

## The contact process with fast voting\*

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### Abstract

Consider a combination of the contact process and the voter model in which deaths occur at rate 1 per site, and across each edge between nearest neighbors births occur at rate  $\lambda$  and voting events occur at rate  $\theta$ . We are interested in the asymptotics as  $\theta \rightarrow \infty$  of the critical value  $\lambda_c(\theta)$  for the existence of a nontrivial stationary distribution. In  $d \geq 3$ ,  $\lambda_c(\theta) \rightarrow 1/(2d\rho_d)$  where  $\rho_d$  is the probability a  $d$  dimensional simple random walk does not return to its starting point. In  $d = 2$ ,  $\lambda_c(\theta)/\log(\theta) \rightarrow 1/4\pi$ , while in  $d = 1$ ,  $\lambda_c(\theta)/\theta^{1/2}$  has  $\liminf \geq 1/\sqrt{2}$  and  $\limsup < \infty$ . The lower bound might be the right answer, but proving this, or even getting a reasonable upper bound, seems to be a difficult problem.

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## 1 Introduction

In this paper we consider a particle system on  $\mathbb{Z}^d$  that is a contact process plus a voter model run at a fast rate  $\theta$ . To define the process precisely, if  $n_i(x)$  is the number of nearest neighbors of  $x$  in state  $i$  then  $x$  changes from  $0 \rightarrow 1$  at rate  $(\lambda + \theta)n_1(x)$  and from  $1 \rightarrow 0$  at rate  $1 + \theta n_0(x)$ . The behavior of the contact process is essentially the same in all dimensions. There is a critical value  $\lambda_c$  (depending on dimension) of the infection parameter  $\lambda$  with the property that the system has a nontrivial stationary distribution for  $\lambda > \lambda_c$  and does not for smaller  $\lambda$ . The voter model, on the other hand, displays a strong dimension dependence. For  $d \leq 2$ , the only extremal invariant measures are the pointmasses on “all zeros” and “all ones”. If  $d \geq 3$ , however, the extremal invariant measures  $\nu_\rho^{\text{vm}}$  are indexed by the density  $\rho$  of ones. For precise statements and proofs, see Chapters V and VI of [11] and Part I of [13].

Now suppose, informally, that the two processes are superimposed on one another in  $d \geq 3$ , with the voter model being run at infinite rate. The effect is to make the distribution  $\mu_t$  at time  $t$  be an invariant measure for the voter model at all times  $t$ .

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Since the density of ones  $\rho(t) = \mu_t(1)$  is not changed by the voter part, the evolution of the density is determined by the contact part:

$$\frac{d}{dt}\rho(t) = 2d\lambda\mu_t(10) - \mu_t(1) = 2d\lambda\nu_{\rho(t)}^{\text{vm}}(10) - \rho(t). \tag{1.1}$$

Here  $\mu_t(1)$  and  $\mu_t(10)$  represent the probabilities that a site takes the value 1, and that two adjacent sites take the values 1,0 at time  $t$  respectively. In the voter equilibrium,  $\nu_{\rho}^{\text{vm}}(10) = \rho(1 - \rho)\rho_d$ , where  $\rho_d$  is the probability that a simple random walk on  $\mathbb{Z}^d$  does not return to the origin. (See page 242 of [11].) Therefore, in the ODE (1.1)

$$\lim_{t \rightarrow \infty} \rho(t) = \begin{cases} 1 - (2d\lambda\rho_d)^{-1} & \text{if } \lambda \geq 1/(2d\rho_d); \\ 0 & \text{if } \lambda \leq 1/(2d\rho_d). \end{cases} \tag{1.2}$$

This suggests that  $\lambda_c(\theta) \rightarrow 1/(2d\rho_d)$  as  $\theta \rightarrow \infty$ . The asymptotics for the critical value and the equilibrium density will be proved in Theorem 1.1 below.

In  $d \leq 2$  this heuristic does not work well, since running the voter model infinitely fast would drive the system to consensus. However, the fact that clustering occurs in the voter model means that an occupied site is usually completely surrounded by other occupied sites preventing births from occurring, and suggests that  $\lambda_c(\theta) \rightarrow \infty$ . Theorems 2–5 prove this and identify the order of magnitude of  $\lambda_c(\theta)$ .

If we let  $\theta = \epsilon^{-2}$  then our model becomes a voter model perturbation in the sense of Cox, Durrett, and Perkins [2]. From their results we get

**Theorem 1.1.** *In  $d \geq 3$ ,  $2d\rho_d\lambda_c(\theta) \rightarrow 1$ .*

*Proof.* The key is the fact proved in Theorem 1.2 (all the results we cite here are from [2]): if  $\theta \rightarrow \infty$ , space is scaled to  $\theta^{-1/2}\mathbb{Z}^d$ , and the initial profile converges to a continuous function  $v(x)$  (in a sense made precise in [2]), then the particle system at each time  $t$  converges (in the same sense) to  $u(t, x)$ , the solution of

$$\frac{\partial u}{\partial t} = \Delta u - u + 2d\lambda\rho_d u(1 - u).$$

If  $\beta = 2d\lambda\rho_d > 1$  results of Aronson and Weinberger (cited and used in a similar way in [2]) imply that Assumption 1 is satisfied, so we can apply Theorem 1.4 to conclude that there is a stationary distribution that concentrates on  $\Omega_{0,1}$ , the configurations with infinitely many 1’s and 0’s, and that in any such stationary distribution sites are occupied with probability close to  $(\beta - 1)/\beta$ , confirming the heuristic calculation in (1.2). Using  $-u + \beta u(1 - u) < (\beta - 1)u$ , we see that if  $\beta < 1$ , then for any initial condition the PDE converges exponentially fast to 0, which verifies Assumption 2, so Theorem 1.5 implies that there is no nontrivial stationary distribution.  $\square$

To prove results in dimensions  $d \leq 2$  we will use the fact that the contact plus voter model, written as  $\xi_t : \mathbb{Z}^d \rightarrow \{0, 1\}$ , is dual to a set-valued branching coalescing random walk,  $\eta_t$ , in which each particle dies at rate 1, jumps to each of its  $2d$  nearest neighbors at rate  $\theta$ , and gives birth onto each neighbor at rate  $\lambda$ . If the transitions in either of the last two possibilities produce two particles on one site, they immediately coalesce to one. Due to work of Harris [9] and Griffeath [8], this is often called the “additive” dual. The precise relationship between the two processes is that for any finite set  $A$

$$P(\xi_t(x) = 0 \text{ for all } x \in A) = P(\xi_0(y) = 0 \text{ for all } x \in \eta_t^A)$$

where  $\eta_t^A$  denotes the dual process starting from  $\eta_0^A = A$ .

To study the dual process, we will adapt arguments in Durrett and Zähle [5] to show that if we let  $\theta \rightarrow \infty$ , scale space to  $\theta^{-1/2}\mathbb{Z}^2$ , let  $\lambda/\log(\theta) \rightarrow \gamma$ , and remove particles that

quickly coalesce with their parents, then on any finite time interval the dual process converges to a branching Brownian motion in which the Brownian motions are run at rate 2, particles give birth at rate  $4\pi\gamma$  and die at rate 1. Once this is done, it is routine to use a block construction to show that if  $4\pi\gamma > 1$  the limiting Brownian motion dominates a supercritical  $M$ -dependent oriented percolation. A weak convergence argument then allows us to extend the comparison with oriented percolation to the dual process with large  $\theta$ , and results from Durrett's St. Flour Notes [4] give the following:

**Theorem 1.2.** *In  $d = 2$ ,*

$$\limsup_{\theta \rightarrow \infty} \lambda_c(\theta) / \log(\theta) \leq \frac{1}{4\pi}.$$

In the other direction we have

**Theorem 1.3.** *Let  $\epsilon = 1/(1 + 2\lambda + 4\theta)$ . If  $d = 2$ , the process dies out if*

$$E^{(1,0)}(1 - \epsilon)^\tau > 1 - \frac{1}{4\lambda},$$

where  $\tau$  is the hitting time of the origin for the lazy simple symmetric random walk on  $Z^2$ , that stays put with probability 1/2. From this it follows that

$$\liminf_{\theta \rightarrow \infty} \lambda_c(\theta) / \log(\theta) \geq \frac{1}{4\pi}.$$

To get the second conclusion from the first, integrate by parts (or use Fubini's theorem) to conclude that if  $F$  is the distribution function of a nonnegative random variable and  $\phi$  is its Laplace transform

$$\int_0^\infty e^{-\lambda x} (1 - F(x)) dx = \frac{1 - \phi(\lambda)}{\lambda}.$$

Using the well-known fact that

$$P^{(1,0)}(\tau > n) \sim \pi / \log n \tag{1.3}$$

(see page 167 of [17] or Lemma 3.1 in [3]) and applying a Tauberian theorem (e.g., Theorem 4 from Section XIII.5 of [6]), we see that

$$1 - \phi(\epsilon) \sim \pi / \log(1/\epsilon).$$

If  $\lambda \sim c \log(\theta)$ , then from the definition of  $\epsilon$  it follows that  $1/\epsilon \sim 4\theta$ , and the inequality will hold for large  $\theta$  if  $c < 1/4\pi$ .

Using techniques similar to the one for Theorem 1.3 we will show:

**Theorem 1.4.** *In  $d = 1$ , the contact process with fast voting dies out if  $\theta > 2\lambda(\lambda - 1)$  and hence*

$$\liminf_{\theta \rightarrow \infty} \lambda_c(\theta) / \theta^{1/2} \geq 1/\sqrt{2}.$$

We note that the Holley-Liggett technique ([10], [12]) can also be used to get upper bounds for  $\lambda_c(\theta)$  – see [14]. However, it gives a weaker bound on the asymptotic growth rate:  $\theta^{2/3}$  instead of  $\theta^{1/2}$ .

The proofs of Theorems 1.3 and 1.4 use similar ideas. Assuming that the process survives, we use duality to write down a harmonic function for the dual process. Of course, this function is not explicit. However, by using various inequalities, we are able to produce from it a function of  $d$  variables that we show is trivial under the assumptions of the two theorems.

To prove an upper bound on the critical value it is convenient to rescale time so that particles die at rate  $\delta = 1/\lambda$ , give birth on each nearest neighbor at rate 1 and jump across each edge at rate  $\nu = \theta/\lambda$ .

**Theorem 1.5.** *In  $d = 1$ ,  $\liminf_{\nu \rightarrow \infty} \nu \delta_c(\nu) > 0$  and hence*

$$\limsup_{\theta \rightarrow \infty} \lambda_c(\theta)/\theta^{1/2} < \infty.$$

To get the second conclusion from the first note that  $\delta \geq c/\nu$  translates into  $\lambda \leq (\theta/c)^{1/2}$ .

The key to the proof is a block construction to compare the process with  $\delta = 0$  with a supercritical one-dependent oriented percolation. To do this, we use an idea from Bramson and Griffeath [1]. We follow two tagged particles in the dual: a white particle that only follows random walk steps and a red particle that follows random walk steps and in addition branchings that take it to the right. The two particles may come together several times and follow the same trajectory for a while, but the drift in the red particle will eventually take it to the right and away from the white particle.

The existence of a comparison with oriented percolation for the process with  $\delta = 0$  immediately implies that  $\delta_c > 0$ . To get the result given in Theorem 1.5, we use the fact that our construction takes place in a space time box that is  $8L\nu$  wide (in space) and  $10L\nu$  high (in time) with  $L$  large, to conclude that the dual process survives with positive probability if  $10L\nu\delta$  is small. Since the proof of Theorem 1.5 is based on a block construction, the constant is very large. This problem was avoided in the proof of Theorem 1.2 by using a large number of particles in the block construction. This is possible because two dimensional Brownian motions do not hit points or each other. This strategy does not seem to be practical in  $d = 1$ .

While this paper has been in the review process, another possible approach to the proofs of Theorems 1.4 and 1.5 has emerged in the work of Newman, Ravishankar, and Schertzer [15]. They define a process on  $\mathcal{L} = \{(x, t) \in \mathbb{Z}^2 : x + t \text{ is even}\}$  that has (i) births (arrows  $(x, t) \rightarrow (x+1, t+1)$  and  $(x, t) \rightarrow (x-1, t+1)$ ) with probability  $b_\beta = e^{-\beta}b$  (ii) deaths (no arrows) with probability  $k_\beta = e^{-\beta}k$ , and (iii) one arrow  $(x, t) \rightarrow (x+1, t+1)$  or  $(x, t) \rightarrow (x-1, t+1)$  with probability  $(1/2)(1 - b_\beta - k_\beta)$  each. If one takes  $e^{-\beta} = \theta^{1/2}$ ,  $b = \lambda$  and  $k = 1$ , this is a discrete time version of our model. They show that as  $\beta \rightarrow \infty$  this process converges to the Brownian net with killing.

If one extends their convergence result to our continuous time system, which the authors of [15] are doing in a paper in preparation, and uses the fact that the limit system has been shown to have a percolation transition, then one can conclude that  $\lambda_c(\theta)/\theta^{1/2}$  converges to some positive finite limit. However, even after this is done our results remain relevant. Theorem 1.4 provides a good bound on the limit, which might be sharp, and the proof of Theorem 1.5 is much simpler than one based on [15].

The remainder of the paper is devoted to proofs. Theorem 1.4 is proved in Section 2 and Theorem 1.3 in Section 3. This follows the order in which the results were first proved, but the proof of Theorem 1.3 is somewhat simpler, so if the reader gets bogged down in the details of Section 2, he might have more success in understanding Section 3. Theorem 1.2 is proved in Section 4, and Theorem 1.5 in Section 5. Section 3 depends heavily on the ideas used in Section 2. However, these sections are otherwise independent. We would like to thank a referee for his careful reading of the original paper and a revision, which corrected a number of errors and made the paper much easier to read.

## 2 Extinction in one dimension

*Proof of Theorem 1.4.* Let  $\mu$  be the upper invariant measure for the process, i.e., the limit starting from all 1's. (This limit exists by monotonicity – see Theorem 2.3 of Chapter III of [11].) For finite  $A \subset \mathbb{Z}$ , let

$$\psi(A) = \mu\{\xi \equiv 0 \text{ on } A\}.$$

From the definition of  $\psi$  it is clear that

$$\begin{aligned}\psi(A) - \psi(A \cup B) &= \mu\{\xi \equiv 0 \text{ on } A, \xi \neq 0 \text{ on } B \setminus A\} \\ &\leq \mu\{\xi \equiv 0 \text{ on } A \cap B, \xi \neq 0 \text{ on } B \setminus A\} = \psi(A \cap B) - \psi(B).\end{aligned}$$

That is,  $\psi$  is *supermodular*:

$$\psi(A \cup B) + \psi(A \cap B) \geq \psi(A) + \psi(B).$$

It is a long known and frequently used fact that invariant measures for a Markov process translate into harmonic functions for the dual process. In the present case, the duality function is  $H(\xi, A) = 1_{\{\xi \equiv 0 \text{ on } A\}}$ . Early examples that show the usefulness of this observation are found in the analysis of the voter model (Section 1 of Chapter V of [11]) and symmetric exclusion process (Section 1 of Chapter VIII of [11]). See Corollary 1.3 of Chapter VIII for an explicit statement of this connection. The discussion there applies equally well to our process and its dual.

At the level of generators, the duality relation asserts that

$$(LH)(\xi, A) = \sum_B q(A, B)[H(\xi, B) - H(\xi, A)],$$

where  $L$  is the generator of the particle system acting on the first coordinate of  $H$ , and  $q(A, B)$  is the  $Q$ -matrix of the dual Markov chain. Integrating both sides with respect to  $\mu$  gives 0 on the left, since  $\mu$  is invariant, so it follows that

$$0 = \sum_B q(A, B)[\psi(B) - \psi(A)]. \tag{2.1}$$

So, this function  $\psi$  is harmonic for the dual chain. Using (2.1) and the shift invariance of  $\mu$ ,

$$\begin{aligned}(1 + 2\lambda)\psi(\{0\}) &= 1 + 2\lambda\psi(\{0, 1\}), \\ (1 + 2\theta + \lambda)\psi(\{0, 1\}) &= (1 + \theta)\psi(\{0\}) + \theta\psi(\{0, 2\}) + \lambda\psi(\{0, 1, 2\}),\end{aligned}$$

To check the second equation, note that while events happen to the pair  $\{0, 1\}$  at rate  $2 + 4\theta + 2\lambda$ , it suffices by symmetry to consider only those that affect the site on the right. Similarly, for  $n \geq 2$ ,

$$\begin{aligned}(1 + 2\theta + 2\lambda)\psi(\{0, n\}) &= \psi(\{0\}) + \theta\psi(\{0, n + 1\}) + \theta\psi(\{0, n - 1\}) \\ &\quad + \lambda\psi(\{0, n, n + 1\}) + \lambda\psi(\{0, n - 1, n\}).\end{aligned}$$

Let  $f(0) = \psi(\{0\})$  and  $f(n) = \psi(\{0, n\})$  for  $n \geq 1$ . Though it is not needed below, we note that  $f(n) \downarrow$  by Theorem 1.9 of Chapter VI of [11]. The proof given there for the contact process applies to our process as well. Changing notation, we have from above

$$(1 + 2\lambda)f(0) = 1 + 2\lambda f(1). \tag{2.2}$$

To simplify the other equations, we use the supermodularity equation

$$\psi(\{0, m, m + 1\}) \geq \psi(\{0, m\}) + \psi(\{m, m + 1\}) - \psi(\{m\})$$

to eliminate  $\psi(A)$  for  $|A| = 3$  in the above harmonicity equations. This gives

$$(1 + 2\theta - \lambda)f(1) \geq (1 + \theta - \lambda)f(0) + \theta f(2), \tag{2.3}$$

and for  $n \geq 2$ ,

$$(1 + 2\theta + \lambda)f(n) + (2\lambda - 1)f(0) \geq 2\lambda f(1) + (\theta + \lambda)f(n - 1) + \theta f(n + 1). \quad (2.4)$$

Now let  $g(n) = f(n) - f(n + 1)$  for  $n \geq 0$ . Using  $\sum_{k=0}^{n-1} g(k) = f(0) - f(n)$ , we see that

$$\theta g(n) \geq (\theta + \lambda)g(n - 1) - 2\lambda g(0) + \sum_{k=0}^{n-1} g(k)$$

for  $n \geq 1$ . In fact, this is (2.3) if  $n = 1$  and (2.4) if  $n \geq 2$ . Furthermore, (2.2) becomes  $g(0) = (1 - f(0))/(2\lambda)$ .

Define  $h(n)$  recursively by  $h(0) = 1, \theta h(1) = 1 + \theta - \lambda$ , and

$$\theta h(n) + 2\lambda = (\theta + \lambda)h(n - 1) + \sum_{k=0}^{n-1} h(k) \quad (2.5)$$

for  $n \geq 2$ . For the future, note that (2.5) also holds when  $n = 1$ . Then

$$g(n) \geq \frac{1 - f(0)}{2\lambda} h(n), \quad n \geq 0. \quad (2.6)$$

To see this, define  $g^*(0) = g(0)$ , and then  $g^*(n)$  by using the recursion satisfied by  $g(n)$  with inequalities replaced by equalities. By induction,  $g(n) \geq g^*(n)$ . Since the recursion is linear,  $g^*(n) = g(0)h(n)$ .

**Lemma 2.1.** *The solution to the recursion (2.5) for  $h$  can be given explicitly:  $h(n) = h^*(n)$ , where*

$$h^*(n) = 1 + \sum_{k=1}^n \sum_{j=0}^k \binom{n+k-j}{2k-j} \binom{k}{j} \frac{k-2j}{k} \frac{\lambda^j}{\theta^k}.$$

*Proof.* To check this, it suffices to show that  $h^*$  satisfies (2.5), together with the initial conditions. The initial conditions are immediate. For the recursion, it suffices to check that after replacing  $h(n)$  in (2.5) by the above expression for  $h^*(n)$ , the coefficient of  $\lambda^j/\theta^k$  for any  $j, k$  is the same on both sides of the resulting equation. To simplify matters, we will use the convention that  $\binom{0}{0} = 1$  and  $\binom{k}{j} = 0$  unless  $0 \leq j \leq k$ . In the computations below, we assume  $k \geq 1$ , leaving the case  $k = 0$  to the reader. The coefficient of  $\lambda^j/\theta^k$  is

$$\binom{n+k-j+1}{2k-j+2} \binom{k+1}{j} \frac{k-2j+1}{k+1} \quad (2.7)$$

for the expression on the left, and

$$\begin{aligned} & \binom{n+k-j}{2k-j+2} \binom{k+1}{j} \frac{k-2j+1}{k+1} + \binom{n+k-j}{2k-j+1} \binom{k}{j-1} \frac{k-2j+2}{k} \\ & + \binom{k}{j} \frac{k-2j}{k} \sum_{l=k}^{n-1} \binom{l+k-j}{2k-j} \end{aligned} \quad (2.8)$$

for the expression on the right. The sum on the right above can be expressed as

$$\sum_{l=k}^{n-1} \binom{l+k-j}{2k-j} = \binom{n+k-j}{2k-j} \frac{n-k}{2k-j+1}. \quad (2.9)$$

To see this, note that the two expressions above agree when  $n = k$ . The differences between the two sides at  $n + 1$  and  $n$  are

$$\binom{n+k-j}{2k-j} \quad \text{and} \quad \binom{n+k-j+1}{2k-j} \frac{n+1-k}{2k-j+1} - \binom{n+k-j}{2k-j} \frac{n-k}{2k-j+1} \quad (2.10)$$

respectively. Writing

$$\binom{n+k-j+1}{2k-j} = \binom{n+k-j}{2k-j} \frac{n+k-j+1}{n-k+1}, \tag{2.11}$$

we see that the two expressions in (2.9) agree. Now replace the sum in (2.8) with the right side of (2.9), and write each of the three summands in (2.8) as a multiple of

$$\frac{(n+k-j)!(k-1)!}{(2k-j+2)!(n-k-1)!j!(k-j+1)!}.$$

The three multiples are

$$k(k-2j+1)(n-k-1), (k-2j+2)(2k-j+2)j, \text{ and } (k-2j)(k-j+1)(2k-j+2)$$

respectively. Since these multiples add up to  $k(k-2j+1)(n+k-j+1)$ , we see that (2.7) and (2.8) agree.  $\square$

In order to determine the behavior of  $h(n)$  for large  $n$ , it is simplest to compute the generating function. To avoid convergence issues in changing of order of summation, one can consider the positive and negative terms separately, but the result is the same. To be precise, one would write  $(k-2j)/k = 1 - 2j/k$  and sum the resulting two series separately, adding the results at the end. Using the Taylor series

$$\sum_{m=0}^{\infty} \binom{m+l}{l} u^m = \frac{1}{(1-u)^{l+1}}, \quad |u| < 1,$$

$$\begin{aligned} \sum_{n=0}^{\infty} u^n h(n) &= 1 + \sum_{k=1}^{\infty} \sum_{j=0}^k \binom{k}{j} \frac{k-2j}{k} \frac{\lambda^j}{\theta^k} \sum_{n=k}^{\infty} \binom{n+k-j}{2k-j} u^n \\ &= 1 + \sum_{k=1}^{\infty} \sum_{j=0}^k \binom{k}{j} \frac{k-2j}{k} \frac{\lambda^j}{\theta^k} u^k (1-u)^{j-2k-1} \\ &= 1 + \frac{u(1-\lambda(1-u))}{\theta(1-u)^3} \sum_{k=1}^{\infty} \left[ \frac{u(1+\lambda(1-u))}{\theta(1-u)^2} \right]^{k-1}, \end{aligned}$$

where the final step comes from the binomial theorem, provided that

$$\left| \frac{u(1+\lambda(1-u))}{\theta(1-u)^2} \right| < 1.$$

Let  $0 < v < 1$  be the solution of  $v(1+\lambda(1-v)) = \theta(1-v)^2$ . (Note that the ratio of the left side to the right side increases from 0 to  $\infty$  as  $v$  goes from 0 to 1, so the solution exists and is unique.) As  $u \uparrow v$ , the final sum above tends to  $\infty$ . Therefore

$$\lim_{u \uparrow v} \sum_{n=0}^{\infty} u^n h(n) = \begin{cases} +\infty & \text{if } \lambda(1-v) < 1; \\ -\infty & \text{if } \lambda(1-v) > 1. \end{cases}$$

Solving the quadratic for  $v$  gives

$$v = \frac{1 + \lambda + 2\theta - \sqrt{(1 + \lambda)^2 + 4\theta}}{2(\lambda + \theta)},$$

and therefore

$$1 - \lambda(1-v) = \frac{2\theta + \lambda(3-\lambda) - \lambda\sqrt{(1+\lambda)^2 + 4\theta}}{2(\lambda + \theta)}.$$

It follows that  $\lambda(1 - v) < 1$  is equivalent to

$$2\theta + \lambda(3 - \lambda) > \lambda\sqrt{(1 + \lambda)^2 + 4\theta}.$$

For this to be true, the left side must be positive, and the inequality obtained by squaring both sides must hold, i.e.,

$$2\theta > \lambda(\lambda - 3) \quad \text{and} \quad \theta > 2\lambda(\lambda - 1).$$

Note that once  $\lambda > \frac{1}{3}$ , the second condition is more restrictive than the first.

To complete the proof, note that if  $\theta > 2\lambda(\lambda - 1)$ , then

$$\lim_{u \uparrow v} \sum_{n=0}^{\infty} u^n h(n) = +\infty.$$

Since  $0 < v < 1$ , this implies that

$$\limsup_{n \rightarrow \infty} h(n) = +\infty,$$

Since  $g(n)$  is bounded, this together with (2.6) implies  $f(0) = 1$ , so the process started with all ones dies out locally, and hence no nontrivial equilibrium distribution exists.  $\square$

*Alternative proof.* For readers who are less adept at computations with Binomial coefficients, we now sketch another approach. We want to solve (2.5),

$$\theta h(n) + 2\lambda = (\theta + \lambda)h(n - 1) + \sum_{k=0}^{n-1} h(k) \quad \text{for } n \geq 1,$$

with  $h(0) = 1$ . If we let  $j(n) = \sum_{k=0}^{n-1} h(k)$  then we have the system that can be written as

$$\begin{pmatrix} h(n) \\ j(n) \end{pmatrix} = \begin{pmatrix} 1 + (\lambda + 1)/\theta & 1/\theta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} h(n - 1) \\ j(n - 1) \end{pmatrix} + \begin{pmatrix} -2\lambda/\theta \\ 0 \end{pmatrix} \quad \text{for } n \geq 1,$$

where  $j(0) = 0$ . Writing  $A$  for the matrix and taking into account the initial condition:

$$\begin{pmatrix} h(n) \\ j(n) \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{m=0}^{n-1} A^m \begin{pmatrix} -2\lambda/\theta \\ 0 \end{pmatrix}.$$

Being a positive matrix, the Perron-Frobenius theorem, Theorem 1.2 in Seneta's book [16] implies  $A_{ij}^n \sim v_i \gamma_1^n w_j$  where  $\gamma_1$  is the largest eigenvalue and  $w$  and  $v$  are associated left and right eigenvectors, which have strictly positive entries and are normalized so that  $\sum_i w_i v_i = 1$ . Here  $a_n \sim b_n$  means that  $a_n/b_n \rightarrow 1$ . Noting that  $\gamma_1 > 1$  since the trace of  $A$  is  $> 2$ , and using the asymptotics for  $A^n$  we see that

$$h(n) \sim v_1 w_1 \left[ \gamma_1^n (1 + (1 - \lambda)/\theta) - \frac{\gamma_1^{n-1}}{1 - 1/\gamma_1} (2\lambda/\theta) \right]. \tag{2.12}$$

To find the eigenvalues we set  $(1 + (\lambda + 1)/\theta - x)(1 - x) - 1/\theta = 0$  which becomes

$$x^2 - (2 + (\lambda + 1)/\theta)x + 1 + \lambda/\theta = 0.$$

Solving we get

$$\gamma_i = \frac{2 + (\lambda + 1)/\theta \pm \sqrt{(2 + (\lambda + 1)/\theta)^2 - 4[1 + \lambda/\theta]}}{2}.$$

Underneath the square root we have

$$4 + \frac{4(\lambda + 1)}{\theta} + \frac{(\lambda + 1)^2}{\theta^2} - 4 - 4\frac{\lambda}{\theta} = \frac{(\lambda + 1)^2}{\theta^2} + \frac{4}{\theta}.$$

Multiplying top and bottom by  $\theta$ , the largest eigenvalue is

$$\gamma_1 = \frac{1 + \lambda + 2\theta + \sqrt{(\lambda + 1)^2 + 4\theta}}{2\theta} > 1.$$

From (2.12) we see that  $h(n) \rightarrow \infty$  if  $\gamma_1 - 1 > 2\lambda/\theta$ . Subtracting 1 from  $\gamma_1$  removes the  $2\theta$  from the numerator. When  $\theta = c\lambda^2$ ,

$$\gamma_1 - 1 \sim \frac{\lambda}{2\theta}(1 + \sqrt{1 + 4c}),$$

so again we find  $\liminf \lambda_c(\theta)/\theta^{1/2} \geq 1/\sqrt{2}$ .

### 3 Extinction in two dimensions

The main ideas in this section are similar to the ones in the previous section, but the details are quite different.

*Proof of Theorem 1.3.* Now let  $f(0, 0) = \psi(\{(0, 0)\})$  and  $f(m, n) = \psi(\{(0, 0), (m, n)\})$  for  $m, n \geq 0, m + n \geq 1$ . To keep formulas neater we will also use

$$g(m, n) = 1 - f(m, n) \quad \text{and} \quad h(m, n) = g(m, n)/2g(0, 0).$$

Imitating the proof in the previous section, we begin by observing that

$$\begin{aligned} (1 + 4\lambda)\psi(\{(0, 0)\}) &= 1 + 2\lambda\psi(\{(0, 0), (1, 0)\}) + 2\lambda\psi(\{(0, 0), (0, 1)\}) \\ (1 + 4\lambda)f(0, 0) &= 1 + 2\lambda f(0, 1) + 2\lambda f(1, 0). \end{aligned} \tag{3.1}$$

Note that we have used reflection symmetry to keep the points in the first quadrant. Next,

$$\begin{aligned} (1 + 3\lambda + 4\theta)\psi(\{(0, 0), (1, 0)\}) &= \psi(\{(0, 0)\})(1 + \theta) + 2\theta\psi(\{(0, 0), (1, 1)\}) \\ &\quad + \theta\psi(\{(0, 0), (2, 0)\}) + 2\lambda\psi(\{(0, 0), (1, 0), (1, 1)\}) + \lambda\psi(\{(0, 0), (1, 0), (2, 0)\}). \end{aligned}$$

Supermodularity implies

$$\psi(\{(0, 0), (1, 0), (1, 1)\}) \geq \psi(\{(0, 0), (1, 0)\}) + \psi(\{(1, 0), (1, 1)\}) - \psi(\{1, 0\}),$$

so switching to the  $f$  notation

$$\begin{aligned} (1 + 3\lambda + 4\theta)f(1, 0) &\geq f(0, 0)(1 + \theta) + 2\theta f(1, 1) + \theta f(2, 0) \\ &\quad + 2\lambda[f(1, 0) + f(0, 1) - f(0, 0)] + \lambda[2f(1, 0) - f(0, 0)], \end{aligned}$$

and rearranging gives

$$(1 - \lambda + 4\theta)f(1, 0) + (3\lambda - \theta - 1)f(0, 0) \geq 2\lambda f(0, 1) + 2\theta f(1, 1) + \theta f(2, 0). \tag{3.2}$$

When  $m \geq 2$  there are terms  $\theta\psi(\{(0, 0), (m - 1, 0)\})$  and  $\lambda\psi(\{(0, 0), (m - 1, 0), (m, 0)\})$ , so switching to the  $f$  notation

$$\begin{aligned} (1 + 4\lambda + 4\theta)f(m, 0) &\geq f(0, 0) + \theta f(m - 1, 0) + 2\theta f(m, 1) + \theta f(m + 1, 0) \\ &\quad + \lambda[f(m - 1, 0) + f(1, 0) - f(0, 0)] + 2\lambda[f(m, 0) + f(0, 1) - f(0, 0)] \\ &\quad + \lambda[f(m, 0) + f(1, 0) - f(0, 0)], \end{aligned}$$

and rearranging gives

$$(1 + \lambda + 4\theta)f(m, 0) + (4\lambda - 1)f(0, 0) \geq 2\lambda f(1, 0) + 2\lambda f(0, 1) + (\lambda + \theta)f(m - 1, 0) + \theta f(m + 1, 0) + 2\theta f(m, 1). \quad (3.3)$$

Similarly, when  $m, n \geq 1$

$$(1 + 2\lambda + 4\theta)f(m, n) + (4\lambda - 1)f(0, 0) \geq 2\lambda f(1, 0) + 2\lambda f(0, 1) + (\lambda + \theta)[f(m - 1, n) + f(m, n - 1)] + \theta[f(m + 1, n) + f(m, n + 1)].$$

Letting  $g(m, n) = 1 - f(m, n)$ , (3.1) becomes

$$(1 + 4\lambda)g(0, 0) = 2\lambda[g(0, 1) + g(1, 0)], \quad (3.4)$$

Multiplying each side of (3.3) by  $-1$ , noting that the coefficients on each side sum to  $5\lambda + 4\theta$  and using the last identity we have

$$(1 + 4\theta + \lambda)g(m, 0) \leq 2g(0, 0) + (\lambda + \theta)g(m - 1, 0) + \theta g(m + 1, 0) + 2\theta g(m, 1). \quad (3.5)$$

Performing these manipulations on (3.2) gives the same result so this equation holds for  $m \geq 1$ . A similar argument shows that for  $m, n \geq 1$

$$(1 + 2\lambda + 4\theta)g(m, n) \leq 2g(0, 0) + (\lambda + \theta)[g(m - 1, n) + g(m, n - 1)] + \theta[g(m, n + 1) + g(m + 1, n)]. \quad (3.6)$$

As a warmup, consider formally the case  $\theta = \infty$ . Then (3.4), (3.5) and (3.6) imply that if  $\bar{S}_k$  is simple random walk on the positive quadrant in  $\mathbb{Z}^2$  with reflection at the boundaries then

$$g(m, n) \leq E^{(m, n)}g(\bar{S}_1). \quad (3.7)$$

Since  $\bar{S}_k$  is recurrent, the bounded subharmonic function  $g$  is constant. By (3.4), this constant is zero, so the process dies out.

Suppose now that the process survives,  $g(0, 0) > 0$ , and let

$$h(m, n) = g(m, n)/2g(0, 0).$$

Then  $h(0, 0) = 1/2$  and (3.4), (3.5) and (3.6) imply that for  $m, n \geq 1$

$$h(0, 1) + h(1, 0) = 1 + 1/4\lambda, \quad (3.8)$$

$$(1 + \lambda + 4\theta)h(m, 0) \leq 1 + (\lambda + \theta)h(m - 1, 0) + \theta h(m + 1, 0) + 2\theta h(m, 1), \quad (3.9)$$

$$(1 + 2\lambda + 4\theta)h(m, n) \leq 1 + (\lambda + \theta)[h(m - 1, n) + h(m, n - 1)] + \theta[h(m, n + 1) + h(m + 1, n)]. \quad (3.10)$$

Clearly,  $f(m, n) \leq f(0, 0)$ . Since the contact process with fast voting has positive correlations (see e.g., Theorem 2.13 of Chapter III of [11]),  $f(m, n) \geq f^2(0, 0)$ , so for all  $m, n$

$$1/2 \leq \frac{1 - f(m, n)}{2(1 - f(0, 0))} \leq \frac{1 - f(0, 0)^2}{2(1 - f(0, 0))} = \frac{1 + f(0, 0)}{2} \leq 1,$$

i.e.,  $1/2 \leq h(m, n) \leq 1$ . Let  $\epsilon = 1/(1 + 2\lambda + 4\theta)$ . To motivate the next definition divide each side of (3.10) by  $(1 + 2\lambda + 4\theta)$  to get

$$h(m, n) \leq \epsilon + (1 - \epsilon) \frac{(\lambda + \theta)}{2\lambda + 4\theta} [h(m - 1, n) + h(m, n - 1)] + (1 - \epsilon) \frac{\theta}{2\lambda + 4\theta} [h(m, n + 1) + h(m + 1, n)], \quad (3.11)$$

and note that

$$\frac{(\lambda + \theta)}{2\lambda + 4\theta} \geq \frac{1}{4} \geq \frac{\theta}{2\lambda + 4\theta}.$$

It is natural to expect that  $h(m + 1, n) \geq h(m - 1, n)$  and  $h(m, n + 1) \geq h(m, n - 1)$ , so we should get an upper bound on  $h$  if we replace the two coefficients by  $1/4$ . For a technical reason (see below) we have to do something a little different. Define  $h_k(m, n)$  for  $k \geq 0$  and  $m, n \in Z$  by  $h_k(0, 0) = 1/2$  for all  $k$ . For  $(m, n) \neq (0, 0)$ , let  $h_0(m, n) = 1$  and

$$h_{k+1}(m, n) = \epsilon + (1 - \epsilon)E^{(m,n)}h_k(\tilde{S}_1), \tag{3.12}$$

where  $\tilde{S}_j$  is the lazy version of the reflecting random walk  $\bar{S}_j$ , defined before (3.7), that stays put with probability  $1/2$ . It is easy to see inductively that  $1/2 \leq h_k(m, n) \leq 1$  for all  $k, m, n$ . Let  $\tau$  be the hitting time of the origin for  $\tilde{S}_k$ . By extrapolating from expressions one obtains for  $h_k$  for small  $k$ , one is led to the following expression:

$$h_k(x) = \frac{1}{2} + \frac{\epsilon}{2} \sum_{j=0}^{k-1} (1 - \epsilon)^j P^x(\tau > j) + \frac{1}{2} (1 - \epsilon)^k P^x(\tau > k), \quad x \neq 0. \tag{3.13}$$

To check that this is in fact correct, we argue by induction on  $k$ . Suppose this expression is correct for  $k$ , and replace  $h_k(x)$  by the right hand side in (3.12). The Markov property yields

$$E^x[P^{\tilde{S}_1}(\tau > j), \tilde{S}_1 \neq 0] = P^x(\tau > j + 1).$$

Using this, one obtains for  $x \neq 0$

$$h_{k+1}(x) = \epsilon + \frac{1 - \epsilon}{2} + \frac{\epsilon}{2} \sum_{j=0}^{k-1} (1 - \epsilon)^{j+1} P^x(\tau > j + 1) + \frac{1}{2} (1 - \epsilon)^{k+1} P^x(\tau > k + 1),$$

which agrees with the right hand side of (3.13) with  $k$  replaced by  $k + 1$ .

We have used the lazy random walk so that

**Lemma 3.1.**  $x \rightarrow P^x(\tau > j)$  is increasing (and hence  $x \rightarrow h_k(x)$  is also) in the usual partial order on the positive quadrant.

*Proof.* To prove this we note that two random walks started at  $(m, n)$  and  $(m', n')$  with  $m' \geq m$  and  $n' \geq n$  can be coupled to preserve the order. The ordering is trivial to maintain while the walk is in the interior of the quadrant. To handle the situation when one walker is at  $(\ell, 1)$  and the other is at  $(\ell, 0)$  note that:

from/to	$(\ell, 2)$	$(\ell, 1)$	$(\ell, 0)$
$(\ell, 1)$	1/8	1/2	1/8
$(\ell, 0)$	0	2/8	1/2

which completes the proof. Note that the coupling would fail for the non-lazy walk.  $\square$

**The equation in the interior.** To begin to compare  $h_k$  with  $h$  note that if  $m, n \geq 1$

$$h_{k+1}(m, n) = \frac{1}{2}[\epsilon + (1 - \epsilon)h_k(m, n)] + \frac{1}{2} \left( \epsilon + \frac{1 - \epsilon}{4} [h_k(m - 1, n) + h_k(m, n - 1) + h_k(m + 1, n) + h_k(m, n + 1)] \right).$$

Since  $h_k(m, n) \leq 1$

$$h_{k+1}(m, n) \geq \frac{1}{2}h_k(m, n) + \frac{\epsilon}{2} + \frac{1}{2} \left( \frac{1-\epsilon}{4} [h_k(m-1, n) + h_k(m, n-1) + h_k(m+1, n) + h_k(m, n+1)] \right).$$

Now  $(1-\epsilon)/4 = (\lambda/2 + \theta)/(1 + 2\lambda + 4\theta) = \epsilon(\lambda/2 + \theta)$  so if we temporarily denote the last four terms by  $h_k^i$ ,  $i = 1, 2, 3, 4$  we have

$$(\lambda/2 + \theta)[h_k^1 + h_k^2 + h_k^3 + h_k^4] = (\lambda + \theta)[h_k^1 + h_k^2] + \theta[h_k^3 + h_k^4] + (\lambda/2)[h_k^3 - h_k^1] + (\lambda/2)[h_k^4 - h_k^2].$$

Dropping the last two terms, which are positive by Lemma 3.1, we have

$$h_{k+1}(m, n) \geq \frac{1}{2}h_k(m, n) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \left( (\lambda + \theta)[h_k(m-1, n) + h_k(m, n-1)] + \theta[h_k(m+1, n) + h_k(m, n+1)] \right). \tag{3.14}$$

Since  $(1-\epsilon)/(2\lambda + 4\theta) = \epsilon$  the recursion (3.11) can be written as

$$h(m, n) \leq \frac{1}{2}h(m, n) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \left( (\lambda + \theta)[h(m-1, n) + h(m, n-1)] + \theta[h(m+1, n) + h(m, n+1)] \right). \tag{3.15}$$

**The equation on the boundary.** We begin by adding  $\lambda h(m, 0)$  to each side of (3.9), and divide by  $1 + 2\lambda + 4\theta$  to rewrite it first as

$$h(m, 0) \leq \epsilon + \epsilon[(\lambda + \theta)h(m-1, 0) + \lambda h(m, 0) + \theta h(m+1, \theta) + 2\theta h(m, 1)],$$

and then as

$$h(m, 0) \leq \frac{1}{2}h(m, 0) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \left( [(\lambda + \theta)h(m-1, 0) + \lambda h(m, 0) + \theta h(m+1, \theta) + 2\theta h(m, 1)] \right). \tag{3.16}$$

From (3.12), the corresponding equation for  $h_{k+1}$  is

$$h_{k+1}(m, 0) = \frac{1}{2}[\epsilon + (1-\epsilon)h_k(m, 0)] + \frac{1}{2} \left( \epsilon + \frac{1-\epsilon}{4} [h_k(m-1, 0) + h_k(m+1, 0) + 2h_k(m, 1)] \right).$$

Using  $(1-\epsilon)/4 = \epsilon(\lambda/2 + \theta)$  and arguing as before we can convert this into

$$h_{k+1}(m, 0) \geq \frac{1}{2}h_k(m, 0) + \frac{\epsilon}{2} + \frac{1}{2}\epsilon \left( (\lambda + \theta)h_k(m-1, 0) + \lambda h_k(m, 0) + \theta h_k(m+1, 0) + 2\theta h_k(m, 1) + \frac{\lambda}{2}[h_k(m+1, 0) - h_k(m-1, 0)] + \lambda[h_k(m, 1) - h_k(m, 0)] \right), \tag{3.17}$$

and drop the last line, which is positive by Lemma 3.1.

**Final details.** Comparing (3.14) with (3.15) and (3.16) with (3.17) we see that  $h_{k+1}$  and  $h$  satisfy similar iterations with  $\geq$  and  $\leq$  respectively. Since  $h_0 \geq h$ , it follows by induction that

$$h_k(m, n) \geq h(m, n). \tag{3.18}$$

Taking the limit in our formula (3.13) for  $h_k$  and using (3.18) gives

$$h(m, n) \leq \frac{1}{2} + \frac{\epsilon}{2} \sum_{j=0}^{\infty} (1 - \epsilon)^j P^{(m,n)}(\tau > j).$$

Since  $h(0, 1) + h(1, 0) = 1 + 1/4\lambda$  by (3.8),

$$1/4\lambda \leq \epsilon \sum_{j=0}^{\infty} (1 - \epsilon)^j P^{(1,0)}(\tau > j).$$

Since this argument was based on the assumption that the process survives, it follows that the process dies out whenever the opposite (strict) inequality holds.

To finish up, we note that the expression on the right is

$$\begin{aligned} &= \sum_{j=0}^{\infty} \epsilon (1 - \epsilon)^j \sum_{k=j+1}^{\infty} P^{(1,0)}(\tau = k) = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \epsilon (1 - \epsilon)^j P^{(1,0)}(\tau = k) \\ &= \sum_{k=1}^{\infty} (1 - (1 - \epsilon)^k) P^{(1,0)}(\tau = k) = 1 - E^{(1,0)}(1 - \epsilon)^\tau \end{aligned}$$

which converts our answer into the one given in Theorem 1.3. □

## 4 Survival in two dimensions

The proof is simple since most of the work has already been done by Durrett and Zähle [5]. It is only necessary to make some minor modifications to their proof presented on pages 1758–1761. However, since we have to give the details, we have taken the opportunity to clarify the proof and sharpen some of the bounds. As in [5], we work with the dual coalescing random walks  $\eta_t$ , but now jumps occur at rate  $\theta$  across each edge between nearest neighbors, births occur at rate  $\lambda = \gamma \log(\theta)$  across each edge, and deaths occur at rate 1 per site. Thus, in contrast to [5], we have no need to speed up the process, only to scale space to  $\theta^{-1/2}\mathbb{Z}^2$ .

Most of the particles that are born in  $\eta_t$  coalesce with their parents. These short intervals in which the size of the dual increases and then decreases are not compatible with weak convergence, so we define a **pruned dual process**  $\bar{\eta}_t$  using the rule that a newly born particle has mass 0 until it has avoided coalescence with its parent for  $1/\log^a(\theta)$  units of time, where  $a$  is the first of several positive constants whose values will be specified later. In the proof we will show that it is very unlikely for a particle to collide with a particle that is not its parent.

Let  $Z_t^1$  and  $Z_t^2$  be independent random walks that make jumps to nearest neighbors at rate 1 and let  $S_t = Z_t^1 - Z_t^2$ . Suppose  $S_0$  is a neighbor of 0. Let  $T_0 = \inf\{t : S_t = 0\}$ . From (1.3),  $P(T_0 > t) \sim \pi/(\log t)$ . Changing to the fast time scale let  $s_t = \theta^{-1/2}S(\theta t)$  and  $t_0 = \inf\{t : s_t = 0\}$ . We use lower case letters to remind us that this process is smaller in space and time. Therefore

$$P(t_0 > 1/\log^a(\theta)) = P(T_0 > \theta/\log^a(\theta)) \sim \frac{\pi}{\log \theta - a \log \log \theta}. \tag{4.1}$$

Let  $R = \theta / \log^a \theta$  and  $r = 1 / \log^a \theta$ . Using a trivial inequality, then (1.3), and Chebyshev's inequality we have

$$\begin{aligned} P(|S_R| \geq \theta^{1/2} / \log^b(\theta) | T_0 > R) &\leq \frac{1}{P(T_0 > R)} P(|S_R| \geq \theta^{1/2} / \log^b(\theta)) \\ &\leq C \log \theta \cdot \frac{\theta}{\log^a \theta} \cdot \frac{\log^{2b}(\theta)}{\theta} = C(\log \theta)^{1+2b-a}. \end{aligned} \quad (4.2)$$

which goes to 0 if  $a > 1 + 2b$ . Here, we have collected all the constants into one placed at the front, a practice we will continue in what follows.

Note that when  $|S_R| \leq \theta^{1/2} / \log^b(\theta)$ , we have  $|s_r| \leq 1 / \log^b(\theta)$ , i.e., the new particle is close to its parent. The next result, which follows from using the local central limit theorem, shows that they are not too close:

$$\begin{aligned} P(|S_R| \leq \theta^{1/2} / \log^c(\theta) | T_0 > R) &\leq \frac{1}{P(T_0 > R)} P(|S_R| \leq \theta^{1/2} / \log^c(\theta)) \\ &\leq C \log(\theta) \cdot \left( \frac{\theta^{1/2}}{\log^c(\theta)} \right)^2 \cdot \frac{\log^a(\theta)}{\theta} = C(\log \theta)^{1+a-2c}. \end{aligned} \quad (4.3)$$

Here the second term on the last line gives the order of magnitude of the number of points  $x$  with  $|x| \leq \theta^{1/2} / \log^c(\theta)$ , and the third is a bound on the order of magnitude of  $P(S_R = x)$ . If  $1 + a < 2c$  this tends to 0.

To prove results about the asymptotic behavior of the pruned dual  $\bar{\eta}_t$  we will consider the **pruned branching random walk**  $\hat{\eta}_t$  in which particles are assigned mass 0 until they have avoided coalescing with their parent for time  $1 / \log^a \theta$ . After this time, we ignore collisions with any other particle (including its parent) until time  $L^2$ , which again is a constant whose value will be specified later. By (4.1) particles in  $\hat{\eta}_t$  are born at rate

$$\sim 4\gamma \log \theta \cdot \frac{\pi}{\log \theta} = 4\gamma\pi,$$

and die at rate 1. From this and (4.2) it is easy to see the following:

**Lemma 4.1.** *As  $\theta \rightarrow \infty$  the pruned branching random walk  $\hat{\eta}_t$ ,  $0 \leq t \leq L^2$ , converges to a branching Brownian motion  $\zeta_t$ ,  $0 \leq t \leq L^2$ , in which births occur at rate  $4\gamma\pi$ , deaths at rate 1, and the Brownian motions run at rate 2.*

To estimate the difference between  $\bar{\eta}_t$  and  $\hat{\eta}_t$ , we need to estimate the probability that a newly born particle in  $\bar{\eta}_t$  collides with another particle before time  $L^2$ . Using (1.3), one can easily show that

$$P(t_0 \leq L^2 | t_0 > 1 / \log^a(\theta)) = P(T_0 \leq L^2 \theta | T_0 > \theta / \log^a(\theta)) \rightarrow 0. \quad (4.4)$$

To do this, we note that if  $\epsilon > 0$  then for large  $\theta$

$$\begin{aligned} P(\theta / \log^a(\theta) < T_0 \leq L^2 \theta) &\leq \frac{\pi(1 + \epsilon)}{\log \theta - a \log \log \theta} - \frac{\pi(1 - \epsilon)}{\log \theta + \log(L^2)} \\ &\leq \frac{2\pi\epsilon}{\log \theta - a \log \log \theta} \sim \frac{2\pi\epsilon}{\log \theta}. \end{aligned}$$

The result in (4.4) implies that after the initial separation we don't have to worry about a particle colliding with its parent before time  $L^2$ . To show that it avoids the other particles, we will first show that if two particles are sufficiently separated then they will not collide by time  $L^2$ . Then we will use induction to prove the desired amount of separation is maintained. The first step is to suppose  $S_0 = x$  with  $|x| \geq \theta^{1/2} / \log^\Delta(\theta)$ ,

with the capital letter  $\Delta$  to suggest this is a large constant, and then estimate the probability  $\alpha(x)$ , that  $S_t$  will hit the ball of radius  $K$  before it exits the ball of radius  $M\theta^{1/2} - 1$ . Our goal is to conclude that

$$\bar{\alpha} = \sup_{|x| \geq \theta^{1/2}/\log^\Delta(\theta)} \alpha(x) \rightarrow 0.$$

If  $S_t$  were a two-dimensional Brownian motion, then we would use the harmonic function  $\log |z|$  to calculate  $\alpha(x)$ . For the random walk, we use the recurrent potential kernel defined by  $A(x) = \sum_{k=0}^{\infty} [q^k(0) - q^k(x)]$  where  $q$  is uniform on the four nearest neighbors and  $q^k$  denotes the  $k$ -fold convolution. It is immediate from the definition that

$$\sum_y q(y-x)A(x) - A(x) = 1_{(x=0)}$$

i.e., the difference is 1 if  $x = 0$  and 0 otherwise. From this we see that  $A(S_t)$  is a martingale until time  $T_0$ . By Theorem 2 of [7] or P3 of Section 12 of [17],

$$A(x) = c_A \log |x| + O(1). \tag{4.5}$$

If  $K$  is large and  $|x| \geq \theta^{1/2}/\log^\Delta(\theta)$ , then applying the optional stopping theorem to  $A(S_t)/c_A$

$$\begin{aligned} (1/2) \log \theta - \Delta \log \log \theta + O(1) &\leq A(x) \\ &\leq \alpha(x)[\log K + O(1)] + (1 - \alpha(x))[(1/2) \log \theta + \log M + O(1)], \end{aligned}$$

where the second inequality comes from the fact that upon exit from the annulus the random walk will jump over the boundary by a distance  $\leq 1$ . Rearranging we have

$$\alpha(x)[(1/2) \log \theta - \log K] \leq \Delta \log \log \theta + \log M + O(1), \tag{4.6}$$

and we see that for any  $\Delta < \infty$ ,  $\bar{\alpha} \rightarrow 0$  as  $\theta \rightarrow \infty$ .

To do the induction argument to bound the spacings between the particles, we can ignore the death of particles. If we do this then, for large  $\theta$ , the growth of the number of particles is bounded by a branching process with births at rate  $5\gamma\pi$ . This implies that the expected number of particles  $N(L^2)$  in the system at time  $L^2$  is finite, so if  $\epsilon > 0$  then we can pick  $N_\epsilon$  so that  $P(N(L^2) > N_\epsilon) \leq \epsilon$ .

We now choose the values of our parameters:

$$b = 1/3, \quad a = 2, \quad c = 2,$$

which satisfy  $a > 1 + 2b$ , needed for (4.2), and  $2c > 1 + a$ , needed for (4.3). With these choices, (4.3) implies that when a new particle is added to  $\bar{\eta}_t$  the newly born particle is with high probability at distance  $\geq 1/\log^2 \theta$  from its parent. Suppose that at time 0 we have  $k$  particles that are separated by  $\geq 1/\log^2 \theta$ . The result in (4.6) implies that none of the  $k$  particles will coalesce by time  $L^2$ . Since, for large  $\theta$ , births in  $\bar{\eta}_t$  occur at rate  $\leq 5\gamma\pi$ , it follows that for any  $\delta > 0$  the first successful birth will with high probability occur after  $R' = \theta/\log^\delta \theta$ . If  $R' \leq t \leq L^2$  then the argument for (4.3) can be repeated to show

$$P(|S_t| \leq \theta^{1/2}/\log^2(\theta)) \leq C \left( \frac{\theta^{1/2}}{\log^2(\theta)} \right)^2 \cdot \frac{\log^\delta(\theta)}{\theta} = C(\log \theta)^{2-\delta}.$$

This implies that when the new particle is added to  $\bar{\eta}_t$  the  $k + 1$  particles are with high probability separated by  $1/\log^2(\theta)$ . Since this conclusion holds with probability  $\geq 1 - \epsilon$  and the argument is only repeated  $N_\epsilon$  times, we have shown that the desired separation is maintained with high probability, and it follows from (4.6) that there is no coalescence. Since this shows that  $\bar{\eta}_t = \hat{\eta}_t$  with high probability, we have:

**Lemma 4.2.** *As  $\theta \rightarrow \infty$  the pruned dual  $\bar{\eta}_t$ ,  $0 \leq t \leq L^2$ , converges to a branching Brownian motion  $\zeta_t$ ,  $0 \leq t \leq L^2$ , in which births occur at rate  $4\gamma\pi$ , deaths at rate 1, and the Brownian motions run at rate 2.*

With Lemma 4.2 established, the rest of the proof is almost identical to that in [5]. Let  $\bar{\zeta}_t$  be a modification of  $\zeta_t$  in which particles are killed when they leave  $[-4L, 4L]^2$ . In [5] this is  $[0, 4L]^2$  but that is because they need to keep their construction in one half-space. We will use the more natural symmetric geometry in the construction here.

If we use  $\bar{\zeta}_t^x(A)$  to denote the number of points in  $A$  when the killed branching Brownian motion  $\bar{\zeta}_t$  starts with one particle at  $x$  and let  $\mu = 4\gamma\pi - 1$  then

$$E[\bar{\zeta}_t^x(A)] = e^{\mu t} P(\bar{B}_t^x \in A),$$

where  $B_t^x$  is a Brownian motion starting from  $x$ , run at rate 2, and killed when it exits  $[-4L, 4L]^2$ . Let  $I_k = (2kL, 0) + [-L, L]^2$ . From the last estimate it follows easily (see page 1760 of [5] for more details) that if  $\mu > 0$  (an assumption that will be in force until we state Lemma 4.3)

$$\inf_{x \in I_0} E[\bar{\zeta}_{L^2}^x(I_i)] \geq 2 \quad \text{for } i = 1, -1,$$

and hence if we let  $\bar{\zeta}_t^A = \sum_{x \in A} \bar{\zeta}_t^x$ , then for  $A \subset I_0$

$$E[\bar{\zeta}_{L^2}^A(I_i)] \geq 2|A| \quad \text{for } i = 1, -1,$$

where  $|A|$  is the number of particles in  $A$ .

Since  $E[\bar{\zeta}_{L^2}^x(I_i)^2] \leq E[\zeta_{L^2}^x(\mathbb{R}^2)^2] \equiv c_L$ , and the  $\bar{\zeta}_t^x$  are independent, it follows that for  $A \subset [-L, L]^2$

$$\text{var}[\bar{\zeta}_{L^2}^A(I_i)] \leq |A|c_L \quad \text{for } i = 1, -1.$$

Using by Chebyshev's inequality that if  $A \subset [-L, L]^2$  has  $|A| \geq K$  then

$$P(\bar{\zeta}_{L^2}^A(I_i) < K) \leq c_L/K \quad \text{for } i = 1, -1.$$

The last conclusion gives us the block event for the branching Brownian motion  $\bar{\zeta}_t$ :

**Lemma 4.3.** *Suppose  $4\gamma\pi > 1$ . Let  $\epsilon > 0$ . If  $L \geq L_0(\epsilon)$  and  $\bar{\zeta}_0(I_0) \geq K$  then we will with probability  $\geq 1 - 2\epsilon$  have  $\bar{\zeta}_{L^2}(I_i) \geq K$  for  $i = -1, 1$ . We do this in a system in which particles are killed when they leave  $[-4L, 4L]^2$  so the events for  $I_0$  and  $I_m$  are independent when  $m \geq 4$ .*

Combining this with Lemma 4.2 it follows that

**Lemma 4.4.** *Suppose  $4\gamma\pi > 1$ . Let  $\epsilon > 0$  and pick  $L \geq L_0(\epsilon)$ . If  $\theta \geq \theta_0(\epsilon, L)$  and we start with  $\geq K$  particles in  $I_0$  in  $\bar{\eta}_t$  that are separated by  $\geq 1/\log^2 \theta$  at time 0 then we will with probability  $\geq 1 - 2\epsilon$  have  $\geq K$  particles in  $I_{-1}$  and  $I_1$  that are separated by  $\geq 1/\log^2 \theta$  at time  $L^2$ . We do this in a system in which particles are killed when they leave  $[-4L, 4L]^2$  so the events for  $I_0$  and  $I_m$  are independent when  $m \geq 4$ .*

With Lemma 4.4 established the existence of a stationary distribution follows from Theorem 4.1 in [4] which shows that dual process starting from a single particle at 0 survives with positive probability.

## 5 Survival in one dimension

In this section we consider the version of the contact plus voter process in which births occur at rate 1. The dual process has deaths at rate  $\delta$ , births across each edge at rate 1, and random walks that jump to each nearest neighbor at rate  $\nu$ , with coalescence

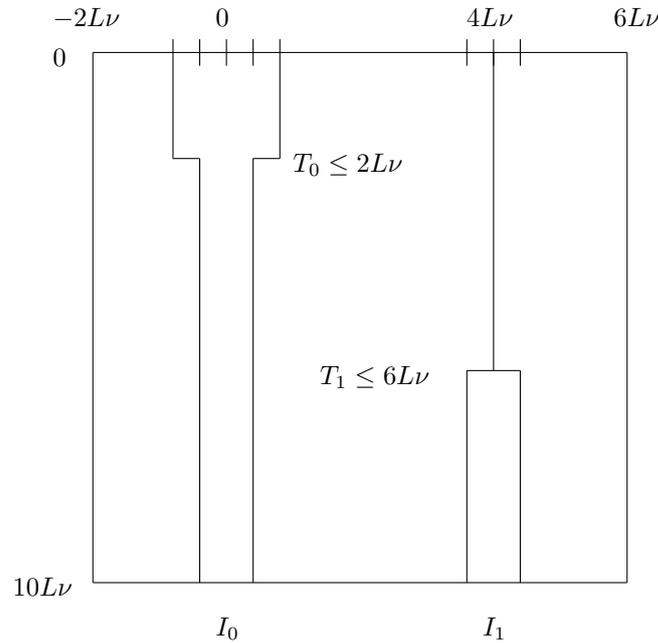


Figure 1: Picture of the block construction in  $d = 1$ . Time runs down the page. The white particle stays in the region on the left. Once the red particle hits  $L\nu$  at time  $T_1$  it stays within the region on the right.

when two walks hit. Let  $L$  be a large constant that will be chosen later. Let  $I_m = 4mL\nu + [-0.2L\nu, 0.2L\nu]$ . We will show for the dual process with  $\delta = 0$  that if we start with one particle in  $I_0$  at time 0 then with high probability we will have one in  $I_0$  and one in  $I_1$  at time  $10L\nu$ . We will do this by following the behavior of two tagged particles in the dual. In order to have the events in our construction one dependent, we will also show that with high probability the particles do not leave  $[-2L\nu, 6L\nu]$ .

**The Key** to the proof is a trick used by Bramson and Griffeath [1]. We follow a tagged particle in the dual that moves according to the following rules: when it is affected by a random walk event it must jump, but when there is a branching event, it follows the birth if and only if the birth takes it to the right. This process, which we call  $X_r(t)$ , makes jumps

$$\begin{aligned} x &\rightarrow x + 1 && \text{at rate } \nu + 1, \\ x &\rightarrow x - 1 && \text{at rate } \nu. \end{aligned}$$

It is easy to check that  $X_r(2\nu t)/2\nu \Rightarrow B_r(t) = B(t) + t$ . Let  $X_\ell$  denote the analogous process that only follows births to the left and has limit  $B_\ell(t) = B(t) - t$ , and let  $X_c$  denote the process that ignores branching arrows and has limit  $B_c(t) = B(t)$ .

**The Construction.** Suppose we have a particle at  $x \in I_0$ . There are three things to prove.

(i) Suppose the particle in  $I_0$  is at  $x$ . We use  $X_r$  if  $x < 0$  or  $X_\ell$  if  $x > 0$  to bring the particle to 0, which happens at time  $T_0$ . We will declare this part of the construction a success if the particle never leaves  $[-0.4L\nu, 0.4L\nu]$  and  $T_0 \leq 2L\nu$ . Having brought our particle to 0, we change our rule so that the particle we are watching, which we call the white particle, only follows random walk arrows.

(ii) At time  $T_0$  we begin to follow a red particle that moves according to  $X_r(t)$ . The red particle may coalesce with the white particle a large number of times, but will keep separating from it because it follows births to the right, while the white particle does not. Since the red particle is approximately a Brownian motion with drift, it will reach  $4L\nu$  at some time  $T_1$ . We declare this part of the construction a success if  $T_1 \leq T_0 + 6L\nu$ . At time  $T_1$ , we change our rule so that the red particle only follows random walk arrows.

(iii) The last thing we need is for the white particle not to leave  $[-0.2L\nu, 0.2L\nu]$  in  $[T_0, 10L\nu]$  and for the red particle not to leave  $4L\nu + [-0.2L\nu, 0.2L\nu]$  in  $[T_1, 10L\nu]$ . Note that between time  $T_0$  and time  $T_1$ , the red particle is always to the right of the white particle, and always to the left of  $4L\nu$  so if the two events in the first sentence happen, the red particle will not leave the interval  $[-0.2L\nu, 4.2L\nu]$ .

**Rescaled Construction.** The processes  $B_r$ ,  $B_\ell$ , and  $B_c$  arise from  $X_r$ ,  $X_\ell$ , and  $X_c$  by scaling space and time by  $2\nu$ . Using the weak convergence of these processes it is enough to show that if  $L$  is large, then the correspondingly scaled events hold for the  $B$ 's with high probability.

- (i) The initial particle starting in  $[-0.1L, 0.1L]$  can be brought to 0 without exiting  $[-0.2L, 0.2L]$  and gets there at time  $\bar{T}_0 \leq L$ .
- (ii) The red particle born at time  $\bar{T}_0$  gets to  $2L$  at time  $\bar{T}_1 \leq \bar{T}_0 + 3L$ .
- (iii) The white particle does not leave  $[-0.1L, 0.1L]$  during  $[\bar{T}_0, 5L]$  and the red particle does not leave  $2L + [-0.1L, 0.1L]$  during  $[\bar{T}_1, 5L]$ .

**Lemma 5.1.** *Suppose  $B_r(0) = x < 0$  and let  $\bar{T}_0 = \inf\{t : B_r(t) = 0\}$ . In this case, as  $L \rightarrow \infty$*

$$\sup_{x \in [-0.1L, 0]} P_x(\bar{T}_0 \geq L) \rightarrow 0 \quad \text{and} \quad \sup_{x \in [-0.1L, 0]} P_x\left(\inf_{t \leq L} B_r(t) \leq -0.2L\right) \rightarrow 0.$$

*Proof.* It suffices to prove the result when  $x = -0.1L$ . As  $L \rightarrow \infty$ ,  $B_r(tL)/L \Rightarrow -0.1 + t$  where  $\Rightarrow$  indicates weak convergence in  $C([0, 1])$ . The two results follow from this.  $\square$

Let  $\tau_1 = \inf\{t : B_r(t) = L\}$ . Repeating the previous proof we see that  $P_0(\tau_1 \geq 2L) \rightarrow 0$ . To check (iii) now, use scaling to conclude that for any  $0 < \epsilon, M < \infty$

$$\lim_{L \rightarrow \infty} P\left(\sup_{t \leq ML} |B_c(t)| \leq \epsilon L\right) = \lim_{L \rightarrow \infty} P\left(\sup_{0 \leq t \leq 1} (ML)^{1/2} B(t) \leq \epsilon L\right) = 1.$$

At this point we have shown that if  $L_0$  is large enough and  $L \geq L_0$  then all three events in the rescaled construction occur with probability  $\geq 1 - \epsilon$ . If we take  $L = L_0$  then the weak convergence of the  $X$ 's to the  $B$ 's implies that if  $\nu \geq \nu_0$  all three events in the construction occur with probability  $\geq 1 - 2\epsilon$ . Up to this point the calculations are for the process with no death. If we pick  $\delta = \eta/L\nu$  with  $\eta$  small then the probability of a death affecting either of our two tagged particles is  $\leq \epsilon$ . The events in the block construction are one dependent, so applying Theorem 4.1 from Durrett's St. Flour Notes [4], we see that the dual process starting from a single particle at 0 survives with positive probability. Consequently, there exists a nontrivial stationary distribution of the contact plus voter process.

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