Erratum: Convergence in law in the second Wiener/Wigner chaos

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Abstract

We correct an error in our paper [1].

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1 Introduction

We use the same notation as in [1] and we assume that the reader is familiar with it. We are indebted to Giovanni Peccati for pointing out, in the most constructive and gentle way, an error in [1, Theorem 3.4] and for providing an explicit counterexample supporting his claim.

2 A correct version of Lemma 3.5

Unfortunately, Lemma 3.5 in [1] is not correct. Our mistake comes from an improper calculation involving a Vandermonde determinant at the end of its proof. To fix the error is not a big deal though: it suffices to replace different by consecutive in the statement of Lemma 3.5, see below for a correct version together with its proof. As a direct consequence of this new version, we should also replace different by consecutive in the assumption (i-c) of both Theorems 3.4 and 4.3 in [1]. We restate these latter results correctly in Section 2 for convenience.

Lemma 3.5. Let $\mu_0 \in \mathbb{R}$, let $a \in \mathbb{N}^*$, let $\mu_1, \ldots, \mu_a \neq 0$ be pairwise distinct real numbers, and let $m_1, \ldots, m_a \in \mathbb{N}^*$. Set

$$Q(x) = x^{2(1+1_{\{\mu_0 \neq 0\}})} \prod_{i=1}^{a} (x - \mu_i)^2.$$
Assume that \( \{ \lambda_j \}_{j \geq 0} \) is a square-integrable sequence of real numbers satisfying
\[
\lambda_0^2 + \sum_{j=1}^\infty \lambda_j^2 = \mu_0^2 + \sum_{i=1}^a m_i \mu_i^2
\]
(2.1)
\[
2^{(1+1(\mu_0 \neq 0)+a)} \sum_{r=2}^{(2(1+1(\mu_0 \neq 0)+a))} \frac{Q^{(r)}(0)}{r!} \sum_{j=1}^\infty \lambda_j^r = \sum_{r=2}^{(2(1+1(\mu_0 \neq 0)+a))} \frac{Q^{(r)}(0)}{r!} \sum_{i=1}^a m_i \mu_i^r
\]
(2.2)
\[
\sum_{j=1}^\infty \lambda_j^r = \sum_{i=1}^a m_i \mu_i^r, \text{ for } 'a' \text{ consecutive values of } r \geq 2(1 + 1(\mu_0 \neq 0)).
\]
(2.3)

Then:
(i) \( |\lambda_0| = |\mu_0| \).
(ii) The cardinality of the set \( S = \{ j \geq 1 : \lambda_j \neq 0 \} \) is finite.
(iii) \( \{ \lambda_j \}_{j \in S} = \{ \mu_i \}_{1 \leq i \leq a} \).
(iv) For any \( i = 1, \ldots, a \), one has \( m_i = \# \{ j \in S : \lambda_j = \mu_i \} \).

Proof. As in the original proof of [1, Lemma 3.5], we divide the proof according to the nullity of \( \mu_0 \).

First case: \( \mu_0 = 0 \). We have \( Q(x) = x^2 \prod_{i=1}^a (x - \mu_i)^2 \). Since the polynomial \( Q \) can be rewritten as
\[
Q(x) = \sum_{r=2}^{2(1+a)} \frac{Q^{(r)}(0)}{r!} x^r,
\]
assumptions (2.1) and (2.2) together ensure that
\[
\lambda_0^2 \prod_{i=1}^a \mu_i^2 + \infty \sum_{j=1}^\infty Q(\lambda_j) = \sum_{i=1}^a m_i Q(\mu_i) = 0.
\]
Because \( Q \) is positive and \( \prod_{i=1}^a \mu_i^2 \neq 0 \), we deduce that \( \lambda_0 = 0 \) and \( Q(\lambda_j) = 0 \) for all \( j \geq 1 \), that is, \( \lambda_j \in \{ 0, \mu_1, \ldots, \mu_a \} \) for all \( j \geq 1 \). This shows claims (i) as well as:
\[
\{ \lambda_j \}_{j \in S} \subset \{ \mu_i \}_{1 \leq i \leq a}.
\]
(2.4)

Moreover, since the sequence \( \{ \lambda_j \}_{j \geq 1} \) is square-integrable, claim (ii) holds true as well. It remains to show (iii) and (iv). For any \( i = 1, \ldots, a \), let \( n_i = \# \{ j \in S : \lambda_j = \mu_i \} \). Also, let \( r \geq 2 \) be such that \( r, r+1, \ldots, r+a-1 \) are ‘a’ consecutive values satisfying (2.3). We then have
\[
\begin{pmatrix}
\mu_1^r \\
\mu_1^{r+1} \\
\vdots \\
\mu_1^{r+a-1}
\end{pmatrix}
\begin{pmatrix}
\mu_2^r \\
\mu_2^{r+1} \\
\vdots \\
\mu_2^{r+a-1}
\end{pmatrix}
\cdots
\begin{pmatrix}
\mu_a^r \\
\mu_a^{r+1} \\
\vdots \\
\mu_a^{r+a-1}
\end{pmatrix}
\begin{pmatrix}
n_1 - m_1 \\
n_2 - m_2 \\
\vdots \\
n_a - m_a
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]

Since \( \mu_1, \ldots, \mu_a \neq 0 \) are pairwise distinct, one has (Vandermonde matrix)
\[
\det \begin{pmatrix}
\mu_1^{r+1} & \mu_2^{r+1} & \cdots & \mu_a^{r+1} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_1^{r+a-1} & \mu_2^{r+a-1} & \cdots & \mu_a^{r+a-1}
\end{pmatrix}
= \prod_{i=1}^a \mu_i^r \times \det \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\mu_1 & \mu_2 & \cdots & \mu_a \\
\vdots & \vdots & \ddots & \vdots \\
\mu_1^{a-1} & \mu_2^{a-1} & \cdots & \mu_a^{a-1}
\end{pmatrix} \neq 0,
\]
from which (iv) follows. Finally, recalling the inclusion (2.4) we deduce (iii).

Second case: $\mu_0 \neq 0$. In this case, one has $Q(x) = x^4 \prod_{i=1}^3 (x - \mu_i)^2$ and claims (ii), (iii) and (iv) may be shown by following the same line of reasoning as above. We then deduce claim (i) by looking at (2.1).

\section{Correct versions of Theorems 3.4 and 4.3}

For convenience, we restate Theorems 3.4 and 4.3 correctly here. Their proofs are unchanged.

\textbf{Theorem 3.4.} Let $f \in L^2_2(\mathbb{R}^2)$ with $0 \leq \text{rank}(f) < \infty$, let $\mu_0 \in \mathbb{R}$ and let $N \sim \mathcal{N}(0, \mu_0^2)$ be independent of the underlying Brownian motion $W$. Assume that $|\mu_0| + \|f\|_{L^2(\mathbb{R}_+)} > 0$ and set

$$Q(x) = x^{2(1+1)}(\mu_0 x^0) \prod_{i=1}^2 (x - \lambda_i(f))^2.$$ 

Let $\{F_n\}_{n \geq 1}$ be a sequence of double Wiener-Itô integrals. Then, as $n \to \infty$, we have

(i) $F_n \overset{\text{law}}{\to} N + I^W_2(f)$

if and only if all the following are satisfied:

(ii-a) $\kappa_2(F_n) \to \kappa_2(N + I^W_2(f)) = \mu_0^2 + 2\|f\|_{L^2(\mathbb{R}^2)}^2$;

(ii-b) $\sum_{r=1}^{\deg Q} Q^{(r)}(0) \frac{\kappa_r(F_n)}{(r-1)!} \to \sum_{r=1}^{\deg Q} Q^{(r)}(0) \frac{\kappa_r(I^W_2(f))}{(r-1)!}^2$;

(ii-c) $\kappa_r(F_n) \to \kappa_r(I^W_2(f))$ for $a(f)$ consecutive values of $r$, with $r \geq 2(1 + 1_{\{\mu_0 \neq 0\}})$.

\textbf{Theorem 4.3.} Let $f \in L^2_2(\mathbb{R}^2)$ with $0 \leq \text{rank}(f) < \infty$, let $\mu_0 \in \mathbb{R}$ and let $A \sim S(0, \mu_0^2)$ be independent of the underlying free Brownian motion $S$. Assume that $|\mu_0| + \|f\|_{L^2(\mathbb{R}_+)} > 0$ and set

$$Q(x) = x^{2(1+1)}(\mu_0 x^0) \prod_{i=1}^2 (x - \lambda_i(f))^2.$$ 

Let $\{F_n\}_{n \geq 1}$ be a sequence of double Wigner integrals. Then, as $n \to \infty$, we have

(i) $F_n \overset{\text{law}}{\to} A + I^S_2(f)$

if and only if all the following are satisfied:

(ii-a) $\tilde{\kappa}_2(F_n) \to \tilde{\kappa}_2(A + I^S_2(f)) = \mu_0^2 + \|f\|_{L^2(\mathbb{R}^2)}^2$;

(ii-b) $\sum_{r=1}^{\deg Q} Q^{(r)}(0) \frac{\tilde{\kappa}_r(F_n)}{(r-1)!} \to \sum_{r=1}^{\deg Q} Q^{(r)}(0) \frac{\tilde{\kappa}_r(I^S_2(f))}{(r-1)!}^2$;

(ii-c) $\tilde{\kappa}_r(F_n) \to \tilde{\kappa}_r(I^S_2(f))$ for $a(f)$ consecutive values of $r$, with $r \geq 2(1 + 1_{\{\mu_0 \neq 0\}})$.

\textbf{References}