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On two-dimensional random walk among heavy-tailed conductances

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Abstract

We consider a random walk among unbounded random conductances on the *two-dimensional* integer lattice. When the distribution of the conductances has an infinite expectation and a polynomial tail, we show that the scaling limit of this process is the fractional kinetics process. This extends the results of the paper [BČ10] where a similar limit statement was proved in dimension $d \geq 3$. To make this extension possible, we prove several estimates on the Green function of the process killed on exiting large balls.

Key words: Random walk among random conductances, functional limit theorems, fractional kinetics, trap models.

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1 Introduction and main results

The main purpose of the present paper is to extend the validity of the quenched non-Gaussian functional limit theorem for random walk among heavy-tailed random conductances on $\mathbb{Z}^d$ to dimension $d = 2$. Analogous limit theorem for $d \geq 3$ was recently obtained in [BČ10].

We recall the model first. Let E^d be the set of all non-oriented nearest-neighbour edges in $\mathbb{Z}^d$ and let $\Omega = (0,\infty)^{E^d}$. On Ω we consider the product probability measure $\mathbb{P}$ under which the canonical coordinates $(\mu_e, e \in E^d)$, interpreted as conductances, are positive i.i.d. random variables. Writing $x \sim y$ if x, y are neighbours in $\mathbb{Z}^d$, and denoting by xy the edge connecting x and y, we set

$$\mu_x = \sum_{y \sim x} \mu_{xy} \qquad \text{for } x \in \mathbb{Z}^d, \tag{1.1}$$

$$p_{xy} = \mu_{xy}/\mu_x \qquad \text{if } x \sim y. \tag{1.2}$$

For a given realisation $\boldsymbol{\mu} = (\mu_e, e \in E^d)$ of the conductances, we consider the continuous-time Markov chain with transition rates p_{xy}. We use $X = (X(t), t \geq 0)$ and $P_x^{\boldsymbol{\mu}}$ to denote this chain and its law on the space $D^d := D([0,\infty),\mathbb{R}^d)$ equipped with the standard Skorokhod J_1-topology. The total transition rate of X from a vertex $x \in \mathbb{Z}^d$ is independent of x: $\sum_{y\sim x} p_{xy} = 1$. Therefore, as in [BD10, BČ10], we call this process the *constant-speed random walk* (CSRW) in the configuration of conductances $\boldsymbol{\mu}$. The CSRW is reversible and μ_x is its reversible measure.

In this paper we assume that the distribution of the conductances is heavy-tailed and bounded from below:

$$\mathbb{P}[\mu_e \geq u] = u^{-\alpha}(1 + o(1)), \text{ as } u \to \infty, \text{ for some } \alpha \in (0,1), \tag{1.3}$$

$$\mathbb{P}[\mu_e > \underline{c}] = 1 \text{ for some } \underline{c} \in (0,\infty). \tag{1.4}$$

Our main result is the following quenched non-Gaussian functional limit theorem.

Theorem 1.1. *Assume* (1.3), (1.4) *and fix* $d = 2$. *Let*

$$X_n(t) = n^{-1} X(t n^{2/\alpha} \log^{1-\frac{1}{\alpha}} n), \qquad t \in [0,\infty), n \in \mathbb{N}, \tag{1.5}$$

be the rescaled CSRW. Then there exists a constant $\mathscr{C} \in (0,\infty)$ *such that* $\mathbb{P}$*-a.s., under* $P_0^{\boldsymbol{\mu}}$*, the sequence of processes* X_n *converges as* $n \to \infty$ *in law on the space* D^2 *equipped with the Skorokhod* J_1*-topology to a multiple of the two-dimensional fractional kinetics process* $\mathscr{C}\,\mathsf{FK}_\alpha$.

The limiting fractional kinetics process FK_α is defined as a time change of a Brownian motion by an inverse of a stable subordinator. More precisely, let BM be a standard two-dimensional Brownian motion started at 0, V_α an α-stable subordinator independent of BM determined by $\mathbb{E}[e^{-\lambda V_\alpha(t)}] = e^{-t\lambda^\alpha}$, and let V_α^{-1} be the right-continuous inverse of V_α. Then

$$\mathsf{FK}_\alpha(s) = \mathsf{BM}(V_\alpha^{-1}(s)), \quad s \in [0,\infty). \tag{1.6}$$

The quenched limit behaviour of the CSRW among unbounded conductances[1] on $\mathbb{Z}^d$ was for the first time investigated in [BD10]. It is proved there that, for all $d \geq 1$ and all distribution of the

[1] For results on CSRW among bounded conductances the reader is referred to [BČ10] and references therein.

conductances satisfying (1.4), the CSRW converges after the normalisation $n^{-1}X(n^2\cdot)$ to a multiple of the d-dimensional Brownian motion, $\sigma\,\mathsf{BM}_d$, $\mathbb{P}$-a.s. The constant σ might be 0, but [BD10] shows that it is positive iff μ_e has a finite $\mathbb{P}$-expectation.

When $\sigma = 0$, that is when $\mathbb{E}[\mu_e] = \infty$, the above scaling is not the right one and the Brownian motion might not be the right scaling limit. In the case when (1.3), (1.4) are satisfied and $d \geq 3$, the paper [BČ10] identifies the fractional kinetics process as the correct scaling limit; the normalisation is as in Theorem 1.1 without the logarithmic correction. The case $d \geq 3$ and $\alpha = 1$ is considered in [BZ10]. Here the Brownian motion is still the scaling limit, however with a normalisation different to [BD10]. Both [BČ10] and [BZ10] do not consider the case $d \leq 2$. Our Theorem 1.1 fills this gap for $d = 2$ and $\alpha \in (0,1)$.

The non-Gaussian limit behaviour of the CSRW is due to trapping that occurs on edges with large conductances: roughly said, the CSRW typically spends at x a time proportional to μ_x before leaving it for a long time. The heavy-tailed distributions of conductances makes the trapping important. The trapping mechanism is very similar to the one considered in the so-called trap models, see [BČ06] and the references therein. Actually, the scaling limit results of [BČ10] and of the present paper are analogous (including the normalisation) to the known scaling behaviour of the trap models [BČ07].

We would also like to point out that the dimension $d = 1$ is rather special for the CSRW (as well as for the trap models). It is not possible to prove any non-degenerated quenched limit theorem when (1.3) holds. The annealed scaling limit is a singular diffusion in a random environment which was defined by Fontes, Isopi and Newman in [FIN02]. As this claim has never appeared in the literature, we prove it in the appendix, adapting the techniques used for the trap models, [BČ05] or Section 3.2 of [BČ06].

The paper [BČ10] considers not only the CSRW but also another important process in a random environment, so-called Bouchaud's trap model (BTM) with the asymmetric dynamics. It shows that for $d \geq 3$ this model has the same scaling behaviour as the CSRW.

The BTM can be briefly defined as follows (for more motivation see [BČ10] again, and [BČ06]). Let $\boldsymbol{\tau} = (\tau_x : x \in \mathbb{Z}^2)$ be a collection of i.i.d. positive random variables on a probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$. Given $\boldsymbol{\tau}$ and $a \in [0,1]$, let $\tilde{P}_x^{\boldsymbol{\tau}}$ be the law of the continuous-time Markov chain $\tilde{X}$ with transition rates $w_{xy} = \tau_x^{a-1}\tau_y^a$ started at x. This process is naturally associated with the random walk among (not i.i.d.) random conductances given by $\tilde{\mu}_{xy} = \tau_x^a \tau_y^a$.

The methods of the present paper can be used with minimal modification (cf. Section 9 of [BČ10]) to show the following scaling limit statement for the two-dimensional BTM. The case $a = 0$ was treated already in [BČM06, BČ07].

Theorem 1.2. *Let $d = 2$, $a \in [0,1]$ and $\alpha \in (0,1)$. Assume that $\tilde{\mathbb{P}}[\tau_x \geq u] = u^{-\alpha}(1 + o(1))$ and that $\tau_x \geq \underline{c} > 0$ $\tilde{\mathbb{P}}$-a.s. Let*

$$\tilde{X}_n(t) = n^{-1}\tilde{X}(tn^{2/\alpha}\log^{1-\frac{1}{\alpha}} n), \qquad t \in [0,\infty), n \in \mathbb{N}, \tag{1.7}$$

be the rescaled BTM. Then there exists a constant $\tilde{\mathscr{C}} \in (0,\infty)$ such that $\tilde{\mathbb{P}}$-a.s., under $\tilde{P}_0^{\boldsymbol{\tau}}$, the sequence of processes $\tilde{X}_n$ converges as $n \to \infty$ in law on the space D^2 equipped with the Skorokhod J_1-topology to a multiple of the two-dimensional fractional kinetics process $\tilde{\mathscr{C}}\,\mathsf{FK}_\alpha$.

Remark 1.3. We emphasize that the topology used in Theorems 1.1 and 1.2 is the usual Skorokhod J_1-topology and not the uniform topology as in [BČ10]. Actually, as pointed out in [Mou10] (see also [Bil68, Chapter 18]), subtle measurability reasons prevent to define the distribution of the processes X_n and $\tilde{X}_n$ on the (non-separable) space D^d equipped with the uniform topology. It is therefore not possible to replace the J_1-topology by the stronger uniform one in our results. The results of [BČ10] should be corrected accordingly.

Let us now give more details on the proof of Theorem 1.1 and, in particular, on the new ingredients which do not appear in [BČ10]. As in [BČ10], we use the fact that the CSRW can be expressed as a time change of another process for which the usual functional limit theorem holds and which can be well controlled. This process, called *variable speed random walk* (VSRW), is a continuous-time Markov chain with transition rates μ_{xy}. We use $Y = (Y(t) : t \geq 0)$ and (with a slight abuse of notation) P^{μ}_x to denote this process and its law. The reversible measure of Y is the counting measure on $\mathbb{Z}^d$.

The time change is as follows. Let the *clock process* S be defined by

$$S(t) = \int_0^t \mu_{Y(t)}\,\mathrm{d}t, \qquad t \in [0,\infty). \tag{1.8}$$

Then, X can be constructed on the same probability space as Y, setting $X(t) = Y(S^{-1}(t))$. Since the behaviour of Y is known (see Proposition 2.1 below), to control the CSRW X we need to know the properties of the clock process S.

Proposition 1.4. *Let*

$$S_n(t) = n^{-2/\alpha}(\log n)^{\frac{1}{\alpha}-1} S(n^2 t), \qquad t \geq 0, n \in \mathbb{N}. \tag{1.9}$$

Then, under the assumptions of Theorem 1.1, there exists constant $\mathscr{C}_S \in (0,\infty)$ such that $\mathbb{P}$-a.s., under P^{μ}_0, S_n converges as $n \to \infty$ to $\mathscr{C}_S V_\alpha$ weakly on the space D^1 equipped with the Skorokhod M_1-topology.

Theorem 1.1 follows from this proposition by the same reasoning as in [BČ10]: The asymptotic independence of the VSRW and the clock process can be proved as in [BČ10] Lemma 6.8. The convergence of the CSRW can be deduced from the joint convergence of the clock process and the VSRW as in Section 8 of [BČ10] or in Section 11 of [Mou10]. (The measurability problems mentioned in Remark 1.3 do not play substantial role here as can be seen by comparing the arguments of [Mou10] and [BČ10].) Therefore in this paper we concentrate on the proof of Proposition 1.4.

To show the convergence of the clock process, the paper [BČ10] uses substantially two properties of the Green function of the VSRW Y killed on exit from a set $A \subset \mathbb{Z}^d$,

$$g^{\mu}_A(x,y) = E^{\mu}_x\Big[\int_0^{\tau_A} \mathbf{1}\{Y(s)=y\}\,\mathrm{d}s\Big], \qquad x,y \in \mathbb{Z}^d, \tag{1.10}$$

where τ_A denotes the exit time of Y from A.

The first property concerns the off-diagonal Green function in balls $B(x,r)$ centred at x with radius r. It roughly states that as r diverges $g^{\mu}_{B(x,r)}(x,y)$ behaves (up to a constant factor) as the Green function of the simple random walk, for many centres x and for all y with distance at least εr to x and to the boundary of $B(x,r)$ (see Proposition 4.3 in [BČ10], cf. also Lemma 3.5 below). This is shown using a combination of the functional limit theorem for the VSRW and the elliptic Harnack inequality which were both proved in [BD10].

In $d=2$ we need finer estimates. We need to consider y with distance of order r^{ξ}, $\xi \in (0,1)$, from x. We will show in Lemma 3.6 that for such y the function $g^{\mu}_{B(x,r)}(x,y)$ also behaves as in the simple random walk case, at least if $x=0$. The reasoning based on the functional limit theorem and the Harnack inequality does not apply here, since $g^{\mu}_{B(x,r)}(x,y)$ is not harmonic at $y=x$. It turns out, however, that by 'patching' together $g^{\mu}_{B(x,r)}$ for many different r's one can control the Green function up to distance r^{ξ} to x (Lemma 3.6).

The second property of the Green function needed in [BČ10] concerns its diagonal behaviour. In rough terms again, we used the fact that for $d \geq 3$ the Green function in balls converges to the infinite volume Green function, $g^{\mu}_{B(x,r)}(x,x) \xrightarrow{r\to\infty} g^{\mu}(x,x)$, that the random quantity $g^{\mu}(x,x)$ has a distribution independent of x, and that $g^{\mu}(x,x)$ and $g^{\mu}(y,y)$ are essentially independent when x and y are not too close.

Such reasoning is rather impossible when $d=2$. First, of course, the CSRW is recurrent and the infinite volume Green function does not exist. We should thus study the killed Green functions exclusively. We will first show that $\mathbb{P}$-a.s.

$$g^{\mu}_{B(x,r)}(x,x) = C_0 \log r(1+o(1)), \qquad \text{as } r\to\infty \text{ for } x=0, \tag{1.11}$$

with some *non-random* constant C_0, see Proposition 3.1. This is proved essentially by integrating the local limit theorem for the VSRW, which can be proved using the same techniques as the local limit theorem for the random walk on a percolation cluster [BH09], see [BD10] Theorem 5.14 .

The next important issue is to extend (1.11) from the origin, $x=0$, to many different centres x. While we believe that (1.11) holds true uniformly for x in $B(0,Kr)$, say, we were not able to show this. The main obstacle is the fact that the speed of the convergence in the local limit theorem is not known, and therefore we cannot extend the local limit theorem to hold uniformly for many different starting points. Note also that the method based on the integration of the functional limit theorem used in [BČ10] to get estimates for the off-diagonal Green function that are uniform over a large ball does not work. This is due to the fact that the principal contribution to the diagonal Green function in balls of radius r comes from visits that occur at (spatial) scales much smaller than r. These scales are not under control in the usual functional limit theorem.

The impossibility to extend (1.11) uniformly to $x \in B(0,Kn)$ appeared to be critical for the techniques of [BČ10]. It however turns out that we do not need to consider so many centres x in (1.11). Inspecting the proof of the convergence of the clock process in [BČ10] (see also [BČM06] for convergence of the two-dimensional trap model), we find out that it is sufficient to have (1.11) for $O(\log r)$ points x in $B(0,Kr)$ only. Moreover these points are typically at distance at least $r' = 2r/\log^2 r$ (Lemma 4.1). Since $g^{\mu}_{B(x,r)}(x,x)$ is well approximated by the Green function in smaller balls, $g^{\mu}_{B(x,r'/2)}(x,x)$ (Lemma 3.4), and the smaller balls are disjoint for the centres of interest, we recover enough independence to proceed similarly as in $d \geq 3$.

Finaly, we would like to draw reader's attention to the recent paper of J.-C. Mourrat [Mou10], where another very nice proof of the scaling limit for the asymmetric ($a \neq 0$) BTM is given for $d \geq 5$. The techniques used in [Mou10] differ considerably from those used here and in [BČ10]. They explore the fact that the clock process (1.8) is an additive functional of the environment viewed by the particle which is a stationary ergodic process under the annealed measure $\tilde{\mathbb{P}} \times \tilde{P}^{\tau}$. Moreover, the variance estimates of [Mou09] imply that this process is sufficiently mixing for $d \geq 5$. This allows to deduce that the clock process must be a subordinator under the annealed measure. An argument in the spirit of [BS02] is then applied to deduce the quenched result; this also requires $d \geq 4$ at least.

The present paper is organised as follows. In Section 2, we recall some known results on the VSRW. Section 3, in some sense the most important part of this paper, gives all necessary estimates on the killed Green function of the VSRW. In Section 4, we prove Theorem 1.1. Since this proof follows the lines of [BČ10], we decided not to give all details here. Instead of this, we will state a sequence of lemmas and propositions corresponding to the main steps of the proof of [BČ10]. The formulation of these lemmas is adapted to the two-dimensional situation. We provide proofs only in the cases when they substantially differ from [BČ10]. In the appendix we discuss the CSRW among the heavy-tailed conductances on the one-dimensional lattice $\mathbb{Z}$.

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2 Preliminaries

We begin by introducing some further notation. Let $B(x,R)$ be the Euclidean ball centred at x of radius R and let $Q(x,R)$ be a cube centred at x with side length R whose edges are parallel to the coordinate axes. Both balls and cubes can be understood either as subsets of $\mathbb{R}^d$, $\mathbb{Z}^d$ or of E^d (an edge is in $B(x,R)$ if both its vertices are), depending on the context. For $A \subset \mathbb{Z}^d$ we write $\partial A = \{y \notin A \exists x \in A, xy \in E^d\}$ and $\bar{A} = A \cup \partial A$. For $A,B \subset \mathbb{Z}^d$ we set $d(A,B) = \inf\{|x-y| : x \in A, y \in B\}$, where $|x-y|$ stands for the Euclidean distance of x and y. For a set $A \subset \mathbb{Z}^d$ we write $B(A,R) = \bigcup_{x\in A} B(x,R)$. We define the exit time of the VSRW Y from A as $\tau_A = \inf\{t \geq 0 : Y(t) \notin A\}$.

We use the convention that all large values appearing below are rounded above to the closest integer, if necessary. It allows us to write that, e.g., $\varepsilon n\mathbb{Z}^d \subset \mathbb{Z}^d$ for $\varepsilon \in (0,1)$ and n large. We use c, c' to denote arbitrary positive and finite constants whose values may change in the computations.

We recall some known facts about the VSRW and its transition density $q_t^{\mu}(x,y) = P_x^{\mu}[Y(t) = y]$, $x,y \in \mathbb{Z}^d$, $t \geq 0$, in $d = 2$.

Proposition 2.1. *Assuming* (1.4) *and* $d = 2$*, the following holds.*

(i) (Functional limit theorem) There exists $\mathscr{C}_Y \in (0,\infty)$ *such that* $\mathbb{P}$*-a.s., under* P_0^{μ}*, the sequence* $Y_n(\cdot) = n^{-1}Y(n^2\cdot)$ *converges as* $n \to \infty$ *in law on* D^2 *to a multiple of a standard two-dimensional Brownian motion,* $\mathscr{C}_Y$ BM.

(ii) (Heat-kernel estimates) There exist a family of random variables $(V_x, x \in \mathbb{Z}^2)$ *on* Ω *and constants* $c_1, c_2 \in (0,\infty)$ *such that*

$$\mathbb{P}[V_x \geq u] \leq c_2 \exp\{-c_1 u^{\eta}\}, \qquad \eta = 1/3, \tag{2.1}$$

$$q_t^{\mu}(x,y) \leq 1 \wedge c_2 t^{-1} \qquad \text{for all } x,y \in \mathbb{Z}^2 \text{ and } t \geq 0, \tag{2.2}$$

$$q_t^{\mu}(x,y) \leq c_2 t^{-1} e^{-\frac{|x-y|^2}{c_2 t}}, \qquad \text{if } t \geq |x-y| \text{ and } |x-y| \vee t^{1/2} \geq V_x, \tag{2.3}$$

$$q_t^{\mu}(x,y) \leq c_2 e^{-c_1|x-y|(1\vee \log\frac{|x-y|}{t})}, \qquad \text{if } t \leq |x-y| \text{ and } |x-y| \vee t^{1/2} \geq V_x, \tag{2.4}$$

$$q_t^{\mu}(x,y) \geq c_1 t^{-1} e^{-\frac{|x-y|^2}{c_1 t}} \qquad \text{if } t \geq V_x^2 \vee |x-y|^{1+\eta}. \tag{2.5}$$

(iii) (Local limit theorem) For all $x \in \mathbb{R}^2$ and $t > 0$ fixed, $\mathbb{P}$-a.s.

$$\lim_{n\to\infty} n^2 q^{\boldsymbol{\mu}}_{n^2 t}(0,[xn]) = \frac{1}{2\pi \mathscr{C}_Y^2 t} \exp\Big\{-\frac{|x|^2}{2t\mathscr{C}_Y^2}\Big\}, \tag{2.6}$$

where $[xn] \in \mathbb{Z}^2$ is the point with coordinates $\lfloor x_1 n\rfloor$, $\lfloor x_2 n\rfloor$.

Proof. Claims (i) and (ii) are parts of Theorems 1.1 and 1.2 of [BD10] specialised to $d = 2$. Claim (iii) is a simple consequence of Theorem 5.14 of [BD10], cf. also Theorems 1.1 and 4.6 of [BH09] where the local limit theorem is shown for the random walk on the super-critical percolation cluster. □

Remark 2.2. Proposition 2.1 is the only place in the proof of Theorem 1.1 where the assumption (1.4) is used explicitly. Hence, if Proposition 2.1 is proved with more general assumptions, then Theorem 1.1 will hold under the same assumptions.

3 Estimates on the Green function

This section contains all estimates on the Green function of the VSRW that we need in the sequel. These estimates might be of independent interest.

3.1 Diagonal estimates

We control the diagonal Green function at the origin first.

Proposition 3.1. *Let $C_0 = (\pi\mathscr{C}_Y^2)^{-1}$, with $\mathscr{C}_Y$ as in Proposition 2.1(i). Then, for $\mathbb{P}$-a.e. environment $\boldsymbol{\mu}$,*

$$\lim_{r\to\infty} \frac{g^{\boldsymbol{\mu}}_{B(0,r)}(0,0)}{C_0 \log r} = 1. \tag{3.1}$$

Proof. We use the local limit theorem (2.6) and the heat-kernel estimates to prove this claim. With $B = B(0,r)$, we write

$$\begin{aligned} g^{\boldsymbol{\mu}}_B(0,0) &= E^{\boldsymbol{\mu}}_0\Big[\int_0^{\tau_B} \mathbf{1}_{Y(t)=0}\,\mathrm{d}t\Big] = E^{\boldsymbol{\mu}}_0\Big[\int_0^{r^2} \mathbf{1}_{Y(t)=0}\,\mathrm{d}t\Big] \\ &\quad + E^{\boldsymbol{\mu}}_0\Big[\int_{r^2}^{\tau_B} \mathbf{1}_{Y(t)=0}\,\mathrm{d}t; \tau_B > r^2\Big] - E^{\boldsymbol{\mu}}_0\Big[\int_{\tau_B}^{r^2} \mathbf{1}_{Y(t)=0}\,\mathrm{d}t; \tau_B < r^2\Big] \\ &=: I_1 + I_2 - I_3. \end{aligned} \tag{3.2}$$

The dominant contribution comes from the term I_1. By the local limit theorem, for every ε there exists $t_0 = t_0(\boldsymbol{\mu},\varepsilon)$ such that for all $t > t_0$

$$q^{\boldsymbol{\mu}}_t(0,0) \in (2\pi\mathscr{C}_Y^2 t)^{-1}(1-\varepsilon, 1+\varepsilon). \tag{3.3}$$

Therefore, for r large enough,

$$\begin{aligned} I_1 = \int_0^{r^2} q_t^{\mu}(0,0) \le \int_0^{t_0} q_t^{\mu}(0,0)\,\mathrm{d}t + (1+\varepsilon)\int_{t_0}^{r^2} (2\pi \mathscr{C}_Y^2 t)^{-1}\mathrm{d}t \\ \le t_0 + (1+\varepsilon)(\pi\mathscr{C}_Y^2)^{-1}\log r. \end{aligned} \tag{3.4}$$

A lower bound on I_1 is obtained analogically, yielding $\lim_{r\to\infty} I_1/\log r = C_0$.
Using the strong Markov property at τ_B and the symmetry of $q_t(\cdot,\cdot)$, it is possible to estimate I_3,

$$I_3 \le \sup_{y\in\partial B} E_y^{\mu}\Big[\int_0^{r^2} \mathbf{1}_{Y(t)=0}\,\mathrm{d}t\Big] = \sup_{y\in\partial B}\int_0^{r^2} q_t^{\mu}(0,y)\,\mathrm{d}t. \tag{3.5}$$

By splitting the last integral on V_0^2 and on $|y| = r(1+o(1))$, using the estimates (2.3) and (2.4), it follows that, as $r\to\infty$,

$$I_3 \le V_0^2 + \int_{V_0^2}^{V_0^2\vee|y|} c\exp(-c|y|)\mathrm{d}t + \int_{V_0^2\vee|y|}^{r^2} ct^{-1}e^{-c|y|^2/t}\mathrm{d}t \le V_0^2 + c. \tag{3.6}$$

To bound I_2 we need the following lemma.

Lemma 3.2. *Let $B = B(0,r)$ and let $q_t^{\mu,B}(x,y) = P_x^{\mu}[Y(t) = y, t < \tau_B]$ be the transition density of the VSRW killed on exiting B. Then there exists $c < \infty$ such that $\mathbb{P}$-a.s. for all $r \in \mathbb{N}$ large enough, all $t \ge r^2$ and $x, y \in B$*

$$q_t^{\mu,B}(x,y) \le \frac{c}{r^2}\exp\Big\{-\frac{t}{cr^2}\Big\}. \tag{3.7}$$

and, in consequence, $\mathbb{P}_x^{\mu}[\tau_B \ge t] \le ce^{-t/cr^2}$.

Proof. An easy consequence of (2.1) is the existence of $C < \infty$ such that (see Lemma 3.3 of [BČ10])

$$\sup_{x\in B(0,r)} V_x \le C\log^{1/\eta} r, \qquad \mathbb{P}\text{-a.s. for all large } r. \tag{3.8}$$

Using the heat-kernel lower bound (2.5) together with (3.8), we obtain for all large r

$$\sup_{x\in B}\sum_{y\in B} q_{r^2}^{\mu}(x,y) = \sup_{x\in B}\Big(1 - \sum_{y\notin B} q_{r^2}^{\mu}(x,y)\Big) \le c < 1. \tag{3.9}$$

Writing $t = kr^2 + s$ for $k\in\mathbb{N}$ and $s\in[0,r^2)$, and using (2.2) together with $q^{\mu,B} \le q^{\mu}$,

$$\begin{aligned} q_t^{\mu,B}(x,y) &= \sum_{z_1,\dots,z_k\in B} q_{r^2}^{\mu,B}(x,z_1)q_{r^2}^{\mu,B}(z_1,z_2)\dots q_{r^2}^{\mu,B}(z_{k-1},z_k)q_s^{\mu,B}(z_k,y) \\ &\le \sum_{z_1,\dots,z_k\in B} q_{r^2}^{\mu}(x,z_1)q_{r^2}^{\mu}(z_1,z_2)\dots q_{r^2}^{\mu}(z_{k-2},z_{k-1})c_2 r^{-2}. \end{aligned} \tag{3.10}$$

Summing over $z_{k-1}, z_{k-2}, \dots z_1$, using (3.9), this yields $q_t^{\mu,B}(x,y) \le c_2 r^{-2}c^{k-1}$, which is equivalent to the right-hand side of (3.7). The second claim of the lemma follows by summing (3.7) over $y\in B$. □

It is now easy to finish the proof of Proposition 3.1. By the previous lemma,

$$I_2 = \int_{r^2}^{\infty} q_t^{\mu,B}(0,0)\,\mathrm{d}t \le \int_{r^2}^{\infty} cr^{-2}e^{-t/cr^2}\mathrm{d}t = O(1). \tag{3.11}$$

Therefore, I_2 and I_3 are $o(\log r)$, and the proof is completed. □

We need rougher estimates on the diagonal Green function, uniform in a large ball.

Lemma 3.3. *There exist $c_1, c_2 \in (0,\infty)$ such that for every $\varepsilon \in (0,1)$, $K > 1$, $\mathbb{P}$-a.s.,*

$$\begin{aligned} c_1 &\le \liminf_{n\to\infty} \inf\{g^{\mu}_{B(x,r)}(x,x)/\log r : x \in B(0,Kn), r \in (\varepsilon n, Kn)\} \\ &\le \limsup_{n\to\infty} \sup\{g^{\mu}_{B(x,r)}(x,x)/\log r : x \in B(0,Kn), r \in (\varepsilon n, Kn)\} \le c_2. \end{aligned} \tag{3.12}$$

Proof. This can be proved exactly by the same argument as Proposition 3.1, replacing the local limit theorem used in (3.4) by the heat-kernel upper and lower bounds from Proposition 2.1(ii), using again the fact (3.8) to control the random variables V_x. □

3.2 Approximation of the diagonal Green function

As discussed in the introduction, to recover some independence required to show Theorem 1.1, we need to approximate the diagonal Green function in large sets by smaller ones.

Lemma 3.4. *Let $k \ge 1$, $K \ge 1$ and $r = n/\log^k n$. Then, $\mathbb{P}$-a.s. as $n \to \infty$, uniformly for all $x \in B(0,Kn)$ and all $A \subset \mathbb{Z}^2$ such that $B(x,r) \subset A \subset B(0,Kn)$,*

$$g^{\mu}_A(x,x) = g^{\mu}_{B(x,r)}(x,x) + O(\log\log n). \tag{3.13}$$

Proof. By the monotonicity of g^{μ}_A in A and the strong Markov property

$$\begin{aligned} g^{\mu}_A(x,x) - g^{\mu}_{B(x,r)}(x,x) &\le g^{\mu}_{B(0,Kn)}(x,x) - g^{\mu}_{B(x,r)}(x,x) \\ &= E^{\mu}_x\Big[\int_{\tau_{B(x,r)}}^{\tau_{B(0,Kn)}} \mathbf{1}_{Y(t)=x}\mathrm{d}t\Big] \le \sup_{y\in\partial B(x,r)} E^{\mu}_y\Big[\int_0^{\tau_{B(0,Kn)}} \mathbf{1}_{Y(t)=x}\mathrm{d}t\Big] \\ &\le \sup_{y\in\partial B(x,r)} \int_0^{n^2\log n} q^{\mu}_t(y,x)\mathrm{d}t + g^{\mu}_{B(0,Kn)}(x,x)P^{\mu}_y[\tau_{B(0,Kn)} \ge n^2\log n], \end{aligned} \tag{3.14}$$

where for the last term we used the fact that

$$E^{\mu}_y\Big[\int_{n^2\log n}^{\tau_{B(0,Kn)}} \mathbf{1}_{Y(t)=x}\mathrm{d}t; \tau_{B(0,Kn)} \ge n^2\log n\Big] \le g^{\mu}_{B(0,Kn)}(x,x)P^{\mu}_y[\tau_{B(0,Kn)} \ge n^2\log n]. \tag{3.15}$$

By Lemmas 3.2 and 3.3, the second term on the right-hand side of (3.14) is $O(n^{-c}\log n) = o(1)$. The first term can be controlled using the heat-kernel estimates again: Observing that, by (3.8), $|x-y| = r \gg \sup\{V_x : x \in B(0,Kn)\}$, we have from (2.3), (2.4)

$$\int_0^{n^2\log n} q^{\mu}_t(y,x)\mathrm{d}t \le \int_0^{|x-y|} ce^{-c|x-y|}\mathrm{d}t + \int_{|x-y|}^{n^2\log n} \frac{c}{u}e^{-|x-y|^2/ct}\mathrm{d}t. \tag{3.16}$$

The first term is clearly $o(1)$. After the substitution $t/|x-y|^2 = u$, the second term is smaller than $\int_0^{(\log n)^{1+2k}} \frac{c}{u}e^{-c/u}\mathrm{d}u = O(\log\log n)$. □

3.3 Off-diagonal estimates

Next, we need off-diagonal estimates on $g^{\boldsymbol{\mu}}_{B(x,r)}(x,y)$. The following lemma provides them for y not too close to x and the boundary of the ball (cf. Proposition 4.3 of [BČ10]).

Lemma 3.5. *Let $K > 1$, $0 < 3\varepsilon_o < \varepsilon_g < K/3$, $\delta > 0$ and $r \in [\varepsilon_g, \varepsilon_g + \varepsilon_o/2)$. Then, $\mathbb{P}$-a.s. for all but finitely many n, for all points $x \in B(0,(K-\varepsilon_g)n)$, and $y \in B(x,(\varepsilon_g - \varepsilon_o)n) \setminus B(x, \varepsilon_o n)$,*

$$1-\delta \le \frac{g^{\boldsymbol{\mu}}_{B(x,rn)}(x,y)}{C_0\big(\log(rn) - \log|x-y|\big)} \le 1+\delta. \tag{3.17}$$

Proof. This lemma can be proved in the same way as Proposition 4.3 of [BČ10], using a suitable integration of the functional limit theorem and the elliptic Harnack inequality. For $d=2$, one should only replace the formula for the Green function $g^{\star}_{B(x,r)}(x,y)$ of the d-dimensional Brownian motion $\mathscr{C}_Y \,\mathsf{BM}_d$ killed on exiting $B(x,r)$ ((4.5) in [BČ10]) with its two-dimensional analogue $g^{\star}_{B(x,r)}(x,y) = C_0(\log r - \log|x-y|)$. Remark also that the condition $\varepsilon_g < 1/2$ appearing in the statement of Proposition 4.3 of [BČ10] is not necessary for the proof, so we omitted it here. □

The previous lemma does not give any estimate on the Green function near to the centre of the ball. When $x = 0$, we can improve it.

Lemma 3.6. *Let $\xi \in (0,1)$ and $\delta > 0$. Then $\mathbb{P}$-a.s. for all but finitely many n, for all $y \in B(0,n/2) \setminus B(0,n^{\xi})$*

$$1-\delta \le \frac{g^{\boldsymbol{\mu}}_{B(0,n)}(0,y)}{C_0\big(\log n - \log|y|\big)} \le 1+\delta. \tag{3.18}$$

Proof. We prove this lemma by patching together the estimate (3.17) (with $r = \varepsilon_g = 1$) on several different scales. Fix $\varepsilon_o < 1$ such that $-\log\varepsilon_o \ge \delta^{-1}$. Due to the previous lemma, we can assume that $|y| \le \varepsilon_o n$, implying that the denominator of (3.18) is larger than $C_0\delta^{-1}$.

Given $\boldsymbol{\mu}$, we choose $n_0 = n_0(\boldsymbol{\mu})$ such that (3.17) holds for all $n > n_0^{\xi}/2$ and we consider $n > n_0$. Let k be the largest integer such that $y \in B(0, 2^{-k}n)$, hence

$$(1-\delta)\frac{\log n - \log|y|}{\log 2} \le k = \Big\lfloor \frac{\log n - \log|y|}{\log 2} \Big\rfloor \le \frac{\log n - \log|y|}{\log 2}. \tag{3.19}$$

Let $r_i = 2^{-i}n$, $i = 0,\dots,k$. By our choice of n, $r_i \ge n_0^{\xi}/2$ for all $i \le k$. We can thus apply Lemma 3.5: For all $z \in B(0,(1-\varepsilon_o)r_i) \setminus B(0,\varepsilon_o r_i)$

$$\Big| g^{\boldsymbol{\mu}}_{B(0,r_i)}(0,z) - C_0\big(\log r_i - \log|z|\big)\Big| \le \delta C_0\big(\log r_i - \log|z|\big). \tag{3.20}$$

By standard properties of the Green functions, the function $h_i(z) = g^{\boldsymbol{\mu}}_{B(0,2r_i)}(0,z) - g^{\boldsymbol{\mu}}_{B(0,r_i)}(0,z)$ is harmonic for the VSRW in $B(0,r_i)$. On $\partial B(0,r_i)$, $g^{\boldsymbol{\mu}}_{B(0,r_i)} \equiv 0$, and, using (3.20),

$$g^{\boldsymbol{\mu}}_{B(0,2r_i)}(0,z) \in C_0\big(\log 2 + O(r_i^{-1})\big)(1-\delta, 1+\delta) \qquad \text{for all } z \in \partial B(0,r_i). \tag{3.21}$$

Therefore, by the maximum principle, $h_i(z) \in C_0(\log 2 + O(r_i^{-1}))(1-\delta, 1+\delta)$ for all $z \in B(0, r_i)$. Iterating this estimate, we obtain

$$g^{\mu}_{B(0,n)}(0,y) - g^{\mu}_{B(0,r_k)}(0,|y|) \in kC_0(\log 2 + O(|y|^{-1}))(1-\delta, 1+\delta), \tag{3.22}$$

and thus, using (3.19),

$$\left| g^{\mu}_{B(0,n)}(0,y) - C_0(\log n - \log|y|) \right| \le g_{B(0,r_k)}(0,y) + 2\delta C_0 k\big(\log 2 + O(|y|^{-1})\big). \tag{3.23}$$

Using $g_{B(0,r_k)}(0,y) \le C_0$ (by (3.20)) for the first term, and (3.19) for the second term on the right-hand side of (3.23), we deduce the lemma. □

For $x \neq 0$ we have the following upper estimate.

Lemma 3.7. *For every $K > 1$, $\xi \in (0,1)$, $\varepsilon_o \in (0,1)$, and $r \in (3\varepsilon_o n, Kn)$ there exists $C > 0$ such that $\mathbb{P}$-a.s for all but finitely many $n \in \mathbb{N}$, $x \in B(0,Kn)$ and $y \in B(x, r - \varepsilon_o n) \setminus B(x, n^{\xi})$*

$$g^{\mu}_{B(x,r)}(x,y) \le C\big(\log(r) - \log|x-y|\big). \tag{3.24}$$

Proof. Due to Lemma 3.5, we should consider only y with $|x-y| \le \varepsilon_o r$. As before,

$$g^{\mu}_{B(x,r)}(x,y) \le \int_0^{r^2} q^{\mu}_t(x,y)\,\mathrm{d}t + \int_{r^2}^{\infty} q^{\mu,B(x,r)}_t(x,y)\,\mathrm{d}t. \tag{3.25}$$

By Lemma 3.2, the second integral is $O(1)$. For the first integral, applying the heat-kernel upper bounds, using $\sup\{V_x : x \in B(0,Kn)\} \ll (rn)^{\xi} < |x-y|$ by (3.8), we get

$$\int_0^{r^2} q^{\mu}_t(x,y)\,\mathrm{d}t \le \int_0^{|x-y|} c' e^{-c|x-y|}\mathrm{d}t + \int_{|x-y|}^{r^2} c' t^{-1} e^{-\frac{|x-y|^2}{ct}}\,\mathrm{d}t. \tag{3.26}$$

The first integral is $o(1)$. By an easy asymptotic analysis, the second integral behaves like $c' \log \frac{cr^2}{|x-y|^2} + O(1) \le C(\log r - \log|x-y|)$ for $|x-y| \le \varepsilon_o r$. □

4 Proof of the main theorem

We now have all estimates required to prove Theorem 1.1. As we have already remarked, it is sufficient to show Proposition 1.4 only. Theorem 1.1 follows from it as in [BČ10]. Moreover, since the proof of Proposition 1.4 mostly follows the lines of [BČ10], we focus on the difficulties appearing for $d = 2$ and we explain the modifications needed to resolve them.

The proof explores the fact that the stable subordinator V_α at time T is well approximated by the sum of a large but finite number of its largest jumps before T. These jumps of the limiting process corresponds in S_n to visits of the VSRW Y to sites with $\mu_x \sim n^{2/\alpha} \log^{-1/\alpha} n$.

To understand this scale heuristically, observe that a fixed time T for S_n corresponds to the time Tn^2 for Y, see (1.9). At this time Y typically visits $N \sim n^2/\log n$ different sites, similarly to the two-dimensional simple random walk. The maximum of N independent variables with the same distribution as μ_e is then of order $N^{1/\alpha} \sim n^{2/\alpha} \log^{-1/\alpha} n$.

We thus define (cf. (6.1) and (6.3) of [BČ10])

$$\begin{aligned} E_n(u,w) &= \{e \in E^2 : \mu_e \in [u,w)n^{2/\alpha}\log^{-1/\alpha} n\}, \\ T_n(u,w) &= \{x \in \mathbb{Z}^d : x \in E_n(u,w), x \notin E_n(w,\infty)\}. \end{aligned} \tag{4.1}$$

Unlike in [BČ10], it is not necessary to define the 'bad' edges (cf. (6.2) of [BČ10]). This is the consequence of the next lemma that shows that the edges of the set $E_n(u,\infty)$ are well separated in $d=2$. This lemma might appear technical, but it is crucial for the applied technique. It implies the independence of $g^{\mu}_{B(x,n/\log^2 n)}(x,x)$ for $x \in T_n(u,\infty)$ not sharing the same edge.

Lemma 4.1. *Let $K>0$, $u>0$. Define $\mathbb{B}_n = B(0,Kn)$. Then there exists a positive constant ι such that $\mathbb{P}$-a.s. for all $n \in \mathbb{N}$ large*

$$\min\{\mathrm{dist}(e,f) : e,f \in E_n(u,\infty)\cap \mathbb{B}_n\} \ge 2n/\log^2 n. \tag{4.2}$$

$$\sup\{\mu_e : e \notin E_n(u,\infty)\cap\mathbb{B}_n, e \textit{ has vertex in } T_n(u,\infty)\} \le n^{-\iota} n^{2/\alpha}\log^{-1/\alpha} n. \tag{4.3}$$

$$B(0,n/\log^2 n)\cap E_n(u,\infty) = \emptyset. \tag{4.4}$$

Proof. Observe first that for $k\ge 2$ and $2^{k-1}\le n \le 2^k$, $E_n(u,\infty)\subset E_{2^k}(2^{-2/\alpha}u,\infty)$. To prove (4.2) it is thus sufficient to show that for all $u'>0$, $\mathbb{P}$-a.s. for all $k\in\mathbb{N}$ large,

$$\min\{\mathrm{dist}(e,f) : e,f \in E_{2^k}(u',\infty)\cap\mathbb{B}_{2^k}\} \ge 2^{k+1}/(k\log 2)^2. \tag{4.5}$$

The probability of the complement of this event is bounded from above by

$$\sum_{e\in\mathbb{B}_{2^k}} \sum_{f\in B(e,2^{k+1}/(k\log 2)^2)} \mathbb{P}[e,f\in E_{2^k}(u',\infty)] \le c(u')k^{-2}, \tag{4.6}$$

where we used the definition of E_n and (1.3) for the last inequality. Borel-Cantelli lemma then implies (4.2). Claims (4.3), (4.4) are proved similarly. □

We now investigate the rescaled clock process S_n. As in [BČ10], we fix ε_s small and treat separately the contributions of vertices from $T_n(0,\varepsilon_s)$, $T_n(\varepsilon_s,\varepsilon_s^{-1})$ and $T_n(\varepsilon_s^{-1},\infty)$ to this process. We first show that the contribution of the visits to the set $T_n(0,\varepsilon_s)$ can be neglected, cf. Proposition 5.1 of [BČ10].

Proposition 4.2. *For every $\delta>0$ there exists ε_s such that for all $K\ge 1$ and $\mathbb{B}_n = B(0,Kn)$, $\mathbb{P}$-a.s. for all but finitely many n,*

$$P_0^{\mu}\Big[K^{-2}n^{-2/\alpha}\log^{\frac{1}{\alpha}-1} n \int_0^{\tau_{\mathbb{B}_n}} \mu_{Y(t)}\mathbf{1}\{Y(t)\in T_n(0,\varepsilon_s)\}\,\mathrm{d}t \ge \delta\Big] \le \delta. \tag{4.7}$$

Proof. The proof resembles to the proof of Proposition 5.1 of [BČ10], but there are some differences caused by the recurrence of the CSRW in $d=2$. We will show that $\mathbb{P}$-a.s. for all large n,

$$E_0^{\mu}\int_0^{\tau_{\mathbb{B}_n}} \mu_{Y(t)}\mathbf{1}\{Y(t)\in T_n(0,\varepsilon_s)\}\,\mathrm{d}t \le K^2 n^{2/\alpha}\log^{1-\frac{1}{\alpha}} n\,\delta^2. \tag{4.8}$$

The proposition then follows by using the Markov inequality. We set $i_{\max} = \min\{i : 2^{-i}\varepsilon_s n^{2/\alpha}\log^{-\frac{1}{\alpha}} n \le \underline{c}\} = O(\log n)$, and

$$H_n(i) = \{e \in \mathbb{B}_n : \mu_e \in \varepsilon_s n^{2/\alpha}\log^{-\frac{1}{\alpha}} n(2^{-i}, 2^{-i+1}]\}. \tag{4.9}$$

Observe that $E_n(0,\varepsilon_s) \subset \cup_{i=1}^{i_{\max}} H_n(i)$. For an edge $e = \{x,y\}$, we set $g_n^{\mu}(e) = g_{\mathbb{B}_n}^{\mu}(0,x) + g_{\mathbb{B}_n}^{\mu}(0,y)$, and for some fixed small ξ we define the deterministic function $\bar{g}_n : E^2 \to [0,\infty)$ as

$$\bar{g}_n(e) = \begin{cases} 4C_0 \log n, & \text{if } \operatorname{dist}(0,e) \le n^{\xi}, \\ 4C_0\{\log n - \log(\operatorname{dist}(0,e))\}, & \text{otherwise.} \end{cases} \tag{4.10}$$

Observe that by Proposition 3.1 and Lemma 3.6, $\mathbb{P}$-a.s. for all n large, $g_n^{\mu}(e) \le \bar{g}_n(e)$ for all $e \in \mathbb{B}_n$. Using this notation, we have $\mathbb{P}$-a.s. for all n large

$$\begin{aligned} E_0^{\mu}\int_0^{\tau_{\mathbb{B}_n}} \mu_{Y(t)} \mathbf{1}\{Y(t) \in T_n(0,\varepsilon_s)\}\,\mathrm{d}t &\le \sum_{e \in E_n(0,\varepsilon_s)} \mu_e g_n^{\mu}(e) \\ &\le \sum_{i=1}^{i_{\max}} \sum_{e \in H_n(i)} 2^{-i+1}\varepsilon_s n^{2/\alpha}\log^{-\frac{1}{\alpha}} n\,\bar{g}_n(e). \end{aligned} \tag{4.11}$$

Setting $p_{n,i} = \mathbb{P}[\mu_e \in H_n(i)] \le C\varepsilon_s^{-\alpha}2^{i\alpha}n^{-2}\log n$, we have for a $C > 0$ and $\lambda > 0$

$$\begin{aligned} &\mathbb{P}\Big[2^{-i+1}\varepsilon_s n^{2/\alpha}\log^{-\frac{1}{\alpha}} n \sum_{e \in H_n(i)} \bar{g}_n(e) \ge C\varepsilon_s^{1-\alpha}2^{i(\alpha-1)}K^2 n^{2/\alpha}\log^{1-\frac{1}{\alpha}} n\Big] \\ &\qquad \le e^{-\lambda C\varepsilon_s^{-\alpha}K^2\log n 2^{i\alpha}} \prod_{e \in \mathbb{B}_n}(1 + p_{n,i}(e^{\lambda\bar{g}_n(e)} - 1)). \end{aligned} \tag{4.12}$$

The logarithm of the last product is bounded by $p_{n,i}\sum_{e \in \mathbb{B}_n} e^{\lambda\bar{g}_n(e)}$. To bound this expression we use the fact that for a small enough λ there is a constant c independent of n such that

$$\sum_{e \in \mathbb{B}_n} e^{\lambda\bar{g}_n(e)} \le cK^2n^2, \tag{4.13}$$

which can be proved as Lemma A.2 of [BČM06]. Therefore, (4.12) is bounded from above by

$$e^{-\lambda C\varepsilon_s^{-\alpha}K^2\log n 2^{i\alpha}} e^{c\varepsilon_s^{-\alpha}2^{i\alpha}n^{-2}\log n K^2n^2} = e^{-\varepsilon_s^{-\alpha}K^2 2^{i\alpha}\log n(\lambda C - c)}. \tag{4.14}$$

Taking C large, this is summable (even after multiplication by $\log n$). The Borel-Cantelli lemma then implies that the complement of the event on the left-hand side of (4.12) holds for all $i \le i_{\max}$, $\mathbb{P}$-a.s. for all large n. Therefore, the left-hand side of (4.8) is bounded by $\sum_{i=1}^{i_{\max}} C\varepsilon^{1-\alpha}2^{i(\alpha-1)}K^2n^{2/\alpha}\log^{1-\frac{1}{\alpha}} n$, which is smaller than the right-hand side, if ε is small enough. □

To treat the dominant contribution of $T_n(\varepsilon_s, \varepsilon_s^{-1})$, we apply the same coarse-graining construction as in [BČ10]. In this construction we observe the VSRW before the exit from the large ball $\mathbb{B}_n$. The VSRW spends a time of order K^2n^2 in this ball and visits only finitely many pairs of sites from $T_n(\varepsilon_s, \varepsilon_s^{-1})$. In every pair it spends a time of order $\log n$. This logarithmic factor explains the additional power of the logarithm appearing in the normalisation of S_n and not in the definition of $E_n(u,v)$.

We now start the construction. Let $v_n = n/\log^2 n$. For $e \in E^d$, $z \in \mathbb{Z}^d$ we set

$$\gamma_n(e) = C_{\text{eff}}[e, B(e, v_n)^c], \tag{4.15}$$

$$\gamma_n(z) = C_{\text{eff}}[z, B(z, v_n+1)^c] = \big(g^{\mu}_{B(z,v_n+1)}(z,z)\big)^{-1}, \tag{4.16}$$

where C_{eff} denotes the effective conductance between two sets, see e.g. [BČ10] (3.8) for the usual definition. We have the following analogue of Lemma 6.2 of [BČ10].

Lemma 4.3. *(i) For all e and n, $\gamma_n(e)$ is independent of μ_e.*
(ii) For every $\varepsilon_c > 0$, $\mathbb{P}$-a.s. for all large n, for all $e \in E_n(u,v) \cap \mathbb{B}_n$ and $z \in e$,

$$(1+\varepsilon_c)\gamma_n(z) \geq \gamma_n(e) \geq \gamma_n(z). \tag{4.17}$$

(iii) For every $e \in E^2$, $C_0 \gamma_n(e) \log n \xrightarrow{n\to\infty} 1$ in $\mathbb{P}$-probability.

Proof. Claims (i), (ii) are proved as in [BČ10]. Claim (iii) follows using the identity $C_{\text{eff}}(z, B(z, A^c)) = g^{\mu}_A(z,z)^{-1}$ valid for any $A \subset \mathbb{Z}^2$, Proposition 3.1, the translation invariance of $\mathbb{P}$, and claim (ii). □

We now split the sets $E_n(u,v)$ according to the value of $\gamma_n(e)$. To this end we choose a sequence h_n so that $h_n \searrow 0$, $b_n := \mathbb{P}[|C_0\gamma_n(e)\log n - 1| > h_n] \searrow 0$, and $b_n \log n \gg \log^{1/2} n$. This is possible by Lemma 4.3(iii). We define

$$\begin{aligned} E_n^0(u,v) &= \{e \in E_n(u,v), |C_0\gamma_n(e)\log n - 1| \leq h_n\}, \\ E_n^1(u,v) &= E_n(u,v) \setminus E_n^0(u,v). \end{aligned} \tag{4.18}$$

We also set $T_n^0(u,v) = \{z \in T_n(u,v), z \in E_n^0(u,v)\}$ and $T_n^1(u,v) = T_n(u,v) \setminus T_n^0(u,v)$. Similarly to Lemma 6.3 of [BČ10] (cf. also (6.41) there), these sets are homogeneously spread over $\mathbb{B}_n$.

Lemma 4.4. *Let $0 < u < v$, $\delta, \varepsilon_b > 0$ be fixed and set $p_n(u,v) = n^{-2}\log n(u^{-\alpha} - v^{-\alpha})$. Then, $\mathbb{P}$-a.s. for all but finitely many n, for all $x \in \varepsilon_b n\mathbb{Z}^2 \cap \mathbb{B}_n$, and all $i \in \{0, \dots, 2\log_2 \log n\}$*

$$|Q(x, \varepsilon_b n) \cap E_n^0(u,v)| \in 2n^2\varepsilon_b^2 p_n(u,v)(1-\delta, 1+\delta), \tag{4.19}$$

$$|Q(x, \varepsilon_b n) \cap E_n^1(u,v)| \leq c b_n n^2 \varepsilon_b^2 p_n(u,v), \tag{4.20}$$

$$|Q(x, 2^{-i}n) \cap E_n(u,v)| \leq c(\log^{1/2} n \vee 2^{-2i} n^2 p_n(u,v)). \tag{4.21}$$

Proof. The proof is a concentration argument for binomial random variables. However, since $p_n(u,v)n^2 = O(\log n)$, we need to work with subsequences in order to apply the Borel-Cantelli lemma.

As the first step, we disregard $\gamma_n(e)$ and show (4.19) for E_n instead of E_n^0:

$$|Q(x, \varepsilon_b n) \cap E_n(u,v)| \in 2n^2\varepsilon_b^2 p_n(u,v)(1-\delta, 1+\delta). \tag{4.22}$$

We start by proving the upper bound. For $n \in \mathbb{N}$ we define $k = k(n) \in \mathbb{N}$ and $s = s(n) \in [1,2)$ by $n = s2^k$. We set $s_i = (1+\frac{\delta}{20})^i$, $i_{\max} = \inf\{i : s_i > 2\}$, $i(n) = \sup\{i : s_i \leq s(n)\}$. It is not difficult to see that

$$E_n(u,v) \cap Q(x, \varepsilon_b n) \subset E_{2^k}(u s_{i(n)-1}^{2/\alpha}, v s_{i(n)+2}^{2/\alpha}) \cap Q(y, \varepsilon_b s_{i(n)+1} 2^k) \tag{4.23}$$

for some $y = y(x,n) \in \frac{1}{20}\delta\varepsilon_b 2^k \mathbb{Z}^2$. Moreover, by definition of $p_n(u,v)$ and (1.3),

$$p_{2^k}(us_{i(n)-1}^{2/\alpha}, vs_{i(n)+1}^{2/\alpha})\varepsilon_b^2 2^{2k} s_{i(n)+1}^2 \le (1+\tfrac{\delta}{2})p_n(u,v)\varepsilon_b^2 n^2. \tag{4.24}$$

Hence, to prove the upper bound of (4.22), it is sufficient to show that $\mathbb{P}$-a.s. for all k large, for all $i_1, i_2, i_3 \in \{0,\dots,i_{\max}\}$, and for all $y \in \frac{1}{20}\delta\varepsilon_b 2^k\mathbb{Z}^2 \cap \mathbb{B}_{2^k}$,

$$\left|E_{2^k}(us_{i_1}^{2/\alpha}, vs_{i_2}^{2/\alpha}) \cap Q(y, \varepsilon_b s_{i_3} 2^k)\right| \le (1+\tfrac{\delta}{2})p_{2^k}(us_{i_1}^{2/\alpha}, vs_{i_2}^{2/\alpha})\varepsilon_b^2 s_{i_3}^2 2^{2k}. \tag{4.25}$$

The number of y's in consideration and $i_{\max}$ are finite. The probability of the complement of (4.25) for given i_1, i_2, i_3, can be bounded using the exponential Chebyshev inequality, using the independence of μ_e's, by $\exp\{-c2^{2k}p_{2^k}(\cdot,\cdot)\} \sim \exp(-ck)$. As this is summable, the upper bound follows. The proof of the lower bound in (4.22) is analogous.

The proof of (4.21) is very similar to the proof of the upper bound of (4.22). In the upper bound on the probability of the complementary event, it is in addition necessary to sum over $0 \le i \le c\log\log 2^k$ and consider $O(\log^2 2^k)$ possible values for y. On the other hand, the term $\log^{1/2} n$ on the right-hand side of (4.21) assures that the Chebyshev inequality gives at least a factor $\exp\{-c\sqrt{\log 2^k}\}$, which assures the summability.

It remains to show (4.20), since, as $b_n \to 0$, (4.19) follows from (4.20) and (4.22). From the proofs of Lemma 4.1 and of (4.22), we know that (4.22) and (4.2) hold out of events whose probabilities are summable along the subsequence $2^k, k \in \mathbb{N}$. Moreover, if (4.2) holds then $(\gamma_n(e) : e \in E_n(u,v) \cap \mathbb{B}_n)$ are independent since $\gamma_n(e)$ depends only on the environment restricted to $B(e,\nu_n)$. We can now use once more the concentration for binomial random variables as before. The fact that $b_n \log n \gg \log^{1/2} n$ and thus $\exp\{-cb_{2^k}p_{2^k}(u,v)2^{2k}\} \ll \exp\{-c\sqrt{k}\}$ assures the summability again. □

As in [BČ10] Lemma 6.4, $\mathbb{P}$-a.s. for all but finitely many n we can define a family of approximate balls $\mathscr{B}_n(x,r) \subset \mathbb{Z}^2$ with the following properties: For all $x \in \mathbb{B}_n$ and $r \in (0, Kn)$

(i) $\mathfrak{B}_n(x,r)$ is simply connected in $\mathbb{Z}^2$.

(ii) $B(x,r) \subset \mathfrak{B}_n(x,r) \subset B(x, r+3\nu_n)$.

(iii) $\partial\mathfrak{B}_n(x,r) \cap \bigcup_{e\in E_n(\varepsilon_s,\infty)} B(e,\nu_n) = \emptyset$.

The existence of these sets follows easily from Lemma 4.1.

We now adapt the definition of the coarse graining from [BČ10]. We use the sets $\mathscr{B}_n(\cdot, \varepsilon_g n)$ to cut the trajectory of Y to several parts whose contribution to S_n we treat separately: Let $\varepsilon_g > 0$, $t_n(0) = 0$, $y_n(0) = 0$ and for $i \ge 1$ let

$$\begin{aligned} t_n(i) &= \inf\{t > t_n(i-1) : Y(t) \notin \mathfrak{B}_n(y_n(i-1), \varepsilon_g n)\}, \\ y_n(i) &= Y(t_n(i)). \end{aligned} \tag{4.26}$$

We define

$$s_n^0(i;u,v) = n^{-2/\alpha}(\log n)^{\frac{1}{\alpha}-1}\int_{t_n(i)}^{t_n(i+1)} \mu_{Y(t)}\mathbf{1}\{Y(t) \in T_n^0(u,v)\}\mathrm{d}t; \tag{4.27}$$

this is the increment of the (normalised) clock process between times $t_n(i)$ and $t_n(i+1)$ caused by sites in $T_n^0(u,v)$.

The behaviour of the sequence $t_n(i)$ is the same as in [BČ10], Lemma 6.5. The distribution of s_n^0 is characterised by the next proposition, cf. Proposition 6.7 of [BČ10].

Proposition 4.5. *Let T, $\varepsilon_s, \varepsilon_g > 0$. Define $s_n^0(i) = s_n^0(i, \varepsilon_s, \varepsilon_s^{-1})$ and let $\mathscr{F}_i^n$ be the σ-algebra generated by $(Y_s, s \le t_n(i))$. Then, there is a constant $\kappa > 0$ independent of $\varepsilon_s, \varepsilon_g$ and T, such that $\mathbb{P}$-a.s. for all n large enough, for all $i \le \varepsilon_g^{-2}T$, and for all intervals $A = (0, a]$ with $a \in (0, \infty)$*

$$\left|P_0^{\mu}[s_n^0(i) = 0|\mathscr{F}_i^n] - (1 - c_{\varepsilon_s}\varepsilon_g^2)\right| \le \kappa c_{\varepsilon_s}^2 \varepsilon_g^4, \tag{4.28}$$

$$\left|P_0^{\mu}[s_n^0(i) \in A|\mathscr{F}_i^n] - \varepsilon_g^2 \nu_{\varepsilon_s}(A)\right| \le \kappa \nu_{\varepsilon_s}(A) c_{\varepsilon_s} \varepsilon_g^4, \tag{4.29}$$

where $c_{\varepsilon_s} = \pi(\varepsilon_s^{-\alpha} - \varepsilon_s^{\alpha})$ and ν_{ε_s} is the measure on $(0, \infty)$ given by

$$\nu_{\varepsilon_s}(\mathrm{d}x) = \int_{\varepsilon_s}^{\varepsilon_s^{-1}} \frac{\pi}{2C_0 u} \exp\left\{ - \frac{x}{2C_0 u}\right\} \alpha u^{-\alpha-1} \mathrm{d}u \mathrm{d}x. \tag{4.30}$$

Remark 4.6. There is an important difference between Proposition 4.5 and Proposition 6.7 of [BČ10]. In [BČ10], it is claimed that the sequence $s_n(i)$ converges to an i.i.d. sequence $s_\infty(i)$ for every fixed ε_g. However, [BČ10] does not show this claim, but only equivalents of our statements (4.28), (4.29).

The i.i.d. sequence $s_\infty(i)$ is used in [BČ10] to prove Lemma 6.8. Its proof can be however modified easily to use (4.28), (4.29) only. We use this occasion to present the correct proof below, see Lemma 4.8

Proof of Proposition 4.5. The proof of this proposition is very similar to the proof of Proposition 6.7 in [BČ10]. One shows that with a probability $1 - c_{\varepsilon_s}\varepsilon_g^2 + O(c_{\varepsilon_s}^2 \varepsilon_g^4)$ none of the sites from $T_n^0(\varepsilon_s, \varepsilon_s^{-1})$ is visited between $t_n(i)$ and $t_n(i+1)$. Otherwise, with probability $c_{\varepsilon_s}\varepsilon_g^2 + O(c_{\varepsilon_s}^2 \varepsilon_g^4)$, Y visits exactly two sites from this set sharing a common edge. More than two sites from $T_n^0(\varepsilon_s, \varepsilon_s^{-1})$ are visited with probability $O(c_{\varepsilon_s}^2 \varepsilon_g^4)$.

If Y visits a site from $T_n^0(\varepsilon_s, \varepsilon_s^{-1})$, it spends there an asymptotically exponentially distributed time, as stated in the next lemma. Its proof is exactly the same as the proof of Lemma 6.6 in [BČ10].

Lemma 4.7. *Let $z \in \mathbb{B}_n^0 = B(0, (K - \varepsilon_g)n)$, $e = xy \in E_n(\varepsilon_s, \varepsilon_s^{-1}) \cap \mathfrak{B}_n(z, \varepsilon_g n)$ be such that $\mu_e = u n^{2/\alpha}$ and $\gamma_n(e)\log n = v$. Then, $\mathbb{P}$-a.s., the distribution of*

$$n^{-2/\alpha}(\log n)^{\frac{1}{\alpha}-1} \int_0^{\tau_{\mathfrak{B}_n(z,\varepsilon_g n)}} \mathbf{1}\{Y(t) \in \{x, y\}\} \mu_{Y(t)} \mathrm{d}t \tag{4.31}$$

under P_x^{μ} and P_y^{μ} converges as $n \to \infty$ to the exponential distribution with mean $2u/v$.

With this lemma at disposition, we need to estimate the probability that a site $x \in T_n^0(\varepsilon_s, \varepsilon_s^{-1})$ is visited between $t_n(i)$ and $t_n(i+1)$. This probability can be written using the Green functions as

$$\frac{g_{\mathscr{B}_n(y_n(i), \varepsilon_g n)}(y_n(i), x)}{g_{\mathscr{B}_n(y_n(i), \varepsilon_g n)}(x, x)}. \tag{4.32}$$

The denominator (4.32) is $C_0 \log(\varepsilon_g n)(1 + o(1))$ by the definition of E_n^0, using also the definition of the approximate balls $\mathscr{B}_n(y_n(i), \varepsilon_g n)$ and Lemma 3.4. The numerator can be estimated using Lemma 3.5 when $|x - y_n(i)| \ge \varepsilon_o n$. Such x's give the principal contribution. For the remaining x's one uses Lemma 3.7 as the upper bound.

The proof of Proposition 4.5 then continues exactly as the proof of Proposition 6.7 of [BČ10] by estimating the probability that $T_n(u,v)$, $\varepsilon_s \le u < v \le \varepsilon_s^{-1}$, is visited between $t_n(i)$ and $t_n(i+1)$ by the sum of probabilities that $e \in E_n(u,v)$ are visited. Due to the homogeneity of $E_n(u,v)$ (Lemma 4.4), this summation can be replaced by an integration with respect to $2p_n(u,v)$ times Lebesgue measure. For the principal contribution coming from e's with $d(e, y_n(i)) \ge \varepsilon_o n$, this leads to the integral

$$\begin{aligned} &P_0^{\mu}[Y \text{ hits } T_n^0(u,v) \text{ between } t_n(i) \text{ and } t_n(i+1)] \\ &\quad\sim 2p_n(u,v) \int_{x\in\mathbb{R}^2:\varepsilon_o n \le |x| \le \varepsilon_g n} \frac{C_0\big(\log(\varepsilon_g n) - \log|x|\big)}{C_0 \log(\varepsilon_g n)}\,\mathrm{d}x \\ &\quad= \frac{p_n \pi \varepsilon_g^2 n^2}{\log n}(1+o(1)) + p_n(u,v) R_n(\varepsilon_o, \varepsilon_g) \\ &\quad= \pi(u^{-\alpha} - v^{-\alpha})\varepsilon_g^2 + p_n(u,v)R_n(\varepsilon_o,\varepsilon_g), \end{aligned} \tag{4.33}$$

where the error term $p_n(u,v)R_n(\varepsilon_o,\varepsilon_g)$ can be made arbitrarily small in comparison to the first term by sending $\varepsilon_o \to 0$ before ε_g. This explains the value of the constant c_{ε_s} in $d=2$. The measure ν_{ε_s} is obtained by combining the previous computation with Lemma 4.7. The technical details of these computations are analogous to [BČ10].

The error terms of (4.28), (4.29) which do not appear in [BČ10] (see Remark 4.6) are explained as follows. With one exception, all the errors in the proof of Proposition 6.7 of [BČ10] can be made arbitrarily small with respect to ε_g^2. The only exception is the error comming from the estimate on the probability that more than two edges from $E_n^0(\varepsilon_s, \varepsilon_s^{-1})$ are visited. This probability can be bounded by the right-hand side of (4.28). The error term in (4.29) then corresponds to the event that the first visited edge of $E_n^0(\varepsilon_s,\varepsilon_s^{-1})$ gives (after the normalisation) a contribution belonging to $A=(0,a]$, and then another edge from $E_n^0(\varepsilon_s,\varepsilon_s^{-1})$ is visited. □

The consequence of Proposition 4.5, is the following lemma replacing Lemma 6.8 of [BČ10]. Since, as explained in Remark 4.6, its proof in [BČ10] uses the incorrectly stated Proposition 6.7, we present the corrected proof here.

Lemma 4.8. *Let $T, \varepsilon_g, \varepsilon_s > 0$, $\ell \in \mathbb{N}$, $\lambda_1,\dots,\lambda_\ell > 0$, $\xi_1,\dots,\xi_\ell \in \mathbb{R}^2$, and $0 \le i_1 < \dots < i_\ell \le T\varepsilon_g^{-2}$. Define $r_n(i) = n^{-1}(y_n(i+1) - y_n(i))$. Then $\mathbb{P}$-a.s. for all large n*

$$\begin{aligned} &E_0^{\mu}\Big[\exp\Big\{-\sum_{j=1}^{\ell}\big[\lambda_j s_n^0(i_j) + \xi_j\cdot r_n(i_j)\big]\Big\}\Big] \\ &\quad= \prod_{j=1}^{\ell}\Big[1+\varepsilon_g^2\Big(\frac{|\xi_j|^2}{4} - c_{\varepsilon_s} + G(\lambda_j)\Big)\Big] + \mathscr{R}, \end{aligned} \tag{4.34}$$

where $G(\lambda) = G^{\varepsilon_s}(\lambda) = \int_0^\infty e^{-\lambda x}\nu_{\varepsilon_s}(\mathrm{d}x)$, and the reminder term $\mathscr{R}$ satisfies for all ε_g small enough $|\mathscr{R}| \le \varepsilon_g^3(1+c_{\varepsilon_s}^2)\sum_{j=1}^{\ell} c(\xi_j,\lambda_j)$.

Proof. We first control the joint Laplace transform of the pair $(s_n^0(i), r_n(i))$, $i \le \varepsilon_g^{-2}T$, given $\mathscr{F}_i^n = \sigma(Y_s, s \le t_n(i))$. The calculation very similar to (6.50)–(6.51) of [BČ10], with the conditional expectations replacing the usual ones, gives

$$\begin{aligned} &E_0^{\boldsymbol{\mu}}\big[e^{-\lambda s_n^0(i)-\xi\cdot r_n(i)}\big|\mathscr{F}_i^n\big] \\ &= E_0^{\boldsymbol{\mu}}\big[e^{-\xi\cdot r_n(i)}\mathbf{1}\{s_n^0(i)=0\}\big|\mathscr{F}_i^n\big] + E_0^{\boldsymbol{\mu}}\big[e^{-\lambda s_n^0(i)-\xi\cdot r_n(i)}\mathbf{1}\{s_n^0(i)\neq 0\}\big|\mathscr{F}_i^n\big] \\ &= E_0^{\boldsymbol{\mu}}\big[e^{-\xi\cdot r_n(i)}\mathbf{1}\{s_n^0(i)=0\}\big|\mathscr{F}_i^n\big] + E_0^{\boldsymbol{\mu}}\big[e^{-\lambda s_n^0(i)}\mathbf{1}\{s_n^0(i)\neq 0\}\big|\mathscr{F}_i^n\big](1+O_\xi(\varepsilon_g)), \end{aligned} \tag{4.35}$$

where $O_\xi(\varepsilon_g)$ denotes an error whose absolute value is bounded by $c(\xi)\varepsilon_g$ for all n large and ε_g small; this estimate comes from $e^{-2\varepsilon_g|\xi|}\le e^{-\xi\cdot r_n(i)}\le e^{2\varepsilon_g|\xi|}$. The first term on the right-hand side of (4.35) can be rewritten using Proposition 4.5 and the fact that, by the functional central limit theorem, the distribution of $r_n(i)$ converges to the uniform distribution on the sphere with radius ε_g,

$$\begin{aligned} &E_0^{\boldsymbol{\mu}}\big[e^{-\xi\cdot r_n(i)}\big|\mathscr{F}_i^n\big] - E_0^{\boldsymbol{\mu}}\big[e^{-\xi\cdot r_n(i)}\mathbf{1}\{s_n^0(i)\neq 0\}\big|\mathscr{F}_i^n\big] \\ &= 1+\frac{\varepsilon_g^2|\xi|^2}{4}+O_\xi(\varepsilon_g^4) - P_0^{\boldsymbol{\mu}}[s_n^0(i)\neq 0|\mathscr{F}_i^n](1+O_\xi(\varepsilon_g)) \\ &= 1+\frac{\varepsilon_g^2|\xi|^2}{4} - c_{\varepsilon_s}\varepsilon_g^2 + (1+c_{\varepsilon_s}^2)O_\xi(\varepsilon_g^4) + c_{\varepsilon_s}O_\xi(\varepsilon_g^3). \end{aligned} \tag{4.36}$$

The second term on the right-hand side of (4.35) can be estimated using Proposition 4.5 again: Itegrating by parts, for all large n,

$$\begin{aligned} &E_0^{\boldsymbol{\mu}}\big[e^{-\lambda s_n^0(i)}\mathbf{1}\{s_n^0(i)\neq 0\}\big|\mathscr{F}_i^n\big] = \int_{(0,\infty)}\lambda e^{-\lambda x}P_0^{\boldsymbol{\mu}}[0<s_n^0(i)\le x]\mathrm{d}x \\ &= \varepsilon_g^2\int_{(0,\infty)}\lambda e^{-\lambda x}\nu_{\varepsilon_s}((0,x])\mathrm{d}x(1+c_{\varepsilon_s}O(\varepsilon_g^2)) = \varepsilon_g^2 G(\lambda)+c_{\varepsilon_s}O_\lambda(\varepsilon_g^4). \end{aligned} \tag{4.37}$$

Putting these two computation together we get for all n large enough

$$E_0^{\boldsymbol{\mu}}\big[e^{-\lambda s_n^0(i)-\xi\cdot r_n(i)}|\mathscr{F}_i^n\big] = 1+\varepsilon_g^2\Big(\frac{|\xi|^2}{4}-c_{\varepsilon_s}+G(\lambda)\Big)+\mathscr{R}'_{\xi,\lambda}, \tag{4.38}$$

where $\mathscr{R}'_{\xi,\lambda}$ is random but its absolute value is bouned by $c_{\xi,\lambda}(1+c_{\varepsilon_s}^2)\varepsilon_g^3$ for all ε_g small enough. Using this statement and the fact that $e^{-\lambda s_n^0(i)+\xi\cdot r_n(i)}\le C(\xi)<\infty$ for all possible values of $s_n^0(i)$ and $r_n(i)$, we can write

$$\begin{aligned} &E_0^{\boldsymbol{\mu}}\Big[e^{-\sum_{j=1}^{\ell}[\lambda_j s_n^0(i_j)+\xi_j\cdot r_n(i_j)]}\Big] \\ &= E_0^{\boldsymbol{\mu}}\Big[e^{-\sum_{j=1}^{\ell-1}[\lambda_j s_n^0(i_j)+\xi_j\cdot r_n(i_j)]}E_0^{\boldsymbol{\mu}}\Big[e^{-[\lambda_\ell s_n^0(i_\ell)+\xi_\ell\cdot r_n(i_\ell)]}|\mathscr{F}_{i_\ell}^n\Big]\Big] \\ &\le \Big[1+\varepsilon_g^2\Big(\frac{|\xi_\ell|^2}{4}-c_{\varepsilon_s}+G(\lambda_\ell)\Big)+c_{\xi_\ell,\lambda_\ell}(1+c_{\varepsilon_s}^2)\varepsilon_g^3\Big]E_0^{\boldsymbol{\mu}}\Big[e^{-\sum_{j=1}^{\ell-1}[\lambda_j s_n^0(i_j)+\xi_j\cdot r_n(i_j)]}\Big]. \end{aligned} \tag{4.39}$$

Iterating the last statement and taking the error terms out of the product (using again ε_g small enough) gives the upper bound for (4.34). The lower bound is obtained analogously. □

To finish the control of the behaviour of S_n we need to estimate the contribution of the sets $T_n(\varepsilon_s^{-1},\infty)$ and $T_n^1(\varepsilon_s,\varepsilon_s^{-1})$ (see below (4.18) for the notation). From the next lemma, which is proved in the same way as Lemmas 7.1, 7.2 of [BČ10], we see that their contribution is zero with a large probability.

Lemma 4.9. *For every $\delta, K > 0$ there exists $\varepsilon_s > 0$ such that, $\mathbb{P}$-a.s., for all but finitely many n,*

$$P_0^{\mu}\left[\sigma_{T_n^1(\varepsilon_s,\varepsilon_s^{-1})\cup T_n(\varepsilon_s^{-1},\infty)} < \tau_{\mathbb{B}}\right] \le \delta, \tag{4.40}$$

where σ_A denotes the hitting time of $A \subset \mathbb{Z}^2$, $\sigma_A = \inf\{t : Y(t) \in A\}$.

Propositions 4.2, 4.5 and Lemma 4.9 characterise the contributions of various sites in $\mathbb{B}_n$ to the clock process S_n. Using these results the proof of Proposition 1.4, and consequently of Theorem 1.1, can be completed as in Section 8 of [BČ10].

A The CSRW on the one-dimensional lattice

In this appendix, we study the CSRW among heavy-tailed random conductances on the one-dimensional lattice. We show that the scaling limit of this process is the singular diffusion in random environment. This diffusion, which is also the scaling limit of the one-dimensional trap model, see [FIN02, BČ05], is defined as follows.

Definition A.1 (Fontes-Isopi-Newman diffusion)**.** Let $(\bar{\Omega}, \bar{\mathbb{P}})$ be a probability space on which we define a standard one-dimensional Brownian motion BM and an inhomogeneous Poisson point process (x_i, v_i) on $\mathbb{R} \times (0,\infty)$ with intensity measure $\mathrm{d}x\, \alpha v^{-1-\alpha}\mathrm{d}v$. Let ρ be the random discrete measure $\rho = \sum_i v_i \delta_{x_i}$. Conditionally on ρ, we define the FIN-diffusion $(Z(s), s \ge 0)$ as a diffusion process (with $Z(0) = 0$) that can be expressed as a time change of BM with the speed measure ρ: Denoting by $\ell(t, y)$ the local time of BM, we set $\phi_\rho(t) = \int_{\mathbb{R}} \ell(t, y)\rho(\mathrm{d}y)$ and $Z(s) = \mathrm{BM}(\phi_\rho^{-1}(s))$.

The following theorem describes the scaling behaviour of the one-dimensional CSRW.

Theorem A.2. *Assume* (1.3), (1.4) *and set $\mathscr{C}_F = \mathbb{E}[\mu_e^{-1}]$,*

$$c_n = \inf\{t \ge 0 : \mathbb{P}(\mu_e > t) \le n^{-1}\} = n^{1/\alpha}(1 + o(1)), \qquad n \ge 1. \tag{A.1}$$

Then, as $n \to \infty$, under $\mathbb{P} \times P_0^{\mu}$, the process

$$X_n(t) := n^{-1} X\left(\mathscr{C}_F n c_n t\right) \tag{A.2}$$

converges in distribution to $Z(t)$.

Proof. The proof follows the lines of [BČ05] and Section 3.2 of [BČ06]. We will construct copies $\bar{X}_n$ of X_n, $n \ge 1$, on the same probability space $(\bar{\Omega}, \bar{\mathbb{P}})$ as the FIN diffusion. On this probability space we then show that $\bar{X}_n$ converges $\bar{\mathbb{P}}$-a.s. To this end we express $\bar{X}_n$ as a time-scale change of BM and show that the speed measures of $\bar{X}_n$ converge to ρ and the scale change is asymptotically negligible.

Let us introduce our notation for the time-scale change first. Consider a locally-finite deterministic discrete measure $\nu(\mathrm{d}x) = \sum_{i \in \mathbb{Z}} w_i \delta_{y_i}(\mathrm{d}x)$. The measure ν is referred to as the speed measure. Let S be a strictly increasing function defined on the set $\{y_i : i \in \mathbb{Z}\}$. We call such S the scaling function. Let us introduce slightly non-standard notation $S \circ \nu$ for the "scaled measure"

$$(S \circ \nu)(\mathrm{d}x) = \sum_{i \in \mathbb{Z}} w_i \delta_{S(y_i)}(\mathrm{d}x). \tag{A.3}$$

With $\ell(t,y)$ denoting the local time of the Brownian motion BM, we define $\phi_{\nu,S}(t) = \int_{\mathbb{R}} \ell(t,y)(S \circ \nu)(\mathrm{d}y)$. Then, the time-scale change of Brownian motion with the speed measure ν and the scale function S is a process $X_{\nu,S}$ defined by

$$X_{\nu,S}(t) = S^{-1}(W(\phi_{\nu,S}^{-1}(t))), \qquad t \geq 0. \tag{A.4}$$

If S is the identity function, we speak about the time change only. The following classical lemma [Sto63] describes the properties of $X_{\nu,S}$ if the set of atoms of ν has no accumulation point.

Lemma A.3. *If the sequence $(y_i, i \in \mathbb{Z})$ has no accumulation point and satisfies (without loss of generality) $y_i < y_j$ for $i < j$, then the process $X_{\nu,S}(t)$ is a continuous time Markov chain with state space $\{y_i\}$ and transition rates ω_{ij} from y_i to y_j given by: $\omega_{ij} = 0$ if $|i-j| \neq 1$,*

$$\begin{aligned} \omega_{i,i-1} &= (2w_i(S(y_i) - S(y_{i-1})))^{-1}. \\ \omega_{i,i+1} &= (2w_i(S(y_{i+1}) - S(y_i)))^{-1}. \end{aligned} \tag{A.5}$$

We can now construct the copies of X_n on the probability space $(\bar{\Omega}, \bar{\mathbb{P}})$. Let $G : [0,\infty) \mapsto [0,\infty)$ be given by

$$\bar{\mathbb{P}}(\rho((0,1]) > G(u)) = \mathbb{P}(\mu_e > u). \tag{A.6}$$

and set

$$\bar{\mu}^n_{x,x+1} = G^{-1}\big(n^{1/\alpha}\rho((x/n,(x+1)/n])\big), \qquad x \in \mathbb{Z}, n \in \mathbb{N}. \tag{A.7}$$

From the definition of the measure ρ, it follows that $\rho((0,1]) \overset{\text{law}}{=} t^{-1/\alpha}\rho((0,t])$. Therefore, for all n, the sequence $(\bar{\mu}^n_{x,x+1} : x \in \mathbb{Z})$ has the same distribution as $(\mu_{x,x+1} : x \in \mathbb{Z})$. For all n we define a measure ν_n on $\mathbb{R}$ and a piece-wise constant function $S_n : \mathbb{R} \mapsto \mathbb{R}$ by

$$\nu_n(\mathrm{d}u) = \frac{1}{2c_n} \sum_{x \in \mathbb{Z}} (\bar{\mu}^n_{x,x-1} + \bar{\mu}^n_{x,x+1})\delta_{x/n}(\mathrm{d}u). \tag{A.8}$$

$$S_n(u) = \begin{cases} n^{-1}\mathscr{C}_F^{-1}\sum_{y=0}^{x-1}(\bar{\mu}^n_{y,y+1})^{-1}, & \text{if } u \in [x/n,(x+1)/n),\ x \in \mathbb{Z}, x \geq 0, \\ n^{-1}\mathscr{C}_F^{-1}\sum_{y=x}^{-1}(\bar{\mu}^n_{y,y+1})^{-1}, & \text{if } u \in [x/n,(x+1)/n),\ x \in \mathbb{Z}, x < 0,. \end{cases} \tag{A.9}$$

We define processes $\bar{X}_n = X_{\nu_n,S_n}$. From Lemma A.3 it follows directly that $\bar{X}_n$ has the same distribution as X_n. The key step in the proof of Theorem A.2 is the following lemma

Lemma A.4. *$\bar{\mathbb{P}}$-a.s., as $n \to \infty$,*

$$S_n \to \mathrm{Id}, \qquad \nu_n \xrightarrow{v} \rho, \qquad S_n \circ \nu_n \xrightarrow{v} \rho, \tag{A.10}$$

where Id *denotes the identity map and $\xrightarrow{v}$ the vague convergence of measures.*

Proof. The first clam can be shown as in [BČ05]; it follows from the law of large numbers for triangular arrays and uses only fact that $\mathbb{E}[\mu_e^{-1}] < \infty$ which is a consequence of (1.4). The second claim is proved in [FIN02, BČ05] for slightly different measures, namely for $\tilde{\nu}_n = c_n^{-1}\sum_{x\in\mathbb{Z}} \bar{\mu}^n_{x,x+1}\delta_{x/n}$. However, this small difference does not play any role for the vague convergence. The third claim is a direct consequence of the first two. □

As a consequence of this lemma we obtain as in [BČ05, BČ06] the following, slightly stronger, convergence result.

Theorem A.5. *Let $T > 0$. Then, $\bar{\mathbb{P}}$-a.s.*

$$\bar{X}_n(t) \xrightarrow{n\to\infty} Z(t) \qquad \textit{uniformly on } [0,T]. \tag{A.11}$$

Theorem A.2 then directly follows from Theorem A.5. □

References

[BD10] M. T. Barlow and J.-D. Deuschel, *Invariance principle for the random conductance model with unbounded conductances*, Ann. Probab. **38** (2010), 234–276. MR 2599199

[BH09] M. T. Barlow and B. M. Hambly, *Parabolic Harnack inequality and local limit theorem for percolation clusters*, Electron. J. Probab. **14** (2009), 1–27. MR 2471657

[BČ10] M. T. Barlow and J. Černý, *Convergence to fractional kinetics for random walks associated with unbounded conductances*, To appear in Probab. Theory Related Fields, 2010.

[BZ10] M. T. Barlow and X. Zheng, *The random conductance model with Cauchy tails*, Ann. Appl. Probab. **20** (2010), 869–889. MR 2680551

[BČ05] G. Ben Arous and J. Černý, *Bouchaud's model exhibits two aging regimes in dimension one*, Ann. Appl. Probab. **15** (2005), 1161–1192. MR 2134101

[BČ06] G. Ben Arous and J. Černý, *Dynamics of trap models*, École d'Été de Physique des Houches, Session LXXXIII "Mathematical Statistical Physics", Elsevier, 2006, pp. 331–394. MR 2581889

[BČ07] G. Ben Arous and J. Černý, *Scaling limit for trap models on $\mathbb{Z}^d$*, Ann. Probab. **35** (2007), 2356–2384. MR 2353391

[BČM06] G. Ben Arous, J. Černý, and T. Mountford, *Aging in two-dimensional Bouchaud's model*, Probab. Theory Related Fields **134** (2006), 1–43. MR 2221784

[Bil68] P. Billingsley, *Convergence of probability measures*, John Wiley & Sons Inc., New York, 1968. MR 0233396

[BS02] E. Bolthausen and A.-S. Sznitman, *On the static and dynamic points of view for certain random walks in random environment*, Methods Appl. Anal. **9** (2002), 345–375. MR 2023130

[FIN02] L. R. G. Fontes, M. Isopi, and C. M. Newman, *Random walks with strongly inhomogeneous rates and singular diffusions: convergence, localization and aging in one dimension*, Ann. Probab. **30** (2002), 579–604. MR 1905852

[Mou09] J.-C. Mourrat, *Variance decay for functionals of the environment viewed by the particle*, arXiv:0902.0204, 2009, to appear in Ann. Inst. H. Poincaré Probab. Statist.

[Mou10] J.-C. Mourrat, *Scaling limit of the random walk among random traps on $\mathbb{Z}^d$*, arXiv:1001.2459, 2010, to appear in Ann. Inst. H. Poincaré Probab. Statist.

[Sto63] C. Stone, *Limit theorems for random walks, birth and death processes, and diffusion processes*, Illinois J. Math. **7** (1963), 638–660. MR 0158440