

Discrete semi-self-decomposability induced by semigroups

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Abstract

A continuous semigroup of probability generating functions $\mathcal{F} := (F_t, t \geq 0)$ is used to introduce a notion of discrete semi-selfdecomposability, or \mathcal{F} -semi-selfdecomposability, for distributions with support on \mathbf{Z}_+ . \mathcal{F} -semi-selfdecomposable distributions are infinitely divisible and are characterized by the absolute monotonicity of a specific function. The class of \mathcal{F} -semi-selfdecomposable laws is shown to contain the \mathcal{F} -semistable distributions and the geometric \mathcal{F} -semistable distributions. A generalization of discrete random stability is also explored.

Key words: composition semigroups, discrete distributions, infinite divisibility, semi-stability, Markov branching processes, weak convergence.

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1 Introduction

A real-valued random variable (rv) X , or its distribution, is said to be self-decomposable if for every $0 < \alpha < 1$ there exists a rv X_α such that

$$(1.1) \quad X \stackrel{d}{=} \alpha X + X_\alpha,$$

where X and X_α are independent. If $X_\alpha \stackrel{d}{=} (1 - \alpha)^{1/\gamma} X$ for some $\gamma > 0$, then X is said to be stable with exponent γ .

Self-decomposable (resp. stable) distributions derive their importance in probability theory from the fact that they arise as weak limits of sequences of normalized partial sums of independent (resp. iid) rv's (Loève, 1977).

A distribution on the real line is self-decomposable if and only if it is infinitely divisible and has an absolutely continuous Lévy measure $\nu(dx) = \frac{g(x)}{|x|} dx$ such that $g(x) \geq 0$ is nondecreasing on $(-\infty, 0)$ and nonincreasing on $(0, \infty)$ (Steutel and van Harn, 2004).

Stable distributions exist only if $\gamma \in (0, 2]$. They are self-decomposable and their characteristic functions admit the canonical form $\ln f(t) = -\lambda|t|^\gamma(1 \pm \text{sign}(t)\theta i)$, for some $\lambda > 0$ and $\theta \in \mathbf{R}$ (Steutel and van Harn, 2004).

van Harn et al. (1982) proposed discrete analogues of self-decomposability and stability for distributions on $\mathbf{Z}_+ := \{0, 1, 2, \dots\}$. The authors first introduced the \mathbf{Z}_+ -valued multiple $\alpha \odot_{\mathcal{F}} X$ for a \mathbf{Z}_+ -valued rv X and $0 < \alpha < 1$ in the following way:

$$(1.2) \quad \alpha \odot_{\mathcal{F}} X = \sum_{k=1}^X Y_k(t) := Z_X(t) \quad (t = -\ln \alpha),$$

where $Y_1(\cdot), Y_2(\cdot), \dots$ are independent copies of a continuous-time Markov branching process, independent of X , such that for every $k \geq 1$, $P(Y_k(0) = 1) = 1$. The processes $(Y_k(\cdot), k \geq 1)$ are driven by a composition semigroup of probability generating functions (pgf's) $\mathcal{F} := (F_t, t \geq 0)$:

$$(1.3) \quad F_s \circ F_t(z) = F_{s+t}(z) \quad (|z| \leq 1; s, t \geq 0).$$

For every $k \geq 1$ and $t \geq 0$, $F_t(z)$ is the pgf of $Y_k(t)$, and the transition matrix $\{p_{ij}(t)\}$ of the Markov process $Y_k(\cdot)$ is determined by the equation

$$(1.4) \quad \sum_{j=0}^{\infty} p_{ij}(t) z^j = \{F_t(z)\}^i \quad (|z| \leq 1; i \geq 0).$$

Note that the process $Z_X(\cdot)$ of (1.2) is itself a Markov branching process driven by \mathcal{F} and starting with X individuals ($Z_X(0) = X$).

Let $P(z)$ be the pgf of X . Then the pgf $P_{\alpha \odot_{\mathcal{F}} X}(z)$ of $\alpha \odot_{\mathcal{F}} X$ is given by

$$(1.5) \quad P_{\alpha \odot_{\mathcal{F}} X}(z) = P(F_t(z)) \quad (t = -\ln \alpha; 0 \leq z \leq 1).$$

As an analogue of scalar multiplication, the operation $\odot_{\mathcal{F}}$ must satisfy some minimal conditions (Steutel and van Harn (2004), Chapter V, Section 8). In particular, the following regularity conditions are imposed on the composition semigroup \mathcal{F} :

$$(1.6) \quad \lim_{t \downarrow 0} F_t(z) = F_0(z) = z, \quad \lim_{t \rightarrow \infty} F_t(z) = 1.$$

The first part of (1.6) implies the continuity of the semigroup \mathcal{F} (by way of (1.3)) and the second part is equivalent to assuming that $m = F'_1(1) \leq 1$, which implies the (sub-)criticality of the continuous-time Markov branching process $Y_k(\cdot)$ in (1.2).

We will restrict ourselves to the subcritical case ($m < 1$) and we will assume without loss of generality that $m = e^{-1}$ (see Remark 3.1 in van Harn et al., 1982).

A \mathbf{Z}_+ -valued rv X , or its distribution, is said to have an \mathcal{F} -self-decomposable distribution if for every $0 < \alpha < 1$,

$$(1.7) \quad X \stackrel{d}{=} \alpha \odot_{\mathcal{F}} X + X_{\alpha} \quad (t = -\ln \alpha),$$

where X_{α} is \mathbf{Z}_+ -valued and X and X_{α} are independent (van Harn et al., 1982).

Equivalently, by (1.5) and (1.7), a distribution on \mathbf{Z}_+ with pgf $P(z)$ is \mathcal{F} -self-decomposable if for every $t > 0$, there exists a pgf $P_t(z)$ such that

$$(1.8) \quad P(z) = P(F_t(z))P_t(z) \quad (0 \leq z \leq 1),$$

where P and P_t are the pgf's of X and X_{α} ($\alpha = e^{-t}$), respectively, in (1.7).

Using (1.3), (1.5) and (1.8), it is easily seen that (1.7) corresponds to an identity that links the distributions of two generations of the Markov branching process $Z_X(\cdot)$ of (1.2):

$$(1.9) \quad Z_X(s) \stackrel{d}{=} Z_X(t+s) + Z'_{X_{\alpha}}(s) \quad (t > 0; s \geq 0; \alpha = e^{-t}),$$

where $Z'_{X_{\alpha}}(\cdot)$ is a continuous-time Markov branching process independent of $Z_X(\cdot)$, driven by \mathcal{F} and such that $Z'_{X_{\alpha}}(0) = X_{\alpha}$.

An \mathcal{F} -self-decomposable distribution is infinitely divisible and is characterized by the following canonical form of its pgf (van Harn et al., 1982):

$$(1.10) \quad P(z) = \exp \left[\int_z^1 \frac{\ln Q(x)}{U(x)} dx \right] \quad (0 \leq z \leq 1),$$

where $U(z)$ is the infinitesimal generator of the semigroup \mathcal{F} (see definition below) and $Q(z)$ is the pgf of an infinitely divisible distribution on \mathbf{Z}_+ .

\mathcal{F} -stable distributions and geometric \mathcal{F} -stable distributions (the definitions are recalled in Section 3 and 4, respectively) form important subclasses of the class of \mathcal{F} -self-decomposable distributions. Steutel et al. (1983) showed that \mathcal{F} -self-decomposable distributions arise as weak limits of continuous-time branching processes with immigration. Aly and Bouzar (2005) discussed the relationship between \mathcal{F} -self-decomposability and \mathbf{Z}_+ -valued first-order autoregressive processes.

Maejima and Naito (1998) introduced the notion of semi-selfdecomposability as follows. Let $\alpha \in (0, 1)$. A real-valued rv X , or its distribution, is said to be semi-selfdecomposable of order α if X satisfies (1.1) for some infinitely divisible rv X_{α} independent of X .

A distribution on the real line is self-decomposable if and only if it is semi-selfdecomposable of all orders $0 < \alpha < 1$ (Maejima and Naito, 1998).

Semi-selfdecomposable distributions are infinitely divisible and are characterized by a specific representation of their Lévy measures (Maejima and Naito, 1998). The class of semi-selfdecomposable

laws coincides with the class of weak limits of subsequences of normalized partial sums of independent rv's. Semistable distributions (Lévy, 1937) and geometric semistable distributions (Borowiecka, 2002) form notable subclasses of the class of semi-selfdecomposable distributions. Several authors have studied the connection between the notions of (semi-)self-decomposability and (semi-)stability and the class of (semi-) selfsimilar additive and Lévy processes. We refer in particular to the work by Choi (1994), Maejima and Sato (2003), and the monograph by Sato (1999), and references therein. The aim of this paper is to introduce and develop a notion of \mathcal{F} -semi-selfdecomposability for distributions with support on \mathbf{Z}_+ that generalizes the concept of \mathcal{F} -self-decomposability. In Section 2, we define \mathcal{F} -semi-selfdecomposability and obtain several properties. Notably, we show that these distributions possess the property of infinite divisibility and are characterized by the absolute monotonicity of a specific function. We establish a relationship between semi-selfdecomposability for distributions on \mathbf{R}_+ and \mathcal{F} -semi-selfdecomposability by way of compound Poisson mixtures. In Section 3, the \mathcal{F} -semistable distributions of Krapavitskaite (1987) are shown to form a subclass of the class of \mathcal{F} -semi-selfdecomposable distributions. We also offer some new results on \mathcal{F} -semistability. The subclass of the geometric \mathcal{F} -semistable distributions, is discussed in Section 4. In Section 5, we explore a generalization of discrete random stability.

In the rest of this section we recall some additional definitions and results about the continuous composition semigroup of pgf's $\mathcal{F} := (F_t, t \geq 0)$. For proofs and further details we refer to van Harn et al. (1982) and Steutel and van Harn (2004) and references therein.

The infinitesimal generator U of the semigroup \mathcal{F} is defined by

$$(1.11) \quad U(z) = \lim_{t \downarrow 0} (F_t(z) - z)/t \quad (|z| \leq 1),$$

and satisfies $U(z) > 0$ for $0 \leq z < 1$.

The related A -function is defined by

$$(1.12) \quad A(z) = \exp\left\{-\int_0^z (U(x))^{-1} dx\right\} \quad (0 \leq z < 1).$$

The functions $U(z)$ and $A(z)$ satisfy for any $t > 0$,

$$(1.13a) \quad \frac{\partial}{\partial t} F_t(z) = U(F_t(z)) = U(z)F_t'(z) \quad (|z| \leq 1),$$

and

$$(1.13b) \quad A(F_t(z)) = e^{-t}A(z) \quad (0 \leq z < 1).$$

A nonnegative function $f(z)$ on $[0, 1)$ is absolutely monotone if it has nonnegative derivatives of all orders. An absolutely monotone function $f(z)$ on $[0, 1)$ is characterized by the power series representation $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k \geq 0$, $k \geq 0$.

Finally, we recall a characterization of infinite divisibility for distributions on \mathbf{Z}_+ .

Theorem 1.1. A distribution $(p_n, n \geq 0)$ on \mathbf{Z}_+ with pgf $P(z)$, $0 < P(0) < 1$, is infinitely divisible if and only if the function $R(z) = P'(z)/P(z)$ is absolutely monotone on $[0, 1)$, with power series expansion

$$(1.14) \quad R(z) = \sum_{n=0}^{\infty} r_n z^n \quad (0 \leq z < 1),$$

where $r_n \geq 0$ for every $n \geq 0$ and, necessarily, $\sum_{n=0}^{\infty} r_n/(n+1) < \infty$.

Following Steutel and van Harn (2004), we will refer to the function $R(z)$ (respectively, the sequence $(r_n, n \geq 0)$) in Theorem 1.1 as the R -function (respectively, the canonical sequence) of $(p_n, n \geq 0)$, or that of its pgf $P(z)$.

2 \mathcal{F} -semi-selfdecomposability

Definition 2.1. Let $0 < \alpha < 1$. A \mathbf{Z}_+ -valued rv X with a nondegenerate distribution is said to be \mathcal{F} -semi-selfdecomposable of order α if X satisfies (1.7) for some infinitely divisible \mathbf{Z}_+ -valued rv X_α independent of X .

By Corollary V.8.4 in Steutel and van Harn (2004), a distribution on \mathbf{Z}_+ is \mathcal{F} -self-decomposable if and only if it is \mathcal{F} -semi-selfdecomposable of all orders $0 < \alpha < 1$.

A \mathbf{Z}_+ -valued rv X is \mathcal{F} -semi-selfdecomposable of order $0 < \alpha < 1$ if and only if its pgf $P(z)$ satisfies (1.8) for some infinitely divisible pgf $P_t(z)$, $t = -\ln \alpha$. In terms of Markov branching processes, this translates as follows. The process $Z_X(\cdot)$ of (1.2) will exhibit the periodic behavior, with period $t = -\ln \alpha$, that for every $s > 0$, $Z_X(s+t)$ and $Z_X(s)$ are linked by the distributional identity (1.9).

\mathcal{F} -semi-selfdecomposability implies infinite divisibility.

Theorem 2.1. An \mathcal{F} -semi-selfdecomposable distribution of order $0 < \alpha < 1$ is infinitely divisible.

Proof: Let $P(z)$ be the pgf of an \mathcal{F} -semi-selfdecomposable distribution of order $0 < \alpha < 1$. By (1.3), (1.8), and an induction argument, we have for all $k \geq 1$,

$$(2.1) \quad P(z) = P(F_{kt}(z)) \prod_{i=0}^{k-1} P_t(F_{it}(z)) \quad (t = -\ln \alpha; 0 \leq z \leq 1),$$

for some infinitely divisible pgf $P_t(z)$. The pgf $P_t(F_s(z))$ is infinitely divisible for any $s > 0$. Closure under convolution of infinite divisibility implies that $\prod_{i=0}^{k-1} P_t(F_{it}(z))$ is an infinitely divisible pgf. Moreover, we have by (1.6) and (2.1)

$$(2.2) \quad P(z) = \lim_{k \rightarrow \infty} \prod_{i=0}^k P_t(F_{it}(z)) \quad (0 \leq z \leq 1).$$

Since the class of infinitely divisible discrete distributions is closed under weak convergence, we conclude from (2.2) that $P(z)$ is infinitely divisible. \square

Next, we obtain some useful characterizations of \mathcal{F} -semi-selfdecomposability.

Theorem 2.2. Let $(p_n, n \geq 0)$ be a distribution on \mathbf{Z}_+ with pgf $P(z)$ and let $0 < \alpha < 1$. The following assertions are equivalent.

- (i) $(p_n, n \geq 0)$ is \mathcal{F} -semi-selfdecomposable of order α .

(ii) $(p_n, n \geq 0)$ is infinitely divisible and the function

$$(2.3) \quad R_t(z) = R(z) - F'_t(z)R(F_t(z)) \quad (t = -\ln \alpha; 0 \leq z < 1),$$

where $R(z)$ is the R -function of $P(z)$, is absolutely monotone on $[0, 1)$.

(iii) $(p_n, n \geq 0)$ is infinitely divisible and its canonical sequence $(r_n, n \geq 0)$ satisfies the following inequality for every $n \geq 0$,

$$(2.4) \quad \sum_{i=0}^{\infty} \left(\sum_{k=0}^n (n-k+1)p_{1,n-k+1}(t)p_{ik}(t) \right) r_i \leq r_n \quad (t = -\ln \alpha),$$

where $(p_{kl}(t), k \geq 0, l \geq 0)$ is the distribution on \mathbf{Z}_+ defined by (1.4).

Proof: Assume that (i) holds. By Theorem 2.1, $(p_n, n \geq 0)$ is infinitely divisible. It is easily shown that $R_t(z)$ of (2.3) is the R -function of $P_t(z)$ and hence (i) \Leftrightarrow (ii) follows by Theorem 1.1. To establish (ii) \Leftrightarrow (iii), we note that if $(r_n, n \geq 0)$ is the canonical sequence of $P(z)$, then by (1.4) and (1.14) the n -th coefficient of the power series expansion of $R(z) - F'_t(z)R(F_t(z))$ is

$$A_n = r_n - \sum_{i=0}^{\infty} \left(\sum_{k=0}^n (n-k+1)p_{1,n-k+1}(t)p_{ik}(t) \right) r_i.$$

Therefore, $R_t(z)$ of (2.3) is absolutely monotone if and only if the inequality (2.4) holds for every $n \geq 0$. \square

Corollary 2.1. The support of an \mathcal{F} -semi-selfdecomposable distribution $(p_n, n \geq 0)$ of order $0 < \alpha < 1$ is equal to \mathbf{Z}_+ . i.e., $p_n > 0$ for every $n \geq 0$.

Proof: By Theorem 2.1, $(p_n, n \geq 0)$ is infinitely divisible. It follows that $p_0 > 0$ (see Section II.1 in Steutel and van Harn (2004)). Let $(r_n, n \geq 0)$ be the canonical sequence of $(p_n, n \geq 0)$. If $r_0 = 0$, then by (2.4) $r_n = 0$ for every $n \geq 1$ and thus $(p_n, n \geq 0)$ would be a degenerate distribution, which is a contradiction. Therefore, $r_0 > 0$ and $p_1 = r_0 p_0 > 0$. The conclusion follows by Corollary II.8.3 in Steutel and van Harn (2004). \square

van Harn et al. (1982) constructed \mathcal{F} -self-decomposable distributions from self-decomposable distributions on \mathbf{R}_+ (see their Theorem 5.3). Their method extends to semi-selfdecomposability.

Let $\phi(\tau)$ be the Laplace-Stieltjes transform (LST) of a distribution on \mathbf{R}_+ and $A(z)$ the A -function of \mathcal{F} (see (1.12)). Then for any $\theta > 0$,

$$(2.5) \quad P_\theta(z) = \phi(\theta A(z)) \quad (0 \leq z \leq 1),$$

is the pgf of a compound Poisson mixture distribution on \mathbf{Z}_+ .

Theorem 2.3. A function $\phi(\tau)$ defined on \mathbf{R}_+ is the LST of a semi-selfdecomposable distribution on \mathbf{R}_+ of order $0 < \alpha < 1$, if and only if for every $\theta > 0$, $P_\theta(z)$ is the pgf of an \mathcal{F} -semi-selfdecomposable distribution of order α .

Proof: The proof is the same as that of Theorem 5.3 in van Harn et al. (1982). \square

An \mathcal{F} -semi-selfdecomposable distribution does not necessarily arise as a mixture of the form (2.5) as the following example shows.

Consider the scaled Sibuya distribution (Sibuya (1979) and Christoph and Schreiber (2000)) with pgf $Q(z) = 1 - c(1-z)^a$, $a, c \in (0, 1]$. Assume $0 < c \leq \frac{1-a}{1+a}$. By Theorem 1 in Christoph and Schreiber (2000), $Q(z)$ is (discrete) self-decomposable with respect to the binomial thinning operator $\odot_{\mathcal{F}^{(0)}}$ of Steutel and van Harn (1979) (see (2.6) below, and the subsequent remark). The function $B(z) = 1 - A(z)$, with $0 \leq z \leq 1$ and $A(z)$ of (1.12), is a pgf (van Harn et al., 1982). Simple calculations (and (1.13b)) show that the pgf $P(z) = Q(B(z)) = 1 - cA(z)^a$, $0 \leq z \leq 1$, satisfies (1.8) and is therefore \mathcal{F} -self-decomposable. If $P(z)$ had the form (2.5) for some LST $\phi(\tau)$ and $\theta > 0$, then $\phi(\tau) = 1 - c\tau^a/\theta$ over a finite interval $[0, \theta]$. This contradicts the fact that $\phi(\tau)$ is an LST.

The subclass of \mathcal{F} - semi-selfdecomposable distributions that have a mixture representation of the type (2.5) coincides with the class of weak limits of subsequences of weighted sums of \mathbf{Z}_+ -valued independent rv's via the multiplier $\odot_{\mathcal{F}}$. The proof is obtained by combining the arguments in the proofs of Theorems 8.4 and 8.5 of van Harn et al. (1982) and by using Theorem 2.1 in Maejima and Naito (1998). We omit the details.

Theorem 2.4. Let $(X_n, n \geq 1)$ be a sequence of independent \mathbf{Z}_+ -valued rv's and $0 < \alpha < 1$. Let $(c_n, n \geq 1)$ be an increasing sequence of real numbers such that $c_n \geq 1$ and $c_n \uparrow \infty$ and let $(k_n, n \geq 1)$ be a strictly increasing sequence in \mathbf{Z}_+ such that $k_n \uparrow \infty$. Assume

$$(i) c_n^{-1} \odot_{\mathcal{F}} \sum_{i=1}^{k_n} X_i \text{ converges weakly to a } \mathbf{Z}_+ \text{-valued rv } X; (ii) \lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} = \alpha; (iii) \lim_{n \rightarrow \infty} \max_{1 \leq i \leq k_n} P(c_n^{-1} \odot_{\mathcal{F}} X_i \geq \epsilon) = 0, \text{ for every } \epsilon > 0.$$

Then X is \mathcal{F} -semi-selfdecomposable of order α . Moreover, X admits the mixture representation (2.5) (for some $\theta > 0$) with a semi-selfdecomposable mixing distribution on \mathbf{R}_+ of order α .

Conversely, if a \mathbf{Z}_+ -valued rv X admits a mixture representation of the type (2.5) with a semi-selfdecomposable mixing distribution of order $0 < \alpha < 1$, then there exist sequences $(X_n, n \geq 1)$, $(c_n, n \geq 1)$ and $(k_n, n \geq 1)$, as defined above, that satisfy (i)-(iii).

van Harn et al. (1982) give some rich examples of continuous composition semigroups of pgf's. We mention the family of semigroups $(\mathcal{F}^{(\theta)}, 0 \leq \theta < 1)$ described by

$$(2.6) \quad F_t^{(\theta)}(z) = 1 - \frac{\bar{\theta} e^{-t}(1-z)}{\bar{\theta} + \theta(1-e^{-t})(1-z)} \quad (t \geq 0, |z| \leq 1, \bar{\theta} = 1 - \theta).$$

In this case we have $m = e^{-1}$, $U^{(\theta)}(z) = \bar{\theta}^{-1}(1-z)(1-\theta z)$ and $A^{(\theta)}(z) = \frac{1-z}{1-\theta z}$. We note that for $\theta = 0$, $\mathcal{F}^{(\theta)}$ corresponds to the standard semigroup $F_t^{(0)}(z) = 1 - e^{-t} + e^{-t}z$ and the multiplication $\odot_{\mathcal{F}^{(0)}}$ becomes the binomial thinning operator of Steutel and van Harn (1979). $\mathcal{F}^{(0)}$ -semi-selfdecomposability was studied by Bouzar (2008).

3 \mathcal{F} -semistability

In this section, we discuss an important subset of the set of \mathcal{F} -semi-selfdecomposable distributions. First, we recall the definition of \mathcal{F} -stability due to van Harn et al. (1982).

A \mathbf{Z}_+ -valued rv X , or its distribution, is said to be \mathcal{F} -stable if for every $n \geq 1$ there exists $c_n \in (0, 1]$ such that

$$(3.1) \quad X \stackrel{d}{=} c_n \odot_{\mathcal{F}} (X_1 + \cdots + X_n),$$

where X_1, \cdots, X_n are iid with $X_1 \stackrel{d}{=} X$.

An \mathcal{F} -stable distribution is \mathcal{F} -selfdecomposable and is characterized by the following canonical form of its pgf (van Harn et al., 1982):

$$(3.2) \quad P(z) = \exp\{-cA(z)^\gamma\} \quad (0 \leq z \leq 1),$$

where $c > 0$, $0 < \gamma \leq 1$, and $A(z)$ is the A -function of \mathcal{F} (see (1.12)). The number γ is called the exponent of the (\mathcal{F} -stable) distribution.

Krapavitskaite (1987) introduced the notion of \mathcal{F} -semistability as follows.

Definition 3.1 A \mathbf{Z}_+ -valued rv X with pgf $P(z)$ is said to be \mathcal{F} -semistable if there exist $\lambda > 0$ and $0 < \alpha < 1$ such that

$$(3.3) \quad P(z) = [P(F_t(z))]^\lambda \quad (t = -\ln \alpha; 0 \leq z \leq 1).$$

Krapavitskaite (1987) showed that if an \mathcal{F} -semistable distribution exists, then α and λ of (3.3) satisfy the inequality $\lambda \alpha \leq 1 < \lambda$. This implies that one can write $\lambda = \alpha^{-\gamma}$ for some $0 < \gamma \leq 1$. We will refer to γ as the exponent of the distribution and α its order.

A distribution on \mathbf{Z}_+ with pgf $P(z)$ is \mathcal{F} -stable if and only if for every $\lambda > 1$, there exists $0 < \alpha < 1$ such that $P(z)$ satisfies (3.3) (van Harn et al., 1982). Therefore, a distribution on \mathbf{Z}_+ is \mathcal{F} -stable distribution if and only if it is \mathcal{F} -semistable for all $\lambda > 0$.

Proposition 3.1. An \mathcal{F} -semistable distribution with exponent $0 < \gamma \leq 1$ and order $0 < \alpha < 1$ is infinitely divisible.

Proof: Let $P(z)$ be the pgf of an \mathcal{F} -semistable distribution with exponent γ and order α . By (1.3), (3.3), and induction, we have for any $n \geq 0$

$$(3.4) \quad [P(z)]^{\alpha^{ny}} = P(F_{nt}(z)) \quad (t = -\ln \alpha; 0 \leq z \leq 1).$$

Let $P_n(z) = \exp\{-\alpha^{-n\gamma}(1 - [P(z)]^{\alpha^{ny}})\}$, $n \geq 0$. By (3.4), $P_n(z)$ is the pgf of a compound Poisson distribution on \mathbf{Z}_+ and is therefore infinitely divisible. Moreover, we have $\lim_{n \rightarrow \infty} P_n(z) = P(z)$, $0 \leq z \leq 1$. Hence, any \mathcal{F} -semistable distribution is the weak limit of a sequence of infinitely divisible distributions and is therefore infinitely divisible. \square

Characterization results for \mathcal{F} -semistability are given next.

Theorem 3.1. Let $0 < \alpha < 1$ and $0 < \gamma \leq 1$, and let $(p_n, n \geq 0)$ be a distribution on \mathbf{Z}_+ with pgf $P(z)$. The following assertions are equivalent.

- (i) $(p_n, n \geq 0)$ is \mathcal{F} -semistable with exponent γ and order α .

(ii) There exists a nonnegative and periodic function $g(\cdot)$ on \mathbf{R}_+ , with period $t = -\ln \alpha$, such that

$$(3.5) \quad P(z) = \exp\{-A(z)^\gamma g(|\ln A(z)|)\} \quad (0 \leq z < 1),$$

where $A(z)$ is the A -function of \mathcal{F} (see (1.12)).

(iii) $(p_n, n \geq 0)$ is infinitely divisible and its R -function has the form

$$(3.6) \quad R(z) = A(z)^\gamma r(|\ln A(z)|)/U(z) \quad (0 \leq z < 1),$$

for some nonnegative and periodic function $r(\cdot)$ on \mathbf{R}_+ , with period $t = -\ln \alpha$.

Proof: (i) \Leftrightarrow (ii): Let $P(z)$ be the pgf of an \mathcal{F} -semistable distribution with exponent γ and order α . Let $f(z) = -A(z)^{-\gamma} \ln P(z)$, $z \in [0, 1]$. Clearly, $f(z) \geq 0$, $z \in [0, 1]$. By (3.3) and (1.13b), $f(F_t(z)) = f(z)$, $t = -\ln \alpha$. Therefore, $P(z)$ has the form

$$(3.7) \quad P(z) = \exp\{-A(z)^\gamma f(z)\} \quad (0 \leq z < 1).$$

Let $g(\tau) = f(A^{-1}(e^{-\tau}))$, $\tau \geq 0$, where $A^{-1}(\cdot)$ is the inverse function of $A(\cdot)$ (note A is one-to-one from $[0, 1]$ onto $[0, 1]$). Clearly, $g(-\ln A(z)) = f(A^{-1}(e^{\ln A(z)})) = f(z)$. Therefore, (3.5) follows from (3.7). Since, by (1.13b), $A[F_t(A^{-1}(e^{-\tau}))] = e^{-\tau-t}$, we have $g(\tau + t) = f(A^{-1}(e^{-\tau-t})) = f(F_t(A^{-1}(e^{-\tau}))) = f(A^{-1}(e^{-\tau})) = g(\tau)$. This implies $g(\tau)$ is periodic with period t . The proof of the converse is straightforward.

(i) \Leftrightarrow (iii): Infinite divisibility follows from Proposition 3.1. By (ii), $\ln P(z) = -A(z)^\gamma g(|\ln A(z)|)$, $z \in [0, 1]$. By differentiation and the fact that $[-U(z)]^{-1} = A'(z)/A(z)$ (see (1.12)), we have $R(z) = A(z)^\gamma (-\gamma g(|\ln A(z)|) + g'(|\ln A(z)|))/U(z)$. The representation (3.6) ensues, with $r(\tau) = \gamma g(\tau) - g'(\tau)$, $\tau \geq 0$. Since $R(z) \geq 0$ on $[0, 1)$ and $g(\tau)$ has period t , it follows that $r(\tau)$ is nonnegative and periodic, with period t . Conversely, by (3.6), (1.13a), (1.13b), and the fact that $r(\tau)$ is periodic with period t , we have for every $0 \leq z < 1$, $R(z) = \alpha^\gamma F_t'(z)R(F_t(z))$. An integration argument leads to (3.3) (with $\lambda = \alpha^{-\gamma}$). \square

The following result is a direct consequence of Theorem 3.1 and equation (1.8).

Corollary 3.1. An \mathcal{F} -semistable distribution with exponent $0 < \gamma \leq 1$, order $0 < \alpha < 1$, and pgf $P(z)$, is \mathcal{F} -semi-selfdecomposable of order α . In this case, the pgf $P_t(z)$ of (1.8), $t = -\ln \alpha$, takes the form

$$(3.8) \quad P_t(z) = [P(z)]^c \quad (c = 1 - \alpha^\gamma; 0 \leq z \leq 1).$$

Corollary 3.1 allows for an interpretation of \mathcal{F} -semistability in terms of Markov branching processes in a way that is similar to the one described for \mathcal{F} -semi-selfdecomposability (see paragraph following Definition 2.1). In this case, the \mathbf{Z}_+ -valued rv X_α in (1.9) will have pgf $P_t(z)$ of (3.8).

\mathcal{F} -semistable distributions can be obtained from their continuous counterparts with support on \mathbf{R}_+ .

Theorem 3.2. A function $\phi(\tau)$ defined on \mathbf{R}_+ is the LST of a semistable distribution on \mathbf{R}_+ with exponent $0 < \gamma \leq 1$ and order $0 < \alpha < 1$ if and only if for every $\theta > 0$, $P_\theta(z)$ of (2.5) is the pgf of an \mathcal{F} -semistable distribution with exponent γ and order α , for some, and then for all, $\theta > 0$.

Proof: The 'only if' part follows from the fact that $\phi(\tau)$ satisfies (Huillet et al., 2001)

$$(3.9) \quad \ln \phi(\tau) = \alpha^{-\gamma} \ln \phi(\alpha\tau) \quad (\tau \geq 0).$$

To prove the 'if' part, assume that $\phi(\tau)$ is an LST with the property that for some $\theta > 0$, $P_\theta(z)$ satisfies (3.3) with $t = -\ln \alpha$ and $\lambda = \alpha^{-\gamma}$. Then by (1.13b), $\ln \phi(\theta A(z)) = \alpha^{-\gamma} \ln \phi(\alpha \theta A(z))$, $z \in [0, 1)$. For $0 \leq \tau \leq \theta$, let $z = A^{-1}((\tau/\theta))$. Noting that $0 \leq z \leq 1$, it follows that $\phi(\tau)$ satisfies (3.9) for every $0 \leq \tau \leq \theta$ and hence for every $\tau \geq 0$ (since an LST is completely determined over finite intervals). The conclusion extends to all $\theta > 0$ by the first part of the theorem. \square

If $(X_n, n \geq 1)$ in Theorem 2.4 is a sequence of iid rv's, then the limiting distribution is \mathcal{F} -semistable (note that in this case condition (iii) of the theorem is necessarily satisfied). The proof is obtained by combining the arguments in the proofs of Theorems 8.4 and 8.5 of van Harn et al. (1982) and by using Theorem 2.1 in Maejima (2001). We omit the details.

Theorem 3.3 Let $(X_n, n \geq 1)$ be a sequence of \mathbf{Z}_+ -valued iid rv's and $0 < \alpha < 1$. Let $(c_n, n \geq 1)$ be an increasing sequence of real numbers such that $c_n \geq 1$ and $c_n \uparrow \infty$ and let $(k_n, n \geq 1)$ be a strictly increasing sequence in \mathbf{Z}_+ such that $k_n \uparrow \infty$. Assume

$$(i) \ c_n^{-1} \odot_{\mathcal{F}} \sum_{i=1}^{k_n} X_i \text{ converges weakly to a } \mathbf{Z}_+ \text{-valued rv } X; \ (ii) \ \lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} = \alpha; \ \text{Then } X \text{ is } \mathcal{F} \text{-semistable}$$

with some exponent $0 < \gamma \leq 1$ and order α . Moreover, X admits the mixture representation (2.5) (for some $\theta > 0$) with a semistable mixing distribution on \mathbf{R}_+ with exponent γ and order α . The sequence $(k_n, n \geq 1)$ necessarily satisfies

$$(3.10) \quad \lim_{n \rightarrow \infty} \frac{k_n}{k_{n+1}} = \gamma.$$

Conversely, if a \mathbf{Z}_+ -valued rv X admits a mixture representation of the type (2.5) with a semistable mixing distribution with exponent γ and order α (for some $0 < \alpha < 1$ and $0 < \gamma \leq 1$), then there exist sequences $(X_n, n \geq 1)$, $(c_n, n \geq 1)$ and $(k_n, n \geq 1)$, as defined above, that satisfy (i)-(ii) and (3.10).

The following result is a direct consequence of Theorems 3.2 and 3.3.

Corollary 3.2. A \mathbf{Z}_+ -valued rv X has an \mathcal{F} -semistable distribution with exponent $0 < \gamma \leq 1$ and order $0 < \alpha < 1$ if and only if X admits the mixture representation (2.5) (for some $\theta > 0$) with a semistable mixing distribution on \mathbf{R}_+ with exponent γ and order α .

Next, we give an example of an \mathcal{F} -semistable distribution.

Let $0 < \alpha < 1$ and $0 < \gamma \leq 1$. Let $\psi_\alpha(x)$ be a continuous bounded nonnegative and periodic function over \mathbf{R}_+ , with period $-\ln \alpha$. The function

$$(3.11) \quad \phi(\tau) = \exp \left\{ - \int_0^\infty (1 - e^{-\tau x}) x^{-1-\gamma} \psi_\alpha(\ln x) dx \right\} \quad (\tau \geq 0)$$

is the LST of an infinitely divisible distribution μ on \mathbf{R}_+ (by Theorem III.4.3 in Steutel and van Harn (2004)). It is easily seen that $\phi(\tau)$ satisfies (3.9). Thus μ is semistable with exponent γ and order

α . By Theorem 3.2,

$$(3.12) \quad P_\theta(z) = \exp\left\{-\int_0^\infty (1 - e^{-\theta A(z)x})x^{-1-\gamma}\psi_\alpha(\ln x) dx\right\} \quad (\theta > 0; 0 \leq z \leq 1)$$

is the pgf of an \mathcal{F} -semistable distribution with exponent γ and order α .

We note that Theorem 3.1 ((i) \Leftrightarrow (ii)) and Theorem 3.3 are essentially due to Krapavitskaite (1987). The proofs we outlined for these results are somewhat simpler than those provided by Krapavitskaite (1987).

4 Geometric \mathcal{F} -semistability

In this section, we introduce a notion of discrete random semistability and discuss its relationship with \mathcal{F} -semistability and \mathcal{F} -semi-selfdecomposability.

We recall (Bouzar, 1999) that a \mathbf{Z}_+ -valued rv X , or its distribution, is said to be geometric \mathcal{F} -stable if for every $0 < p < 1$, there exists $0 < \alpha < 1$ such that

$$(4.1) \quad X \stackrel{d}{=} \alpha \odot_{\mathcal{F}} \sum_{i=1}^{N_p} X_i,$$

where $(X_i, i \geq 1)$ is a sequence of iid rv's, $X_i \stackrel{d}{=} X$, N_p has the geometric distribution with parameter p (i.e., $P(N_p = k) = p(1-p)^{k-1}$, $k \geq 1$), and $(X_i, i \geq 1)$ and N_p are independent.

A geometric \mathcal{F} -stable distribution is \mathcal{F} -selfdecomposable and is characterized by the following form of its pgf (Bouzar, 1999):

$$(4.2) \quad P(z) = (1 + cA(z)^\gamma)^{-1} \quad (0 \leq z \leq 1),$$

where $c > 0$, $0 < \gamma \leq 1$, and $A(z)$ is the A -function of \mathcal{F} (see (1.12)). The number γ is called the exponent of the (geometric \mathcal{F} -stable) distribution.

Definition 4.1. A \mathbf{Z}_+ -valued rv X is said to be geometric \mathcal{F} -semistable if it satisfies (4.1) for some $0 < p < 1$ and $0 < \alpha < 1$.

It follows from the definitions above that a \mathbf{Z}_+ -valued rv X is geometric \mathcal{F} -stable if and only if it is geometric \mathcal{F} -semistable for every $0 < p < 1$.

It is easily shown that a \mathbf{Z}_+ -valued rv with pgf $P(z)$ is geometric \mathcal{F} -semistable if and only if there exist $p, \alpha \in (0, 1)$ such that

$$(4.3) \quad P(F_t(z)) = \frac{P(z)}{p + (1-p)P(z)} \quad (t = -\ln \alpha; 0 \leq z \leq 1).$$

Klebanov *et al.* (1984) introduced random infinite divisibility as follows. A rv X is said to be geometric infinitely divisible if for any $0 < p < 1$, there exists a sequence of iid rv's $(X_i^{(p)}, i \geq 1)$ and a rv N_p with a geometric distribution (as in (4.1)), and independent of $(X_i^{(p)}, i \geq 1)$, such that

$$(4.4) \quad X \stackrel{d}{=} \sum_{i=1}^{N_p} X_i^{(p)}.$$

Proposition 4.1. A geometric \mathcal{F} -semistable distribution is geometric infinitely divisible, and hence infinitely divisible.

Proof: Let $P(z)$ be the pgf of a geometric \mathcal{F} -semistable distribution. Using (4.3) inductively (for $p, \alpha \in (0, 1)$), along with (1.3) and (1.6), it can be shown that $P(z) = \lim_{n \rightarrow \infty} P_n(z)$, where $P_n(z) = \left(1 + p^{-n}(1 - P(F_{nt}(z)))\right)^{-1}$ and $t = -\ln \alpha$. By Aly and Bouzar (2000), $P_n(z)$ is the pgf of a geometric infinitely divisible distribution for every $n \geq 1$. Therefore, any geometric \mathcal{F} -semistable distribution is the weak limit of a sequence of geometric infinitely divisible distributions and, hence, must itself be geometric infinitely divisible (Klebanov et al., 1984). The second part follows by Aly and Bouzar (2000). \square

A useful link exists between \mathcal{F} -semistability and geometric \mathcal{F} -semistability.

Theorem 4.1. A distribution on \mathbf{Z}_+ with pgf $P(z)$ is geometric \mathcal{F} -semistable for some $p, \alpha \in (0, 1)$ if and only if

$$(4.5) \quad H(z) = \exp\left\{1 - \frac{1}{P(z)}\right\} \quad (0 \leq z \leq 1)$$

is the pgf of an \mathcal{F} -semistable distribution with exponent $\gamma = \ln p / \ln \alpha$ and order α .

Proof: If $P(z)$ satisfies (4.3) for $p, \alpha \in (0, 1)$, then by Proposition 4.1, $P(z)$ is geometric infinitely divisible. By Aly and Bouzar (2000), $H(z)$ of (4.5) is an infinitely divisible pgf. Moreover, we have by (4.3) that $H(z)$ satisfies (3.3) with $\lambda = 1/p$. Therefore $H(z)$ is \mathcal{F} -semistable with exponent $\gamma = \ln p / \ln \alpha$ and order α . This proves the 'only if' part. For the 'if' part, assume $H(z)$ of (4.5) is \mathcal{F} -semistable with exponent $0 < \gamma \leq 1$ and order $0 < \alpha < 1$. Then $P(z)$ satisfies (4.3) with $p = \alpha^\gamma$. \square

By Theorem 4.1, the parameters $p, \alpha \in (0, 1)$ of a geometric \mathcal{F} -semistable distribution are linked by the equation $p = \alpha^\gamma$, $0 < \gamma \leq 1$. We will refer to γ as the exponent of the distribution and α its order.

Corollary 4.1. A distribution on \mathbf{Z}_+ with pgf $P(z)$ is geometric \mathcal{F} -semistable with exponent $0 < \gamma \leq 1$ and order $0 < \alpha < 1$ if and only if

$$(4.6) \quad P(z) = (1 - \ln H(z))^{-1} \quad (0 \leq z \leq 1),$$

where $H(z)$ is the pgf of an \mathcal{F} -semistable distribution with exponent γ and order α .

Combining Theorem 3.1 and Corollary 4.1 yields the following representation result.

Corollary 4.2. A distribution on \mathbf{Z}_+ with pgf $P(z)$ is geometric \mathcal{F} -semistable with exponent $0 < \gamma \leq 1$ and order $0 < \alpha < 1$ if and only if there exists a nonnegative and periodic function $g(\cdot)$ on \mathbf{R}_+ , with period $t = -\ln \alpha$, such that

$$(4.7) \quad P(z) = (1 + A(z)^\gamma g(|\ln A(z)|))^{-1} \quad (0 \leq z < 1),$$

where $A(z)$ is the A -function of \mathcal{F} (see (1.12)).

From Corollary 4.2 and equation (1.8), we deduce

Corollary 4.3. A geometric \mathcal{F} -semistable distribution with exponent $0 < \gamma \leq 1$, order $0 < \alpha < 1$, and pgf $P(z)$, is \mathcal{F} -semi-selfdecomposable of order α . In this case, the pgf $P_t(z)$ of (1.8) takes the form

$$(4.8) \quad P_t(z) = 1 - c + cP(z) \quad (c = 1 - \alpha^\gamma; 0 \leq z \leq 1).$$

Corollary 4.3 allows for an interpretation of geometric \mathcal{F} -semistability in terms of Markov branching processes in a way that is similar to the one described for \mathcal{F} -semi-selfdecomposability (see paragraph following Definition 2.1). By (4.8), the rv X_α in (1.9) satisfies $X_\alpha \stackrel{d}{=} IX'$, where I and X' are independent, I is Bernoulli $(1 - \alpha^\gamma)$, and $X' \stackrel{d}{=} X$.

By Theorem 3.2 and Corollary 4.1 (and its continuous counterpart), we obtain

Corollary 4.4. A function $\phi(\tau)$ defined on \mathbf{R}_+ is the LST of a geometric semistable distribution on \mathbf{R}_+ with exponent $0 < \gamma \leq 1$ and order $0 < \alpha < 1$, if and only if $P_\theta(z)$ of (2.5) is the pgf of a geometric \mathcal{F} -semistable distribution with exponent γ and order $t = -\ln \alpha$, for some, and then for all, $\theta > 0$.

Theorem 4.1, Corollary 3.2, and Corollary 4.1 (and its continuous counterpart) yield

Corollary 4.5. A \mathbf{Z}_+ -valued rv X has a geometric \mathcal{F} -semistable distribution with exponent $0 < \gamma \leq 1$ and order $0 < \alpha < 1$ if and only if X admits the mixture representation (2.5) (for some $\theta > 0$) with a geometric semistable mixing distribution on \mathbf{R}_+ with exponent γ and order α .

Combining Corollary 4.1 with equation (3.12) yields an example of a geometric \mathcal{F} -semistable distribution with pgf

$$P(z) = \left(1 + \int_0^\infty (1 - e^{-\theta A(z)}) x^{-1-\gamma} \psi_\alpha(\ln x) dx \right)^{-1} \quad (\theta > 0; 0 \leq z \leq 1).$$

5 \mathcal{N} -semistability on \mathbf{Z}_+

The concept of geometric infinite divisibility of Klebanov et al. (1984) encountered in Section 4 was generalized independently by Klebanov and Rachev (1996) and Bunge (1996). The definition follows.

Let I be a subset of $(0, 1)$ and $\mathcal{N} := \{N_p, p \in I\}$ a family of \mathbf{Z}_+ -valued rv's such that $E(N_p) = 1/p$ for any $p \in I$, and

$$(5.1) \quad K_{p_1} \circ K_{p_2}(z) = K_{p_2} \circ K_{p_1}(z) \quad (p_1, p_2 \in I; 0 \leq z \leq 1),$$

where $K_p(z)$ is the pgf of N_p .

Definition 5.1. A rv X is said to be \mathcal{N} -infinitely divisible if it satisfies (4.1) for any $N_p \in \mathcal{N}$.

The commutative property (5.1) implies (see the proof of Theorem 4.6.1 in Gnedenko and Korolev (1996)) the existence of an LST $\varphi(\tau)$ that satisfies $\varphi(0) = -\varphi'(0) = 1$ and

$$(5.2) \quad \varphi(\tau) = K_p(\varphi(p\tau)) \quad (p \in I; \tau \geq 0).$$

The function $\varphi(\tau)$ links \mathcal{N} -infinite divisibility with its classical counterpart. We cite the main result in the discrete case (Aly and Bouzar, 2000).

A \mathbf{Z}_+ -valued rv X is \mathcal{N} -infinitely divisible if and only if its pgf $P(z)$ has the form

$$(5.3) \quad P(z) = \varphi(-\ln H(z)) \quad (0 \leq z \leq 1),$$

where $H(z)$ is the pgf of an infinitely divisible rv.

We now define a related concept of discrete random semistability.

Definition 5.2. A \mathbf{Z}_+ -valued rv X is said to have an $\mathcal{N}(F)$ -semistable distribution if it satisfies (4.1) for some $p \in I$, $0 < \alpha < 1$, and $N_p \in \mathcal{N}$.

Using the following equivalent formulation of (5.2),

$$(5.4) \quad K_p(z) = \varphi(\varphi^{-1}(z)/p) \quad (p \in I; 0 \leq z \leq 1),$$

it is easily shown that a distribution on \mathbf{Z}_+ with pgf $P(z)$ is $\mathcal{N}(F)$ -semistable for some $p \in I$ and $0 < \alpha < 1$ if and only if

$$(5.5) \quad P(F_t(z)) = \varphi(p\varphi^{-1}(P(z))) \quad (t = -\ln \alpha; 0 \leq z \leq 1).$$

Let $P(z)$ be the pgf of an $\mathcal{N}(F)$ -semistable distribution for some $p \in I$ and $0 < \alpha < 1$. Using (5.5) inductively, along with (1.3), yields

$$(5.6) \quad P(F_{nt}(z)) = \varphi(p^n \varphi^{-1}(P(z))) \quad (t = -\ln \alpha; 0 \leq z \leq 1),$$

for all $n \geq 1$. Let $P_n(z) = \varphi(p^{-n}(1 - P(F_{nt}(z))))$, $n \geq 1$. It follows by (5.6), and the basic properties of $\varphi(\tau)$ that $\lim_{n \rightarrow \infty} P_n(z) = P(z)$. By (5.3), $P_n(z)$ is the pgf of an \mathcal{N} -infinitely divisible distribution. The closure property of \mathcal{N} -infinite divisibility under weak convergence (see Gnedenko and Korolev (1996), Section 4.6) implies the following result.

Proposition 5.1. Any $\mathcal{N}(F)$ -semistable distribution is \mathcal{N} -infinitely divisible.

Next, we describe a useful link between $\mathcal{N}(F)$ -semistability and \mathcal{F} -semistability.

Theorem 5.1. A distribution on \mathbf{Z}_+ with pgf $P(z)$ is $\mathcal{N}(F)$ -semistable for some $p \in I$ and $0 < \alpha < 1$ if and only if

$$(5.7) \quad H(z) = \exp\{-\varphi^{-1}(P(z))\} \quad (0 \leq z \leq 1)$$

is the pgf of an \mathcal{F} -semistable distribution with exponent $\gamma = \ln p / \ln \alpha$ and order α .

Proof: If $P(z)$ is $\mathcal{N}(F)$ -semistable for some $p \in I$ and $0 < \alpha < 1$, then by (5.5)

$$(5.8) \quad P(z) = \varphi(\varphi^{-1}(P(F_t(z)))/p) \quad (0 \leq z \leq 1).$$

By Proposition 5.1, $P(z)$ has the form (5.3) for some infinitely divisible pgf $H(z)$. Solving for $H(z)$ in (5.3) yields (5.7). By (5.5), (5.7), and (5.8), we deduce that $H(z)$ satisfies (3.3) with $\lambda = 1/p$. Therefore $H(z)$ is \mathcal{F} -semistable with exponent $\gamma = \ln p / \ln \alpha$ and order α . Conversely, if $H(z)$ of (5.7) is $\mathcal{N}(F)$ -semistable with exponent $0 < \gamma \leq 1$ and order $0 < \alpha < 1$, then $P(z)$ satisfies (5.5) with $p = \alpha^\gamma$. \square

By Theorem 5.1, the parameters p and α of a geometric $\mathcal{N}(F)$ -semistable distribution are linked by the equation $p = \alpha^\gamma$, $0 < \gamma \leq 1$. We will refer to γ as the exponent of the distribution and α its order.

We now gather some characterizations of $\mathcal{N}(F)$ -semistability.

Theorem 5.2. Let $0 < \gamma \leq 1$ and $0 < \alpha < 1$, and let $(p_n, n \geq 0)$ be a distribution on \mathbf{Z}_+ with pgf $P(z)$. The following assertions are equivalent.

- (i) $(p_n, n \geq 0)$ is $\mathcal{N}(F)$ -semistable with exponent γ and order α .
- (ii) $P(z)$ has the form (5.3), where $H(z)$ is the pgf of an \mathcal{F} -semistable distribution with exponent γ and order α .
- (iii) There exists a nonnegative and periodic function $g(\cdot)$ on \mathbf{R}_+ , with period $t = -\ln \alpha$, such that

$$(5.9) \quad P(z) = \varphi(A(z)^\gamma g(|\ln A(z)|)) \quad (0 \leq z < 1),$$

where $A(z)$ is the A -function of \mathcal{F} (see (1.12)).

Proof: (i) \Rightarrow (ii) follows from Theorem 5.1. (ii) \Rightarrow (iii) is shown by applying Theorem 3.1 to $H(z)$. Finally, using (1.13b), it is easily verified that $P(z)$ of (5.9) satisfies (5.5) with $p = \alpha^\gamma$, which proves (iii) \Rightarrow (i). \square

Corollary 5.1. Let $(p_n, n \geq 0)$ be an $\mathcal{N}(F)$ -semistable distribution with exponent $0 < \gamma \leq 1$ and order $0 < \alpha < 1$. If $\varphi(\tau)$ of (5.2) is the LST of a self-decomposable distribution on \mathbf{R}_+ , then $(p_n, n \geq 0)$ is \mathcal{F} -semi-selfdecomposable of order α .

Proof: Let $P(z)$ be the pgf of $(p_n, n \geq 0)$. By Theorem 5.1, $P(z) = \phi(-\ln H(z))$ for some \mathcal{F} -semistable pgf $H(z)$ with exponent γ and order α . It is easy to show that if $\varphi(\tau)$ is self-decomposable then $P(z)$ is \mathcal{F} -semi-selfdecomposable of order α . \square

Definition 5.3. A \mathbf{Z}_+ -valued rv X is said to have an $\mathcal{N}(F)$ -stable distribution if for any $p \in I$, there exists $0 < \alpha < 1$ and $N_p \in \mathcal{N}$ such that (4.1) holds.

The same argument that led to Theorem 5.1 can be used to show that a distribution on \mathbf{Z}_+ with pgf $P(z)$ is $\mathcal{N}(F)$ -stable if and only if $H(z)$ of (5.7) is the pgf of an \mathcal{F} -stable distribution. Using (3.2), we obtain the following representation result for $\mathcal{N}(F)$ -stability.

Theorem 5.3. A distribution on \mathbf{Z}_+ with pgf $P(z)$ is $\mathcal{N}(F)$ -stable if and only if there exist $0 < \gamma \leq 1$ and $c > 0$ such that

$$P(z) = \varphi(cA(z)^\gamma) \quad (0 \leq z \leq 1),$$

where $A(z)$ is the A -function of \mathcal{F} (see (1.12)).

We note that classical (resp. geometric) infinite divisibility corresponds to the family of rv's \mathcal{N} where $N_p = \frac{1}{p}$ with probability 1, $p \in I = \{\frac{1}{n} : n \geq 1\}$ (resp. N_p has the geometric distribution with parameter $p \in I = (0, 1)$). In the classical case $\varphi(\tau) = e^{-\tau}$, whereas in the geometric case, $\varphi(\tau) = (1 + \tau)^{-1}$.

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