

Fractional elliptic, hyperbolic and parabolic random fields ^{*}

N.N. Leonenko[†]

M.D. Ruiz-Medina[‡]

M.S. Taqqu[§]

Abstract

This paper introduces new classes of fractional and multifractional random fields arising from elliptic, parabolic and hyperbolic equations with random innovations derived from fractional Brownian motion. The case of stationary random initial conditions is also considered for parabolic and hyperbolic equations.

Key words: Cylindrical fractional Brownian motion; elliptic, hyperbolic, parabolic random fields; fractional Bessel potential spaces; fractional Holder spaces; fractional random fields; multifractional random fields; spectral representation.

AMS 2010 Subject Classification: Primary 60G60, 60G18, 60G20, 60G22; Secondary: 35J15, 35K10, 35L10.

Submitted to EJP on November 27, 2010, final version accepted May 6, 2011.

^{*}This work has been supported in part by projects MTM2009-13393 of the DGI, MEC, P09-FQM-5052 of the Andalusian CICE, Spain, and by grant of the European commission PIRSES-GA-2008-230804 (Marie Curie). Murad S. Taqqu was supported by the NSF grant DMS-1007616 at Boston University

[†]Cardiff School of Mathematics, Senghennydd Road. Cardiff CF24 4AG, U.K., LeonenkoN@cardiff.ac.uk

[‡]Department of Statistics and Operations Research, University of Granada, Campus de Fuente Nueva s/n, E-18071 Granada, Spain, mruiz@ugr.es

[§]Charles River, Boston University Campus, Boston, MA 02215, USA, bumastat@gmail.com

1 Introduction

There has been some recent interest in studying stochastic partial differential equations driven by a fractional noise (see Duncan *et al.*, 2002; Tindel *et al.*, 2003; Muller and Tribe 2004; Hu *et al.*, 2004; Maslowski and Nualart, 2005; Hu and Nualart, 2009a, 2009b; Sanz-Solé and Torrecilla, 2009; Sanz-Solé and Vailermot, 2010, among others). In this paper, we provide a structure for developing mean-square weak-sense (generalized) and strong-sense (pointwise definition) solutions to stochastic elliptic, hyperbolic and parabolic equations driven by fractional Gaussian noise, whose integral is fractional Brownian motion.

Linear stochastic evolution equations driven by an additive cylindrical fractional Brownian motion with Hurst parameter H were studied by Duncan *et al.* (2002) in the case $H \in (1/2, 1)$. Similar result holds when one adds nonlinearity of a special form (see Maslowski and Nualart, 2005). Other important and relevant papers are Hu (2001) and Mueller and Tribe (2004). Hu *et al.* (2004) present a white noise calculus for the d -parametric fractional Brownian motion $W_H(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, with general d -dimensional Hurst parameter $H = (H_1, \dots, H_d) \in (0, 1)^d$, and separable covariance function

$$E [W_H(\mathbf{x})W_H(\mathbf{y})] = \frac{1}{2^d} \prod_{j=1}^d c(H_j) (|x_j|^{2H_j} + |y_j|^{2H_j} - |x_j - y_j|^{2H_j}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d,$$

where

$$c(H) = \frac{1}{2\pi} \int_{\mathbb{R}} \left| \frac{\exp(i\lambda) - 1}{i\lambda} \right|^2 |\lambda|^{-2H+1} d\lambda. \quad (1)$$

As illustration, they solved the stochastic Poisson problem

$$\begin{aligned} \Delta u(\mathbf{x}) &= -(W_H(\mathbf{x}))', \quad \mathbf{x} \in \mathcal{D}, \\ u(\mathbf{x}) &= 0, \quad \mathbf{x} \in \partial \mathcal{D}, \end{aligned} \quad (2)$$

where the potential $(W_H)'$ is d -parametric fractional white noise defined as

$$(W_H)'(\mathbf{x}) = \frac{\partial^d W_H(\mathbf{x})}{\partial x_1 \dots \partial x_d},$$

and $\mathcal{D} \subset \mathbb{R}^d$ is a given smooth domain. Hu and Nualart (2009a) study the stochastic heat equation with a multiplicative Gaussian noise which is white in space, and has the covariance of a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ in time. Two types of equations are considered, in the Itô-Skorokhod sense, and in the Stratonovich sense. An explicit chaos expansion for the equation is obtained. The rough path analysis (see Lions and Qian, 2002, and the references therein) is also applicable to the fractional calculus (see Gubinelle *et al.*, 2006; Hu and Nualart, 2009b, and the references therein). Mild solutions for a class of fractional SPDEs have been developed for elliptic and parabolic problems by Sanz-Solé and Torrecilla (2009), and Sanz-Solé and Vuillermot (2010). They defined the stochastic convolution integrals of the Green function with fractional noise as Wiener integrals.

In this paper, we provide an overview of the mean-square solution of stochastic elliptic, hyperbolic and parabolic problems driven by fractional Gaussian random fields. We interpret the corresponding stochastic integrals of non-random Green functions with respect to fractional noise as Wiener

integrals in the spectral domain. This approach gives us an opportunity, for relatively simple situations, to obtain an explicit parabolic, hyperbolic and elliptic parametric family of models involving fractional Gaussian random fields. Fractional Gaussian random fields constitute an important area of research in modeling homogeneous/heterogeneous fractality, as well as long-range dependence in the self-similar case.

Elliptic fractional and multifractional Gaussian random fields have been extensively studied in the last and a half decade (see, for example, Anh, Angulo and Ruiz-Medina, 1999; Benassi, Jaffard and Roux, 1997; Kelbert, Leonenko and Ruiz-Medina, 2005; Ruiz-Medina, Angulo and Anh, 2002; 2003; 2006, and Ruiz-Medina, Anh and Angulo, 2004a; 2004b; 2010). The cited references provide several examples of Gaussian random fields with reproducing kernel Hilbert space (RKHS) having inner product defined in terms of a fractional or multifractional bilinear form (defined between suitable fractional Sobolev or Besov spaces). The special case where the RKHS is isomorphic to a fractional/multifractional Sobolev space has been treated, in a generalized random field framework, in Ruiz-Medina, Angulo and Anh (2002; 2003; 2006) and Ruiz-Medina, Anh and Angulo (2004a; 2004b; 2010). Additionally, under suitable conditions, a weak-sense elliptic fractional pseudodifferential representation in terms of Gaussian white noise innovations can be derived (see Ruiz-Medina, Anh, and Angulo, 2004b). The strong-sense equality, in the sample-path sense, holds for mean-square continuous Gaussian random fields (see Adler, 1981). The mentioned class of elliptic fractional/multifractional Gaussian random fields includes as particular cases homogeneous/heterogeneous fractal Gaussian random fields satisfying elliptic fractional/multifractional pseudodifferential equations with Gaussian white noise innovations.

Parabolic fractional and multifractional Gaussian random fields have also been extensively studied in the context of Gaussian white noise and Lévy-type innovations (see Angulo, Ruiz-Medina, Anh and Grecksch 2000; Angulo, Anh, McVinish and Ruiz-Medina, 2005; Kelbert, Leonenko and Ruiz-Medina, 2005; Ruiz-Medina, Angulo and Anh, 2008, among others). Random evolution equations, fractional in time and in space, with random initial conditions, interpolating parabolic and wave equations, are introduced, for instance, in Anh and Leonenko (2001). The spatial local mean quadratic variation properties of these random fields can be characterized in terms of fractional Hölder exponents. Also, heavy-tailed behaviors of spatial covariance functions can be represented in this framework.

Stochastic hyperbolic equations have been studied in the two-parameter diffusion process context, e.g. Ornstein-Uhlenbeck-type random fields (see the pioneering work by Nualart and Sanz-Solé, 1979), and in the random initialized hyperbolic equation context (see, for example, Kozachenko and Slivka, 2007). In the fractional random field framework, the structural properties of hyperbolic fractional random fields on fractal domains have been investigated in Ruiz-Medina, Angulo and Anh (2006), considering Gaussian white noise innovations.

In this paper, families of elliptic, parabolic and hyperbolic fractional and multifractional Gaussian random fields are introduced, with fractional Brownian motion type innovations. Specifically, the spectral analysis of the solution to elliptic, parabolic and hyperbolic equations, with random innovations defined in terms of the weak-sense derivatives of fractional Brownian motion, is undertaken. Exact formulae in the temporal and spatial domains are also established in some special cases. The generalized random field framework and the RKHS theory are used to formulate suitable conditions for the definition of the solution. Some extensions related to fractional and multifractional pseudodifferential equations are established, including the case of random initial conditions in the parabolic and hyperbolic cases.

For other approaches to stochastic integration with respect to fractional noise, see the recent books by Biagini, Hu, Øksendal and Zhang (2008) or Mishura (2008), and the references therein. New Green functions for the case of the heat equation with quadratic potential were constructed in Leonenko and Ruiz-Medina (2006, 2008).

2 Fractional Brownian motion and stochastic integration

We start with the one-dimensional case. Let W_H be a stochastic process defined as fractional Brownian motion (FBM), i.e., we consider that $\{W_H(x), x \in \mathbb{R}\}$ is a zero-mean Gaussian process with covariance function

$$B_{W_H}(x, y) = E[W_H(x)W_H(y)] = \frac{c(H)}{2} (|x|^{2H} + |y|^{2H} - |x - y|^{2H}), \quad H \in (0, 1),$$

and $c(H)$ is defined as in (1). When $H = 1/2$, $W_H(x) = W_{1/2}(x)$ is Brownian motion. The spectral representation of the process W_H is given by (see Taqqu, 1979, 2003)

$$W_H(x) = \int_{\mathbb{R}} \frac{\exp(i\lambda x) - 1}{i\lambda} (i\lambda)^{-H+1/2} Z(d\lambda), \quad (3)$$

where $Z(\cdot)$ is a complex Gaussian white noise spectral measure such that $\overline{Z(d\lambda)} = Z(-d\lambda)$, and

$$E|Z(d\lambda)|^2 = \frac{1}{2\pi} d\lambda.$$

Its temporal domain representation is

$$W_H(x) = \frac{1}{\Gamma(H + 1/2)} \int_{-\infty}^0 [(x - y)^{H-1/2} - (-y)^{H-1/2}] dB(y) + \int_0^x (x - y)^{H-1/2} dB(y), \quad (4)$$

with B being standard Brownian motion. From (3), we obtain the following weak-sense definition of the *derivative process*, i.e, the following definition in the generalized random field sense. Thus,

$$\frac{dW_H(x)}{dx} = (W_H)'(x) \underset{\text{w.s.}}{=} \int_{\mathbb{R}} \exp(i\lambda x) (i\lambda)^{-H+1/2} Z(d\lambda) \quad (5)$$

where $\underset{\text{w.s.}}{=}$ denotes the weak-sense identity, that is,

$$(W_H)'(\psi) = \int_{\mathbb{R}} (W_H)'(x) \psi(x) dx = \sqrt{2\pi} \int_{\mathbb{R}} \widehat{\psi}(\lambda) (i\lambda)^{-H+1/2} Z(d\lambda), \quad \psi \in [\mathcal{H}_{(W_H)'}]^*,$$

where $[\mathcal{H}_{(W_H)'}]^* = [\mathcal{H}_{dW_H}]^*$ denotes the dual of the RKHS \mathcal{H}_{dW_H} of process $(W_H)'$ expressed as dW_H .

Remark 1. Note that the functions in the RKHS \mathcal{H}_{dW_H} of dW_H are not continuous. Thus, the process dW_H is not continuous in the mean-square sense, and, since we are in the Gaussian case, its trajectories are not continuous (see Adler, 1981). Therefore, the identity (5) cannot be established in the strong-sense (pointwise), and it must be established in the weak sense, as an integral with respect to a suitable test function ψ .

The integration of a non-random function $G(x)$ with respect to $(W_H(x))' \equiv dW_H$ is then defined as follows. First, formally,

$$\int_{\mathbb{R}} G(x) dW_H(x) = \int_{\mathbb{R}} G(x) (W_H)'(x) dx = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \exp(i\lambda x) G(x) dx \right] (i\lambda)^{-H+1/2} Z(d\lambda).$$

The precise meaning of the above identities can be obtained from the following definition (see Iglói and Terdik, 1999).

Definition 1. Let $G : \mathbb{R} \rightarrow \mathbb{R}$, $G \in L^2(\mathbb{R})$, and

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \exp(i\lambda x) G(x) dx \right|^2 |\lambda|^{-2H+1} d\lambda < \infty.$$

Then,

$$\int_{\mathbb{R}} G(x) dW_H(x) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \exp(i\lambda x) G(x) dx \right] (i\lambda)^{-H+1/2} Z(d\lambda) = \sqrt{2\pi} \int_{\mathbb{R}} \widehat{G}(\lambda) (i\lambda)^{-H+1/2} Z(d\lambda), \quad (6)$$

where $\widehat{G}(\lambda)$ denotes the Fourier transform of G in the sense of tempered distributions, i.e., $\widehat{G}(\lambda)$ is given by

$$\widehat{G}(\varphi) = G(\widehat{\varphi}),$$

for all test function $\varphi \in \mathcal{S}$, with \mathcal{S} denoting the Schwartz function space, and $\widehat{\varphi}$ the Fourier transform of φ , in the ordinary sense (see, for example, Dautray and Lions, 1985a).

Remark 2. The condition

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \exp(i\lambda x) G(x) dx \right|^2 |\lambda|^{-2H+1} d\lambda < \infty$$

means that function G belongs to the dual $[\mathcal{H}_{dW_H}]^*$ of the RKHS of the process dW_H . Thus,

$$\int_{\mathbb{R}} G(x) \phi(x) dx < \infty,$$

for all function $\phi \in \mathcal{H}_{dW_H}$, i.e., for all function ϕ satisfying that

$$\int_{\mathbb{R}} |\widehat{\phi}(\lambda)|^2 |\lambda|^{2H-1} d\lambda < \infty,$$

where $\widehat{\phi}$ denotes the Fourier transform of ϕ .

Note also that, asymptotically in the spectral domain, the decay velocity of the Fourier transform of functions in the space $[\mathcal{H}_{dW_H}]^*$ coincides to the one of functions in the fractional Sobolev space $\mathbf{H}^{-H+1/2}(\mathbb{R})$.

In the two-dimensional case, fractional Brownian motion is introduced as a zero-mean Gaussian random field with covariance function

$$B_{W_H}(\mathbf{x}, \mathbf{y}) = E[W_H(\mathbf{x})W_H(\mathbf{y})] = \frac{1}{2^2} \prod_{j=1}^2 c(H_j) \left(|x_j|^{2H_j} + |y_j|^{2H_j} - |x_j - y_j|^{2H_j} \right).$$

Similarly, the two-dimensional fractional Brownian motion can be defined in the spectral domain as follows:

$$W_H(x, y) = \int_{\mathbb{R}^2} \left[\frac{\exp(i\lambda_1 x) - 1}{i\lambda_1} \right] \left[\frac{\exp(i\lambda_2 y) - 1}{i\lambda_2} \right] (i\lambda_1)^{-H_1+1/2} (i\lambda_2)^{-H_2+1/2} Z(d\lambda_1, d\lambda_2), \quad (7)$$

for $0 < H_1 < 1$, $0 < H_2 < 1$, $H = (H_1, H_2)$, where Z is a complex Gaussian white noise satisfying that

$$E|Z(d\lambda_1, d\lambda_2)|^2 = \frac{1}{(2\pi)^2} d\lambda_1 d\lambda_2.$$

Consider then the generalized random field

$$\frac{\partial^2 W_H}{\partial x \partial y}(x, y) \stackrel{w.s.}{=} \int_{\mathbb{R}^2} \exp(i\lambda_1 x + i\lambda_2 y) (i\lambda_1)^{-H_1+1/2} (i\lambda_2)^{-H_2+1/2} Z(d\lambda_1 d\lambda_2). \quad (8)$$

That is,

$$\begin{aligned} \frac{\partial^2 W_H}{\partial x \partial y}(\psi) &= \int_{\mathbb{R}^2} \frac{\partial^2 W_H}{\partial x \partial y}(x, y) \psi(x, y) dx dy \\ &= 2\pi \int_{\mathbb{R}^2} \widehat{\psi}(\lambda_1, \lambda_2) (i\lambda_1)^{-H_1+1/2} (i\lambda_2)^{-H_2+1/2} Z(d\lambda_1, d\lambda_2), \end{aligned} \quad (9)$$

for all $\psi \in [\mathcal{H}_{\partial^2 W_H}]^*$.

The local regularity properties of the square-integrable functions belonging to the RKHS $\mathcal{H}_{\partial^2 W_H}$ coincide, for $H_i > 1/2$, $i = 1, 2$, with the ones displayed by functions in anisotropic fractional Bessel potential spaces $\mathbf{H}_2^{s, \mathbf{a}}(\mathbb{R}^2) \equiv \mathbf{H}_2^{s/\mathbf{a}}(\mathbb{R}^2)$ (see, for example, Dachkovski, 2003). These spaces are defined in the Appendix. In particular, the parameters s and $\mathbf{a} = (a_1, a_2)$ are given as follows (see Proposition 1 in the Appendix):

$$\begin{aligned} s &= \frac{2(H_1 - 1/2)(H_2 - 1/2)}{H_1 + H_2 - 1} \\ a_1 &= \frac{2(H_2 - 1/2)}{H_1 + H_2 - 1} \\ a_2 &= \frac{2(H_1 - 1/2)}{H_1 + H_2 - 1}. \end{aligned} \quad (10)$$

The following definition provides a stochastic integration formula, in the mean-square sense, with respect to $\partial^2 W_H$, for functions in the space $[\mathcal{H}_{\partial^2 W_H}]^*$.

Definition 2. Let $G : \mathbb{R}^2 \rightarrow \mathbb{R}$, with $G \in L^2(\mathbb{R}^2)$, and

$$\|G\|_{\mathcal{H}_{\partial^2 W_H}^*}^2 = \int_{\mathbb{R}^2} |\widehat{G}(\lambda_1, \lambda_2)|^2 |\lambda_1|^{-2H_1+1} |\lambda_2|^{-2H_2+1} d\lambda_1 d\lambda_2 < \infty.$$

That is, $G \in [\mathcal{H}_{\partial^2 W_H}]^*$, with the Fourier transform \widehat{G} of G , as before, to be interpreted in the dual sense, i.e., in the sense of tempered distributions. Then,

$$\int_{\mathbb{R}^2} G(x, y) dW_H(x, y) = 2\pi \int_{\mathbb{R}^2} \widehat{G}(\lambda_1, \lambda_2) (i\lambda_1)^{-H_1+1/2} (i\lambda_2)^{-H_2+1/2} Z(d\lambda_1, d\lambda_2).$$

3 The elliptic, hyperbolic and parabolic cases

We first consider the fractional stochastic differential equation defined by

$$\mathcal{L} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) u(x, y) = \frac{\partial^2 W_H}{\partial x \partial y}(x, y), \quad (x, y) \in \mathbb{R}^2, \quad (11)$$

where $H = (H_1, H_2) \in (0, 1) \times (0, 1)$. The elliptic, hyperbolic and parabolic cases will be introduced in terms of some special cases of operator \mathcal{L} .

- (i) *Elliptic case:*

$$\mathcal{L} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \gamma^2, \quad \gamma > 0, \quad (12)$$

or, in a more general form

$$\mathcal{L} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = \frac{1}{a^2} \frac{\partial^2}{\partial x^2} + \frac{1}{b^2} \frac{\partial^2}{\partial y^2} - 1. \quad (13)$$

The fractional pseudodifferential case can be studied, for example, in terms of the following equation:

$$\mathcal{L} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = f((I - \Delta)^{\beta/2}),$$

where f is a continuous function, and $(I - \Delta)^{\beta/2}$ is the pseudodifferential operator defined in terms of the inverse of the Bessel potential of order $\beta \in (0, 2)$, with, as usual, $(-\Delta)$ denoting the negative Laplacian operator. It is well-known that operators $(I - \Delta)^{\beta/2}$, $\beta \in \mathbb{R}$, generate the norm of isotropic fractional Bessel potential spaces, where solutions of elliptic fractional pseudodifferential equations can be found (see Appendix). Non-linear continuous transformations f of these operators can also be defined via the Spectral Representation Theorem for self-adjoint operators (see, for example, Dautray and Lions, 1985b). In fact the operator $(I - \Delta)^{\beta/2}$ can be replaced in the above equation by a fractional pseudodifferential operator with continuous spectrum given in terms, for instance, of a positive elliptic fractional rational function (see Ramm, 2005).

- (ii) *Hyperbolic case:*

$$\mathcal{L} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = \frac{\partial^2}{\partial x \partial y} + \theta_1 \frac{\partial}{\partial x} + \theta_2 \frac{\partial}{\partial y} + \theta_1 \theta_2, \quad \theta_1 > 0, \theta_2 > 0, \quad (14)$$

or

$$\mathcal{L} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = \left(\frac{\partial}{\partial x} + \alpha \right) \left(\frac{\partial}{\partial y} + \beta \right) + \gamma^2, \quad \alpha > 0, \beta > 0, \gamma > 0. \quad (15)$$

- (iii) *Parabolic case* ($y = t$), $t > 0$, where t can be interpreted as *time*:

$$\mathcal{L} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) = \frac{\partial}{\partial t} - \theta_1 \frac{\partial^2}{\partial x^2} + \theta_2, \quad \theta_1 > 0. \quad (16)$$

In this case, fractional versions of the above equation can also be considered, for example, in terms of fractional powers of the negative Laplacian, i.e.,

$$(-\Delta)^{\beta/2} = \left(-\frac{\partial^2}{\partial x^2} \right)^{\beta/2}, \quad \beta \in (0, 1),$$

that is,

$$\mathcal{L} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) = \frac{\partial}{\partial t} + \theta_1 \left(-\frac{\partial^2}{\partial x^2} \right)^{\beta/2} + \theta_2, \quad \theta_1 > 0.$$

The Green function $G(x, y)$ of the corresponding deterministic problem, in equation (11), satisfies the identity

$$\mathcal{L} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) G(x, y) = \delta(x)\delta(y), \quad (17)$$

where δ denotes the Dirac delta distribution. Therefore, since the Green function is a distribution, its Fourier transform

$$\widehat{G}(\lambda_1, \lambda_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp(-i\lambda_1 x - i\lambda_2 y) G(x, y) dx dy.$$

is interpreted in the weak sense. Thus, the general solution to (11) is formally given by

$$\begin{aligned} u(x, y) &= \int_{\mathbb{R}^2} G(x - u, y - v) dW_H(u, v) \\ &= 2\pi \int_{\mathbb{R}^2} \exp(i\lambda_1 x + i\lambda_2 y) \widehat{G}(\lambda_1, \lambda_2) (i\lambda_1)^{-H_1+1/2} (i\lambda_2)^{-H_2+1/2} Z(d\lambda_1, d\lambda_2). \end{aligned} \quad (18)$$

Its distributional and strong-sense definitions, in term of anisotropic fractional Bessel potential spaces, is provided in the Appendix.

The covariance function of u is then formally given by

$$\begin{aligned} B_u(x, y; x', y') = \text{Cov}(u(x, y), u(x', y')) &= \int_{\mathbb{R}^2} \exp(i(\lambda_1(x - x') + \lambda_2(y - y'))) \\ &\quad \times |\widehat{G}(\lambda_1, \lambda_2)|^2 |\lambda_1|^{-2H_1+1} |\lambda_2|^{-2H_2+1} d\lambda_1 d\lambda_2 \\ &= \|G\|_{[\mathcal{H}_{\partial^2 W_H}]^*}^2, \end{aligned}$$

for $H_i \in (0, 1)$ and $1 - 2H_i \in (-1, 1)$, with $i = 1, 2$.

The corresponding spectral density is

$$f(\lambda_1, \lambda_2) = |\widehat{G}(\lambda_1, \lambda_2)|^2 |\lambda_1|^{-2H_1+1} |\lambda_2|^{-2H_2+1}, \quad (\lambda_1, \lambda_2) \in \mathbb{R}^2,$$

which can be interpreted as the spectral density of a continuous stationary Gaussian random field u under the conditions stated in the Appendix. When these conditions do not hold, the Fourier

transform of the covariance function is interpreted in the sense of distributions, as well as B_u , which is defined as

$$B_u(\psi, \varphi) = E[u(\psi)u(\varphi)] = \int_{\mathbb{R}^2} \psi(x, y)B_u(x - u, y - v)\varphi(u, v)dudvdxdy,$$

for ψ and φ in a suitable test function family related to the anisotropic fractional Bessel potential space $\mathbf{H}_2^{-\beta/b}(\mathbb{R}^2)$ (see Appendix). Thus, the generalized random field framework must be considered in the derivation of a formal solution.

4 Elliptic fractional Brownian field

For the elliptic model given in equation (12), the Green function of the corresponding deterministic problem (see Heine, 1955; Mohapl, 1999) is of the form

$$G_\gamma(x, y) = \frac{-1}{2\pi}K_0(-\gamma\sqrt{x^2 + y^2}),$$

while for the operator (13), the Green function is defined as

$$G_{a,b}(x, y) = \frac{-1}{2\pi}K_0(-\gamma\sqrt{a^2x^2 + b^2y^2}),$$

with K_0 denoting the modified Bessel function of second kind and order zero. Its Fourier transform (Matérn class) is of the form

$$\widehat{G}_\gamma(\lambda_1, \lambda_2) = \frac{1}{\gamma^2 + \lambda_1^2 + \lambda_2^2},$$

which is not integrable. Thus, in view of (8), the elliptic fractional Brownian motion field can be written in the space and spectral domains as

$$\begin{aligned} u(x, y) &= \int_{\mathbb{R}^2} \frac{-1}{2\pi}K_0\left(-\gamma\sqrt{(x-u)^2 + (y-v)^2}\right) dW_H(u, v) \\ &= 2\pi c(\gamma) \int_{\mathbb{R}^2} \exp(i(\lambda_1x + \lambda_2y)) \frac{1}{(\lambda_1^2 + \lambda_2^2 + \gamma^2)} \\ &\quad \times (i\lambda_1)^{-H_1+1/2} (i\lambda_2)^{-H_2+1/2} Z(d\lambda_1, d\lambda_2), \end{aligned}$$

with covariance function

$$\begin{aligned} \text{Cov}(u(x, y)u(x', y')) &= c^2(\gamma) \int_{\mathbb{R}^2} \exp(i(\lambda_1(x-x') + \lambda_2(y-y'))) \left(\frac{1}{\lambda_1^2 + \lambda_2^2 + \gamma^2}\right)^2 \\ &\quad \times |\lambda_1|^{-2H_1+1} |\lambda_2|^{-2H_2+1} d\lambda_1 d\lambda_2. \end{aligned} \tag{19}$$

If $H_1 \in (1/2, 1)$ and $H_2 \in (1/2, 1)$, then $-2H_i + 1 \in (-1, 0)$, for $i = 1, 2$. Thus, the spectral density

$$f(\lambda_1, \lambda_2) = c^2(\gamma) \left(\frac{1}{\lambda_1^2 + \lambda_2^2 + \gamma^2}\right)^2 |\lambda_1|^{-2H_1+1} |\lambda_2|^{-2H_2+1} \in L^2(\mathbb{R}^2).$$

For $H_1 = 1/2$ and $H_2 = 1/2$, the Heine (1955)'s formula provides the solution (see also Mohapl, 1999)

$$u(x, y) = \frac{-1}{2\pi} \int_{\mathbb{R}^2} K_0 \left(-\gamma \sqrt{(x-u)^2 + (y-v)^2} \right) Z(du, dv),$$

with covariance function

$$R(x, y) = \frac{1}{4\pi\gamma} \sqrt{x^2 + y^2} K_1 \left(\gamma \sqrt{x^2 + y^2} \right),$$

where K_1 denotes the modified Bessel function of the second kind and order one, and the corresponding spectral density of Matérn class

$$f(\lambda_1, \lambda_2) = c^2(\gamma) \left(\frac{1}{\gamma^2 + \lambda_1^2 + \lambda_2^2} \right)^2,$$

which is absolutely integrable.

For $H_i \in (1/2, 1)$, $i = 1, 2$, the square-integrable functions in the RKHS \mathcal{H}_u of the solution u belong to the anisotropic fractional Bessel potential space $\mathbf{H}_2^{\beta/b}(\mathbb{R}^2)$ with

$$\begin{aligned} \beta &= \frac{2(H_1 + 3/2)(H_2 + 3/2)}{H_2 + H_1 + 3} \\ b_1 &= \frac{2(H_2 + 3/2)}{H_2 + H_1 + 3} \\ b_2 &= \frac{2(H_1 + 3/2)}{H_2 + H_1 + 3} \end{aligned} \quad (20)$$

(see Proposition 2(ii) in the Appendix). Similar formulae can be obtained for (13) (see Guyon, 1987) for the case $H_i = 1/2$, for $i = 1, 2$.

5 Hyperbolic fractional Brownian field

For the operator given in equation (15), the Green function in (17) is defined as (see Heine, 1955)

$$G(x, y) = G_\gamma(x, y) = \exp(-\alpha|x| - \beta|y|) J_0 \left(2\gamma \sqrt{|xy|} \right), \quad (x, y) \in \mathbb{R}^2, \quad (21)$$

where $\alpha > 0$, $\beta > 0$, and

$$J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n n!} \quad (22)$$

is the Bessel function of the first kind and zero order. In particular, for $\gamma = 0$, we have an Ornstein-Uhlenbeck covariance structure

$$G_0(x, y) = \exp(-\alpha|x| - \beta|y|), \quad (x, y) \in \mathbb{R}^2,$$

which has Fourier transform

$$\widehat{G}_0(\lambda_1, \lambda_2) = \frac{\alpha\beta}{\pi^2(\alpha^2 + \lambda_1^2)(\beta^2 + \lambda_2^2)}, \quad (\lambda_1, \lambda_2) \in \mathbb{R}^2.$$

Equivalently, from the above equations, hyperbolic fractional Brownian motion can be formally defined as

$$u(x, y) = 2\pi \int_{\mathbb{R}^2} \exp(i(\lambda_1 x + \lambda_2 y)) \widehat{G}_\gamma(\lambda_1, \lambda_2) (i\lambda_1)^{-H_1+1/2} (i\lambda_2)^{-H_2+1/2} Z(d\lambda_1, d\lambda_2),$$

and for $\gamma = 0$, we have

$$u(x, y) = 2\pi \int_{\mathbb{R}^2} \exp(i(\lambda_1 x + \lambda_2 y)) \frac{\alpha\beta}{\pi^2(\alpha^2 + \lambda_1^2)(\beta^2 + \lambda_2^2)} \times (i\lambda_1)^{-H_1+1/2} (i\lambda_2)^{-H_2+1/2} Z(d\lambda_1, d\lambda_2). \quad (23)$$

Thus, the covariance function of hyperbolic fractional Brownian motion is then given by

$$\begin{aligned} \text{Cov}(u(x, y), u(x', y')) &= \int_{\mathbb{R}^2} \exp(i(\lambda_1(x - x') + \lambda_2(y - y'))) \\ &\quad \times |\widehat{G}_\gamma(\lambda_1, \lambda_2)|^2 |\lambda_1|^{-2H_1+1} |\lambda_2|^{-2H_2+1} d\lambda_1 d\lambda_2, \\ &= \int_{\mathbb{R}^2} \exp(i(\lambda_1(x - x') + \lambda_2(y - y'))) \end{aligned} \quad (24)$$

$$\begin{aligned} &\quad \times |-\lambda_1\lambda_2 + \beta(i\lambda_1) + \alpha(i\lambda_2) + \alpha\beta + \gamma^2|^{-2} |\lambda_1|^{-2H_1+1} |\lambda_2|^{-2H_2+1} d\lambda_1 d\lambda_2, \\ &= \int_{\mathbb{R}^2} \exp(i(\lambda_1(x - x') + \lambda_2(y - y'))) \\ &\quad \times \frac{|\lambda_1|^{-2H_1+1} |\lambda_2|^{-2H_2+1}}{(\lambda_1\lambda_2)^2 + 2(\alpha\beta + \gamma^2)\lambda_1\lambda_2 + \beta^2\lambda_1^2 + \alpha^2\lambda_2^2 + 2\alpha\beta\lambda_1\lambda_2 + (\alpha\beta + \gamma^2)^2} d\lambda_1 d\lambda_2. \end{aligned} \quad (25)$$

Therefore, for $H_i \in (1/2, 1)$, $i = 1, 2$, the spectral density of u is absolutely integrable, i.e., u is a Gaussian stationary random field. While for $H_i \in (0, 1/2)$, $i = 1, 2$, random field u is introduced as a generalized random field, which can be defined on a subspace of $\mathcal{L}(\mathbf{H}^{-s/\mathbf{a}}(\mathbb{R}^2))$, with parameters s and \mathbf{a} given as in equation (10), and \mathcal{L} being the hyperbolic operator (15) (see Proposition 3(iii) in the Appendix). Moreover, in the ordinary case ($H_i \in (1/2, 1)$, $i = 1, 2$), the square integrable functions in the RKHS \mathcal{H}_u also belong to the anisotropic fractional Bessel potential space $\mathbf{H}_2^{\nu/\mathbf{c}}(\mathbb{R}^2)$, with ν and $\mathbf{c} = (c_1, c_2)$ as follows (see Proposition 3(ii) in the Appendix)

$$\begin{aligned} \nu &= \frac{2(H_1 + 1/2)(H_2 + 1/2)}{H_1 + H_2 + 1} \\ c_1 &= \frac{2(H_2 + 1/2)}{H_1 + H_2 + 1} \\ c_2 &= \frac{2(H_1 + 1/2)}{H_1 + H_2 + 1}. \end{aligned} \quad (26)$$

For $\gamma = 0$, we have

$$\begin{aligned} \text{Cov}(u(x, y), u(x', y')) &= \int_{\mathbb{R}^2} \exp(i(\lambda_1(x - x') + \lambda_2(y - y'))) \left| \frac{\alpha\beta}{\pi^2(\alpha^2 + \lambda_1^2)(\beta^2 + \lambda_2^2)} \right|^2 \\ &\quad \times |\lambda_1|^{-2H_1+1} |\lambda_2|^{-2H_2+1} d\lambda_1 d\lambda_2. \end{aligned}$$

For the particular case, $H_i = 1/2$, for $i = 1, 2$ (see Heine, 1955 and Guyon, 1987), the following expression is obtained for the covariance function of u :

$$\begin{aligned} \text{Cov}(u(0, 0), u(x, y)) &= \exp(-\alpha|x| - \beta|y|) \\ &\quad \times \int_0^\infty \exp(-\delta u) J_0(2\gamma(x + u \cos(\theta))^{1/2}) (y + u \sin(\theta))^{1/2} du, \end{aligned} \quad (27)$$

where $\delta = 2\alpha\beta(\alpha^2 + \beta^2)^{-1/2}$; $\tan(\theta) = 4\beta$, and where J_0 is given in (22).

Our main interest relies on the definition of exact and asymptotic formulae of the Fourier transform of the Green function. Specifically, from equation (21), one can compute the Fourier transform of G :

$$\begin{aligned} \hat{G}_\gamma(\lambda_1, \lambda_2) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp(-i\lambda_1 x - i\lambda_2 y - \alpha|x| - \beta|y|) \mathcal{I}_0(2\gamma\sqrt{|xy|}) dx dy \\ &= \frac{1}{2\pi} \sum_{n=0}^\infty \int_{\mathbb{R}^2} \exp(-i\lambda_1 x - i\lambda_2 y - \alpha|x| - \beta|y|) \frac{(-1)^n (2\gamma)^{2n} |xy|^n}{2^n n!} dx dy \\ &= \frac{1}{2\pi} \sum_{n=0}^\infty \frac{(2\gamma)^{2n} (-1)^n}{2^n n!} \left[\int_{\mathbb{R}} \exp(-i\lambda_1 x - \alpha|x|) |x|^n dx \right] \\ &\quad \times \left[\int_{\mathbb{R}} \exp(-i\lambda_2 y - \beta|y|) |y|^n dy \right] \\ &= \frac{1}{2\pi} \sum_{n=0}^\infty 2^{n+1} (-1)^n (\gamma)^{2n} n! \frac{(\alpha - i\lambda_1)^{n+1} + (\alpha + i\lambda_1)^{n+1}}{(\alpha^2 + \lambda_1^2)^{n+1}} \times \\ &\quad \times \frac{(\beta - i\lambda_2)^{n+1} + (\beta + i\lambda_2)^{n+1}}{(\alpha^2 + \lambda_2^2)^{n+1}}, \end{aligned} \quad (28)$$

which is defined in the sense of distributions, over the space of infinitely differentiable functions with compact support contained in \mathbb{R}^2 .

Mohapl (1999, equation (55)) provides, for the model

$$\left(\frac{\partial^2}{\partial x \partial y} + \theta_2 \frac{\partial}{\partial y} + \theta_1 \frac{\partial}{\partial x} + \theta_1 \theta_2 \right) u(x, y) = \frac{\partial^2 W_{1/2, 1/2}}{\partial x \partial y}(x, y), \quad (29)$$

the following solution:

$$\begin{aligned} u(x, y) &= q_1 \exp(-\theta_1 x) + q_2 \exp(-\theta_2 y) + q_3 \exp(-\theta_1 x - \theta_2 y) \\ &\quad + \gamma \int_0^x \int_0^y \exp(-\theta_1(y - v) - \theta_2(x - u)) Z(du, dv), \end{aligned} \quad (30)$$

with covariance function

$$R(x, y) = \frac{1}{q_1 q_2 \theta_1 \theta_2} \exp(-\theta_1 |x| - \theta_2 |y|),$$

where $q_i, i = 1, 2$, are i.i.d. Gaussian random variables with zero mean and variance $1/(\theta_1 \theta_2)$.

6 Parabolic fractional Brownian field

In the simplest case, for $y = t > 0$, and for $\theta > 0$, we have the classical heat equation:

$$\left(\frac{\partial}{\partial t} - \theta \frac{\partial^2}{\partial x^2} \right) u(t, x) = \frac{\partial^2 W_H}{\partial t \partial x}(t, x), \quad (31)$$

where $H_j \in (0, 1), j = 1, 2$. Its solution can be expressed as

$$u(t, x) = \sqrt{2\pi} \int_{\mathbb{R}} \exp(ix\lambda) \int_0^t \exp(-\theta(t-s)|\lambda|^2) (i\lambda)^{-H_2+1/2} Z_s(d\lambda), \quad (32)$$

where H_2 denotes the Hurst index in space, and

$$\frac{\partial^2 W_H}{\partial t \partial x}(t, x) = \int_{\mathbb{R}} \exp(i\lambda x) (i\lambda)^{-H_2+1/2} Z_t(d\lambda). \quad (33)$$

Thus, $Z_t(d\lambda)$ is defined, in the Gaussian context, as a generalized random field, in the temporal domain, and as a random white noise measure, in the spatial spectral domain, satisfying

$$E [Z_s(d\omega) Z_t(d\lambda)] \underset{t-w.s.}{=} B'(s, t) \frac{1}{2\pi} \delta(\omega - \lambda) ds d\lambda, \quad (34)$$

with

$$B'(s, t) \underset{st-w.s.}{=} \frac{\partial^2}{\partial s \partial t} \left(\frac{c(H_1)}{2} (|s|^{2H_1} + |t|^{2H_1} - |s-t|^{2H_1}) \right),$$

defined in terms of the temporal Hurst index H_1 . Here, $\underset{st-w.s.}{=}$ stands for the weak-sense identity in the temporal parameters s and t , that is, for the identity in the sense of tempered distributions in time, i.e.,

$$\begin{aligned} f(\cdot) \underset{t-w.s.}{=} g(\cdot) &\iff \int_{\mathbb{R}_+} g(s) \phi(s) ds = \int_{\mathbb{R}_+} f(s) \phi(s) ds, \\ B(\cdot, \cdot) \underset{st-w.s.}{=} K(\cdot, \cdot) &\iff \int_{\mathbb{R}_+ \times \mathbb{R}_+} B(s, t) \phi(s, t) ds dt = \int_{\mathbb{R}_+ \times \mathbb{R}_+} K(s, t) \phi(s, t) ds dt, \end{aligned}$$

for all test function ϕ in the dual Hilbert space \mathbf{H}^* of \mathbf{H} (respectively, in the dual of $\mathbf{H} \otimes \mathbf{H}$), the Hilbert space where f and g belong to (respectively, where B and K belong to).

In the spatiotemporal domain, the Green function G of the corresponding deterministic problem is given by

$$G(t, x) = \frac{1}{\sqrt{4\pi\theta t}} \exp\left(-\frac{|x|^2}{4\theta t}\right).$$

Therefore, the solution u to problem (31) can also be formally expressed as

$$u(t, x) = \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\theta(t-s)}} \exp\left(-\frac{|x-y|^2}{4\theta(t-s)}\right) \partial^2 W_H(ds, dy). \quad (35)$$

In the more general case (Mohapl, 1999)

$$\left(\frac{\partial}{\partial t} - \theta_1 \frac{\partial^2}{\partial x^2} + \theta_2\right) u(t, x) = \frac{\partial^2 W_H}{\partial t \partial x}(t, x),$$

the Green function is defined as

$$G(t, x) = \frac{1}{\sqrt{4\pi\theta_1 t}} \exp\left(-\frac{x^2}{4\theta_1 t} - \theta_2 t\right).$$

Then,

$$\widehat{G}(t, \lambda) = \exp\left(-\theta_1 t |\lambda|^2 - \theta_2 t\right), \quad t > 0.$$

In Mohapl (1999), the associated covariance function for the case $H_i = 1/2$, for $i = 1, 2$, is obtained as

$$B(t, x) = \frac{1}{\sqrt{4\pi\theta_1 t}} \int_{\mathbb{R}'} \exp\left(-\frac{(x-y)^2}{4\theta_1 t} - \theta_2 t\right) \rho(y) dy,$$

where

$$\rho(y) = \frac{\sigma^2}{2\sqrt{\theta_2\theta_1}} \exp\left(-|y| \sqrt{\frac{\theta_2}{\theta_1}}\right).$$

The above derivation of an explicit solution of equation (31), given by (32), in the spatial spectral domain, and by (35), in the spatiotemporal domain, is based on the semigroup approach. Under this approach, from the differential geometry of the random string processes, Wu and Xiao (2006) also obtain the characterization of the sample path properties of the solution of equation (31), randomly initialized, for the case $H_i = 1/2$, for $i = 1, 2$. The book by Chow (2007) provides an overview on the treatment of stochastic partial differential equations, and, in particular, on stochastic parabolic equations, under the semigroup approach, including the case of bounded domains where the point spectra approximation can be considered.

Alternatively to the semigroup approach, a stationary increment solution can also be explicitly derived on \mathbb{R}^2 , for $H_i = 1/2$, $i = 1, 2$, and $\theta_1 = 1$, $\theta_2 = 0$, as follows (see, for example, Robeva and Pitt, 2007):

$$u(t, x) = 2\pi \int_{\mathbb{R}^2} \frac{[\exp(i \langle (t, x), (\omega, \lambda) \rangle) - 1]}{i\omega + \lambda^2} Z(d\omega, d\lambda),$$

since

$$\int_{\mathbb{R}^2} \frac{\omega^2 + \lambda^2}{1 + \omega^2 + \lambda^2} \frac{1}{\omega^2 + \lambda^4} d\omega d\lambda < \infty$$

(see Yaglom, 1957). Here, $\langle (t, x), (\omega, \lambda) \rangle = t\omega + x\lambda$ and Z represents a Gaussian white noise measure. For $H_i \neq 1/2$, $i = 1, 2$, and $\theta_1 = 1$, $\theta_2 = 0$, a stationary increment solution can be defined as

$$u(t, \mathbf{x}) = 2\pi \int_{\mathbb{R}^2} \frac{[\exp(i \langle (t, \mathbf{x}), (\omega, \lambda) \rangle) - 1]}{i\omega + \lambda^2} (i\omega)^{-H_1+1/2} (i\lambda)^{-H_2+1/2} Z(d\omega, d\lambda),$$

in the generalized random field sense, on the space of infinitely differentiable functions which vanish, together with all their derivatives, outside of a compact domain (see Yaglom, 1986, pp.437-438, on generalized locally homogeneous fields). Specifically, for H_1 and H_2 such that there exists an integer p with

$$\int_{\mathbb{R}^2} \frac{\omega^2 + \lambda^2}{(1 + \omega^2 + \lambda^2)^{p+1}} \frac{1}{\omega^2 + \lambda^4} \omega^{-2H_1+1} \lambda^{-2H_2+1} d\omega d\lambda < \infty,$$

the solution can be derived in the generalized random field setting stated in Yaglom (1986). Within this generalized random field solution framework, in the Gaussian innovation case (see, for example, Kelbert, Leonenko and Ruiz-Medina, 2005), the scale of anisotropic Bessel potential spaces also provides a suitable context for the definition of the weak-sense solution of the heat equation (31) on \mathbb{R}^2 as follows:

$$U(\phi) = 2\pi \int_{\mathbb{R}^2} \phi(t, \mathbf{x}) \int_{\mathbb{R}^2} \exp(i \langle (t, \mathbf{x}), (\omega, \lambda) \rangle) \frac{1}{i\omega + \lambda^2} (i\omega)^{-H_1+1/2} (i\lambda)^{-H_2+1/2} Z(d\omega, d\lambda) dt d\mathbf{x},$$

for $\theta = 1$, and for every $\phi \in \mathcal{D} \left(\left[\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \left(\frac{\partial}{\partial t} \right)^{H_1-1/2} \left(\frac{\partial}{\partial x} \right)^{H_2-1/2} \right]^{-1} \right)$, i.e., for any function ϕ in the domain of operator

$$\left[\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \left(\frac{\partial}{\partial t} \right)^{H_1-1/2} \left(\frac{\partial}{\partial x} \right)^{H_2-1/2} \right]^{-1}$$

on \mathbb{R}^2 . Specifically, we can select a subspace of $\mathcal{L}(\mathbf{H}^{-s/\mathbf{a}}(\mathbb{R}^2))$ as test function space for the generalized random field solution, with parameters s and \mathbf{a} given as in equation (10), and \mathcal{L} being the parabolic operator defining equation (31) (see Proposition 4(iii) in the Appendix). Furthermore, for $H_i \in (1/2, 1)$, $i = 1, 2$, the square integrable functions in the RKHS \mathcal{H}_u belong to the anisotropic fractional Bessel potential space $\mathbf{H}_2^{-r/\mathbf{e}}(\mathbb{R}^2)$, where parameter r and \mathbf{e} are given by (see Proposition 4(ii) in the Appendix)

$$\begin{aligned} r &= \frac{2(H_1 + 1/2)(H_2 + 3/2)}{H_1 + H_2 + 2} \\ e_1 &= \frac{2(H_2 + 3/2)}{H_1 + H_2 + 2} \\ e_2 &= \frac{2(H_1 + 1/2)}{H_1 + H_2 + 2}. \end{aligned} \tag{36}$$

Similar arguments can be applied to the multidimensional case, that is, to the case where the following parabolic equation is considered:

$$\left(\frac{\partial}{\partial t} + \mathcal{L} \right) u(t, \mathbf{x}) = \frac{\partial^{d+1} W_H}{\partial t \partial x_1 \dots \partial x_d} (t, \mathbf{x}), \tag{37}$$

where \mathcal{L} is an elliptic operator with constant coefficients on \mathbb{R}^d . In this case, the Gaussian generalized random field solution is defined as

$$\begin{aligned} U(\phi) &= (2\pi)^{(d+1)/2} \int_{\mathbb{R}^{d+1}} \phi(t, \mathbf{x}) \int_{\mathbb{R}^{d+1}} \exp(i \langle (t, \mathbf{x}), (\omega, \boldsymbol{\lambda}) \rangle) \frac{1}{i\omega + \mathcal{P}(\boldsymbol{\lambda})} \\ &\quad \times (i\omega)^{-H_1+1/2} (i\lambda_1)^{-H_2+1/2} \dots (i\lambda_d)^{-H_{d+1}+1/2} Z(d\omega, d\boldsymbol{\lambda}) dt d\mathbf{x}, \end{aligned} \tag{38}$$

with \mathcal{P} denoting the characteristic polynomial of operator \mathcal{L} , Z being a Gaussian white noise measure on \mathbb{R}^{d+1} , and ϕ representing, as before, a suitable test function in the domain of operator

$$\left[\left(\frac{\partial}{\partial t} + \mathcal{L} \right) \left(\frac{\partial}{\partial t} \right)^{H_1-1/2} \left(\frac{\partial}{\partial x_1} \right)^{H_2-1/2} \cdots \left(\frac{\partial}{\partial x_d} \right)^{H_{d+1}-1/2} \right]^{-1}$$

on \mathbb{R}^{d+1} . The scale of anisotropic fractional Bessel potential spaces again provides an appropriate functional space scale, in the selection procedure of the space where the test functions lie for derivation of a generalized random field solution. An element of this scale is chosen according to the order of the characteristic polynomial of \mathcal{L} with respect to each independent spatial variable.

Note also that, under the above general setting, a stationary increment Gaussian solution on \mathbb{R}^2 ,

$$u(t, \mathbf{x}) = (2\pi)^{(d+1)/2} \int_{\mathbb{R}^{d+1}} \frac{[\exp(i \langle (t, \mathbf{x}), (\omega, \lambda) \rangle) - 1]}{i\omega + \mathcal{P}(\lambda)} (i\omega)^{-H_{d+1}+1/2} \\ \times (i\lambda_1)^{-H_1+1/2} \cdots (i\lambda_d)^{-H_d+1/2} Z(d\omega, d\lambda)$$

can be defined under the assumption that \mathcal{L} has characteristic polynomial \mathcal{P} such that the following condition holds (see Yaglom, 1957):

$$\int_{\mathbb{R}^{d+1}} \frac{\omega^2 + \|\lambda\|^2}{1 + \omega^2 + \|\lambda\|^2} \frac{1}{\omega^2 + [\mathcal{P}(\lambda)]^2} \omega^{-2H_{d+1}+1} (\lambda_1)^{-2H_1+1} \cdots (\lambda_d)^{-2H_d+1} d\omega d\lambda < \infty.$$

7 Parabolic equations with a spatial diffusion operator with variable coefficients

Interesting alternative examples of parabolic equations can be introduced in terms of the d -dimensional equation

$$\left[\frac{\partial}{\partial t} + \mathcal{L}(t, x) \right] u(t, \mathbf{x}) = \frac{\partial^{d+1} W_H}{\partial t \partial x_1 \cdots \partial x_d}(t, \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \quad (39)$$

when some special cases of operator $\mathcal{L}(t, x)$ are considered, including the case of temporal variable coefficients, continuous functions of the negative Laplacian operator, and multifractional elliptic operators.

1. We first suppose that $\mathcal{L}(t, x) = \mathcal{L}(t)$, that is, consider

$$\mathcal{L}(t) = - \sum_{j,k=1}^d a_{j,k}(t) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{k=1}^d b_k(t) \frac{\partial}{\partial x_k} + c(t),$$

$$H = (H_1, \dots, H_{d+1}) \in (0, 1)^{d+1},$$

and the $d + 1$ -dimensional fractional Brownian motion can be defined in the spatial spectral domain as follows:

$$W_H(t, x_1, \dots, x_d) = \int_{\mathbb{R}^{d+1}} \prod_{j=1}^d \left[\frac{\exp(i\lambda_j x_j) - 1}{i\lambda_j} (i\lambda_j)^{-H_j+1/2} \right] Z_t(d\lambda_1, \dots, d\lambda_d),$$

where, for each $t \in \mathbb{R}_+$, Z_t is a complex Gaussian white noise on the d -dimensional spatial spectral domain satisfying

$$E|Z_s(d\omega_1, \dots, d\omega_d)Z_t(d\lambda_1, \dots, d\lambda_d)|_{t-w.s.} = B'(s, t) \frac{1}{(2\pi)^d} d\lambda_1 \dots d\lambda_d,$$

with, as before,

$$B'(s, t) \underset{st-w.s.}{=} \frac{\partial^2}{\partial s \partial t} \left(\frac{c(H_{d+1})}{2} (|s|^{2H_{d+1}} + |t|^{2H_{d+1}} - |s-t|^{2H_{d+1}}) \right). \quad (40)$$

The coefficients $\{a_{j,k}(t)\}$, $\{b_k(t)\}$, $c(t)$ of $\mathcal{L}(t)$ are assumed to be continuous functions on the half-line $[0, \infty)$, and

$$a_{j,k}(t) = a_{k,j}(t).$$

We assume local parabolicity of the equation, that is, for every $T > 0$, there exists a $K_T > 0$ such that

$$\operatorname{Re} \left(\sum_{j,k=1}^d a_{j,k}(t) z_j z_k^* \right) \geq K_T |z|^2$$

for any $t \in [0, T]$ and $z = (z_1, \dots, z_d) \in \mathbb{C}^d$.

In addition, we shall assume that the following oscillation condition holds: for some $A > 0$

$$\left| \det \left(\operatorname{Im} \left(\int_0^t a_{j,k}(s) ds \right) \right) \right| \geq A |t|^d, \quad t \in [0, \infty).$$

We shall also assume that

$$\int_0^\infty \left| \operatorname{Re} (a_{j,k}(t)) \right| dt < \infty, \quad j, k = 1, \dots, d; \quad \int_0^\infty |\operatorname{Re}(c(t))| dt < \infty.$$

Under the above assumptions, the Green function of the equation

$$\left[\frac{\partial}{\partial t} + \mathcal{L}(t) \right] u(t, \mathbf{x}) = 0$$

is of the form

$$G(t, \mathbf{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle - A(t, \boldsymbol{\lambda})) d\boldsymbol{\lambda},$$

where

$$A(t, \boldsymbol{\lambda}) = \sum_{j,k} \lambda_j \lambda_k \int_0^t a_{jk}(s) ds - i \sum_{k=1}^d \lambda_k \int_0^t b_k(s) ds + \int_0^t c(s) ds,$$

and satisfies

$$|G(t, x)| < c' t^{-d/2}, \quad t > 0, \quad x \in \mathbb{R}^d,$$

$$|D_x^m G(t, x)| < c_{T,m} t^{-(d+|m|)/2} \exp\left(-c^d \frac{|x|^2}{t}\right), \quad t \in [0, T], \quad x \in \mathbb{R}^d.$$

Thus, G can be defined in the spectral domain as

$$\widehat{G}(t, \lambda) = \exp\{-A(t, \lambda)\} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-i \langle \lambda, \mathbf{x} \rangle) G(t, \mathbf{x}) d\mathbf{x}.$$

Moreover, under the above conditions,

$$u(t, \mathbf{x}) = \int_0^t (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \mathbf{x}, \lambda \rangle) \exp\left\{-\int_s^t A(u, \lambda) du\right\} \prod_{j=1}^d (i\lambda_j)^{-H_j+\frac{1}{2}} Z_s(d\lambda_1, \dots, d\lambda_d).$$

and

$$\begin{aligned} \text{Cov}(u(t, \mathbf{x}), u(s, \mathbf{y})) &= \int_{\mathbb{R}^d} \exp(i \langle \mathbf{x} - \mathbf{y}, \lambda \rangle) \exp\left(-\int_{u \vee v}^{t \wedge s} A(\xi, \lambda) d\xi\right) \\ &\quad \times \prod_{j=1}^d |\lambda_j|^{-2H_j+1} B'(u, v) d\lambda_1 \dots d\lambda_d, \end{aligned} \quad (41)$$

where $u \vee v$ denotes the maximum of u and v , and $t \wedge s$ denotes the minimum of t and s . Here, the temporal covariance function B' is defined in the weak-sense as in equation (40).

2. Extensions of equation (31) can also be defined considering as spatial diffusion operator, a continuous function of the negative Laplacian operator on \mathbb{R}^d , i.e.,

$$\left(\frac{\partial}{\partial t} + f(-\Delta)\right) u(t, \mathbf{x}) = \frac{\partial^{d+1} W_H}{\partial t \partial x_1 \dots \partial x_d}(t, \mathbf{x}), \quad (42)$$

where, for example,

$$f(-\Delta) = \frac{P(-\Delta)}{Q(-\Delta)}. \quad (43)$$

Here, P and Q denote positive elliptic polynomials of respective fractional orders p and q , with $p, q \in \mathbb{R}_+$. One may also consider $f(-\Delta) = (-\Delta)^{\alpha/2} (I - \Delta)^{\beta/2}$, $\alpha, \beta > 0$. The parabolic equation with this spatial diffusion operator has been studied in detail in Angulo, Ruiz-Medina, Anh and Grecksch (2000), for the case $H_i = 1/2$, for $i = 1, 2$. Alternatively, for the same case $H_i = 1/2$, for $i = 1, 2$, in Kelbert, Leonenko and Ruiz-Medina (2005), fractional extensions of equation (31) are formulated in terms of fractional powers of operator $\left(\frac{\partial}{\partial t} + \theta_1(-\Delta) + \theta_2\right)$, i.e., considering the fractional Helmholtz equation

$$\left(\frac{\partial}{\partial t} + \theta_1(-\Delta) + \theta_2\right)^v u(t, \mathbf{x}) = \varepsilon(t, \mathbf{x}), \quad v > 0,$$

where ε denotes spatiotemporal white noise.

The solution to equation (42) is defined as

$$u(t, \mathbf{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \mathbf{x}, \boldsymbol{\lambda} \rangle) \int_0^t \exp(-(t-s)f(\|\boldsymbol{\lambda}\|^2)) \prod_{j=1}^d (i\lambda_j)^{-H_j+1/2} Z_s(d\lambda_1, \dots, d\lambda_d), \quad (44)$$

where $Z_s(d\lambda_1, \dots, d\lambda_d)$ is given as in equations (33)-(34), considering, in the spectral domain, a d -dimensional white noise measure.

3. The case where \mathcal{L} is an elliptic multifractional pseudodifferential operator is now studied. Specifically, in the equation

$$\left(\frac{\partial}{\partial t} + \mathcal{L} \right) u(t, \mathbf{x}) = \frac{\partial^{d+1} W_H}{\partial t \partial x_1 \dots \partial x_d}(t, \mathbf{x}),$$

operator \mathcal{L} is defined as

$$\mathcal{L}(\phi)(\mathbf{x}) = \mathcal{P}(\phi)(\mathbf{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \mathbf{x}, \boldsymbol{\lambda} \rangle) p(\mathbf{x}, \boldsymbol{\lambda}) \widehat{\phi}(\boldsymbol{\lambda}) d\boldsymbol{\lambda}, \quad (45)$$

where \mathcal{P} is a pseudodifferential operator of variable order (see, for instance, Jacob, 2005 and Leopold, 1991) with symbol p in the space of C^∞ functions whose derivatives of all orders are bounded, and satisfying, for any multi-indices α and β , that there exists a positive constant $C_{\alpha,\beta}$ such that

$$|D_{\mathbf{x}}^\alpha D_{\boldsymbol{\lambda}}^\beta p(\mathbf{x}, \boldsymbol{\lambda})| \leq C_{\alpha,\beta} \langle \boldsymbol{\lambda} \rangle^{\sigma(\mathbf{x}) - \rho|\alpha| + \delta|\beta|}, \quad (46)$$

where $0 \leq \delta < \rho \leq 1$, and σ is a real-valued function in $\mathcal{B}^\infty(\mathbb{R}^d)$, the set of all C^∞ -functions whose derivatives of all orders are bounded. Here,

$$\langle \boldsymbol{\lambda} \rangle = (1 + \|\boldsymbol{\lambda}\|^2)^{1/2}. \quad (47)$$

The solution is then defined as (see Ruiz-Medina, Angulo and Anh, 2008, for the case $H_i = 1/2$, for $i = 1, \dots, d+1$)

$$u(t, \mathbf{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \mathbf{x}, \boldsymbol{\lambda} \rangle) \int_0^t \exp(-(t-s)p(\mathbf{x}, \boldsymbol{\lambda})) \prod_{j=1}^d (i\lambda_j)^{-H_j+1/2} Z_s(d\lambda_1, \dots, d\lambda_d), \quad (48)$$

where $Z_s(d\boldsymbol{\lambda})$ is given as in equations (33)-(34), considering the d -dimensional spatial spectral case.

4. An alternative multifractional version of equation (42) is obtained when time-dependent pseudodifferential operators are studied. Explicitly, the following multifractional operator can be considered:

$$\mathcal{L}(\phi)(\mathbf{x}) = \mathcal{P}(\phi)(t, \mathbf{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \mathbf{x}, \boldsymbol{\lambda} \rangle) p(t, \boldsymbol{\lambda}) \widehat{\phi}(\boldsymbol{\lambda}) d\boldsymbol{\lambda},$$

where the symbol p is again in the space of C^∞ functions whose derivatives of all orders are bounded, and satisfies similar regularity conditions, as above, with respect to the independent variables t and $\boldsymbol{\lambda}$. The solution is then defined as

$$u(t, \mathbf{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \int_0^t \exp(i \langle \mathbf{x}, \boldsymbol{\lambda} \rangle) \exp\left(-\int_s^t p(u, \boldsymbol{\lambda}) du\right) \prod_{j=1}^d (i\lambda_j)^{-H_j+1/2} Z_s(d\lambda_1, \dots, d\lambda_d). \quad (49)$$

In particular, we can consider the multifractional heat-type (temporal-multifractional Riesz-Bessel) equation defined as

$$\left(\frac{\partial}{\partial t} + (I - \Delta)^{\beta(t)/2} (-\Delta)^{\gamma(t)/2} \right) u(t, \mathbf{x}) = \frac{\partial^{d+1} W_H}{\partial t \partial x_1 \dots \partial x_d} (t, \mathbf{x}),$$

where the symbol p defining the multifractional pseudodifferential operator is given by

$$p(t, \boldsymbol{\lambda}) = (1 + \|\boldsymbol{\lambda}\|^2)^{\beta(t)/2} \|\boldsymbol{\lambda}\|^{\gamma(t)}.$$

8 Fractional both in time and in space equations

Let us now consider the equation

$$\frac{\partial^\beta u}{\partial t^\beta} (t, \mathbf{x}) = -\mu (I - \Delta)^{\alpha/2} (-\Delta)^{\gamma/2} u(t, \mathbf{x}) + \frac{\partial^{d+1} W_H}{\partial t \partial x_1 \dots \partial x_d} (t, \mathbf{x}), \quad t > 0, \mathbf{x} \in \mathbb{R}^d,$$

where $\beta \in (0, 2]$, $\gamma \geq 0$, $\alpha > 0$ are fractional parameters. The fractional derivative in time is taken in the Caputo-Djrbashian sense:

$$\frac{\partial^\beta u}{\partial t^\beta} = \begin{cases} \frac{\partial^m u(t, \mathbf{x})}{\partial t^m}, & \beta = m \in \{1, 2\}, \\ \frac{1}{\Gamma(m-\beta)} \int_0^t (t-\tau)^{m-\beta-1} \frac{\partial^m u(\tau, \mathbf{x})}{\partial \tau^m} d\tau, & m-1 < \beta < m, m \in \{1, 2\}. \end{cases}$$

Here, Δ is the d -dimensional Laplace operator, and the operators $-(I - \Delta)^{\gamma/2}$, $\gamma \geq 0$, and $(-\Delta)^{\alpha/2}$, $\alpha > 0$, are interpreted as inverses of the Bessel and Riesz potentials respectively. Both Bessel and Riesz potentials are considered to be defined acting on the tempered distributions in the frequency domain, as it is usual, in the framework of fractional Bessel potential spaces (see Triebel, 1978).

The spatial Fourier transform of the Green function of the corresponding deterministic problem is defined as

$$\widehat{G}(t, \boldsymbol{\lambda}) = E_\beta \left(-\mu t^\beta \|\boldsymbol{\lambda}\|^\alpha (1 + \|\boldsymbol{\lambda}\|^2)^{\gamma/2} \right),$$

where

$$E_\beta(-x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^j}{\Gamma(j\beta + 1)},$$

is the Mittag-Leffler function, and

$$G(t, \mathbf{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle) E_\beta \left(-\mu t^\beta \|\boldsymbol{\lambda}\|^\alpha (1 + \|\boldsymbol{\lambda}\|^2)^{\gamma/2} \right) d\boldsymbol{\lambda}.$$

Note that

$$E_\beta(-x) = \begin{cases} \exp(-x), & \beta = 1 \\ \sim \frac{c}{x}, & 0 < \beta < 1, \end{cases}$$

when $x \rightarrow \infty$.

The solution then is

$$u(t, \mathbf{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \mathbf{x}, \boldsymbol{\lambda} \rangle) \int_0^t E_\beta \left(-\mu(t-s)^\beta \|\boldsymbol{\lambda}\|^\alpha (1 + \|\boldsymbol{\lambda}\|^2)^{\gamma/2} \right) \times \prod_{j=1}^d (i\lambda_j)^{-H_j+1/2} Z_s(d\lambda_1, \dots, d\lambda_d), \quad (50)$$

where, for each $s \in \mathbb{R}_+$, Z_s is defined as in the previous section. The anisotropic fractional Bessel potential spaces can also be considered here to characterize, under suitable conditions (see Appendix), the mean smoothness index s of the functions in the RKHS of the solution u , as well as their directional smoothness indexes. Specifically, for $H_i \in (1/2, 1)$, $i = 1, \dots, d+1$, the square-integrable functions in the RKHS \mathcal{H}_u of the solution u have mean smoothness index s given from the formula

$$\frac{1}{s} = \frac{1}{d+1} \left[\frac{1}{H_{d+1} - 1/2 + \beta} + \sum_{i=1}^d \frac{1}{H_i - 1/2 + \alpha + \gamma} \right],$$

where the temporal smoothness index is $H_{d+1} - 1/2 + \beta$, and the spatial smoothness indexes are $H_i - 1/2 + \alpha + \gamma$, for $i = 1, \dots, d$.

Fractional interpolation is possible from the equation

$$\frac{\partial^\beta u}{\partial t} - \mu(-\Delta)^{\gamma/2} u = \frac{\partial^{d+1} W_H}{\partial t \partial x_1 \dots \partial x_d}(t, \mathbf{x}),$$

with formal solution

$$u(t, \mathbf{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \mathbf{x}, \boldsymbol{\lambda} \rangle) \int_0^t E_\beta \left(-\mu(t-s)^\beta \|\boldsymbol{\lambda}\|^\gamma \right) \prod_{j=1}^d (i\lambda_j)^{-H_j+1/2} Z_s(d\lambda_1, \dots, d\lambda_d),$$

since for $0 < \beta \leq 1$, we have a fractional parabolic equation, for $1 < \beta < 2$, we have fractional parabolic-hyperbolic equation, and for $\beta = 2$, the hyperbolic case is recovered. Note that for $\beta = 0$, we have the elliptic equation.

9 General cases

We now consider, in equation (31), the case where the initial behavior of the solution u is defined in terms of a spatial stationary process. Specifically, the following problem is studied:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) u(t, x) = \frac{\partial^2 W_H}{\partial t \partial x}(t, x), \quad x \in \mathbb{R}, \quad t > 0, \quad (51)$$

$$u(0, x) = h(x) = \sqrt{2\pi} \int_{\mathbb{R}} \exp(i\lambda x) \sqrt{f_h(\lambda)} Z_h(d\lambda), \quad (52)$$

with formal solution (see (32)) given by

$$\begin{aligned}
u(t, x) &= \int_{\mathbb{R}} G(t, x - y)h(y)dy + \int_0^t \int_{\mathbb{R}} G(t - s; x - y)\partial^2 W_H(ds, dy), \\
&= \sqrt{2\pi} \int_{\mathbb{R}} \exp(i\lambda x) \exp(-t\lambda^2) \sqrt{f_h(\lambda)} Z_h(d\lambda) \\
&+ \sqrt{2\pi} \int_0^t \int_{\mathbb{R}} \exp(i\lambda x) \exp(-(t - s)\lambda^2) (i\lambda)^{-H_2+1/2} Z_s(d\lambda),
\end{aligned} \tag{53}$$

where $Z_s(d\lambda)$ is defined as in equations (33)-(34), and f_h and Z_h respectively denote the spectral density and the white noise measure associated with the spectral representation of the stationary random initial condition h . Note that for the suitable definition of equation (53), h must be such that the Green function G , as a function of the spatial component, is in the intersection of the dual spaces of the spatial reproducing kernel Hilbert spaces of processes h and $\frac{\partial^2 W_H}{\partial t \partial x}(t, \cdot)$, i.e., $G_t \in [\mathcal{H}_h]^* \cap \left[\mathcal{H}_{\frac{\partial^2 W_H}{\partial t \partial x}(t, \cdot)} \right]^*$.

9.1 Multidimensional wave equation

In the hyperbolic case, the following extended formulation is considered:

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(t, \mathbf{x}) &= -(I - \Delta)u(t, \mathbf{x}) + \frac{\partial^{d+1} W_H}{\partial t \partial x_1 \dots \partial x_d}(t, \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, t > 0 \\
u(0, \mathbf{x}) &= g(\omega, \mathbf{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle) \sqrt{f_g(\boldsymbol{\lambda})} Z_g(d\boldsymbol{\lambda}), \quad \omega \in \Omega, \mathbf{x} \in \mathbb{R}^d \\
\frac{\partial u}{\partial t}(0, \mathbf{x}) &= h(\omega, \mathbf{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle) \sqrt{f_h(\boldsymbol{\lambda})} Z_h(d\boldsymbol{\lambda}), \quad \omega \in \Omega, \mathbf{x} \in \mathbb{R}^d,
\end{aligned} \tag{54}$$

with Ω denoting the sample space involved in the construction of the basic probabilistic space (Ω, \mathcal{A}, P) , where the random fields h , g , and u are defined. Here, f_g and f_h are respectively the spectral densities of stationary random fields g and h , and Z_g and Z_h represent the respective spectral white noise measures involved in the spectral representation of such random fields.

The solution to problem (54) can be defined as

$$\begin{aligned}
u(t, \mathbf{x}) &= (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle) \frac{\partial \widehat{G}}{\partial t}(t, \boldsymbol{\lambda}) \widehat{g}(\boldsymbol{\lambda}) d\boldsymbol{\lambda} + (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle) \widehat{G}(t, \boldsymbol{\lambda}) \widehat{h}(\boldsymbol{\lambda}) d\boldsymbol{\lambda} \\
&+ (2\pi)^{d/2} \int_0^t \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle) \widehat{G}(t-s, \boldsymbol{\lambda}) \prod_{j=1}^d (i\lambda_j)^{-H_j+1/2} Z_s(d\boldsymbol{\lambda}) \\
&= (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle) \frac{\partial \widehat{G}}{\partial t}(t, \boldsymbol{\lambda}) \sqrt{f_g(\boldsymbol{\lambda})} Z_g(d\boldsymbol{\lambda}) \\
&+ (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle) \widehat{G}(t, \boldsymbol{\lambda}) \sqrt{f_h(\boldsymbol{\lambda})} Z_h(d\boldsymbol{\lambda}) \\
&+ (2\pi)^{d/2} \int_0^t \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle) \widehat{G}(t-s, \boldsymbol{\lambda}) \prod_{j=1}^d (i\lambda_j)^{-H_j+1/2} Z_s(d\boldsymbol{\lambda}),
\end{aligned} \tag{55}$$

where $Z_s(d\boldsymbol{\lambda})$ is given as in equations (33)-(34), considering the d -dimensional spatial domain case, i.e., in terms of a d -dimensional Gaussian white noise measure in the spectral domain. Here, the Fourier transform of the Green function G associated with the corresponding deterministic problem is given by

$$\widehat{G}(t, \boldsymbol{\lambda}) = \frac{\sin(\langle \boldsymbol{\lambda} \rangle t)}{\langle \boldsymbol{\lambda} \rangle},$$

with $\langle \boldsymbol{\lambda} \rangle$ defined as in equation (47). For the suitable definition of u , the random fields g and h must be such that $G_t \in [\mathcal{H}_g]^* \cap [\mathcal{H}_h]^* \cap \left[\mathcal{H}_{\frac{\partial^2 w_H}{\partial t \partial x}}(t, \cdot) \right]^*$. Note that here the framework of anisotropic fractional Bessel potential spaces can also be introduced for characterization of the local regularity properties of the solution, according to the mean smoothness index, and directional temporal and spatial smoothness indexes of the functions in its RKHS.

Consider now the so-called d'Alembert solution to the equation

$$\begin{aligned}
u_{tt} &= c^2 u_{xx}, \quad x \in \mathbb{R}, \quad t > 0 \\
u(0, x) &= g(\omega, x) = \int_{\mathbb{R}} \exp(i\lambda x) \sqrt{f_g(\lambda)} Z_g(d\lambda), \quad \omega \in \Omega \\
u_t(0, x) &= h(\omega, x) = \int_{\mathbb{R}} \exp(i\lambda x) \sqrt{f_h(\lambda)} Z_h(d\lambda), \quad \omega \in \Omega.
\end{aligned}$$

The d'Alembert solution u is

$$u(t, x) = \frac{1}{2} [g(x-ct) + g(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} h(y) dy,$$

which is given here by

$$\begin{aligned}
u(t, x) &= \frac{1}{2} \left[\exp(-i\lambda ct) \int_{\mathbb{R}} \exp(i\lambda x) \sqrt{f_g(\lambda)} Z_g(d\lambda) + \exp(i\lambda ct) \int_{\mathbb{R}} \exp(i\lambda x) \sqrt{f_g(\lambda)} Z_g(d\lambda) \right] \\
&+ \frac{1}{2c} \int_{\mathbb{R}} \sqrt{f_h(\lambda)} \left(\int_{x-ct}^{x+ct} \exp(i\lambda y) dy \right) Z_h(d\lambda) \\
&= \frac{1}{2} \left[\exp(-i\lambda ct) \int_{\mathbb{R}} \exp(i\lambda x) \sqrt{f_g(\lambda)} Z_g(d\lambda) + \exp(i\lambda ct) \int_{\mathbb{R}} \exp(i\lambda x) \sqrt{f_g(\lambda)} Z_g(d\lambda) \right] \\
&+ \frac{1}{2c} \int_{\mathbb{R}} \frac{\exp(i\lambda(x+ct)) - \exp(i\lambda(x-ct))}{i\lambda} \sqrt{f_h(\lambda)} Z_h(d\lambda).
\end{aligned}$$

9.1.1 Fractional and multifractional versions

The formulation in the previous section can be extended to the case where the spatial diffusion operator belongs to the family of fractional pseudodifferential operators considered in equation (42). The case where the spatial diffusion operator is a pseudodifferential operator of variable order can also be studied in a similar way. Specifically, the following fractional and multifractional hyperbolic equations are studied:

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(t, \mathbf{x}) &= -f((I - \Delta)^{1/2})u(t, \mathbf{x}) + \frac{\partial^{d+1} W_H}{\partial t \partial x_1 \dots \partial x_d}(t, \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0 \\
u(0, \mathbf{x}) &= g(\omega, \mathbf{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle) \sqrt{f_g(\boldsymbol{\lambda})} Z_g(d\boldsymbol{\lambda}), \quad \omega \in \Omega, \quad \mathbf{x} \in \mathbb{R}^d \\
\frac{\partial u}{\partial t}(0, \mathbf{x}) &= h(\omega, \mathbf{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle) \sqrt{f_h(\boldsymbol{\lambda})} Z_h(d\boldsymbol{\lambda}), \quad \omega \in \Omega, \quad \mathbf{x} \in \mathbb{R}^d,
\end{aligned}$$

where f is a continuous function, which can be, for example, a fractional elliptic polynomial or positive rational function. The solution is then defined as

$$\begin{aligned}
u(t, \mathbf{x}) &= (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle) \frac{\partial \widehat{K}}{\partial t}(t, \boldsymbol{\lambda}) \sqrt{f_g(\boldsymbol{\lambda})} Z_g(d\boldsymbol{\lambda}) \\
&+ (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle) \widehat{K}(t, \boldsymbol{\lambda}) \sqrt{f_h(\boldsymbol{\lambda})} Z_h(d\boldsymbol{\lambda}) \\
&+ (2\pi)^{d/2} \int_0^t \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle) \widehat{K}(t-s, \boldsymbol{\lambda}) \prod_{j=1}^d (i\lambda_j)^{-H_j+1/2} Z_s(d\boldsymbol{\lambda}), \quad (56)
\end{aligned}$$

where

$$\widehat{K}(t, \boldsymbol{\lambda}) = \frac{\sin \left(t [f(\langle \boldsymbol{\lambda} \rangle)]^{1/2} \right)}{[f(\langle \boldsymbol{\lambda} \rangle)]^{1/2}}.$$

Similarly, one can consider the multifractional hyperbolic equation

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}(t, \mathbf{x}) &= -\mathcal{P}u(t, \mathbf{x}) + \frac{\partial^{d+1} W_H}{\partial t \partial x_1 \dots \partial x_d}(t, \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0 \\ u(0, \mathbf{x}) &= g(\omega, \mathbf{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle) \sqrt{f_g(\boldsymbol{\lambda})} Z_g(d\boldsymbol{\lambda}), \quad \omega \in \Omega, \quad \mathbf{x} \in \mathbb{R}^d \\ \frac{\partial u}{\partial t}(0, \mathbf{x}) &= h(\omega, \mathbf{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(i \langle \boldsymbol{\lambda}, \mathbf{x} \rangle) \sqrt{f_h(\boldsymbol{\lambda})} Z_h(d\boldsymbol{\lambda}), \quad \omega \in \Omega, \quad \mathbf{x} \in \mathbb{R}^d,\end{aligned}$$

where \mathcal{P} is a pseudodifferential operator of variable order, defined as in equation (45), in terms of symbol p satisfying the regularity conditions given in the previous section (see equation (46)). The Green function is then given by

$$G(t, \mathbf{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \exp(\langle \boldsymbol{\lambda}, \mathbf{x} \rangle) \frac{\sin(t[p(\mathbf{x}, \boldsymbol{\lambda})]^{1/2})}{[p(\mathbf{x}, \boldsymbol{\lambda})]^{1/2}} d\boldsymbol{\lambda}. \quad (57)$$

Thus, the corresponding solution u is defined as in equation (56) in terms of function G of equation (57) instead of function \widehat{K} .

10 Final comments

This paper provides the necessary elements for the introduction of random field models in the context of elliptic, parabolic and hyperbolic equations driven by fractional Gaussian random fields. The temporal, spatial and spatiotemporal Gaussian random field models considered here can be extended to a more general, not necessarily stationary, random innovation setting. Specifically, the innovation process can be defined in terms of the weak-sense second-order derivatives of a Gaussian generalized random field with RKHS isomorphic to a fractional (isotropic or anisotropic) Bessel potential space on the temporal, spatial or spatiotemporal domain considered. This isomorphic relationship ensures a covariance factorization. Then, the solution to the corresponding random elliptic, parabolic or hyperbolic equation can be expressed as a weak-sense integral, involving the convolution of the corresponding Green kernel and the kernel factorizing the covariance operator of the innovation process. The case of random initial condition can be similarly addressed using the covariance factorization of the Gaussian innovation process.

Appendix

The basic elements and results on anisotropic fractional Bessel potential and Hölder spaces, needed in the characterization of the local regularity properties of the solutions to the formulated elliptic, hyperbolic and parabolic equations driven by fractional Gaussian random processes, are now introduced.

We first consider the definition of anisotropic fractional Bessel potential spaces:

Definition 3. (see Dachkovski, 2003) The anisotropic fractional Bessel potential space $\mathbf{H}_2^{s, \mathbf{a}}(\mathbb{R}^d) \equiv \mathbf{H}_2^{s/\mathbf{a}}(\mathbb{R}^d)$ of vectorial order $s/\mathbf{a} = \left(\frac{s}{a_1}, \dots, \frac{s}{a_d}\right)$, with $s \in \mathbb{R}$ and $a_i > 0$, for $i = 1, \dots, d$, is defined

as the space of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ having square integrable weak-sense directional derivatives of order $\left(\frac{s}{a_1}, \dots, \frac{s}{a_d}\right)$, i.e., $f \in \mathcal{S}'(\mathbb{R}^d)$ is such that

$$\sum_{i=1}^d \sqrt{\int_{\mathbb{R}^d} \left| (1 + \lambda_i^2)^{s/2a_i} \widehat{f}(\lambda_1, \dots, \lambda_d) \right|^2 d\lambda_1 \dots d\lambda_d} < \infty.$$

Here, as before, \widehat{f} denotes the Fourier transform of f in the sense of the tempered distributions.

In the above definition the parameter s represents the mean smoothness of functions in the space $\mathbf{H}_2^{s/\mathbf{a}}(\mathbb{R}^d)$. Specifically, the parameter s is computed as

$$\frac{1}{s} = \frac{1}{d} \left(\frac{1}{l_1} + \dots + \frac{1}{l_d} \right), \quad (58)$$

from the vector

$$(l_1, \dots, l_d) = \left(\frac{s}{a_1}, \dots, \frac{s}{a_d} \right), \quad a_1 + \dots + a_d = d,$$

providing the directional smoothness of functions in the space $\mathbf{H}_2^{s/\mathbf{a}}(\mathbb{R}^d)$. Note that, the classical isotropic fractional Bessel potential space corresponds to $\mathbf{a} = \mathbf{1} = (1, \dots, 1)$. The following identity defines the duality between these anisotropic Bessel potential spaces:

$$[\mathbf{H}_p^{s/\mathbf{a}}(\mathbb{R}^d)]' = \mathbf{H}_p^{-s/\mathbf{a}}(\mathbb{R}^d).$$

Since we are considering the case $p = 2$ in this paper, the above identity also leads to the duality between Hilbert spaces here, i.e.,

$$[\mathbf{H}_2^{s/\mathbf{a}}(\mathbb{R}^d)]^* = \mathbf{H}_2^{-s/\mathbf{a}}(\mathbb{R}^d).$$

After the introduction of anisotropic Bessel potential spaces, the anisotropic fractional Hölder spaces are now defined. Denote by $\mathbb{R}^{(d, \widehat{d}_i)}$, for $1 \leq i \leq l$, the set

$$\mathbb{R}^{(d, \widehat{d}_i)} := \mathbb{R}^{d_1} \times \dots \times \widehat{\mathbb{R}^{d_i}} \times \dots \times \mathbb{R}^{d_l},$$

where $\sum_{k=1}^l d_k = d$, and with the hat over a factor (or component) meaning that the corresponding entry is absent. Thus, $x_{\widehat{i}} := (x_1, \dots, \widehat{x_i}, \dots, x_l)$, and for a function u on \mathbb{R}^d we write

$$u(x_{\widehat{i}}, \cdot) := u(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_l).$$

Let $\mathcal{C}_0(\mathbb{R}^d)$ be the space of continuous functions on \mathbb{R}^d vanishing at infinity. Then, define, for $u \in \mathcal{C}_0(\mathbb{R}^d)$, the corresponding U_i , for $1 \leq i \leq l$, defined as follows:

$$U_i : \mathcal{C}_0(\mathbb{R}^d) \longrightarrow \mathcal{C}_0(\mathbb{R}^{(d, \widehat{d}_i)}, \mathcal{C}_0(\mathbb{R}^{d_i})),$$

given by

$$U_i(u) := (x_{\widehat{i}} \longrightarrow u(x_{\widehat{i}}, \cdot)) \in \mathcal{C}_0(\mathbb{R}^{(d, \widehat{d}_i)}, \mathcal{C}_0(\mathbb{R}^{d_i})).$$

The following definition introduces anisotropic fractional Hölder spaces.

Definition 4. We define the anisotropic Hölder space $\mathcal{C}_0^{t/\mathbf{a}}(\mathbb{R}^d)$ of vectorial order $\left(\frac{t}{a_1}, \dots, \frac{t}{a_d}\right)$, $t > 0$, as

$$\mathcal{C}_0^{t/\mathbf{a}}(\mathbb{R}^d) := \bigcap_{i=1}^l \mathcal{C}_0 \left(\mathbb{R}^{(d, d_i)}, \mathcal{C}_0^{t/a_i}(\mathbb{R}^{d_i}) \right),$$

with U_i being defined, as before, from $x_{\hat{i}} := (x_1, \dots, \hat{x}_i, \dots, x_l)$, for $i = 1, \dots, l$. Here, $C_0^r(\mathbb{R}^{d_i})$, $i = 1, \dots, l$, $r \in \mathbb{R}_+$, is defined, as usual, as a member of the fractional Besov space scale on \mathbb{R}^{d_i} , $i = 1, \dots, l$ (see, for example, Triebel, 1978). That is, the space $C_0^r(\mathbb{R}^{d_i})$ contains continuous functions on \mathbb{R}^{d_i} with continuous fractional derivatives up to order r vanishing at infinity, for $i = 1, \dots, l$.

Note that, along this paper we have considered the case $d_i = 1$, for $i = 1, \dots, l$, with $d = l$. The following results provides the continuous injection of anisotropic fractional Bessel potential spaces into anisotropic fractional Hölder spaces on \mathbb{R}^d .

Theorem 1. (see Theorem 3.9.1, Amann, 2009)

For $1 < p < \infty$, if $s > t + |\mathbf{a}|/p$, and $\mathbf{a} \neq a(1, \dots, 1)$, for any $a > 0$, the following embedding holds

$$\mathbf{H}_p^{s/\mathbf{a}}(\mathbb{R}^d) \hookrightarrow \mathcal{C}_0^{t/\mathbf{a}}(\mathbb{R}^d).$$

Here, $|\mathbf{a}| = a_1 + \dots + a_d$.

Remark 3. In the formulation of Theorem 1 we have applied Theorem 3.9.1. of Amann (2009) using the fact that every finite-dimensional Banach space, in particular \mathbb{R}^d , is so-called 'UMD', and has the so-called property (α) (see Amann, 2009, p.43). These are conditions which are assumed in Theorem 3.9.1 of Amann (2009).

The application of Theorem 1 with $p = 2$ allows us to establish conditions under which the functions in the RKHS of the solution of fractional elliptic, hyperbolic and parabolic equations, driven by fractional Gaussian white noise, are continuous.

1. First, we consider the study of local regularity properties of functions in the RKHS $\mathcal{H}_{\partial^2 W_H}$ of $\partial^2 W_H$. By definition, every function g belonging to the RKHS $\mathcal{H}_{\partial^2 W_H}$ of the weak-sense derivative of fractional Brownian motion $\frac{\partial^2 W_H}{\partial x \partial y}$ satisfies

$$\|g\|_{\mathcal{H}_{\partial^2 W_H}}^2 = \int_{\mathbb{R}^2} |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 < \infty,$$

where \widehat{g} denotes, as before, the Fourier transform of g .

The interrelation between subspaces of functions in $\mathcal{H}_{\partial^2 W_H}$ and $\mathbf{H}_2^{s/\mathbf{a}}(\mathbb{R}^2)$ is provided in the following proposition, where the parameters s and \mathbf{a} are specified.

Proposition 1. The following assertions hold:

(i) For $H_1 = H_2 = 1/2$,

$$\mathcal{H}_{\partial^2 W_H} = L^2(\mathbb{R}^2) = \mathbf{H}_2^{0/\mathbf{a}}(\mathbb{R}^2).$$

(ii) For $H_1, H_2 \in (1/2, 1)$, the set of square-integrable functions in $\mathcal{H}_{\partial^2 W_H}$ is included in the anisotropic fractional Bessel potential space $\mathbf{H}_2^{s/a}(\mathbb{R}^2)$ (see Definition 3), for $s_1 = \frac{s}{a_1} = H_1 - 1/2$, and $s_2 = \frac{s}{a_2} = H_2 - 1/2$, that is, according to equation (58),

$$s = \frac{2(H_1 - 1/2)(H_2 - 1/2)}{H_1 + H_2 - 1}. \quad (59)$$

(iii) For $H_1, H_2 \in (0, 1/2)$, the subspace of absolutely integrable functions in $\mathbf{H}_2^{s/a}(\mathbb{R}^2)$ is included in the space $\mathcal{H}_{\partial^2 W_H}$, with, as before, $s_1 = \frac{s}{a_1} = H_1 - 1/2$, and $s_2 = \frac{s}{a_2} = H_2 - 1/2$.

Proof. The proof of (i) follows directly from the definition of spaces $\mathcal{H}_{\partial^2 W_H}$, for $H_1 = H_2 = 1/2$, and $\mathbf{H}_2^{0/a}(\mathbb{R}^2)$, which coincides with $L^2(\mathbb{R}^2)$.

Regarding assertion (ii), since we are considering the set of square integrable functions, for every function $g \in \mathcal{H}_{\partial^2 W_H}$, we have

$$\int_{\varepsilon_R(0) \times \varepsilon_R(0)} (1 + |\lambda_i|^2)^{H_i - 1/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 < \infty, \quad i = 1, 2,$$

where $\varepsilon_R(0) \subset \mathbb{R}$ denotes a one-dimensional neighborhood of zero frequency of radius $R > 0$. Thus,

$$\begin{aligned} & \int_{\varepsilon_R(0) \times \varepsilon_R(0)} (1 + |\lambda_1|^2)^{H_1 - 1/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 \\ & \quad + \int_{\varepsilon_R(0) \times \varepsilon_R(0)} (1 + |\lambda_2|^2)^{H_2 - 1/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 < \infty. \end{aligned} \quad (60)$$

Additionally, for every function $g \in \mathcal{H}_{\partial^2 W_H}$,

$$\begin{aligned} & \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} (1 + |\lambda_1|^2)^{H_1 - 1/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 \\ & = \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} \frac{|\lambda_1|^{2H_1 - 1} |\lambda_2|^{2H_2 - 1} (1 + |\lambda_1|^2)^{H_1 - 1/2}}{|\lambda_1|^{2H_1 - 1} |\lambda_2|^{2H_2 - 1}} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 \\ & \leq M \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} \frac{|\lambda_1|^{2H_1 - 1} |\lambda_2|^{2H_2 - 1}}{|\lambda_2|^{2H_2 - 1}} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 \\ & \leq MC \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} |\lambda_1|^{2H_1 - 1} |\lambda_2|^{2H_2 - 1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 < \infty, \end{aligned} \quad (61)$$

and similarly,

$$\begin{aligned} & \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} (1 + |\lambda_2|^2)^{H_2 - 1/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 \\ & \leq MC \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} |\lambda_1|^{2H_1 - 1} |\lambda_2|^{2H_2 - 1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 < \infty. \end{aligned} \quad (62)$$

Therefore, from equations (60)-(62), $g \in \mathbf{H}_2^{s/a}(\mathbb{R}^2)$, with $s_1 = \frac{s}{a_1} = H_1 - 1/2$, and $s_2 = \frac{s}{a_2} = H_2 - 1/2$. Since $a_1 + a_2 = d = 2$, we get by (58),

$$\frac{1}{s} = \frac{1}{2} \left(\frac{1}{s_1} + \frac{1}{s_2} \right) = \frac{1}{2} \left(\frac{1}{H_1 - 1/2} + \frac{1}{H_2 - 1/2} \right),$$

which yields (59). Thus, assertion (ii) holds.

Note that, in the derivation of inequalities in equations (61)-(62), we have applied the fact that, there exists a constant $M > 0$ such that

$$\left[\frac{(1 + |\lambda_i|^2)}{|\lambda_i|^2} \right]^{H_i - 1/2} \leq M, \quad i = 1, 2,$$

for $\lambda_i \in \mathbb{R} \setminus \varepsilon_R(0)$, with R sufficiently large. Additionally, we have used the continuity of the Riesz operator on the space of square integrable functions (see Theorem 9:5:10(a), p. 660, of Edwards, 1965). That is, there exists a constant $C > 0$, such that, for every function $h \in L^2(T)$, with T an interval of \mathbb{R} ,

$$\|(-\Delta)^{-(H_j - 1/2)}(h)\|_{L^2(T)}^2 \simeq \int_{\mathbb{R} \setminus \varepsilon_R(0)} \frac{|\widehat{h}(\lambda)|^2}{|\lambda|^{2H_j - 1}} d\lambda \leq C \|\widehat{h}\|_{L^2(T)}^2 < \infty, \quad j = 1, 2,$$

where in equations (61)-(62), we have considered

$$\widehat{h}_{\lambda_i}(\lambda) = |\lambda_i|^{2H_i - 1} |\lambda|^{2H_j - 1} \widehat{g}_{\lambda_i}(\lambda), \quad i = 1, 2, \quad i \neq j,$$

for each fixed value of $\lambda_i \in \mathbb{R} \setminus \varepsilon_R(0)$, since $g \in \mathcal{H}_{\partial^2 W_H}$.

Finally, for the proof of assertion (iii), we first apply the fact that for any absolutely integrable functions $g \in \mathbf{H}_2^{s/a}(\mathbb{R}^2)$, with $H_i \in (0, 1/2)$, we also have

$$\int_{\varepsilon_R(0) \times \varepsilon_R(0)} |\lambda_1|^{2H_1 - 1} |\lambda_2|^{2H_2 - 1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 < \infty.$$

Furthermore, for $H_i \in (0, 1/2)$, for $i = 1, 2$, for every $g \in \mathbf{H}_2^{s/a}(\mathbb{R}^2)$,

$$\begin{aligned} & \int_{\mathbb{R}^2 \setminus \varepsilon_R(0) \times \varepsilon_R(0)} |\lambda_1|^{2H_1 - 1} |\lambda_2|^{2H_2 - 1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 \\ &= \int_{\mathbb{R}^2 \setminus \varepsilon_R(0) \times \varepsilon_R(0)} \frac{|\lambda_1|^{2H_1 - 1} |\lambda_2|^{2H_2 - 1} (1 + |\lambda_i|^2)^{H_i - 1/2}}{(1 + |\lambda_i|^2)^{H_i - 1/2}} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 \\ &\leq M \int_{\mathbb{R}^2 \setminus \varepsilon_R(0) \times \varepsilon_R(0)} (1 + |\lambda_i|^2)^{H_i - 1/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 < \infty, \quad i = 1, 2. \end{aligned} \tag{63}$$

Then, $g \in \mathcal{H}_{\partial^2 W_H}$, and assertion (iii) holds. Note that in the derivation of equation (63) we have applied that there exists a positive constant M such that

$$\left[\frac{|\lambda_i|^2}{(1 + |\lambda_i|^2)} \right]^{H_i - 1/2} \leq M, \quad i = 1, 2,$$

for $\lambda_i \in \mathbb{R} \setminus \varepsilon_R(0)$, with R sufficiently large. Moreover, we have considered the fact that $|\lambda_i|^{2H_i-1} \leq 1$, $H_i \in (0, 1/2)$, for $\lambda_i \in \mathbb{R} \setminus \varepsilon_R(0)$, with R sufficiently large, for $i = 1, 2$. \square

It follows from Theorem 1, that the functions in the space $\mathbf{H}_2^{s/a}(\mathbb{R}^2)$ belong to $\mathcal{C}_0^{t/a}(\mathbb{R}^2)$ if $s > t + |\mathbf{a}|/2 = t + 1$, for certain $t > 0$. Considering the parameters s and \mathbf{a} specified in Proposition 1, this inequality means that

$$s = \frac{2(H_1 - 1/2)(H_2 - 1/2)}{H_1 + H_2 - 1} > t + |\mathbf{a}|/2 = t + 1,$$

that is, H_1 and H_2 must be such that

$$\frac{2(H_1 - 1/2)(H_2 - 1/2)}{H_1 + H_2 - 1} - 1 > t,$$

i.e.,

$$t < \frac{2H_1H_2 - 2[H_1 + H_2] + 3/2}{H_1 + H_2 - 1}, \tag{64}$$

for certain $t > 0$.

From Proposition 1(ii), the set of square-integrable functions in $\mathcal{H}_{\partial^2 W_H}$ are then continuous for $H_i \in (1/2, 1)$, $i = 1, 2$, such $t > 0$ satisfying (64).

2. In the case of the general solution (18), when H_i , $i = 1, 2$, and \mathcal{L} are such that

$$\left[\mathcal{L} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \right]^{-1} \left[\frac{\partial}{\partial x} \right]^{-H_1+1/2} \left[\frac{\partial}{\partial x} \right]^{-H_2+1/2} : L^2(\mathbb{R}^2) \longrightarrow \mathbf{H}_2^{\beta/\mathbf{b}}(\mathbb{R}^2),$$

for a suitable $\beta \in \mathbb{R}$, and $\mathbf{b} = (b_1, b_2)$, where, as before, $\mathbf{H}_2^{\beta/\mathbf{b}}(\mathbb{R}^2)$ denotes the anisotropic fractional Bessel potential space of vectorial order β/\mathbf{b} on \mathbb{R}^2 . The strong-sense (pointwise) definition of u , in the mean-square sense, as a continuous Gaussian random field, follows from the continuous injection of anisotropic Bessel potential spaces into anisotropic Hölder spaces, under suitable conditions. Specifically, from Theorem 1, for $\beta > t + \frac{b_1+b_2}{2} = t + 1$, for certain $t > 0$, $\mathbf{H}_2^{\beta/\mathbf{b}}(\mathbb{R}^2) \hookrightarrow \mathcal{C}_0^{t/\mathbf{b}}(\mathbb{R}^2)$, and a strong-sense (pointwise) definition of u can be established. When this cannot be done, a weak-sense definition of u must be adopted, in terms of the tempered distributions belonging to the anisotropic Bessel potential space $\mathbf{H}_2^{-\beta/\mathbf{b}}(\mathbb{R}^2)$, that is,

$$u(\psi) = \int_{\mathbb{R}^2} u(x, y)\psi(x, y)dx dy, \quad \psi \in \mathbf{H}_2^{-\beta/\mathbf{b}}(\mathbb{R}^2).$$

We now refer to the specific conditions satisfied by H_1 and H_2 , in the elliptic, hyperbolic and parabolic cases, that allow the identification of the local regularity properties of the functions in the RKHS of their solutions, in terms of fractional Bessel potential spaces.

3. In the *elliptic* case, we study the relationship of the RKHS \mathcal{H}_u of the solution u to equation (12) with the anisotropic fractional Bessel potential space with smoothness parameters given in equation (20). These parameters are derived from Proposition 1 by considering the domain of the elliptic

differential operator (12). Since in the Fourier domain $\frac{\partial}{\partial x}$ acts as $i\lambda$, the operator (12) acts as $|i\lambda_1|^2 + |i\lambda_2|^2 + \gamma^2$. Hence, the RKHS \mathcal{H}_u of the solution u made of the functions g such that

$$\int_{\mathbb{R}^2} |(\lambda_1^2 + \lambda_2^2 + \gamma^2)|^2 |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 < \infty. \quad (65)$$

Consequently,

Proposition 2. *The following equalities and inclusions hold:*

(i) For $H_1 = H_2 = 1/2$, and $\gamma = 1$,

$$\mathcal{H}_u = \mathbf{H}_2^2(\mathbb{R}^2),$$

where $\mathbf{H}_2^2(\mathbb{R}^2)$ denotes the isotropic Bessel potential space of integer order 2.

(ii) For $H_1, H_2 \in (1/2, 1)$, the set of square-integrable functions in the RKHS \mathcal{H}_u is included in the anisotropic fractional Bessel potential space $\mathbf{H}_2^{\beta/b}(\mathbb{R}^2)$ (see Definition 3), for $\beta_1 = H_1 - 1/2 + 2 = H_1 + 3/2$, and $\beta_2 = H_2 - 1/2 + 2 = H_2 + 3/2$, that is, according to equation (20),

$$\beta = \frac{2(H_1 + 3/2)(H_2 + 3/2)}{H_1 + H_2 + 3}.$$

(iii) For $H_1, H_2 \in (0, 1/2)$, the subspace of absolutely integrable functions in $\mathcal{L}^{-1}(\mathbf{H}_2^{s/a}(\mathbb{R}^2))$, with s and \mathbf{a} defined as in Proposition 1, and \mathcal{L} given in equation (12), is included in the RKHS space \mathcal{H}_u , where

$$\langle f, g \rangle_{\mathcal{L}^{-1}(\mathbf{H}_2^{s/a}(\mathbb{R}^2))} = \langle \mathcal{L}(f), \mathcal{L}(g) \rangle_{\mathbf{H}_2^{s/a}(\mathbb{R}^2)}, \quad f, g \in \mathcal{L}^{-1}(\mathbf{H}_2^{s/a}(\mathbb{R}^2)).$$

Proof. (i) It directly follows from the definition of the spaces \mathcal{H}_u , in equation (65), and the isotropic Bessel potential space $\mathbf{H}_2^2(\mathbb{R}^2)$.

(ii) For every square-integrable function $g \in \mathcal{H}_u$,

$$\begin{aligned} & \int_{\varepsilon_R(0) \times \varepsilon_R(0)} (1 + |\lambda_1|^2)^{H_1+3/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 \\ & \quad + \int_{\varepsilon_R(0) \times \varepsilon_R(0)} (1 + |\lambda_2|^2)^{H_2+3/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 < \infty. \end{aligned} \quad (66)$$

Now, for every function $g \in \mathcal{H}_u$,

$$\begin{aligned} & \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} (1 + |\lambda_1|^2)^{H_1+3/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 \\ & = \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} \frac{|(\lambda_1^2 + \lambda_2^2 + \gamma^2)|^2 |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} (1 + |\lambda_1|^2)^{H_1+3/2}}{|(\lambda_1^2 + \lambda_2^2 + \gamma^2)|^2 |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1}} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 \\ & \leq \tilde{M} \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} \frac{|(\lambda_1^2 + \lambda_2^2 + \gamma^2)|^2 |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1}}{|\lambda_2|^{2H_2-1}} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 \\ & \leq \tilde{M} C \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} |(\lambda_1^2 + \lambda_2^2 + \gamma^2)|^2 |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 < \infty, \end{aligned} \quad (67)$$

and similarly,

$$\begin{aligned} & \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} (1 + |\lambda_2|^2)^{H_2+3/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 \\ & \leq \widetilde{M} C \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} |(\lambda_1^2 + \lambda_2^2 + \gamma^2)|^2 |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 < \infty. \end{aligned} \quad (68)$$

Therefore, from equations (66)-(68), $g \in \mathbf{H}_2^{\beta/\mathbf{b}}(\mathbb{R}^2)$, with $\beta_1 = \frac{\beta}{b_1} = H_1 + 3/2$, and $\beta_2 = \frac{\beta}{b_2} = H_2 + 3/2$. Thus, assertion (ii) holds. In the derivation of inequalities (67)-(68), we have applied the fact that

$$\frac{(1 + |\lambda_i|^2)^{H_i+3/2}}{|(\lambda_1^2 + \lambda_2^2 + \gamma^2)|^2 |\lambda_i|^{2H_i-1}} \leq \widetilde{M}, \quad i = 1, 2,$$

for certain positive constant \widetilde{M} , and $\lambda_i \in \mathbb{R} \setminus \varepsilon_R(0)$, $i = 1, 2$, and, as in Proposition 1, the continuity of the Riesz operator on $L^2(T)$ is applied.

(iii) We first remark that for any absolutely integrable functions $g \in \mathcal{L}^{-1}(\mathbf{H}_2^{s/\mathbf{a}}(\mathbb{R}^2))$, with $H_i \in (0, 1/2)$, $i = 1, 2$,

$$\int_{\varepsilon_R(0) \times \varepsilon_R(0)} |(\lambda_1^2 + \lambda_2^2 + \gamma^2)|^2 |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 < \infty.$$

Furthermore, for $H_i \in (0, 1/2)$, $i = 1, 2$, for every $g \in \mathcal{L}^{-1}(\mathbf{H}_2^{s/\mathbf{a}}(\mathbb{R}^2))$, similarly to Proposition 1(iii), we obtain

$$\begin{aligned} & \int_{\mathbb{R}^2 \setminus \varepsilon_R(0) \times \varepsilon_R(0)} |(\lambda_1^2 + \lambda_2^2 + \gamma^2)|^2 |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 \\ & = \int_{\mathbb{R}^2 \setminus \varepsilon_R(0) \times \varepsilon_R(0)} \frac{|(\lambda_1^2 + \lambda_2^2 + \gamma^2)|^2 |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} (1 + |\lambda_i|^2)^{H_i-1/2}}{(1 + |\lambda_i|^2)^{H_i-1/2}} \\ & \quad \times |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 \\ & \leq M \int_{\mathbb{R}^2 \setminus \varepsilon_R(0) \times \varepsilon_R(0)} |(\lambda_1^2 + \lambda_2^2 + \gamma^2)|^2 (1 + |\lambda_i|^2)^{H_i-1/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 < \infty, \end{aligned} \quad (69)$$

for $i = 1, 2$. Then, $g \in \mathcal{H}_u$, and assertion (iii) holds. \square

It follows from Theorem 1, that the functions in the space $\mathbf{H}_2^{\beta/\mathbf{b}}(\mathbb{R}^2)$ belong to $\mathcal{C}_0^{t/\mathbf{b}}(\mathbb{R}^2)$ if $\beta > t + |\mathbf{b}|/2 = t + 1$, for certain $t > 0$. Considering the parameters β and \mathbf{b} specified in Proposition 2, we then have that, for H_1 and H_2 such that there exists a certain positive t , with

$$\frac{2(H_1 + 3/2)(H_2 + 3/2)}{H_1 + H_2 + 3} - 1 > t,$$

i.e.

$$t < \frac{2H_1H_2 + 2[H_1 + H_2] + 3/2}{H_1 + H_2 + 3}, \quad (70)$$

the functions in $\mathbf{H}_2^{\beta/b}(\mathbb{R}^2)$ are continuous. In addition, from Proposition 2(ii), for $H_i \in (1/2, 1)$, $i = 1, 2$, the set of square-integrable functions in \mathcal{H}_u are then continuous for $t > 0$ satisfying (70).

4. We now consider the *hyperbolic* case, and, in particular, establish (26).

Proposition 3. *The following inclusions hold:*

(i) For $H_1 = H_2 = 1/2$,

$$\mathcal{H}_u \subset \mathbf{W}_2^2(\mathbb{R}^2),$$

where $\mathbf{W}_2^2(\mathbb{R}^2)$ denotes the classical Sobolev space of integer order 2.

(ii) For $H_1, H_2 \in (1/2, 1)$, the set of square-integrable functions in the RKHS \mathcal{H}_u is included in the anisotropic fractional Bessel potential space $\mathbf{H}_2^{v/c}(\mathbb{R}^2)$ (see Definition 3), for $v_1 = H_1 - 1/2 + 1 = H_1 + 1/2$, and $v_2 = H_2 - 1/2 + 1 = H_2 + 1/2$, and hence

$$v = \frac{2(H_1 + 1/2)(H_2 + 1/2)}{H_1 + H_2 + 1}.$$

(iii) For $H_1, H_2 \in (0, 1/2)$, the subspace of absolutely integrable functions in $\mathcal{L}^{-1}(\mathbf{H}_2^{s/a}(\mathbb{R}^2))$, with s and \mathbf{a} defined as in Proposition 1, and \mathcal{L} given in equation (15), is included in the space \mathcal{H}_u , where, as before,

$$\langle f, g \rangle_{\mathcal{L}^{-1}(\mathbf{H}_2^{s/a}(\mathbb{R}^2))} = \langle \mathcal{L}(f), \mathcal{L}(g) \rangle_{\mathbf{H}_2^{s/a}(\mathbb{R}^2)}, \quad f, g \in \mathcal{L}^{-1}(\mathbf{H}_2^{s/a}(\mathbb{R}^2)).$$

Proof. We first remark that the norm in the RKHS \mathcal{H}_u of the solution to the hyperbolic problem (15) is given by

$$\begin{aligned} \|g\|_{\mathcal{H}_u}^2 &= \int_{\mathbb{R}^2} |(i\lambda_1 + \alpha)(i\lambda_2 + \beta) + \gamma^2|^2 |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 \\ &= \int_{\mathbb{R}^2} |-\lambda_1\lambda_2 + \beta(i\lambda_1) + \alpha(i\lambda_2) + \alpha\beta + \gamma^2|^2 |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2. \end{aligned}$$

(i) This is straightforward from the definition of the space \mathcal{H}_u , in the above equation, and the definition of the classical Sobolev space $\mathbf{W}_2^2(\mathbb{R}^2)$.

(ii) For every square-integrable function $g \in \mathcal{H}_u$,

$$\begin{aligned} &\int_{\varepsilon_R(0) \times \varepsilon_R(0)} (1 + |\lambda_1|^2)^{H_1+1/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 \\ &\quad + \int_{\varepsilon_R(0) \times \varepsilon_R(0)} (1 + |\lambda_2|^2)^{H_2+1/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 < \infty. \end{aligned} \quad (71)$$

Now, for every function $g \in \mathcal{H}_u$,

$$\begin{aligned}
& \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} (1 + |\lambda_1|^2)^{H_1+1/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 \\
&= \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} \frac{|-\lambda_1 \lambda_2 + \beta(i\lambda_1) + \alpha(i\lambda_2) + \alpha\beta + \gamma^2|^2}{|-\lambda_1 \lambda_2 + \beta(i\lambda_1) + \alpha(i\lambda_2) + \alpha\beta + \gamma^2|^2 |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} \\
&\quad \times |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} (1 + |\lambda_1|^2)^{H_1+1/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1} \\
&\leq \widetilde{M} \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} \frac{|-\lambda_1 \lambda_2 + \beta(i\lambda_1) + \alpha(i\lambda_2) + \alpha\beta + \gamma^2|^2 |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1}}{|\lambda_2|^{2H_2-1}} \\
&\quad \times |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 \\
&\leq \widetilde{M} C \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} |-\lambda_1 \lambda_2 + \beta(i\lambda_1) + \alpha(i\lambda_2) + \alpha\beta + \gamma^2|^2 |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} \\
&\quad \times |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 < \infty, \tag{72}
\end{aligned}$$

and similarly,

$$\begin{aligned}
& \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} (1 + |\lambda_2|^2)^{H_2+1/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 \\
&\leq \widetilde{M} C \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} |-\lambda_1 \lambda_2 + \beta(i\lambda_1) + \alpha(i\lambda_2) + \alpha\beta + \gamma^2|^2 |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} \\
&\quad \times |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 < \infty. \tag{73}
\end{aligned}$$

Therefore, from equations (71)-(73), $g \in \mathbf{H}_2^{v/c}(\mathbb{R}^2)$, with $v_1 = \frac{v}{c_1} = H_1+1/2$, and $v_2 = \frac{v}{c_2} = H_2+1/2$. Thus, assertion (ii) holds. In the derivation of inequalities (72)-(73), we have applied the fact that

$$\frac{(1 + |\lambda_i|^2)^{H_i+1/2}}{|-\lambda_1 \lambda_2 + \beta(i\lambda_1) + \alpha(i\lambda_2) + \alpha\beta + \gamma^2|^2 |\lambda_i|^{2H_i-1}} \leq \widetilde{M}, \quad i = 1, 2,$$

for $\lambda_i \in \mathbb{R} \setminus \varepsilon_R(0)$, $i = 1, 2$, and, as in Proposition 1(ii), the continuity of the Riesz operator on $L^2(T)$ is also applied.

(iii) We first remark that for any absolutely integrable functions $g \in \mathcal{L}^{-1}(\mathbf{H}_2^{s/a}(\mathbb{R}^2))$, with $H_i \in (0, 1/2)$, $i = 1, 2$,

$$\int_{\varepsilon_R(0) \times \varepsilon_R(0)} |-\lambda_1 \lambda_2 + \beta(i\lambda_1) + \alpha(i\lambda_2) + \alpha\beta + \gamma^2|^2 |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 < \infty.$$

Furthermore, for $H_i \in (0, 1/2)$, for $i = 1, 2$, for every $g \in \mathcal{L}^{-1}(\mathbf{H}_2^{s/a}(\mathbb{R}^2))$, similarly to Proposition 2(iii), we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^2 \setminus \varepsilon_R(0) \times \varepsilon_R(0)} \left| -\lambda_1 \lambda_2 + \beta(i\lambda_1) + \alpha(i\lambda_2) + \alpha\beta + \gamma^2 \right|^2 |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 \\
&= \int_{\mathbb{R}^2 \setminus \varepsilon_R(0) \times \varepsilon_R(0)} \frac{\left| -\lambda_1 \lambda_2 + \beta(i\lambda_1) + \alpha(i\lambda_2) + \alpha\beta + \gamma^2 \right|^2}{(1 + |\lambda_i|^2)^{H_i-1/2}} \\
&\quad \times |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} (1 + |\lambda_i|^2)^{H_i-1/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 \\
&\leq M \int_{\mathbb{R}^2 \setminus \varepsilon_R(0) \times \varepsilon_R(0)} \left| -\lambda_1 \lambda_2 + \beta(i\lambda_1) + \alpha(i\lambda_2) + \alpha\beta + \gamma^2 \right|^2 (1 + |\lambda_i|^2)^{H_i-1/2} \\
&\quad \times |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 < \infty,
\end{aligned} \tag{74}$$

for $i = 1, 2$. Then, $g \in \mathcal{H}_u$, and assertion (iii) holds. \square

The square integrable functions in the RKHS \mathcal{H}_u of the solution u to equation (11), considering operator (15), are continuous for $H_i \in (1/2, 1)$, $i = 1, 2$, and $\nu > t + |\mathbf{c}|/2 = t + 1$, which, from Proposition 3 leads to the inequality

$$\frac{2(H_1 + 1/2)(H_2 + 1/2)}{H_1 + H_2 + 1} - 1 > t,$$

that is, the square-integrable functions in \mathcal{H}_u are continuous for $H_i \in (1/2, 1)$, $i = 1, 2$, such that

$$t < \frac{2H_1H_2 - 1/2}{H_1 + H_2 + 1}.$$

5. Finally, we consider the *parabolic* equation driven by fractional Gaussian white noise, defined in terms of operator (16), with $\theta_1 = 1$, and $\theta_2 = 0$. In the Fourier domain, the operator $\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ acts as $[i\lambda_1 - |\lambda_2|^2]$ whose modulus square equals $\lambda_1^2 + \lambda_2^4$. The RKHS of the solution is thus made up by functions g such that

$$\int_{\mathbb{R}^2} [\lambda_1^2 + \lambda_2^4] |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 < \infty. \tag{75}$$

The following result provides the relationship between \mathcal{H}_u and the anisotropic fractional Bessel potential spaces and establishes (36).

Proposition 4. *The following inclusions hold:*

- (i) For $H_1, H_2 \in (1/2, 1)$, the set of square-integrable functions in the RKHS \mathcal{H}_u is included in the anisotropic fractional Bessel potential space $\mathbf{H}_2^{r/e}(\mathbb{R}^2)$ (see Definition 3), for $r_1 = H_1 + 1/2$, and $r_2 = H_2 + 3/2$, and thus

$$r = \frac{2(H_1 + 1/2)(H_2 + 3/2)}{H_1 + H_2 + 2}.$$

(ii) For $H_1, H_2 \in (0, 1/2)$, the subspace of absolutely integrable functions in $\mathcal{L}^{-1}(\mathbf{H}_2^{s/a}(\mathbb{R}^2))$, with s and \mathbf{a} defined as in Proposition 1, and \mathcal{L} given in equation (16), is included in the space \mathcal{H}_u , where, as before,

$$\langle f, g \rangle_{\mathcal{L}^{-1}(\mathbf{H}_2^{s/a}(\mathbb{R}^2))} = \langle \mathcal{L}(f), \mathcal{L}(g) \rangle_{\mathbf{H}_2^{s/a}(\mathbb{R}^2)}, \quad f, g \in \mathcal{L}^{-1}(\mathbf{H}_2^{s/a}(\mathbb{R}^2)).$$

Proof. (i) For every square-integrable function $g \in \mathcal{H}_u$,

$$\begin{aligned} & \int_{\varepsilon_R(0) \times \varepsilon_R(0)} (1 + |\lambda_1|^2)^{H_1+1/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 \\ & \quad + \int_{\varepsilon_R(0) \times \varepsilon_R(0)} (1 + |\lambda_2|^2)^{H_2+3/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 < \infty. \end{aligned} \quad (76)$$

Now, for every function $g \in \mathcal{H}_u$,

$$\begin{aligned} & \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} (1 + |\lambda_1|^2)^{H_1+1/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 \\ & = \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} \frac{[\lambda_1^2 + \lambda_2^4] |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} (1 + |\lambda_1|^2)^{H_1+1/2}}{[\lambda_1^2 + \lambda_2^4] |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1}} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 \\ & \leq \widetilde{M} \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} \frac{[\lambda_1^2 + \lambda_2^4] |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1}}{|\lambda_2|^{2H_2-1}} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 \\ & \leq \widetilde{M} C \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} [\lambda_1^2 + \lambda_2^4] |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 < \infty, \end{aligned} \quad (77)$$

and similarly,

$$\begin{aligned} & \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} (1 + |\lambda_2|^2)^{H_2+3/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 \\ & \leq \widetilde{M} C \int_{\mathbb{R} \setminus \varepsilon_R(0) \times \mathbb{R} \setminus \varepsilon_R(0)} [\lambda_1^2 + \lambda_2^4] |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_2 d\lambda_1 < \infty. \end{aligned} \quad (78)$$

Therefore, from equations (76)-(78), $g \in \mathbf{H}_2^{r/e}(\mathbb{R}^2)$, with $r_1 = H_1 + 1/2$, and $r_2 = H_2 + 3/2$. Thus, assertion (ii) holds.

(ii) We first remark that for any absolutely integrable functions $g \in \mathcal{L}^{-1}(\mathbf{H}_2^{s/a}(\mathbb{R}^2))$, with $H_i \in (0, 1/2)$, $i = 1, 2$,

$$\int_{\varepsilon_R(0) \times \varepsilon_R(0)} [\lambda_1^2 + \lambda_2^4] |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 < \infty.$$

Furthermore, for $H_i \in (0, 1/2)$, $i = 1, 2$, for every $g \in \mathcal{L}^{-1}(\mathbf{H}_2^{s/a}(\mathbb{R}^2))$, similarly to Proposition 2(iii), we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^2 \setminus \varepsilon_R(0) \times \varepsilon_R(0)} [\lambda_1^2 + \lambda_2^4] |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 \\
&= \int_{\mathbb{R}^2 \setminus \varepsilon_R(0) \times \varepsilon_R(0)} \frac{[\lambda_1^2 + \lambda_2^4] |\lambda_1|^{2H_1-1} |\lambda_2|^{2H_2-1} (1 + |\lambda_i|^2)^{H_i-1/2}}{(1 + |\lambda_i|^2)^{H_i-1/2}} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 \\
&\leq M \int_{\mathbb{R}^2 \setminus \varepsilon_R(0) \times \varepsilon_R(0)} [\lambda_1^2 + \lambda_2^4] (1 + |\lambda_i|^2)^{H_i-1/2} |\widehat{g}(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2 < \infty, \quad i = 1, 2.
\end{aligned} \tag{79}$$

Then, $g \in \mathcal{H}_u$, and assertion (ii) holds. \square

As in the previous cases considered, from Theorem 1 and Proposition 4, the square integrable functions in \mathcal{H}_u are continuous for $H_i \in (1/2, 1)$, $i = 1, 2$, such that

$$t < \frac{2H_1H_2 + 2H_1 - 1/2}{H_1 + H_2 + 2}.$$

References

- [1] ADLER, R.J. (1981). *The Geometry of Random Fields*. Wiley, Chichester. MR0611857
- [2] AMANN, H. (2009). *Anisotropic Function Spaces and Maximal Regularity for Parabolic Problems. Part 1: Function Spaces*. Jindrich Necas Center for Mathematical Modeling Lecture Notes, Prague, Volume 6.
- [3] ANGULO, J.M., ANH, V.V., MCVINISH, R. AND RUIZ-MEDINA, M.D. (2005). Fractional kinetic equation driven by gaussian or infinitely divisible noise. *Advances in Applied Probability* **37**, 366–392. MR2144558
- [4] ANGULO, J.M. RUIZ-MEDINA, M.D., ANH, V.V. AND GRECKSCH, W. (2000). Fractional diffusion and fractional heat equation. *Advances in Applied Probability* **32**, 1077-1099. MR1808915
- [5] ANH, V.V., ANGULO J.M. AND RUIZ-MEDINA, M.D. (1999). Possible long-range dependence in fractional random fields. *Journal of Statistical Planning and Inference* **80**, 95–110. MR1713795
- [6] ANH, V.V. AND LEONENKO, N.N. (2001). Spectral analysis of fractional kinetic equations with random data. *J. Statis. Phys.* **104**, 1349–1387. MR1859007
- [7] BENASSI, A. JAFFARD, S AND ROUX, D. (1997). Elliptic Gaussian random processes. *Rev. Mat. Iberoamericana* **13** 19–90. MR1462329
- [8] BIAGINI, F. , HU, Y., ØKSENDAL, AND ZHANG, T. (2008), *Stochastic calculus for fractional Brownian motion and applications*. Springer-Verlag, London. MR2387368
- [9] CHOW, P-L. (2007). *Stochastic partial differential equations*. Chapman & Hall. MR2295103
- [10] DACHKOVSKI, S. (2003). Anisotropic function spaces and related semilinear hypoelliptic equations. *Math. Nachr.* **248-249**, 40-61. MR1950714
- [11] DAUTRAY, R. AND LIONS, J.L. (1985a). *Mathematical analysis and numerical methods for science and technology*. Volume 2 of *Functional and variational methods*. Springer-Verlag, New York. MR0969367
- [12] DAUTRAY, R. AND LIONS, J. L. (1985b). *Mathematical Analysis and Numerical Methods for Science and Technology*, Volume 3 of *Spectral Theory and Applications*. Springer-Verlag, New York. MR1064315

- [13] DUNCAN, T.E., MASLOWSKI, B. AND PASIK-DUNCAN, B. (2002). Fractional Brownian motion and stochastic equations in Hilbert space. *Stoch. Dyn.* **2**, 225-250. MR1912142
- [14] EDWARDS, R.E. (1965). *Functional Analysis*. Holt, Rinehart and Winston, New York. MR0221256
- [15] GIBINELLE, M., LEJAY, A. AND TINDEL, S. (2006). Young integrals and SPDE. *Pot. Analysis* **25**, 307-326. MR2255351
- [16] GUYON, X. (1987). Variations de champs gaussiens stationnaires: application à l'identification. *Probability Theory and Related Fields* **75**, 179–193. MR0885461
- [17] HEINE, V. (1955). Models for two-dimensional stationary stochastic processes. *Biometrika* **42**, 170–178. MR0071673
- [18] HU, Y. (2001). Heat equation with fractional white noise potentials. *Appl. Math. Optim.* **43**, 221-243. MR1885698
- [19] HU, Y. AND NUALART, D. (2009a). Stochastic heat equation driven by fractional noise and local time. *Probab. Theory Relat. Fields* **143**, 285-328. MR2449130
- [20] HU, Y. AND NUALART, D. (2009b). Rough path analysis via fractional calculus. *Trans. Amer. Math. Soc.* **361**, 2689-2718. MR2471936
- [21] HU, Y., OKSENDAL, B. AND ZHANG, T. (2004). General fractional multiparameter noise theory and stochastic partial differential equations. *Commun. Partial Differemt. Equations* **29** 1-23. MR2038141
- [22] IGLÓI, E. AND TERDIK, G. (1999). Bilinear stochastic systems with fractional Brownian motion input. *Ann. Appl. Probab.* **9**, 46–77. MR1682600
- [23] JACOB, N. (2005). Pseudodifferential operators: Markov processes III. Markov processes and applications, Imperial College Press. MR2158336
- [24] KELBERT, M. LEONENKO, N. AND RUIZ-MEDINA, M.D. (2005). Fractional Random fields associated with stochastic fractional heat equations. *Advances in Applied Probability* **37**, 108-133. MR2135156
- [25] KIKUCHI, K. AND NEGORO, A. (1997). On Markov processes generated by pseudodifferential operator of variable order. *Osaka Journal of Mathematics* **34**, 319–335. MR1483853
- [26] KOZACHENKO, YU. V. AND SLIVKA, G. I. (2007). Modelling a solution of a hyperbolic equation with random initial conditions *Theor. Probability and Math. Statist.* **74**, 59-75. MR2336779
- [27] LEONENKO, N. N. AND RUIZ-MEDINA, M. D. (2006) Scaling laws for the multidimensional Burgers equation with quadratic external potential. *J. Stat. Phys.*, **124**, no. 1, 191–205. MR2256621
- [28] LEONENKO, N. N. AND RUIZ-MEDINA, M. D. (2008) Gaussian scenario for the heat equation with quadratic potential and weakly dependent data with applications. *Methodol. Comput. Appl. Probab.*, **10**, no. 4, 595–620 MR2443082
- [29] LEOPOLD, H.-G. (1991). On function spaces of variable order of differentiation. *Forum Mathematicum* **3**, 1–21. MR1085592
- [30] LIONS, T. AND QIAN, Z. (2002). *System control and rough paths* Oxford Mathematical Monographs, Oxford University Press, Oxford. MR2036784
- [31] MALOWSKI, B. AND NUALART, D. (2005). Evolution equations driven by a fractional Brownian motion. *J. Func. Anal.* **209**, 277-305. MR1994773
- [32] MISHURA, Y.S. (2008). Stochastic calculus for fractional Brownian motion and related processes. Lecture Notes in Mathematics, 1929, Springer-Verlag, Berlín. MR2378138
- [33] MOHAPL, J. (1999). On estimation in random fields generated by linear stochastic partial differential equations. *Mathematica Slovaca* **49**, 95–115. MR1804478
- [34] MUELLER, C. AND TRIBE, R. (2004). A singular parabolic Anderson model. *Electron J. Probab.* **9**, 98-144. MR2041830

- [35] NUALART, D. AND SANZ-SOLÉ, M. (1979). A Markov property for two-parameter Gaussian processes. *Stochastica* **3**, 1–16. MR0562437
- [36] RAMM, A.G. (2005). Random fields estimation. World Scientific, Singapore.
- [37] RATANOV, N.E., SHUHOV, A.G. AND SUHOV, YU M. (1991). Stabilization of the statistical solution of the parabolic equation. *Acta Applicandae Mathematicae* **22**, 103-115. MR1100768
- [38] ROBEVA, R.S. AND PITT, L.D. (2007). On the equality of sharp and germ σ -fields for Gaussian processes and fields. *Pliska Stud. Math. Bulgar.* **16**, 183-205. MR2070315
- [39] RUIZ-MEDINA, M.D., ANGULO, J.M. AND ANH, V.V. (2002). Stochastic fractional-order differential models on fractals. *Theory of Probability and Mathematical Statistics* **67**, 130–146. MR1956627
- [40] RUIZ-MEDINA, M.D., ANGULO, J.M. AND ANH, V.V. (2003). Fractional Generalized Random Fields on Bounded Domains. *Stochastic Analysis and Applications* **21**, 465–492. MR1967723
- [41] RUIZ-MEDINA, M.D., ANGULO, J.M. AND ANH, V.V. (2006). Spatial and spatiotemporal Karhunen-Loève-type representations on fractal domains. *Stochastic Analysis and Applications* **24**, 195–219. MR2198541
- [42] RUIZ-MEDINA, M.D., ANGULO, J.M. AND ANH, V.V. (2008). Multifractality in space-time statistical models. *Stochastic Environmental Research and Risk Assessment* **22**, 81–86. MR2418414
- [43] RUIZ-MEDINA, M.D., ANH, V.V. AND ANGULO, J.M. (2004a). Fractional generalized random fields of variable order. *Stochastic Analysis and Applications* **22**, 775–800. MR2047278
- [44] RUIZ-MEDINA, M.D., ANH, V.V. AND ANGULO, J.M. (2004b). Fractal random fields on domains with fractal boundary. *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **7**, 395–417. MR2085640
- [45] RUIZ-MEDINA, M.D., ANH, V.V. AND ANGULO, J.M. (2010). Multifractal Markov processes in heterogeneous domains. *Stochastic Analysis and Applications* **29**, 15–47.
- [46] SANZ-SOLÉ, M. AND TORRECILLA, I. (2009). A fractional Poisson equation: existence, regularity and approximation of the solutions. *Stochastics and Dynamics* **9**, 519-548. MR2589036
- [47] SANZ-SOLÉ, M. AND VUILERMOT, P.A. (2010). Mild solutions for a class of fractional SPDs and their sample paths. Preprint. MR2731272
- [48] TAQQU, M. S. (1979). Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrsch. verw. Gebiete* **50**, 53-83. MR0550123
- [49] TAQQU, M. S. (2003) Fractional Brownian motion and long-range dependence. *Theory and applications of long-range dependence*, 5–38, Birkhäuser Boston, Boston, MA. MR1956042
- [50] TINDEL, S., TUDOR, C. AND VIENS, F. (2003). Stochastic evolution equations with the drift in the first fractional chaos. *Stoch. Anal. Appl.* **22**, 1209-1233.
- [51] TRIEBEL, H. (1978). *Interpolation Theory, Function Spaces, Differential Operators*. North-Holland Publishing Co. Amsterdam. MR0503903
- [52] VECCHIA, A. V. (1985). A general class of models for stationary two-dimensional random processes. *Biometrika* **72**, 281-291. MR0801769
- [53] WU, D. AND XIAO, Y. (2006). Fractal properties of the random string processes. *IMS Lecture Notes-Monograph Series. High Dimensional Probability* **51**, 128-147. MR2387765
- [54] YAGLOM. A.M. (1957). Some classes of random fields in n-dimensional space related to stationary random processes. *Theor. Prob. Appl.* **3**, 273-320.
- [55] YAGLOM. A.M. (1986). *Correlation Theory of Stationary and Related Random Functions I. Basic Results*. Springer-Verlag, New-York. MR0893393