = p_t(y-x)$ is closely comparable with (1.12) and (1.17), (1.18). When $\frac{|x-y|}{t}$ is large, (1.17), (1.18) directly provide (3.31) with g_t replaced by

$$\exp\left(-|x| \ln^{\delta} \frac{|x|}{t}\right).$$

On the other hand, one can show easily that on every bounded set the function $\mathscr{K}_Z$ is bounded and bounded away from 0, and the function $\mathscr{D}_Z$ satisfies

$$-d_1 x^2 \leq \mathscr{D}_Z(x) \leq -d_2 x^2$$

with positive constants d_1, d_2. Thus, when $\frac{|x-y|}{t}$ is bounded, (1.12) provides (3.31) with g_t replaced by

$$\frac{1}{t^{1/2}} \exp\left(-\frac{|x|^2}{t}\right).$$

Note that (1.17) and (1.18) are somewhat more precise than (3.31): by choosing $\frac{x}{t}$ large enough, one can make the constants c_1, c_2 therein to be arbitrarily close to a given constant c_*, while in (3.31) respective constants C_2 and C_4 are different and fixed.

Although having a non-trivial intersection, the classes of processes, treated in our case and in [24] and [18], are substantially different. Our approach, based on the Fourier transform technique, is not applicable to the class of symmetric jump processes from [24] and [18] in the whole generality. On the other hand, this approach is applicable to particularly interesting processes which can not be studied by the technique of [24], [18], including non-symmetric Markov processes (like the Lévy driven Ornstein-Uhlenbeck process) and non-Markov processes (like the fractional Lévy motion).

4 Explicit conditions: time-dependent setting

Our further aim is to consider conditions of Theorem 2.1 in the general, i.e. time-dependent, setting. To make the exposition reasonably short, we address this problem in a particular case of the self-similar kernel f; that is, we assume that

$$f(t,s) = \chi(t) f\left(\frac{s}{\theta(t)}\right), \quad t \in \mathbb{T}, \quad s \in I \tag{4.1}$$

with some functions $f:\mathbb{R}\to\mathbb{R}$ and $\chi,\theta:\mathbb{T}\to(0,+\infty)$. Assumption (4.1) is satisfied for particularly interesting processes like the Lévy process Z and the fractional Lévy motion Z^H. In these cases we have, respectively,

$$f(s)=\mathbb{1}_{[0,1]}(s),\quad \chi(t)=1,\quad \theta(t)=t; \tag{4.2}$$

$$f(s)=\frac{1}{\Gamma(H+1/2)}\left[(1-s)_+^{H-1/2}-(-s)_+^{H-1/2}\right],\quad \chi(t)=t^{H-1/2},\quad \theta(t)=t. \tag{4.3}$$

For the function $f(s)$ we keep our standard standing assumptions: it is bounded and satisfies (2.3), (2.5). For the Lévy measure μ we assume (1.11) and (1.5) to hold true, as before.

Similarly to Section 2, denote

$$\Theta(z,A)=\iint_{\{(s,u)\in\mathbb{R}\times\mathbb{R}:\, f(s)u\in A\}}(1-\cos(f(s)zu))\mu(du)ds,\quad z\in\mathbb{R},$$

$$\Psi(z)=\int_{\mathbb{R}}\int_{\mathbb{R}}\left(e^{-izf(s)u}-1+izf(s)u\right)\mu(du)ds,\quad z\in\mathbb{C},$$

$$H(y,z)=iyz+\Psi(z),\quad \mathscr{M}_k(\zeta)=\frac{\partial^k}{\partial\zeta^k}\Psi(i\zeta),\quad k\geq 1,\quad y\in\mathbb{R},\quad \zeta\in\mathbb{R}.$$

Denote by $\zeta(y)\in\mathbb{R}$ the unique solution to the equation

$$\frac{\partial}{\partial\zeta}H(y,i\zeta)=0, \tag{4.4}$$

and put

$$\mathscr{D}(y)=H(y,i\zeta(y)),\quad \mathscr{K}(y)=\mathscr{M}_2(\zeta(y))=\frac{\partial^2}{\partial\zeta^2}H(y,i\zeta)\Big|_{\zeta=\zeta(y)}. \tag{4.5}$$

Denote $\tau(t)=\chi(t)\theta(t)$. Further in this section we assume θ and χ to be bounded on every segment $[a,b]\subset(0,+\infty)$, and to be bounded away from 0 on the whole $\mathbb{T}$. Clearly, the functions θ,χ in (4.2) and in (4.3) with $H>1/2$ satisfy these assumptions. In addition, we assume that

$$\begin{aligned}&\theta(t)\to+\infty,\quad \ln\Big((\ln\chi(t))\vee 1\Big)\ll\ln\theta(t),\quad t\to+\infty;\\ &\liminf_{t\to+\infty}\chi(t)>0.\end{aligned} \tag{4.6}$$

in the cases (4.2) and (4.3) this assumption holds true.

Theorem 4.1. *Assume that the measure μ satisfies (T_1) and (T_2). Assume also that μ satisfies one of the conditions (N_i) and, respectively, f satisfies one of the assumptions (F_i), $i=1,\dots,4$. In the case $i=1$, assume additionally that μ satisfies condition (C).*

Then for every $t>0$ the law of Y_t has a distribution density $p_t\in C_b^\infty$, and for every $t_0>0$

$$p_t(x)\sim\frac{1}{\tau(t)}\sqrt{\frac{\theta(t)}{2\pi\mathscr{K}(x/\tau(t))}}e^{\theta(t)\mathscr{D}(x/\tau(t))},\quad t+x\to\infty,\quad (t,x)\in[t_0,+\infty)\times\mathbb{R}^+. \tag{4.7}$$

Remark 4.1. The expression on the the right hand side of (4.7) is self-similar in the sense that the variable x, rescaled by $\tau(t)$, is involved in this expression only as an argument of given functions $\mathscr{K}$ and $\mathscr{D}$. Note that the Lévy measure μ is not assumed to have a self-similarity property, and therefore, in general, the family of distributions Y_t, $t>0$ is not self-similar. Thus, although the assumption (4.1) on the kernel f itself does not provide self-similarity for the *distribution densities* of Y_t, $t>0$, it is powerful enough to provide self-similarity for the *asymptotic relation* for these densities.

Proof. The relations below follow easily from the self-similarity assumption (4.1):

$$H(t,x,z)=\theta(t)H\left(\frac{x}{\tau(t)},\chi(t)z\right),\quad \mathscr{M}_k(t,\xi)=\chi^k(t)\theta(t)\mathscr{M}_k(\chi(t)\xi),\quad k\geq 1. \tag{4.8}$$

By the first relation in (4.8), we can rewrite the relation (2.10), which determines $\xi=\xi(t,x)$, as

$$\chi(t)\theta(t)\frac{\partial}{\partial\zeta}H\left(\frac{x}{\tau(t)},i\zeta\right)\Big|_{\zeta=\chi(t)\xi}=0.$$

This means that $\chi(t)\xi$ solves (4.4) with $y=x/\tau(t)$, and therefore

$$\xi(t,x)=\chi^{-1}(t)\zeta\left(\frac{x}{\tau(t)}\right).$$

Combined with (4.8), this relation gives

$$D(t,x)=\theta(t)\mathscr{D}\left(\frac{x}{\tau(t)}\right),\quad K(t,x)=\chi^2(t)\theta(t)\mathscr{K}\left(\frac{x}{\tau(t)}\right)=\frac{\tau^2(t)}{\theta(t)}\mathscr{K}\left(\frac{x}{\tau(t)}\right).$$

Thus (4.7) would follow from (2.13) with $\mathscr{A}=[t_0,+\infty)\times\mathbb{R}^+$, provided that conditions (H_1) – (H_4) are verified.

In Section 3 we proved that under assumptions imposed on the Lévy measure μ and the function $f(s)$, conditions $(\hat{H}_1)$, $(\hat{H}_2)$, $(\hat{H}_3^s)$, and $(\hat{H}_4)$ hold true. Now we show that these conditions yield (H_1) – (H_4) with $\mathscr{T}=[t_0,+\infty)$, $\mathscr{B}=[t_0,+\infty)\times\mathbb{R}^+$, and with the function $\theta(t)$ replaced by $\vartheta\theta(t)$ (the constant ϑ comes from $(\hat{H}_4)$).

The second relation in (4.8) gives

$$\frac{\mathscr{M}_4(t,\xi)}{\mathscr{M}_2^2(t,\xi)}=\frac{1}{\theta(t)}\frac{\mathscr{M}_4(\chi(t)\xi)}{\mathscr{M}_2^2(\chi(t)\xi)}. \tag{4.9}$$

Observe that, under our assumptions on θ and χ,

$$t+\xi\to\infty\quad\text{implies}\quad\theta(t)\to+\infty\quad\text{or}\quad\chi(t)\xi\to+\infty. \tag{4.10}$$

Therefore, (H_1) follows from $(\hat{H}_1)$ and (4.9).

By the second relation in (4.8),

$$\frac{\mathscr{M}_4(t,\xi)}{\mathscr{M}_2(t,\xi)}=\chi^2(t)\frac{\mathscr{M}_4(\chi(t)\xi)}{\mathscr{M}_2(\chi(t)\xi)}, \tag{4.11}$$

which together with $(\hat{H}_2)$ and (4.10) gives

$$\ln\left(\left(\chi^{-2}(t)\frac{\mathscr{M}_4(t,\xi)}{\mathscr{M}_2(t,\xi)}\right)\vee 1\right)\ll\ln\theta(t)+\chi(t)\xi,\quad t+\xi\to+\infty.$$

Similarly,

$$\ln\Big(\Big(\ln \mathscr{M}_2(t,\xi)\Big)\vee 1\Big) = \ln\Big(\ln\Big(\chi^2(t)\theta(t)\mathscr{M}_2(\chi(t)\xi)\Big)\vee 1\Big)$$
$$= \ln\Big(\Big(\ln\chi^2(t)+\ln\theta(t)+\ln\mathscr{M}_2(\chi(t)\xi)\Big)\vee 1\Big).$$

By $(\hat{H}_2)$, (4.10) and (4.6) one has

$$\ln\Big(\Big(\ln \mathscr{M}_2(t,\xi)\Big)\vee 1\Big) \ll \ln\theta(t)+\chi(t)\xi, \quad t+\xi\to+\infty.$$

This completes the proof of (H_2).

By $(\hat{H}_3^s)$, for every $\varkappa>0$ there exists $Q>0$ such that

$$\Theta(z,\mathbb{R}^+)\ge \varkappa\ln|z|, \quad |z|\ge Q.$$

By the self-similarity assumption (4.1), we have

$$\Theta(t,z,A)=\theta(t)\Theta\left(\chi(t)z,\frac{1}{\chi(t)}A\right).$$

Denote $\theta_*=\inf_t\theta(t)$, $\chi_*=\inf_t\chi(t)$. Then taking $\varkappa=\theta_*^{-1}(1+\delta)$ and $R=\chi_*^{-1}Q$, we obtain (H_3).

Finally, by $(\hat{H}_4)$ we have

$$\inf_{|z|>\varepsilon}\Theta(t,z,[q\chi(t),+\infty)) = \theta(t)\inf_{|z|>\varepsilon}\Theta(\chi(t)z,[q,+\infty))$$
$$= \theta(t)\inf_{|z'|>\chi(t)\varepsilon}\Theta(z',[q,+\infty))\ge\vartheta\theta(t)\Big((\chi(t)\varepsilon)^2\wedge 1\Big).$$

Thus, (H_4) holds true with $r=q$ and $\theta(t)$ replaced by $\vartheta\theta(t)$. Clearly, such a change of the function $\theta(t)$ does not spoil conditions (H_1) – (H_3) proved above. □

Corollary 4.1. *Assume the Lévy measure of the noise satisfy (1.11), (1.5), and "tail" conditions (T_1), (T_2). Then*

(1) For the Lévy process Z, assuming additionally μ to satisfy (N_1) and (C), one has (1.12).

(2) For the fractional Lévy motion Z^H, one has (1.13).

5 Explicit asymptotic expressions as $x\to+\infty$

Theorem 2.1, Corollary 3.1, and Theorem 4.1 describe the asymptotic behaviour of a distribution density precisely, but in an implicit form: functions $K(t,x)$, $D(t,x)$, $\mathscr{K}(x)$, $\mathscr{D}(x)$, involved in (2.13), (3.21) and (4.7), are defined in terms of the solutions to equations (2.10) or (4.4). In this section we study the asymptotic behavior of these functions as $x\to+\infty$, and deduce explicit asymptotic expressions for the distribution densities.

In what follows, we mainly discuss the behavior of the functions $\mathscr{K}(x)$ and $\mathscr{D}(x)$ under additional assumptions on the Lévy measure μ; without any essential change of the argument, similar results can be obtained for the functions $K(t,x)$, $D(t,x)$ with a fixed variable t. To simplify the argument, we assume in the sequel f to be non-negative. This assumption is satisfied, for instance, for the Lévy process and the fractional Lévy motion (respective functions f are given in (4.2) and (4.3)). To shorten the exposition, we restrict ourselves to the cases where the Lévy measure μ is either "truncated" (i.e. supported in a bounded set, see Example 3.1) or "exponentially damped" (i.e. having its "tails" satisfying (3.23), see Example 3.2).

We keep the notation introduced in Example 3.1, Example 3.2, and Section 4; in particular, $F = \operatorname{ess\,sup} f(s)$, and σ_+ is the extreme right point of the support of μ.

Theorem 5.1. *Assume the kernel $f(t,s)$ to be of the form (4.1) with θ and χ satisfying (4.6). Assume that the measure μ satisfies one of the conditions (N_i), and the respective function $f(s)$ in (4.1) satisfies one of the assumptions (F_i), $i = 1,\dots,4$. In the case $i = 1$, assume additionally μ to satisfy condition (C).*

1. If μ is truncated, then for any constants $c_1 > 1/(\sigma_+ F)$ and $c_2 < 1/(\sigma_+ F)$ there exists $y = y(c_1, c_2)$ such that, for $x/\tau(t) > y$,

$$\exp\left(-c_1\left(\frac{x}{\chi(t)}\right)\ln\left(\frac{x}{\tau(t)}\right)\right) \le p_t(x) \le \exp\left(-c_2\left(\frac{x}{\chi(t)}\right)\ln\left(\frac{x}{\tau(t)}\right)\right). \tag{5.1}$$

2. If μ is exponentially damped, then for any constants

$$c_2 < \left(\beta^{-\frac{1}{\beta-1}} - \beta^{-\frac{\beta}{\beta-1}}\right)^{-\frac{\beta-1}{\beta}} b^{\frac{1}{\beta}} F^{-1} < c_1$$

there exists $y = y(c_1, c_2)$ such that, for $x/\tau(t) > y$,

$$\exp\left(-c_1\left(\frac{x}{\chi(t)}\right)\ln^{\frac{\beta-1}{\beta}}\left(\frac{x}{\tau(t)}\right)\right) \le p_t(x) \le \exp\left(-c_2\left(\frac{x}{\chi(t)}\right)\ln^{\frac{\beta-1}{\beta}}\left(\frac{x}{\tau(t)}\right)\right). \tag{5.2}$$

Proof. We consider in detail the case of a truncated Lévy measure, and then outline the changes in the proof that should be made in the case of an exponentially damped Lévy measure.

Denote

$$\mathscr{M}_0(\zeta) = \Psi(i\zeta) = \int_{\mathbb{R}}\int_{\mathbb{R}} \left(e^{\zeta f(s)u} - 1 - \zeta f(s)u\right)\mu(du)ds.$$

Similarly to (3.3), one has

$$\mathscr{M}_0(\zeta) = \int_{\mathbb{R}} M_0(f(s)\zeta)\,ds, \quad M_0(\xi) := \int_{\mathbb{R}} (e^{\xi u} - 1 - \xi u)\mu(du).$$

To describe the asymptotic behavior of $\mathscr{K}$, $\mathscr{D}$, we need to analyze the behavior of $\mathscr{M}_k$, $k = 0,1,2$. For this, we analyze first the behavior of M_k, $k = 0,1,2$.

One can easily see that (3.22) holds true for $k = 0$ as well. From (3.22) we have for any $k \ge 0$

$$M_k(\xi) \sim \sigma_+^k M_0(\xi), \quad \xi \to +\infty. \tag{5.3}$$

Moreover, the first relation in (3.22) provides that for every $\varepsilon > 0$

$$\mathscr{M}_k(\zeta) \sim \int_{f(s)\ge F-\varepsilon} f^k(s) M_k(f(s)\zeta)\,ds, \quad \zeta \to +\infty, \tag{5.4}$$

(recall that we assume f to be non-negative), which together with (5.3) yields

$$\mathcal{M}_k(\zeta) \sim F^k \sigma_+^k \mathcal{M}_0(\zeta), \quad \zeta \to +\infty. \tag{5.5}$$

Recall that $\mathcal{K}(y) = \mathcal{M}_2(\zeta(y))$, and

$$\mathcal{D}(y) = -y\zeta(y) + \mathcal{M}_0(\zeta(y)).$$

The function $\zeta(y)$ is defined by the equation $\mathcal{M}_1(\zeta(y)) = y$ and, under our fixed assumption (1.11), we have $\zeta(y) \to +\infty$ as $y \to +\infty$. Hence, by (5.5),

$$\mathcal{K}(y) \sim F\sigma_+ y, \quad \mathcal{M}_0(\zeta(y)) \sim (1/F\sigma_+)y, \quad y \to +\infty. \tag{5.6}$$

The second relation in the above formula yields

$$\mathcal{D}(y) \sim -y\zeta(y), \quad y \to +\infty. \tag{5.7}$$

Similarly to (5.5), one can deduce from (3.22) that for any $\varepsilon > 0$

$$e^{(\sigma_+ F - \varepsilon)\zeta} \ll \mathcal{M}_1(\zeta) \ll e^{(\sigma_+ F + \varepsilon)\zeta}, \quad \zeta \to +\infty,$$

and consequently

$$\zeta(y) \sim \frac{1}{\sigma_+ F} \ln y, \quad y \to +\infty. \tag{5.8}$$

Let us prove the lower bound in (5.1), the proof of the upper bound is similar and omitted. It follows from (5.7) and (5.8) that for any $c > 1/(\sigma_+ F)$ we have for $x/\tau(t)$ large enough

$$e^{\theta(t)\mathcal{D}(x/\tau(t))} \geq \exp\left(-c\left(\frac{\theta(t)x}{\tau(t)}\right)\ln\left(\frac{x}{\tau(t)}\right)\right) = \exp\left(-c\left(\frac{x}{\chi(t)}\right)\ln\left(\frac{x}{\tau(t)}\right)\right), \tag{5.9}$$

(recall that $\tau(t) = \theta(t)\chi(t)$).

Since μ is supported in a bounded set, it satisfies "tail" conditions (T_1), (T_2) (see Example 3.1), and thus we can apply Theorem 4.1. By Theorem 4.1 and (5.9), to prove the first inequality in (5.1) it is enough to take $c \in \left(1/(\sigma_+ F), c_1\right)$ and prove that for $x/\tau(t)$ large enough,

$$\frac{1}{\tau(t)}\sqrt{\frac{\theta(t)}{2\pi\mathcal{K}(x/\tau(t))}} \geq \exp\left((c - c_1)\left(\frac{x}{\chi(t)}\right)\ln\left(\frac{x}{\tau(t)}\right)\right). \tag{5.10}$$

By (5.6) and (4.6), we have for any $q > 0$,

$$\frac{1}{\tau(t)}\sqrt{\frac{\theta(t)}{2\pi\mathcal{K}(x/\tau(t))}} \geq \frac{1}{\theta(t)}e^{-qx/\tau(t)} \tag{5.11}$$

for $x/\tau(t)$ large enough. On the other hand, for a fixed $y > 1$ and $x/\tau(t) \geq y$,

$$\exp\left((c - c_1)\left(\frac{x}{\chi(t)}\right)\ln\left(\frac{x}{\tau(t)}\right)\right) \leq e^{-q\theta(t)(x/\tau(t))} \quad \text{with} \quad q = (c_1 - c_2)\ln y > 0.$$

Hence (5.10) follows from the inequality

$$\frac{1}{a}e^{-qb} = e^{-\ln a - qb} \geq e^{-q(a+b)} \geq e^{-qab},$$

valid for a, b large enough.

Let us discuss briefly the changes that should be made when the measure μ satisfies (3.23). Clearly, for any $\sigma > 0$ and $k > j$,

$$M_k^\sigma(\xi) \geq \sigma^{k-j} M_j^\sigma(\xi), \quad \xi \geq 0$$

(see the notation in Example 3.2). This means that instead of (5.3) and (5.5) we have now

$$M_k(\xi) \gg M_j(\xi), \quad \xi \to +\infty, \tag{5.12}$$

$$\mathscr{M}_k(\zeta) \gg \mathscr{M}_j(\zeta), \quad \zeta \to +\infty \tag{5.13}$$

for any $k > j$. The latter relation with $k = 1, j = 0$ yields (5.7). From (3.26) and (5.4) it follows that for every $\varepsilon > 0$ for ζ large enough

$$e^{(C_* - \varepsilon)\zeta^{\frac{\beta}{\beta-1}}} \leq \mathscr{M}_1(\zeta) \leq e^{(C_* + \varepsilon)\zeta^{\frac{\beta}{\beta-1}}},$$

where $C_* = \left(\beta^{-\frac{1}{\beta-1}} - \beta^{-\frac{\beta}{\beta-1}}\right) b^{-\frac{1}{\beta-1}} F^{\frac{\beta}{\beta-1}}$. Consequently,

$$\zeta(y) \sim \left(\frac{1}{C_*}\ln y\right)^{\frac{\beta-1}{\beta}}, \quad \mathscr{D}(y) \sim -y\left(\frac{1}{C_*}\ln y\right)^{\frac{\beta-1}{\beta}}, \quad y \to +\infty, \tag{5.14}$$

which means that the analogue of (5.2), with $e^{\theta(t)\mathscr{D}(x/\tau(t))}$ instead of $p_t(x)$, holds true, and the only thing we need to verify is that the term

$$\frac{1}{\tau(t)}\sqrt{\frac{\theta(t)}{2\pi\mathscr{K}(x/\tau(t))}}$$

is negligible. Note that this term is bounded:

$$\sup_t \frac{1}{\tau(t)}\sqrt{\frac{\theta(t)}{2\pi}} = \sup_t \frac{1}{\sqrt{2\pi\chi^2(t)\theta(t)}} < +\infty$$

because θ and χ are assumed to be separated from 0, and by (5.13)

$$\mathscr{K}(x/\tau(t)) = \mathscr{M}_2(\zeta(x/\tau(t))) \gg \mathscr{M}_1(\zeta(x/\tau(t))) = x/\tau(t), \quad x/\tau(t) \to +\infty.$$

This observation provides the upper bound in (5.2).

On the other hand, it follows from (3.26) that

$$\ln\frac{M_2(\xi)}{M_1(\xi)} \ll \xi, \quad \xi \to +\infty$$

(cf. (5.12)). Similarly to the proof of Lemma 3.2, one can deduce from this relation that

$$\ln\frac{\mathscr{M}_2(\zeta)}{\mathscr{M}_1(\zeta)} \ll \zeta, \quad \zeta \to +\infty,$$

and consequently

$$\ln \mathscr{K}(y) \ll \zeta(y) + \ln y, \quad y \to +\infty.$$

Together with (4.6) and the first relation in (5.14), this implies (5.11). Repeating the argument after (5.11), we obtain the lower bound in (5.2). □

For the Lévy process Z and the fractional Lévy motion Z^H, Theorem 5.1 gives the following. Denote

$$c_* = 1/\sigma_+ \tag{5.15}$$

in the case of the truncated Lévy measure μ, and

$$c_* = \left(\beta^{-\frac{1}{\beta-1}} - \beta^{-\frac{\beta}{\beta-1}}\right)^{-\frac{\beta-1}{\beta}} b^{\frac{1}{\beta}} \tag{5.16}$$

in the case of the exponentially damped Lévy measure μ.

Corollary 5.1. *Assume that the Lévy measure satisfies (N_1) and (C), then for the distribution density of the Lévy process Z the following estimates hold.*

1. If μ is truncated, then for any constants $c_1 > c_*$ and $c_2 < c_*$ there exists $y = y(c_1, c_2)$, such that for $x/t > y$ (1.17) holds true.

2. If μ is exponentially damped, then for any constants $c_1 > c_*$ and $c_2 < c_*$ there exists $y = y(c_1, c_2)$, such that for $x/t > y$, (1.18) holds true.

Corollary 5.2. *Assume that the Lévy measure satisfies (1.11), then for the distribution density of the fractional Lévy motion Z^H the following estimates hold.*

1. If μ is truncated, then for any constants $c_1 > c_*$ and $c_2 < c_*$ there exists $y = y(c_1, c_2)$, such that for $x/t^{H+1/2} > y$ (1.19) holds true.

2. If μ is exponentially damped, then for any constants $c_1 > c_*$ and $c_2 < c_*$ there exists $y = y(c_1, c_2)$, such that for $x/t^{H+1/2} > y$ (1.20) holds true.

We have mentioned in the beginning of the section that for fixed t the functions $K(t,x)$, $D(t,x)$ can be analyzed in the same way as $\mathscr{K}(x)$ and $\mathscr{D}(x)$. Respectively, the analogue of Theorem 5.1 can be proved for the density $p_t(x)$ with fixed t without the self-similarity assumption (4.1). Let us formulate one statement of such a kind.

Consider the stationary version X of a Lévy driven Ornstein-Uhlenbeck process, and assume that μ satisfies (N_3). Then the distribution of X_t, in fact, does not depend on t, and by Proposition 2.1 has a C^∞ distribution density p, which we call the *invariant distribution density* of the respective Ornstein-Uhlenbeck process. Moreover, assuming the “tail” conditions (T_1) and (T_2) to hold, we have by Corollary 3.1 the asymptotic relation for this density, which after trivial transformations can be written in the form

$$p(x) \sim \frac{1}{\sqrt{2\pi\mathscr{K}(x)}} e^{\mathscr{D}(x)}, \quad x \to +\infty, \tag{5.17}$$

where $\mathscr{K}$, $\mathscr{D}$ are defined by (4.5) with $f(s) = e^{\gamma s}\mathbb{1}_{s\le 0}$. Similarly to Theorem 5.1, one can deduce from (5.17) the following statement (the proof is omitted).

Proposition 5.1. *Assume that the Lévy measure μ satisfies (1.5) and (N_3). Then for the invariant distribution density of the Ornstein-Uhlenbeck process (1.7) the following estimates hold.*

1. If μ is truncated, then for any constants $c_1 > c_*$ and $c_2 < c_*$ there exists $y = y(c_1, c_2)$, such that for $x > y$,

$$\exp\left(-c_1 x \ln x\right) \le p(x) \le \exp\left(-c_2 x \ln x\right).$$

2. If μ is exponentially damped, then for any constants $c_1 > c_*$ and $c_2 < c_*$ there exists $y = y(c_1, c_2)$ such that for $x > y$,

$$\exp\left(-c_1 x \ln^{\frac{\beta-1}{\beta}} x\right) \le p(x) \le \exp\left(-c_2 x \ln^{\frac{\beta-1}{\beta}} x\right).$$

Here c_* depends on μ only, and is defined, respectively, in (5.15) or (5.16).

As we mentioned in the Introduction, there is a particular theoretical interest in studying the ratio (1.14) of the values of the invariant distribution density. One can see that the statement of Proposition 5.1 is not strong enough to provide an exact estimate for the ratio (1.14) because of different constants c_1 and c_2, involved in respective estimates. In the theorem below we provide the exact estimate for the ratio (1.14).

Theorem 5.2. *Assume that the Lévy measure μ satisfies (T_1), (T_2), and (N_3).*

Then for every bounded set $A \subset \mathbb{R}$

$$r_a(x) \sim e^{-a\zeta(x)}, \quad x \to +\infty, \tag{5.18}$$

uniformly in $a \in A$.

In particular, for any constants $c_1 > c_*$ and $c_2 < c_*$ there exists $y = y(c_1, c_2, A)$, such that for $x > y$, $a \in A$,

$$x^{-c_1 a} \le r_a(x) \le x^{-c_2 a} \tag{5.19}$$

when μ is truncated, and

$$x^{-c_1 a \ln^{-\frac{1}{\beta}} x} \le r_a(x) \le x^{-c_2 a \ln^{-\frac{1}{\beta}} x} \tag{5.20}$$

when μ is exponentially damped. Here c_* is defined respectively in (5.15) or (5.16).

Proof. By the inverse function theorem,

$$\frac{d}{dx}\zeta(x) = \left[\frac{d}{d\zeta}\mathscr{M}_1(\zeta)\right]^{-1}\Big|_{\zeta=\zeta(x)} = [\mathscr{M}_2(\zeta(x)]^{-1}. \tag{5.21}$$

Then

$$\frac{d}{dx}\ln \mathscr{K}(x) = \frac{\mathscr{M}_3(\zeta(x))}{\mathscr{M}_2(\zeta(x))}\frac{d}{dx}\zeta(x) = \frac{\mathscr{M}_3(\zeta(x))}{\mathscr{M}_2^2(\zeta(x))}.$$

If μ is supported in a bounded set, then $\mathscr{M}_3(\zeta) \sim (1/\sigma_+)\mathscr{M}_4(\zeta)$, $\zeta \to +\infty$ (see (5.5)). If μ is not supported in a bounded set, then $\mathscr{M}_3(\zeta) \ll \mathscr{M}_4(\zeta)$, $\zeta \to +\infty$ (see (5.13)). In both cases, we have

$$\frac{d}{dx}\ln \mathscr{K}(x) \to 0, \quad x \to \infty$$

because (T_1) provides $(\hat{H}_1)$ (Lemma 3.1). Thus, for any bounded set A,

$$\frac{\mathcal{K}(x+a)}{\mathcal{K}(x)} = \exp\left(\int_x^{x+a} \frac{d}{dy} \ln \mathcal{K}(y)\, dy\right) \to 0, \quad x \to +\infty$$

uniformly in $a \in A$. Therefore, by (5.17),

$$r_a(x) \sim e^{\mathcal{D}(x+a)-\mathcal{D}(x)}, \quad x \to +\infty$$

uniformly in $a \in A$.

We have

$$\mathcal{D}(x+a) - \mathcal{D}(x) = -a\zeta(x) - (x+a)[\zeta(x+a) - \zeta(x)] + \mathcal{M}_0(\zeta(x+a)) - \mathcal{M}_0(\zeta(x)).$$

Since $\frac{d}{d\zeta}\mathcal{M}_0(\zeta) = \mathcal{M}_1(\zeta)$ and $\mathcal{M}_1(\zeta(y)) = y$, we get

$$(x+a)[\zeta(x+a) - \zeta(x)] - \mathcal{M}_0(\zeta(x+a)) - \mathcal{M}_0(\zeta(x)) = (x+a)\int_x^{x+a} \zeta'(y)\, dy - \int_x^{x+a} y\zeta'(y)\, dy$$
$$= \int_x^{x+a} (x+a-y)\zeta'(y)\, dy = \int_x^{x+a} \int_y^{x+a} \zeta'(y)\, dv dy = \int_0^a \int_r^a \zeta'(x+r)\, ds dr.$$

Since μ satisfies (N_3) we have (1.11), and therefore $\mathcal{M}_2(\zeta) \to +\infty$, $\zeta \to +\infty$. By (5.21), this yields

$$\zeta'(x) \to 0, \quad x \to +\infty.$$

From the above relations we deduce that

$$\mathcal{D}(x+a) - \mathcal{D}(x) \to -a\zeta(x),, \quad x \to +\infty$$

uniformly in $a \in A$, which completes the proof of (5.18).

From (5.18) and (5.8) we deduce (5.19). From (5.18) and the first relation in (5.14) we obtain (5.20). □

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