

## Geometric stable processes and related fractional differential equations

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### Abstract

We are interested in the differential equations satisfied by the density of the Geometric Stable processes  $\{\mathcal{G}_\alpha^\beta(t); t \geq 0\}$ , with stability index  $\alpha \in (0, 2]$  and symmetry parameter  $\beta \in [-1, 1]$ , both in the univariate and in the multivariate cases. We resort to their representation as compositions of stable processes with an independent Gamma subordinator. As a preliminary result, we prove that the latter is governed by a differential equation expressed by means of the shift operator. As a consequence, we obtain the space-fractional equation satisfied by the transition density of  $\mathcal{G}_\alpha^\beta(t)$ . For some particular values of  $\alpha$  and  $\beta$ , we get some interesting results linked to well-known processes, such as the Variance Gamma process and the first passage time of the Brownian motion.

**Keywords:** Symmetric Geometric Stable law; Geometric Stable subordinator; Shift operator; Riesz-Feller fractional derivative; Gamma subordinator.

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## 1 Introduction and notation

The Geometric Stable (hereafter GS) random variable (r.v.) is usually defined through its characteristic function: let  $\mathcal{G}_\alpha^\beta$  be a GS r.v. with stability index  $\alpha \in (0, 2]$ , symmetry parameter  $\beta \in [-1, 1]$ , position parameter  $\mu \in \mathbb{R}$ , scale parameter  $\sigma > 0$ , then

$$\Phi_{\mathcal{G}_\alpha^\beta}(\theta) := \mathbb{E}e^{i\theta\mathcal{G}_\alpha^\beta} = \frac{1}{1 + \sigma^\alpha |\theta|^\alpha \omega_{\alpha,\beta}(\theta) - i\mu\theta}, \quad \theta \in \mathbb{R}, \quad (1.1)$$

where

$$\omega_{\alpha,\beta}(\theta) := \begin{cases} 1 - i\beta \operatorname{sign}(\theta) \tan(\pi\alpha/2), & \text{if } \alpha \neq 1 \\ 1 + 2i\beta \operatorname{sign}(\theta) \log|\theta|/\pi, & \text{if } \alpha = 1 \end{cases},$$

(see e.g. [10]). Moreover the following relationship holds (see [13], [6])

$$\Phi_{\mathcal{G}_\alpha^\beta}(\theta) = \frac{1}{1 - \log \Phi_{S_\alpha^\beta}(\theta)}$$

where

$$\Phi_{S_\alpha^\beta}(\theta) := \mathbb{E}e^{i\theta S_\alpha^\beta} = \exp\{i\theta\mu - \sigma^\alpha |\theta|^\alpha \omega_{\alpha,\beta}(\theta)\}, \quad \theta \in \mathbb{R}, \quad (1.2)$$

is the characteristic function of a stable r.v.  $S_\alpha^\beta$  with the same parameters  $\alpha, \beta, \mu, \sigma$ . We will consider, for simplicity, the case  $\mu = 0$ ; then we will refer only to strictly stable r.v.'s, if  $\alpha \neq 1$ .

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The main features of the GS laws are the heavy tails and the unboundedness at zero. These two characteristics, together with their stability properties (with respect to geometric summation) and domains of attraction, make them attractive in modelling financial data, as shown, for example, in [16]. As particular cases, when the symmetry parameter  $\beta$  is equal to 1, the support of the GS r.v. is limited to  $\mathbb{R}^+$  and its law coincides, for  $0 < \alpha \leq 1$ , with the Mittag-Leffler distribution, as shown in [10] and [13]. Moreover the GS distribution is sometimes referred to as "asymmetric Linnik distribution", since it can be considered a generalization of the latter (to which it reduces for  $\beta = \mu = 0$ , see [14], [8]). The Linnik distribution exhibits fat tails, finite mean for  $1 < \alpha \leq 2$  and also finite variance only for  $\alpha = 2$  (when it takes the name of Laplace distribution, see [12]) and is applied in particular to model temporal changes in stock prices (see [2]).

We will denote by  $\{\mathcal{G}_\alpha^\beta(t), t \geq 0\}$  the univariate GS process; it is well-known that it is a Lévy process (see, for example, [27]) and thus infinitely divisible for each  $t$ , so that we can write its characteristic function as

$$\Phi_{\mathcal{G}_\alpha^\beta(t)}(\theta) = \left[ \Phi_{\mathcal{G}_\alpha^\beta(1)}(\theta) \right]^t = e^{t\eta_{\mathcal{G}_\alpha^\beta}(\theta)}. \tag{1.3}$$

We will consider the most general case where the characteristic exponent in (1.3) is given by

$$\eta_{\mathcal{G}_\alpha^\beta}(\theta) := \frac{1}{t} \log \Phi_{\mathcal{G}_\alpha^\beta(t)}(\theta) = -a \log \left( 1 + \frac{\sigma^\alpha}{b} |\theta|^\alpha \omega_{\alpha,\beta}(\theta) \right), \quad \theta \in \mathbb{R}.$$

The parameters  $a, b > 0$  are referred to the following representation of the GS process, (see [9]), i.e.

$$\mathcal{G}_\alpha^\beta(t) := S_\alpha^\beta(\Gamma(t)), \quad t \geq 0, \tag{1.4}$$

where  $\{\Gamma(t), t \geq 0\}$  is a Gamma subordinator, with shape parameter  $a$  and scale parameter  $1/b$  (see (2.1) below), and  $S_\alpha^\beta(t)$  is an independent stable process with characteristic function

$$\Phi_{S_\alpha^\beta(t)}(\theta) = \exp\{-t|\theta|^\alpha \sigma^\alpha \omega_{\alpha,\beta}(\theta)\}, \quad \theta \in \mathbb{R}. \tag{1.5}$$

We note that, for  $\beta = 0$ , the process  $\mathcal{G}_\alpha^\beta(t)$  reduces to a symmetric GS process (that we will denote simply as  $\mathcal{G}_\alpha(t)$ ), while, for  $\beta = 1$ , it is called GS subordinator (since it is increasing and Lévy); we will denote it as  $\mathcal{G}'_\alpha(t)$ .

The space-fractional differential equation that we obtain here, as governing equations of  $\mathcal{G}_\alpha^\beta(t)$ , are expressed in terms of Riesz and Riesz-Feller derivatives. We recall that the Riesz fractional derivative  ${}^R\mathcal{D}_x^\alpha$  is defined through its Fourier transform, which reads, for  $\alpha > 0$  and for an infinitely differentiable function  $u$ ,

$$\mathcal{F} \{ {}^R\mathcal{D}_x^\alpha u(x); \theta \} = -|\theta|^\alpha \mathcal{F} \{ u(x); \theta \}, \tag{1.6}$$

where the Fourier transform is defined as  $\mathcal{F} \{ u(x); \theta \} := \int_{-\infty}^{+\infty} e^{i\theta x} u(x) dx$  (see [20] and [11], p.131). Alternatively it can be explicitly represented as follows, for  $\alpha \in (0, 2]$ ,

$${}^R\mathcal{D}_x^\alpha u(x) := -\frac{1}{2 \cos(\alpha\pi/2)} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{+\infty} \frac{u(z)}{|x-z|^\alpha} dz \tag{1.7}$$

(see [24]). The more general Riesz-Feller definition is given by

$$\mathcal{F} \{ {}^{RF}\mathcal{D}_{x,\beta}^\alpha u(x); \theta \} = \psi_\beta^\alpha(\theta) \mathcal{F} \{ u(x); \theta \}, \quad \alpha \in (0, 2], \tag{1.8}$$

where

$$\psi_\beta^\alpha(\theta) := -|\theta|^\alpha e^{i \operatorname{sign} \theta \arctan[-\beta \tan \frac{\pi\alpha}{2}]} \tag{1.9}$$

(see [11], p.359 and [20]). Note that  $\psi_\beta^\alpha(\theta)$  coincides with (minus) the characteristic exponent of the stable random variable  $S_\alpha^\beta$ , in the Feller parametrization, for  $\gamma = \frac{2}{\pi} \arctan[-\beta \tan \frac{\pi\alpha}{2}]$  and  $|\gamma| \leq \min\{\alpha, 2 - \alpha\}$ . Indeed (1.2) can be rewritten (for  $\mu = 0$ ) as

$$\Phi_{S_\alpha^\beta}(\theta) = \exp\{c\psi_\beta^\alpha(\theta)\}, \quad \theta \in \mathbb{R}, \quad c = \sigma^\alpha [\cos(\pi\gamma/2)]^{-1}. \tag{1.10}$$

We recall now the following result on stable processes proved in [20] (in the special case  $c = 1$ ), which will be used later: let  $p_\alpha^\beta(x, t)$ ,  $x \in \mathbb{R}, t \geq 0$ , be the transition density of the stable process  $S_\alpha^\beta(t)$ , then  $p_\alpha^\beta(x, t)$  satisfies the following space-fractional differential equation, for  $\alpha \in (0, 2]$ ,  $x \in \mathbb{R}, t \geq 0$ :

$$\begin{cases} {}^{RF}\mathcal{D}_{x,\beta}^\alpha p_\alpha^\beta(x, t) = \frac{1}{c} \frac{\partial}{\partial t} p_\alpha^\beta(x, t) \\ p_\alpha^\beta(x, 0) = \delta(x) \\ \lim_{|x| \rightarrow \infty} p_\alpha^\beta(x, t) = 0 \end{cases}, \tag{1.11}$$

and the additional condition  $\frac{\partial}{\partial t} p_\alpha^\beta(x, t)|_{t=0} = 0$ , if  $\alpha > 1$ .

Our main result concerns the space-fractional equation satisfied by the transition density  $g_\alpha^\beta(x, t)$ ,  $x \in \mathbb{R}, t \geq 0$ , of the GS process  $\mathcal{G}_\alpha^\beta(t)$ . As a preliminary step we derive the partial differential equation satisfied by the transition density  $f_\Gamma(x, t)$ ,  $x, t \geq 0$ , of the Gamma subordinator  $\Gamma(t)$  and then we resort to the representation (1.4) of the GS process. Indeed we prove that  $f_\Gamma(x, t)$  satisfies

$$\frac{\partial}{\partial x} f_\Gamma = -b(1 - e^{-\frac{1}{a}\partial_t})f_\Gamma, \quad x, t \geq 0, \tag{1.12}$$

where  $a$  and  $b$  are the shape and rate parameters of the Gamma distribution respectively (see (2.1) below) and  $e^{-\frac{1}{a}\partial_t}$  is a particular case (for  $k = 1/a$ ) of the shift operator, defined as

$$e^{-k\partial_t} f(t) := \sum_{n=0}^{\infty} \frac{(-k\partial_t)^n}{n!} f(t) = f(t - k), \quad k \in \mathbb{R}, \tag{1.13}$$

for any analytical function  $f : \mathbb{R} \rightarrow \mathbb{R}$ . As a consequence, we show that  $g_\alpha^\beta(x, t)$  satisfies, for  $x \in \mathbb{R}, t \geq 0, \alpha \in (0, 2]$ , the following Cauchy problem

$$\begin{cases} {}^{RF}\mathcal{D}_{x,\beta}^\alpha g_\alpha^\beta(x, t) = \frac{b}{c}(1 - e^{-\frac{1}{a}\partial_t})g_\alpha^\beta(x, t) \\ g_\alpha^\beta(x, 0) = \delta(x) \\ \lim_{|x| \rightarrow \infty} g_\alpha^\beta(x, t) = 0 \end{cases}. \tag{1.14}$$

In the  $n$ -dimensional case, we prove that the governing equation of the GS vector process in  $\mathbb{R}^n$  is analogous to (1.14), but the Riesz-Feller fractional derivative is substituted, in this case, by the fractional derivative operator  $\nabla_M^\alpha$  defined by

$$\mathcal{F}\{\nabla_M^\alpha u(\mathbf{x}); \theta\} = - \left[ \int_{S^n} (-i \langle \mathbf{z}, \theta \rangle)^\alpha M(d\mathbf{z}) \right] \mathcal{F}\{u(\mathbf{x}); \theta\}, \quad \theta, \mathbf{x} \in \mathbb{R}^n, \alpha \in (0, 2], \alpha \neq 1, \tag{1.15}$$

where  $S^n := \{\mathbf{s} \in \mathbb{R}^n : \|\mathbf{s}\| = 1\}$  and  $M$  is the spectral measure (see [21], with a change of sign due to the different definition of Fourier transform). The multivariate GS law has been first introduced in [1] (in the isotropic case) and called multivariate Linnik distribution.

As special cases of the previous results the governing equations of some well-known processes are obtained: indeed, in the symmetric case and for  $\alpha = 2$ , the GS process reduces to the Variance Gamma process, while, for  $\alpha = 1$ , it coincides with a Cauchy process subordinated to a Gamma subordinator. On the other hand, in the positively

asymmetric case,  $\mathcal{G}_\alpha^\beta(t)$  reduces to a GS subordinator, which is used in particular as random time argument of the subordinated Brownian motion, via successive iterations (see [6], [27]) Moreover, for  $\alpha = 1/2$ , we can obtain, as a corollary, the fractional equation satisfied by the density  $g'_{1/2}(x, t)$  of the first-passage time of a standard Brownian motion  $B(t)$  through a Gamma distributed random barrier, i.e.

$$g'_{1/2}(x, t) := P \left\{ \inf_{s>0} \{B(s) = \Gamma(t)\} \in dx \right\}, \quad x, t \geq 0.$$

Indeed we prove that  $g'_{1/2}(x, t)$  satisfies the space-fractional equation

$$\frac{\partial^{1/2}}{\partial|x|^{1/2}} g'_{1/2}(x, t) = \sqrt{2}b(1 - e^{-\frac{1}{a}\partial t})g'_{1/2}(x, t), \quad x, t \geq 0, \tag{1.16}$$

where  $\partial^{1/2}/\partial|x|^{1/2} := {}^{RF}\mathcal{D}_{x,1}^{1/2}$ , with the conditions in (1.14).

Finally we consider a Gamma-subordinated process more general than the GS, defined as  $Y(t) = L(\Gamma(t))$ ,  $t \geq 0$ , where  $L(t)$  is a Lévy process with distribution function  $F_L(\cdot, t)$  and generator  $\mathcal{A}_L$ . As the GS process, also  $Y(t)$  is, by definition, a Lévy process and we prove that its generator is given by

$$\mathcal{A}_Y = -\log(1 - \mathcal{A}_L). \tag{1.17}$$

The process  $Y(t)$  turns out to be relevant in the fluctuation theory for Lévy processes, which studies their behavior in the neighborhood of their suprema (or infima), see, for example, [7]. In particular the following result, known as Wiener-Hopf decomposition, holds true (see [17]):

$$Y(1) := L(\Gamma(1)) \stackrel{d}{=} L^+(\Gamma^+(1)) + L^-(\Gamma^-(1)), \tag{1.18}$$

where  $\stackrel{d}{=}$  means "equality in distribution" and  $L^+$  and  $L^-$  are defined as

$$L^+(t) := \sup_{0 \leq s \leq t} L(s), \quad L^- := \inf_{0 \leq s \leq t} L(s).$$

Here  $\Gamma^+(1)$  and  $\Gamma^-(1)$  represent independent, exponentially distributed random times and  $L^+(\Gamma^+(1))$ ,  $L^-(\Gamma^-(1))$  are themselves independent, as well as infinitely divisible.

Moreover, it has been proved in [23] that the generator  $\mathcal{A}_Y$  of the subordinated process  $Y(t) = L(\Gamma(t))$ ,  $t \geq 0$ , can be written as

$$\mathcal{A}_Y = \mathcal{A}_Y^+ + \mathcal{A}_Y^-,$$

where, for  $u \in C^\infty$ ,

$$\begin{aligned} \mathcal{A}_Y^+ u(x) &= \int_{0+}^{+\infty} [u(x-y) - u(x)] \nu_{a,b}(dy) \\ \mathcal{A}_Y^- u(x) &= \int_{-\infty}^{0-} [u(x-y) - u(x)] \nu_{a,b}(dy) \end{aligned}$$

and

$$\nu_{a,b}(dy) = a \int_0^{+\infty} e^{-bt} t^{-1} F_Y(dx, t), \quad x \neq 0.$$

We prove in Proposition 7 below that, in the special case where  $L(t)$  coincides with a symmetric  $\alpha$ -stable process, the generator  $\mathcal{A}_Y$  is given by the following fractional operator:

$$\mathcal{P}_{k,x}^\alpha u(x) := \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l k^l} {}^R\mathcal{D}_x^{\alpha l} u(x), \quad x \geq 0, k \in \mathbb{R},$$

where  ${}^R\mathcal{D}_x^\nu$  is the Riesz derivative of order  $\nu > 0$ . The study of the generators  $\mathcal{A}_Y^+$ ,  $\mathcal{A}_Y^-$  (which, in the stable case, should be fractional as well) is left as an important open issue for future research.

## 2 Preliminary results

We start by deriving the differential equation satisfied by the density of the Gamma subordinator, since it will be applied in the study of the equation governing the GS process (thanks to the representation (1.4)).

The one-dimensional density of the Gamma subordinator  $\{\Gamma_{a,b}(t), t \geq 0\}$ , of parameters  $a, b > 0$  is given by

$$f_{\Gamma_{a,b}}(x, t) := \Pr \{ \Gamma_{a,b}(t) \in dx \} = \begin{cases} \frac{b^{at}}{\Gamma(at)} x^{at-1} e^{-bx}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \quad t \geq 0. \quad (2.1)$$

(see, for example, [3], p.54). Hereafter we will denote for brevity  $\Gamma_{a,b} := \Gamma$ . The Fourier transform of (2.1) is given by

$$\widehat{f}_{\Gamma}(\theta, t) := \mathcal{F} \{ f_{\Gamma}(x, t); \theta \} = \mathbb{E} e^{i\theta \Gamma(t)} = \left( 1 - \frac{i\theta}{b} \right)^{-at}, \quad \theta \in \mathbb{R}. \quad (2.2)$$

**Lemma 2.1.** *The density (2.1) of the Gamma subordinator satisfies the following equation*

$$\frac{\partial}{\partial x} f_{\Gamma} = -b(1 - e^{-\frac{1}{a}\partial_t}) f_{\Gamma}, \quad x, t \geq 0, \quad (2.3)$$

where  $e^{-\partial_t}$  is the partial derivative version of the shift operator defined in (1.13), for  $k = 1/a$ . The initial and boundary conditions are the following

$$\begin{cases} f_{\Gamma}(x, 0) = \delta(x) \\ \lim_{|x| \rightarrow +\infty} f_{\Gamma}(x, t) = 0, \quad t \geq 0 \end{cases}. \quad (2.4)$$

*Proof.* The first condition in (2.4) can be checked easily by considering (2.2) and the definition of the Dirac delta function, i.e.  $\delta(x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\theta x} d\theta$ . The second one is immediately satisfied by (2.1). As far as equation (2.3) is concerned, the Fourier transform of its left-hand side, with respect to  $x$ , is given by

$$\begin{aligned} & \mathcal{F} \left\{ \frac{\partial}{\partial x} f_{\Gamma}(x, t); \theta \right\} \\ &= \text{[by (2.4)]} = -i\theta \widehat{f}_{\Gamma}(\theta, t) = -i\theta \left( \frac{b}{b - i\theta} \right)^{at}. \end{aligned} \quad (2.5)$$

For the right-hand side of (2.3) we have that

$$\begin{aligned} -b \widehat{f}_{\Gamma}(\theta, t) + b e^{-\frac{1}{a}\partial_t} \widehat{f}_{\Gamma}(\theta, t) &= -b \left( \frac{b}{b - i\theta} \right)^{at} + b e^{-\frac{1}{a}\partial_t} \left( \frac{b}{b - i\theta} \right)^{at} \\ &= -b \left( \frac{b}{b - i\theta} \right)^{at} + b \left( \frac{b}{b - i\theta} \right)^{at-1}, \end{aligned}$$

which coincides with (2.5). □

An alternative result on the differential equation satisfied by  $f_{\Gamma}$  can be obtained by considering the following differential operator: for any given infinitely differentiable function  $f(x)$ ,

$$\mathcal{P}_{k,x} f(x) := \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j k^j} D_x^j f(x), \quad x \geq 0, \quad k \in \mathbb{R}. \quad (2.6)$$

We could use for (2.6) the formalism  $\mathcal{P}_{k,x} f(x) = \log(1 + D_x/k)$ .

If moreover  $D_x^j f(x)|_{|x|=\infty} = 0$ , for any  $j \geq 0$ , the Fourier transform of (2.6) can be written as follows:

$$\begin{aligned} \mathcal{F}\{\mathcal{P}_{k,x}f(x); \theta\} &= \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l k^l} \int_{-\infty}^{+\infty} e^{i\theta x} D_x^l f(x) dx \\ &= \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l k^l} (-i\theta)^l \widehat{f}(\theta) \\ &= \log\left(1 - \frac{i\theta}{k}\right) \widehat{f}(\theta). \end{aligned} \tag{2.7}$$

**Lemma 2.2.** *The following differential equation is satisfied by the density of the Gamma subordinator:*

$$\frac{\partial}{\partial t} f_{\Gamma} = -a\mathcal{P}_{b,x}f_{\Gamma}, \quad x, t \geq 0, \tag{2.8}$$

with the conditions

$$\begin{cases} f_{\Gamma}(x, 0) = \delta(x) \\ \lim_{|x| \rightarrow \infty} D_x^l f_{\Gamma}(x, t) = 0, \quad l = 0, 1, \dots \end{cases} \tag{2.9}$$

*Proof.* The conditions (2.9) are immediately verified by (2.1). Moreover, by taking the Fourier transform of the l.h.s. of (2.8), we get

$$\begin{aligned} \mathcal{F}\left\{\frac{\partial}{\partial t} f_{\Gamma}(x, t); \theta\right\} &= \frac{\partial}{\partial t} \left(1 - \frac{i\theta}{b}\right)^{-at} \\ &= -a \left(1 - \frac{i\theta}{b}\right)^{-at} \log\left(1 - \frac{i\theta}{b}\right) \\ &= -a \widehat{f}_{\Gamma}(\theta, t) \log\left(1 - \frac{i\theta}{b}\right) \\ &= -a\mathcal{F}\{\mathcal{P}_{b,x}f_{\Gamma}(x, t); \theta\}. \end{aligned}$$

□

**Remark 2.3.** *From the previous Lemma we can conclude that the infinitesimal generator of the Gamma process can be written as  $\mathcal{A}_{\Gamma} = -a \log(1 + \frac{D_x}{b})$ , while usually it is expressed in the following integral form*

$$\mathcal{A}_{\Gamma}f(x) = \int_0^{+\infty} [f(x+y) - f(x)] \frac{e^{-y}}{y} dy$$

(see, for example, [18]).

### 3 Main results

#### 3.1 Univariate GS process

By resorting to the representation (1.4) and applying the previous results, we can obtain the differential equation satisfied by the density of the univariate GS process  $\mathcal{G}_{\alpha}^{\beta}(t)$ . This can be done, for  $t > 1/a$ , by considering Lemma 1 together with the result (1.11) on  $S_{\alpha}^{\beta}(t)$ , as follows: by (1.4), we can write

$$g_{\alpha}^{\beta}(x, t) = \int_0^{\infty} p_{\alpha}^{\beta}(x, z) f_{\Gamma}(z, t) dz. \tag{3.1}$$

By (2.3) we get

$$\begin{aligned}
 & b(1 - e^{-\frac{1}{a}\partial_t})g_\alpha^\beta(x, t) \\
 = & b \int_0^\infty p_\alpha^\beta(x, z)(1 - e^{-\frac{1}{a}\partial_t})f_\Gamma(z, t)dz \\
 = & - \int_0^\infty p_\alpha^\beta(x, z) \frac{\partial}{\partial z} f_\Gamma(z, t)dz \\
 = & -[p_\alpha^\beta(x, z)f_\Gamma(z, t)]_{z=0}^\infty + \int_0^\infty \frac{\partial}{\partial z} p_\alpha^\beta(x, z)f_\Gamma(z, t)dz \\
 = & c^{RF} \mathcal{D}_{x,\beta}^\alpha \int_0^\infty p_\alpha^\beta(x, z)f_\Gamma(z, t)dz = c^{RF} \mathcal{D}_{x,\beta}^\alpha g_\alpha^\beta(x, t).
 \end{aligned}$$

In the last step we have applied the first equation in (1.11) and we have considered that, for  $t > 1/a$ ,  $f_\Gamma(0, t) = 1$ . In the next theorem we prove the same result in an alternative way, which can be applied for any  $t \geq 0$ .

**Proposition 3.1.** *The density  $g_\alpha^\beta$  of the GS process  $\mathcal{G}_\alpha^\beta(t)$  satisfies the following equation, for any  $x, t \geq 0$  and  $\alpha \in (0, 2]$ ,*

$$c^{RF} \mathcal{D}_{x,\beta}^\alpha g_\alpha^\beta(x, t) = \frac{b}{c}(1 - e^{-\frac{1}{a}\partial_t})g_\alpha^\beta(x, t), \tag{3.2}$$

with conditions

$$\begin{cases} g_\alpha^\beta(x, 0) = \delta(x) \\ \lim_{|x| \rightarrow \infty} g_\alpha^\beta(x, t) = 0 \end{cases}, \tag{3.3}$$

where  $c > 0$  is the spreading rate of dispersion defined in (1.10).

*Proof.* By (3.1) and (1.10) we can write the characteristic function of  $\mathcal{G}_\alpha^\beta(t)$  as

$$\begin{aligned}
 \mathbb{E}e^{i\theta \mathcal{G}_\alpha^\beta(t)} &= \int_0^{+\infty} \exp\{cz\psi_\beta^\alpha(\theta)\}f_\Gamma(z, t)dz \\
 &= \frac{b^{at}}{\Gamma(at)} \int_0^\infty \exp\{cz\psi_\beta^\alpha(\theta)\}z^{at-1}e^{-bz}dz \\
 &= \left(\frac{b}{b - c\psi_\beta^\alpha(\theta)}\right)^{at},
 \end{aligned} \tag{3.4}$$

where  $\psi_\beta^\alpha(\theta)$  is defined in (1.9); thus the Fourier transform of the space-fractional differential equation (3.2) can be written as

$$\begin{aligned}
 & \mathcal{F}\{c^{RF} \mathcal{D}_{x,\beta}^\alpha g_\alpha^\beta(x, t); \theta\} \\
 = & \text{[by (1.8)]} = \psi_\beta^\alpha(\theta) \mathcal{F}\{g_\alpha^\beta(x, t); \theta\} \\
 = & \psi_\beta^\alpha(\theta) \left(\frac{b}{b - c\psi_\beta^\alpha(\theta)}\right)^{at}.
 \end{aligned} \tag{3.5}$$

On the other hand we get

$$\begin{aligned}
 \frac{b}{c}(1 - e^{-\frac{1}{a}\partial_t})\mathcal{F}\{g_\alpha^\beta(x, t); \theta\} &= \frac{b}{c}(1 - e^{-\frac{1}{a}\partial_t}) \left(\frac{b}{b - c\psi_\beta^\alpha(\theta)}\right)^{at} \\
 &= \frac{b}{c} \left(\frac{b}{b - c\psi_\beta^\alpha(\theta)}\right)^{at} - \frac{b}{c} \left(\frac{b}{b - c\psi_\beta^\alpha(\theta)}\right)^{at-1},
 \end{aligned}$$

which coincides with (3.5). The conditions (3.3) are clearly satisfied since

$$g_\alpha^\beta(x, 0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\theta x} \left( \frac{b}{b - c\psi_\beta^\alpha(\theta)} \right)^{at} \Big|_{t=0} d\theta = \delta(x)$$

and  $\lim_{|x| \rightarrow \infty} g_\alpha^\beta(x, t) = 0$  (by (1.11) and (3.1)). □

### 3.1.1 Symmetric GS process

In the special case of a symmetric GS process  $\mathcal{G}_\alpha(t)$  we can easily derive from Proposition 4 the following result, which is expressed in terms of the Riesz derivative  ${}^R\mathcal{D}_x^\alpha$ , defined in (1.6). In its regularized form, for  $\alpha \in (0, 2]$ , the derivative  ${}^R\mathcal{D}_x^\alpha$  can be explicitly represented as

$${}^R\mathcal{D}_x^\alpha u(x) = \frac{\Gamma(1 + \alpha) \sin(\pi\alpha/2)}{\pi} \int_0^\infty \frac{u(x + y) - 2u(x) + u(x - y)}{y^{1+\alpha}} dy, \quad (3.6)$$

(see [20]).

**Corollary 3.2.** *The density  $g_\alpha$  of the symmetric GS process  $\mathcal{G}_\alpha(t)$  satisfies the following equation, for any  $x, t \geq 0$  and  $\alpha \in (0, 2]$ ,*

$${}^R\mathcal{D}_x^\alpha g_\alpha(x, t) = \frac{b}{c} (1 - e^{-\frac{1}{a}\partial_t}) g_\alpha(x, t), \quad (3.7)$$

where  $c = \sigma^\alpha$  and with conditions

$$\begin{cases} g_\alpha(x, 0) = \delta(x) \\ \lim_{|x| \rightarrow \infty} g_\alpha(x, t) = 0 \end{cases} . \quad (3.8)$$

**Remark 3.3.** *We consider now some interesting special cases of the previous results. For  $\alpha = 1$ , we have, from the previous corollary, that the density  $g_1(x, t)$  of a Cauchy process  $\mathcal{C}(t)$  subordinated to an independent Gamma subordinator (i.e. the process defined as  $\{\mathcal{C}(\Gamma(t)), t \geq 0\}$ ) satisfies the following equation, for any  $x, t \geq 0$ :*

$$\frac{\partial}{\partial|x|} g_1(x, t) = \frac{b}{c} (1 - e^{-\frac{1}{a}\partial_t}) g_1(x, t),$$

with conditions (3.8) and  $\partial/\partial|x| := {}^R\mathcal{D}_x^1$ . For  $\alpha = 2$ , we derive the governing equation of the density  $g_2(x, t)$  of the Variance Gamma process, since the latter can be represented as a standard Brownian motion  $B(t)$  subordinated to an independent Gamma subordinator, i.e. as  $\{B(\Gamma(t)), t \geq 0\}$ . In this case the parameters of the Gamma distribution must be specified as follows:  $a = b = 1/\nu$  while  $c = \sigma^2$ , if we follow the usual parametrization  $(\sigma, \theta, \nu, \mu)$  (see, for example formula (6) in [19]); moreover we consider the case where  $\mu = 1$  and  $\theta = 0$  (since the Brownian motion has no drift, in our case). Under these assumptions, we get that  $g_2(x, t)$  satisfies, for any  $x, t \geq 0$ , the second order differential equation

$$\frac{\partial^2}{\partial x^2} g_2(x, t) = \frac{1}{\nu\sigma^2} (1 - e^{-\nu\partial_t}) g_2(x, t),$$

with conditions (3.8).

We derive now another equation satisfied by the density of the symmetric GS process, which, unlike (3.7), involves a standard time derivative and a space fractional

differential operator which generalizes (2.6). Let us define the fractional version of  $\mathcal{P}_{k,x}$ , for any  $\alpha > 0$ , as

$$\mathcal{P}_{k,x}^\alpha f(x) := \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{lk^l} {}^R\mathcal{D}_x^{\alpha l} f(x), \quad x \geq 0, k \in \mathbb{R}, \tag{3.9}$$

where  ${}^R\mathcal{D}_x^\nu$  is the Riesz derivative of order  $\nu > 0$ . We note that in the non-symmetric case (i.e. for  $\beta \neq 0$ ) we can not define the analogue to (3.9) since the Riesz-Feller derivative is not defined for a fractional order greater than 2.

**Proposition 3.4.** *The density  $g_\alpha$  of the symmetric GS process  $\mathcal{G}_\alpha(t)$  satisfies the following equation, for any  $x, t \geq 0$  and  $\alpha \in (0, 2]$ ,*

$$\frac{\partial}{\partial t} g_\alpha(x, t) = a\mathcal{P}_{b/c,x}^\alpha g_\alpha(x, t), \tag{3.10}$$

where  $c = \sigma^\alpha$  and with conditions

$$\begin{cases} g_\alpha(x, 0) = \delta(x) \\ \lim_{|x| \rightarrow \infty} \frac{\partial^l}{\partial x^l} g_\alpha(x, t) = 0, \quad l = 0, 1, \dots \end{cases} \tag{3.11}$$

*Proof.* The Fourier transform of (3.9) is given by

$$\begin{aligned} \mathcal{F}\{\mathcal{P}_{k,x}^\alpha f(x); \theta\} &= \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{lk^l} \int_{-\infty}^{+\infty} e^{i\theta x} {}^R\mathcal{D}_x^{\alpha l} f(x) dx \\ &= -\sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{lk^l} |\theta|^{\alpha l} \mathcal{F}\{f(x); \theta\} \\ &= -\log\left(1 + \frac{|\theta|^\alpha}{k}\right) \mathcal{F}\{f(x); \theta\}. \end{aligned} \tag{3.12}$$

Therefore we get

$$\begin{aligned} \mathcal{F}\left\{\mathcal{P}_{b/c,x}^\alpha g_\alpha(x, t); \theta\right\} &= \log\left(\frac{b}{b + c|\theta|^\alpha}\right) \mathcal{F}\{g_\alpha(x, t); \theta\} \\ &= \log\left(\frac{b}{b + c|\theta|^\alpha}\right) \left(\frac{b}{b + c|\theta|^\alpha}\right)^{at}, \end{aligned} \tag{3.13}$$

since, for  $\beta = 0$ , the characteristic function (3.4) reduces to

$$\mathbb{E}e^{i\theta\mathcal{G}_\alpha(t)} = \left(\frac{b}{b + c|\theta|^\alpha}\right)^{at}.$$

The expression (3.13) clearly coincides, up to the constant  $a$ , with the Fourier transform of the left-hand side of (3.10). □

The previous result agrees with the expression of the infinitesimal generator  $\mathcal{A}_{\mathcal{G}_\alpha}$  of the GS process (with  $a = b = 1$ ), which is given by  $\mathcal{A}_{\mathcal{G}_\alpha} = -\log\left[1 + \left(-\frac{d^2}{dx^2}\right)^{\alpha/2}\right]$  (see [9]). We can generalize it to the case of a more general Gamma-subordinated process defined as

$$Y(t) = L(\Gamma(t)), \quad t \geq 0, \tag{3.14}$$

where  $L(t)$  is an Lévy process with generator  $\mathcal{A}_L$ . As the GS process also  $Y$  is, by definition, a Lévy process (by Theorem 30.1, in [26], p.197) and we evaluate its generator as follows. We put, for simplicity,  $a = b = 1$ .

**Proposition 3.5.** *The generator  $\mathcal{A}_Y$  of the process defined in (3.14) is given by*

$$\mathcal{A}_Y = -\log(1 - \mathcal{A}_L),$$

where  $\mathcal{A}_L$  is the generator of the Lévy process  $L(t)$ .

*Proof.* Let  $f_Y(x, t)$ ,  $x \in \mathbb{R}, t \geq 0$  be the transition density of  $Y(t)$  and  $f_L(x, t)$ ,  $x \in \mathbb{R}, t \geq 0$  be the transition density of  $L(t)$ . Then we can write

$$\begin{aligned} \frac{\partial}{\partial t} f_Y(x, t) &= \int_0^{+\infty} f_L(x, z) \frac{\partial}{\partial t} f_\Gamma(z, t) dz \\ &= \text{[by Lemma 2]} \\ &= - \int_0^{+\infty} f_L(x, z) \mathcal{P}_{1,z} f_\Gamma(z, t) dz \\ &= - \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \int_0^{+\infty} f_L(x, z) \frac{\partial^j}{\partial z^j} f_\Gamma(z, t) dz \\ &= \text{[by successive integrations by parts]} \\ &= - \sum_{j=1}^{\infty} \frac{(-1)^{2j+1}}{j} \int_0^{+\infty} \frac{\partial^j}{\partial z^j} f_L(x, z) f_\Gamma(z, t) dz \\ &= -\log(1 - \mathcal{A}_L) \int_0^{+\infty} f_L(x, z) f_\Gamma(z, t) dz. \end{aligned}$$

□

### 3.1.2 GS subordinator

In the positively asymmetric case, i.e. for  $\beta = 1$ , the process  $\mathcal{G}_\alpha^\beta(t)$  reduces to a GS subordinator (we will denote it as  $\mathcal{G}'_\alpha(t)$ ). We can simply derive the differential equation satisfied by its transition density from Proposition 4, by taking into account that, in this case, the scale parameter in (1.10) reduces to  $c = \sigma^\alpha (\cos(\pi\alpha/2))^{-1}$ . Indeed, for  $\beta = 1$ , we get  $\gamma = \frac{2}{\pi} \arctan[-\tan \frac{\pi\alpha}{2}] = -\alpha$ . Moreover formula (1.9) reduces to  $\psi_\beta^\alpha(\theta) = -|\theta|^\alpha e^{-i \operatorname{sign} \theta \alpha} = -(i|\theta|)^\alpha \operatorname{sign}(\theta)$ .

**Corollary 3.6.** *The density  $g'_\alpha(x, t)$  of the GS subordinator  $\mathcal{G}'_\alpha(t)$  satisfies the following equation, for any  $x, t \geq 0$  and  $\alpha \in (0, 2]$ ,*

$${}^{RF}\mathcal{D}_{x,1}^\alpha g'_\alpha(x, t) = \frac{b}{c} (1 - e^{-\frac{1}{a}\partial t}) g'_\alpha(x, t), \tag{3.15}$$

where  $c = \sigma^\alpha (\cos(\pi\alpha/2))^{-1}$  and with conditions

$$\begin{cases} g'_\alpha(x, 0) = \delta(x) \\ \lim_{|x| \rightarrow \infty} g'_\alpha(x, t) = 0 \end{cases} . \tag{3.16}$$

and  ${}^{RF}\mathcal{D}_{x,1}^\alpha$  is the Riesz-Feller derivative defined by

$$\mathcal{F} \{ {}^{RF}\mathcal{D}_{x,1}^\alpha u(x); \theta \} = -(-i|\theta|)^\alpha \operatorname{sign}(\theta) \mathcal{F} \{ u(x); \theta \}.$$

**Remark 3.7.** *We now consider the special case  $\alpha = 1/2$  of the previous result. It is well-known that the stable law with parameters  $\alpha = 1/2, \mu = 0, \beta = 1, \sigma > 0$  coincides with the Lévy density. Moreover if we define as*

$$\mathcal{T}_z := \inf_{s>0} \{ B(s) = z \}, \quad z \geq 0,$$

the first-passage time of a standard Brownian motion  $B(t)$ , we have that

$$P\{\mathcal{T}_z \in dx\} = p'_{1/2}(x, z), \quad x, z \geq 0,$$

since  $\mathcal{T}_z$  is equal in distribution to a stable random variable  $S'_{1/2}(z)$  of index  $1/2$  and scaling parameter  $z^2$  (whose density is denoted as  $p'_{1/2}(x, z)$ ). Therefore, from the previous corollary, we can derive that the density of the process  $\{\mathcal{T}_{\Gamma(t)}, t \geq 0\}$ , given by

$$g'_{1/2}(x, t) = \int_0^\infty p'_{1/2}(x, z) f_\Gamma(z, t) dz$$

satisfies the following equation for any  $x, t \geq 0$ :

$$\frac{\partial^{1/2}}{\partial |x|^{1/2}} g'_{1/2}(x, t) = \sqrt{2}b(1 - e^{-\frac{1}{a}\partial t})g'_{1/2}(x, t), \quad (3.17)$$

with conditions (3.16) and  $\partial^{1/2}/\partial |x|^{1/2} := {}^{RF}\mathcal{D}_{x,1}^{1/2}$ . The constant in (3.17) can be derived by considering that, in this case,  $c = (\cos(\pi/4))^{-1}$ . The process  $\mathcal{T}_{\Gamma(t)}$  can be interpreted as the first-passage time of a standard Brownian motion through a random barrier, represented by a Gamma process of parameters  $a, b$ . Thus we can conclude that

$$P\left\{\inf_{s>0}\{B(s) = \Gamma(t)\} \in dx\right\}, \quad x, t \geq 0$$

satisfies the space-fractional equation (3.17).

### 3.2 Multivariate GS process

The multivariate GS distribution was first defined in [22] and applied later to model multivariate financial portfolios of securities, in [15].

In the  $n$ -dimensional case, we denote by  $\{\mathbf{G}_\alpha^n(t), t \geq 0\}$  a multivariate GS process with stability index  $\alpha \in (0, 2]$ , position parameter  $\mu = \mathbf{0}$  (for simplicity) and spectral measure  $M$ , then its characteristic function can be written as

$$\mathbb{E}e^{i\langle \theta, \mathbf{G}_\alpha^n(t) \rangle} = \left[1 + \int_{S^n} |\langle \theta, \mathbf{z} \rangle|^\alpha \omega_{\alpha,1}(\langle \theta, \mathbf{z} \rangle) M(d\mathbf{z})\right]^{-t}, \quad \theta \in \mathbb{R}^n, \quad (3.18)$$

where  $S^n := \{\mathbf{s} \in \mathbb{R}^n : \|\mathbf{s}\| = 1\}$ ,  $\langle \theta, \mathbf{z} \rangle = \sum_{j=1}^n \theta_j z_j$  and

$$\omega_{\alpha,1}(\langle \theta, \mathbf{z} \rangle) := \begin{cases} 1 - i \operatorname{sign}(\langle \theta, \mathbf{z} \rangle) \tan(\pi\alpha/2), & \text{if } \alpha \neq 1 \\ 1 + 2i \operatorname{sign}(\langle \theta, \mathbf{z} \rangle) \log |\langle \theta, \mathbf{z} \rangle|/\pi, & \text{if } \alpha = 1 \end{cases}.$$

Moreover, as in the univariate case, the following relationship holds for the r.v.  $\mathbf{G}_\alpha^n := \mathbf{G}_\alpha^n(1)$ :

$$\mathbb{E}e^{i\theta \mathbf{G}_\alpha^n} = \frac{1}{1 - \log \Phi_{\mathbf{S}_\alpha^n}(\theta)}, \quad \theta \in \mathbb{R}^n$$

(see [15]), where

$$\Phi_{\mathbf{S}_\alpha^n}(\theta) := \mathbb{E}e^{i\langle \theta, \mathbf{S}_\alpha^n \rangle} = \exp\left\{-\int_{S^n} |\langle \theta, \mathbf{z} \rangle|^\alpha \omega_{\alpha,1}(\langle \theta, \mathbf{z} \rangle) M(d\mathbf{z})\right\}, \quad \theta \in \mathbb{R}^n$$

is the characteristic function of a stable multivariate r.v.  $\mathbf{S}_\alpha^n$  with  $\mu = \mathbf{0}$  and spectral measure  $M$  (see e.g. [25], p.65).

Let the process  $\{\mathbf{S}_\alpha^n(t), t \geq 0\}$  be defined by its characteristic function, i.e.

$$\Phi_{\mathbf{S}_\alpha^n(t)}(\theta) := \mathbb{E}e^{i\langle \theta, \mathbf{S}_\alpha^n(t) \rangle} = \exp\left\{-t \int_{S^n} |\langle \theta, \mathbf{z} \rangle|^\alpha \omega_{\alpha,1}(\langle \theta, \mathbf{z} \rangle) M(d\mathbf{z})\right\}, \quad \theta \in \mathbb{R}^n.$$

Then the transition density  $p_\alpha^n(\mathbf{x}, t)$  of  $\mathbf{S}_\alpha^n(t)$  satisfies the initial value problem, for  $\alpha \in (0, 2], \alpha \neq 1$ ,

$$\begin{cases} \nabla_M^\alpha p_\alpha^n(\mathbf{x}, t) = \frac{1}{c} \frac{\partial}{\partial t} p_\alpha^n(\mathbf{x}, t) \\ p_\alpha^n(\mathbf{x}, 0) = \delta(\mathbf{x}) \end{cases}, \quad \mathbf{x} \in \mathbb{R}^n, t \geq 0, \quad (3.19)$$

where  $c = (\cos(\pi\alpha/2))^{-1}$  (see [21], being careful with the signs, for the different definition of Fourier transform) and  $\nabla_M^\alpha$  is the fractional derivative operator defined in (1.15).

The results of the previous section can be generalized to the  $n$ -dimensional case, as follows.

**Proposition 3.8.** *The transition density  $g_\alpha^n(\mathbf{x}, t)$  of the  $n$ -dimensional GS process  $\mathbf{G}_\alpha^n(t)$  satisfies the following Cauchy problem, for  $\alpha \in (0, 2], \alpha \neq 1, c = (\cos(\pi\alpha/2))^{-1}$ ,*

$$\begin{cases} \nabla_M^\alpha g_\alpha^n(\mathbf{x}, t) = \frac{1}{c}(1 - e^{-\partial t})g_\alpha^n(\mathbf{x}, t) \\ g_\alpha^n(\mathbf{x}, 0) = \delta(\mathbf{x}) \\ \lim_{\|\mathbf{x}\| \rightarrow \infty} g_\alpha^n(\mathbf{x}, t) = 0 \end{cases}, \quad \mathbf{x} \in \mathbb{R}^n, t \geq 0. \quad (3.20)$$

*Proof.* The Fourier transform of the space-fractional differential equation in (3.20) can be written as

$$\begin{aligned} & \mathcal{F} \{ \nabla_M^\alpha g_\alpha^n(\mathbf{x}, t); \theta \} \\ &= [\text{by (1.15)}] = - \left[ \int_{S^n} (-i \langle \theta, \mathbf{z} \rangle)^\alpha M(d\mathbf{z}) \right] \mathcal{F} \{ g_\alpha^n(\mathbf{x}, t); \theta \} \\ &= -\cos(\pi\alpha/2) \left[ \int_{S^n} |\langle \theta, \mathbf{z} \rangle|^\alpha \omega_{\alpha,1}(\langle \theta, \mathbf{z} \rangle) M(d\mathbf{z}) \right] \left[ 1 + \int_{S^n} |\langle \theta, \mathbf{z} \rangle|^\alpha \omega_{\alpha,1}(\langle \theta, \mathbf{z} \rangle) M(d\mathbf{z}) \right]^{-t} \\ &= [\text{by (3.18)}] \\ &= \cos(\pi\alpha/2)(1 - e^{-\partial t}) \mathcal{F} \{ g_\alpha^n(\mathbf{x}, t); \theta \}, \end{aligned}$$

by considering that

$$(-i \langle \theta, \mathbf{z} \rangle)^\alpha = |\langle \theta, \mathbf{z} \rangle|^\alpha \cos(\pi\alpha/2) \omega_{\alpha,1}(\langle \theta, \mathbf{z} \rangle).$$

The first condition in (3.20) is verified since the characteristic function of  $\mathbf{G}_\alpha^n(t)$ , given in (3.18), reduces to 1 for  $t = 0$ , while for the second one we must consider that

$$g_\alpha^n(\mathbf{x}, t) = \int_0^\infty p_\alpha^n(\mathbf{x}, z) f_\Gamma(z, t) dz$$

and that  $\lim_{\|\mathbf{x}\| \rightarrow \infty} p_\alpha^n(\mathbf{x}, z) = 0$ . □

**Remark 3.9.** *If we consider the special case of an isotropic  $n$ -dimensional GS process  $\{\mathbf{G}_\alpha(t), t \geq 0\}$ , the previous results can be considerably simplified. Indeed in this case we can use the fractional Laplace operator defined by*

$$\mathcal{F} \{ (-\Delta)^\alpha u(\mathbf{x}); \theta \} = -\|\theta\|^\alpha \mathcal{F} \{ u(\mathbf{x}); \theta \}, \quad \mathbf{x}, \theta \in \mathbb{R}^n \quad (3.21)$$

(where  $\|\cdot\|$  denotes the Euclidean norm) or, by the Bochner representation, as

$$(-\Delta)^\alpha = -\frac{\sin(\pi\alpha)}{\pi} \int_0^{+\infty} z^{\alpha-1} (z - \Delta)^{-1} \Delta dz \quad (3.22)$$

(see [5] and [4]). Moreover the  $n$ -dimensional isotropic GS process is defined through its characteristic function

$$\mathbb{E} e^{i \langle \theta, \mathbf{G}_\alpha(t) \rangle} = \left( \frac{1}{1 + \|\theta\|^\alpha} \right)^t, \quad \theta \in \mathbb{R}^n \quad (3.23)$$

(see [15]) and its marginals coincide with the multivariate Linnik distributions introduced in [1]. The process  $\mathbf{G}_\alpha(t)$  can be represented as

$$\mathbf{G}_\alpha(t) := \mathbf{S}_\alpha(\Gamma(t)), \quad t \geq 0, \quad (3.24)$$

where  $\Gamma(t)$  is now assumed, for simplicity, to have parameters  $a = b = 1$  and  $\{\mathbf{S}_\alpha(t), t > 0\}$  is an independent isotropic stable vector, with characteristic function

$$\mathbb{E}e^{i\langle \theta, \mathbf{S}_\alpha(t) \rangle} = \exp\{-t\|\theta\|^\alpha\}, \quad \theta \in \mathbb{R}^n.$$

Then it is well-known that the density  $p_\alpha(\mathbf{x}, t)$  of  $\mathbf{S}_\alpha(t)$  satisfies the equation

$$\begin{cases} (-\Delta)^\alpha p_\alpha(\mathbf{x}, t) = \frac{1}{c} \frac{\partial}{\partial t} p_\alpha(\mathbf{x}, t) \\ p_\alpha(\mathbf{x}, 0) = \delta(\mathbf{x}) \\ \lim_{\|\mathbf{x}\| \rightarrow \infty} p_\alpha(\mathbf{x}, t) = 0 \end{cases}, \quad (3.25)$$

for  $x \in \mathbb{R}^n$ ,  $t \geq 0$ ,  $c = (\cos(\pi\alpha/2))^{-1}$  and  $\alpha \in (0, 2]$ . Therefore by Proposition 4, we can conclude that the density  $g_\alpha(\mathbf{x}, t)$  of  $\mathbf{G}_\alpha(t)$  satisfies the following Cauchy problem, for any  $\mathbf{x} \in \mathbb{R}^n$ ,  $t \geq 0$ :

$$\begin{cases} (-\Delta)^\alpha g_\alpha(\mathbf{x}, t) = \frac{1}{c}(1 - e^{-\partial_t})g_\alpha(\mathbf{x}, t) \\ g_\alpha(\mathbf{x}, 0) = \delta(\mathbf{x}) \\ \lim_{\|\mathbf{x}\| \rightarrow \infty} g_\alpha(\mathbf{x}, t) = 0 \end{cases}, \quad (3.26)$$

where  $(-\Delta)^\alpha$  is the fractional Laplace operator defined in (3.21).

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