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Convergence of clock process in random environments and aging in Bouchaud's asymmetric trap model on the complete graph

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Abstract

In this paper the celebrated arcsine aging scheme of Ben Arous and Černý is taken up. Using a brand new approach based on point processes and weak convergence techniques, this scheme is implemented in a broad class of Markov jump processes in random environments that includes Glauber dynamics of discrete disordered systems. More specifically, conditions are given for the underlying clock process (a partial sum process that measures the total time elapsed along paths of a given length) to converge to a subordinator, and consequences for certain time correlation functions are drawn. This approach is applied to Bouchaud's asymmetric trap model on the complete graph for which aging is for the first time proved, and the full, optimal picture, obtained. Application to spin glasses are carried out in follow up papers.

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1 Introduction

The term aging qualifies dynamics whose transients towards equilibrium become increasingly slower as time elapses. This phenomenon is measured in the anomalous behavior of certain time correlation functions. Discovered in the physics of spin glasses, aging was successfully accounted for, on a theoretical level, using simple phenomenological models – the so called trap models of Bouchaud [15, 18, 33, 17, 16]. These are Markov jump processes that describe the behavior of spin glass dynamics on long time scales in terms of thermally activated barrier crossing in landscapes made of random 'traps'.

The first rigorous connection between the microscopic dynamics of a spin system and a trap model was established in [3, 4] where it is proved that a particular Glauber dynamics of the REM, known as the Random Hopping Dynamics (hereafter RHD), has the same aging behavior as Bouchaud's symmetric trap model on the complete graph. Meanwhile, in another direction of research, trap models on $\mathbb{Z}^d$ with i.i.d. heavy tailed

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landscapes where studied in depth [7, 8, 10, 25]. From this it emerged that aging in $\mathbb{Z}^d$, $d \geq 2$, also is the same as in Bouchaud's symmetric trap model on the complete graph. In all these situations a common arcsine law governs the relevant time correlation functions

In a landmark paper [9], Ben Arous and Černý proposed a scheme that explains this apparent universality by linking aging to the arcsine law for subordinators through the (asymptotic) nature of a partial sum process called the *clock process*. In the present paper we take a new look at this scheme and extend it to a general framework that covers Glauber dynamics of general disordered systems. As a first, simple but important application we study Bouchaud's asymmetric trap model on the complete graph. Applications to spin glasses and general Glauber dynamics, that are not of RHD type, are carried out in follow up papers [19, 20, 27, 28].

1.1 Setting.

Let us describe the Markov jump processes in random environments that we are interested in. Let $G_n(\mathcal{V}_n, \mathcal{E}_n)$, $n \in \mathbb{N}$, be a sequence of connected graphs with set of vertices $\mathcal{V}_n$ and set of edges $\mathcal{E}_n$. We assume that $|\mathcal{V}_n| \uparrow \infty$ as $n \uparrow \infty$ (possibly, $\mathcal{V}_n = \mathcal{V}_\infty$). A *random landscape* on $\mathcal{V}_n$, or *random environment*, is a family $(\tau_n(x), x \in \mathcal{V}_n)$ of non-negative random variables defined on some common probability space $(\Omega^\tau, \mathcal{F}^\tau, \mathbb{P})$. Note that we do not assume independence.

On $\mathcal{V}_n$ we consider a continuous time Markov chain $(X_n(t), t \geq 0)$ with graph $G_n(\mathcal{V}_n, \mathcal{E}_n)$, whose infinitesimal generator $\Lambda_n \equiv (\lambda_n(x,y))_{x,y\in\mathcal{V}_n}$ is a random matrix on $(\Omega^\tau, \mathcal{F}^\tau, \mathbb{P})$. Of course, setting $\lambda_n(x,x) \equiv -\lambda_n(x)$, we must have

$$\lambda_n(x) \equiv \sum_{y\in\mathcal{V}_n, y\neq x} \lambda_n(x,y), \quad \forall x \in \mathcal{V}_n . \tag{1.1}$$

We will assume that $\sup_{x\in\mathcal{V}_n} \lambda_n(x) < C$ $\mathbb{P}$-a.s. for some $0 < C < \infty$. A specially important class of such matrices is obtained by choosing the $\lambda_n(y,x)$'s such that

$$\tau_n(x)\lambda_n(x,y) = \tau_n(y)\lambda_n(y,x), \quad \forall\, (x,y) \in \mathcal{E}_n\,, x \neq y\,, \tag{1.2}$$

and $\lambda_n(y,x) = 0$ for all $(x,y) \notin \mathcal{E}_n\,, x \neq y$. This implies that $X_n(t)$ is reversible w.r.t. the random measure on $\mathcal{V}_n$ that assigns to x the mass $\tau_n(x)$. Glauber dynamics in particular belong to this class.

Given a family $(e_{n,i}\,, n \in \mathbb{N}, i \in \mathbb{N})$ of independent mean one exponential r.v.'s the *clock process* is the partial sum process defined by

$$\widetilde{S}_n(k) = \sum_{i=0}^{k} \lambda_n^{-1}(J_n(i))e_{n,i}\,, \quad k \in \mathbb{N}\,. \tag{1.3}$$

Here $(J_n(k)\,, k \in \mathbb{N})$ is the *jump chain* of X_n, namely, the discrete time Markov chain with one-step transition probabilities

$$p_n(x,y) = \begin{cases} \lambda_n(x,y)/\lambda_n(x) & \text{if } (x,y) \in \mathcal{E}_n\,, x \neq y, \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

Note that $\widetilde{S}_n(k)$ gives the total time spent by X_n along the first k steps of J_n. Thus, if X_n has initial distribution μ_n, J_n has initial distribution μ_n and

$$X_n(t) = J_n(\widetilde{S}_n^{\leftarrow}(t))\,, \quad t > 0. \tag{1.5}$$

(Here $\widetilde{S}_n^{\leftarrow}$ denotes the right continuous inverse of $\widetilde{S}_n$.)

The last expression places the clock process in the limelight. The idea behind the arcsine aging scheme is that if, after appropriate rescaling, the clock process converges to a stable subordinator, then anomalous slowdown of the long term dynamics can be explained in terms of the arcsine law for stable subordinators. To put this scheme into practice one faces two difficulties: the clock process is a random process on the probability space of the environment, and, for fixed realization of the environment, it is a partial sum process whose summands are made dependent through the chain J_n.

In [9] this problem is solved for dynamics of RHD type, that is, denoting by d_x the degree of x in the graph $G_n(\mathcal{V}_n, \mathcal{E}_n)$, for the rates

$$\lambda_n(y,x) = (d_x \tau_n(x))^{-1} \text{ if } (x,y) \in \mathcal{E}_n \,. \tag{1.6}$$

Using a detailed knowledge of the potential theory of the chain J_n (reduced here to the symmetric random walk on $G_n(\mathcal{V}_n, \mathcal{E}_n)$) and properties of the environments, a set of abstract conditions is derived that ensure that the clock process converges to a stable subordinator for $\mathbb{P}$-almost all environments. Although independence of the $\tau_n(x)'s$ is not assumed a priori, this approach was only applied to such environments. In particular, it did not allow to deal with the p-spin SK spin glass model. This was done in [2] where, using approximations techniques for Gaussian processes, it is proved that on a certain range of times scales and temperature, and for the rates (1.6), the clock process converges to a stable subordinator, but in $\mathbb{P}$-law only.

In the present paper we adopt yet another approach that allows to both implement the arcsine aging scheme in the general setting of Markov jump processes in random environments described above, and obtain results in the strongest possible convergence mode with respect to the law $\mathbb{P}$ of the environment. Our approach is based on a powerful and illuminating method developed by Durrett and Resnick [21] to prove functional limit theorems for dependent variables. By extending the framework of [21] to our random setting, and specializing it to processes of the form (1.3), we give simple sufficient conditions for the properly rescaled sequence S_n to converge to a subordinator. This is the content of Subsection 1.2 below. Consequences for aging are drawn in Subsection 1.3

For later reference we denote by $\mathcal{P}_{\mu_n}$ the law of X_n and by P_{μ_n} the law of J_n with initial distribution μ_n. In view of taking $n \uparrow \infty$ limits we assume that the sequences of chains X_n, resp. J_n, can be constructed on a common probability space $(\Omega^X, \mathcal{F}^X, \mathcal{P})$, resp. $(\Omega^J, \mathcal{F}^J, P)$. Expectation with respect to $\mathbb{P}$, P, and $\mathcal{P}$ will be denoted respectively by $\mathbb{E}$, E, and $\mathcal{E}$

1.2 Convergence of the clock process to a subordinator.

The first increment of the clock process plays a special role. For this reason we define

$$\sigma_n = c_n^{-1} \widetilde{S}_n(0)\,, \quad \overline{S}_n(k) = \begin{cases} \sum_{i=1}^{k} \lambda_n^{-1}(J_n(i)) e_{n,i} & \text{if } k \geq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.7}$$

Given a positive sequences c_n and a_n we then set, for $t \geq 0$,

$$S_n(t) = c_n^{-1} \overline{S}_n(\lfloor a_n t \rfloor)\,, \tag{1.8}$$

and

$$\widehat{S}_n(t) = \sigma_n + S_n(t)\,. \tag{1.9}$$

The re-scaled clock processes $S_n(t)$ and $\widehat{S}_n(t)$ will be called *pure* and *delayed*, respectively.

We now state three conditions, (A1)-(A3), that ensure that the sequence of pure processes S_n converges to a subordinator. Because this process is a random variable on the probability space $(\Omega^\tau, \mathcal{F}^\tau, \mathbb{P})$ of the landscape (our random environment) we must first decide in which sense to seek convergence on that space. The relevant convergence modes (those that convey the most useful information in applications) are almost sure convergence and convergence in probability. This means that one of the following statements should be in force:

Almost sure convergence: There exists a subset $\widetilde{\Omega}^\tau \subset \Omega^\tau$ such that $\mathbb{P}(\widetilde{\Omega}^\tau) = 1$ and such that, for all $\omega \in \widetilde{\Omega}^\tau$, for all large enough n, (A1)-(A3) are verified.

Convergence in probability: There exists a sequence $\widetilde{\Omega}^\tau_n \subset \Omega^\tau$ such that $\lim_{n\to\infty} \mathbb{P}(\widetilde{\Omega}^\tau_n) = 1$ and such that, for all large enough n, (A1)-(A3) are verified for all $\omega \in \widetilde{\Omega}^\tau_n$.

We now state our three conditions for fixed ω and make this explicit by adding the superscript ω to landscape dependent quantities. These conditions depend on the initial distribution μ_n, and on the sequences a_n and c_n. We suppose them fixed.

Condition (A1). There exists a σ-finite measure ν on $(0,\infty)$ satisfying $\int_{(0,\infty)}(1 \wedge u)\nu(du) < \infty$ such that, for all $t > 0$ and all $u > 0$,[1]

$$P^\omega\left(\left|\sum_{j=1}^{\lfloor a_n t\rfloor}\sum_{x\in\mathcal{V}_n} p_n^\omega(J_n^\omega(j-1),x)e^{-uc_n\lambda_n^\omega(x)} - t\nu(u,\infty)\right| < \varepsilon\right) = 1 - o(1)\,, \quad \forall\varepsilon > 0\,. \quad (1.10)$$

Condition (A2). For all $u > 0$ and all $t > 0$,

$$P^\omega\left(\sum_{j=1}^{\lfloor a_n t\rfloor}\left[\sum_{x\in\mathcal{V}_n} p_n^\omega(J_n^\omega(j-1),x)e^{-uc_n\lambda_n^\omega(x)}\right]^2 < \varepsilon\right) = 1 - o(1)\,, \quad \forall\varepsilon > 0\,. \quad (1.11)$$

Condition (A3). There exists a sequence of functions $\varepsilon_n \geq 0$ satisfying $\lim_{\delta\to 0}\limsup_{n\to\infty}\varepsilon_n(\delta) = 0$ such that for some $0 < \delta_0 \leq 1$, for all $0 < \delta \leq \delta_0$ and all $t > 0$,

$$E^\omega\left(\int_0^\delta du \sum_{j=1}^{\lfloor a_n t\rfloor}\sum_{x\in\mathcal{V}_n} p_n^\omega(J_n^\omega(j-1),x)e^{-uc_n\lambda_n^\omega(x)}\right) \leq t\varepsilon_n(\delta)\,. \quad (1.12)$$

Theorem 1.1. *For all sequences of initial distributions μ_n and all sequences a_n and c_n for which Conditions (A1), (A2), and (A3) are verified, either $\mathbb{P}$-almost surely or in $\mathbb{P}$-probability, the following holds w.r.t. the same convergence mode: let $\{(t_k,\xi_k)\}$ be the points of a Poisson random measure of intensity measure $dt \times d\nu$; then,*

$$S_n(\cdot) \Rightarrow S(\cdot) \equiv \sum_{t_k \leq \cdot} \xi_k\,, \quad (1.13)$$

where convergence holds weakly in the space $D([0,\infty))$ of càdlàg functions on $[0,\infty)$ equipped with the Skorohod J_1-topology [2].

Remark 1.2. *Although we do not make this explicit in the notation, note that the Lévy measure ν of the limiting subordinator S may remain a random variable on the probability space $(\Omega^\tau, \mathcal{F}^\tau, \mathbb{P})$ of the random landscape. We will see an example of this in the context of the asymmetric trap model on the complete graph (cf. Proposition 3.9).*

To obtain convergence of the delayed re-scaled clock process $\widehat{S}_n$ of (1.9), we still need to control the initial increment σ_n. For this we introduce a separate condition.

[1] The set $\widetilde{\Omega}^\tau$ (respectively the sequence of sets $\widetilde{\Omega}^\tau_n$) for which convergence w. r. t. the environment holds almost surely (respectively in probability) is (are) the same for all $t > 0$ and $u > 0$.

[2] see e. g. [36] p. 83 for the definition of convergence in $D([0,\infty))$.

Condition (A0). There exists a continuous distribution function F^ω on $[0,\infty)$ such that, for all $v \geq 0$,

$$\left| \sum_{x\in\mathcal{V}_n} \mu_n^\omega(x) e^{-vc_n\lambda_n^\omega(x)} - (1 - F^\omega(v)) \right| = o(1) \,. \tag{1.14}$$

Theorem 1.3. *For all sequences of initial distributions μ_n and all sequences a_n and c_n for which Conditions (A0), (A1), (A2), and (A3) are verified, either $\mathbb{P}$-almost surely or in $\mathbb{P}$-probability, the following holds w.r.t. the same convergence mode: let σ denote the random variable of (possibly random) distribution function F; then, for S defined in (1.13),*

$$\widehat{S}_n \Rightarrow \widehat{S} = \sigma + S \,, \tag{1.15}$$

where $\Rightarrow$ has the same meaning as in (1.13).

1.3 Aging.

We now show how the clock process convergence obtained in Theorem (1.3) is useful for deriving aging information, and in particular, for proving the existence of an arcsine aging regime.

We begin with a few definitions. The aging behavior of X_n is quantified using a time correlation function, namely, a two-time function $\mathcal{C}_n(t,s)$, $t,s \geq 0$, that measures the dependence of $X_n(c_n(t+s))$ and $X_n(c_nt)$. We then say that:

Definition 1.4. *A time correlation function $\mathcal{C}_n$ exhibits normal aging on time scale c_n if one of the following three relations is verified:*

$$\lim_{t\to 0}\lim_{n\to\infty} \mathcal{C}_n(t,\rho t) = \mathcal{C}_\infty(\rho) \,, \tag{1.16}$$

$$\lim_{n\to\infty} \mathcal{C}_n(t,\rho t) = \mathcal{C}_\infty(\rho) \,, \quad t > 0 \text{ arbitrary,} \tag{1.17}$$

$$\lim_{t\to \infty}\lim_{n\to\infty} \mathcal{C}_n(t,\rho t) = \mathcal{C}_\infty(\rho) \,, \tag{1.18}$$

for all $\rho \geq 0$, some non trivial limiting function $\mathcal{C}_\infty : [0,\infty) \mapsto [0,1]$, and for some convergence mode w.r.t. the probability law $\mathbb{P}$ of the random landscape.

In virtually all situations where normal aging was proved so far, the limiting time correlation function is the distribution function of the generalized arcsine law with parameter $0 < \alpha < 1$,

$$\mathrm{Asl}_\alpha(u) = \frac{\sin \alpha\pi}{\pi} \int_0^u (1-x)^{-\alpha} x^{\alpha-1} dx \,, \quad 0 \leq u \leq 1 \,. \tag{1.19}$$

This motivates the next definition.

Definition 1.5. *The process X_n has an arcsine aging regime of parameter α whenever one can find a time correlation function $\mathcal{C}_n$ that exhibits normal aging with*

$$\mathcal{C}_\infty(\rho) = \mathrm{Asl}_\alpha(1/1+\rho) \,. \tag{1.20}$$

While the choice of $\mathcal{C}_n(t,s)$ is model dependent it turns out that the most commonly used time correlation functions (see e.g. [9], [19]) can be approximated, up to error terms that vanish as $n \to \infty$, by the following one

$$\mathcal{C}_n(t,s) = \mathcal{P}_{\mu_n}\left(\left\{ \widehat{S}_n(u) \,, u > 0 \right\} \cap (t, t+s) = \emptyset \right) \,, \quad 0 \leq t < t+s \,. \tag{1.21}$$

This is the probability that the range of the re-scaled clock process $\widehat{S}_n$ does not intersect the time interval $(t, t+s)$. For this choice we have,

Theorem 1.6. *Let the assumptions and the notation be as in Theorem 1.3. Set*

$$\mathcal{C}_\infty(t,s)=\mathcal{P}\left(\{S(u)\,,u>0\}\cap(t,t+s)=\emptyset\right)\,,\quad 0\le t<t+s\,. \tag{1.22}$$

If, for each $\omega\in\Omega^\tau$, σ and S in (1.15) are independent random variables on $(\Omega^X,\mathcal{F}^X,\mathcal{P})$, then, for all $0\le t<t+s$, w.r.t. the same convergence mode as in (1.15),

$$\lim_{n\to\infty}\mathcal{C}_n(t,s)=1-F(t+s)+\int_0^t\mathcal{C}_\infty(t-v,s)dF(v)\,. \tag{1.23}$$

In particular, if $\sigma=0$,

$$\lim_{n\to\infty}\mathcal{C}_n(t,s)=\mathcal{C}_\infty(t,s)\,. \tag{1.24}$$

Remark 1.7. *In line with the remark following Theorem 1.1, σ and/or S may be random variables on $(\Omega^\tau,\mathcal{F}^\tau,\mathbb{P})$. Thus both the limiting functions in (1.23) and (1.24) may be random variables on that space. (We will see an instance of this in Theorem 3.5.)*

By (1.13) of Theorem 1.1 S is a subordinator with Lévy measure ν and zero drift, and for such processes (1.22) is well understood. The theorem below, that can be seen as Dynkin-Lamperti Theorem in continuous time, is classical (see page 81 of [11] for the first half and Theorem 1 of [12] for the second half).

Theorem 1.8.

(i) [Arcsine law] $\mathcal{C}_\infty(t,\rho t)$ converges as $t\to\infty$ (respectively $t\to 0+$) if and only if $\nu(x,\infty)$ is regularly varying at $0+$ (respectively at ∞) with index $\alpha\in[0,1]$. When $0<\alpha<1$ the limiting function is given by $\mathrm{Asl}_\alpha(1/1+\rho)$. If $\nu(x,\infty)=\kappa x^{-\alpha}$ for some constant $\kappa>0$ and $0<\alpha<1$[3] then

$$C_\infty(t,\rho t)=\mathrm{Asl}_\alpha(1/1+\rho)\textit{for all} t>0\,. \tag{1.25}$$

(ii) [Finite mean life time renewal] If $\int_0^\infty\nu(x,\infty)dx=m<\infty$ and S is not a compound Poisson process then, for each fixed $s>0$,

$$\lim_{t\to\infty}\mathcal{C}_\infty(t,s)=\frac{1}{m}\int_s^\infty\nu(x,\infty)dx\,. \tag{1.26}$$

Combing Theorem 1.3, Theorem 1.6 and Theorem 1.8 gives sufficient conditions for the process X_n to have, or not to have, an arcsine aging regime. This extends the arcsine aging scheme of [9] to situations where the limiting clock process is not necessarily a stable subordinator. This happens for instance in Bouchaud's asymmetric trap model on the complete graph (see Proposition 3.9 and Lemma 3.10) and in the REM [27], when X_n is observed on time scales that are of the order of the time scale of stationarity. Let us finally stress that the form of the relation (1.23), where the role of the initial distribution μ_n is made explicit, is new. The effect of μ_n on the aging phenomenon will be studied elsewhere.

The remainder of the paper is organized as follows. Section 2 contains the proofs of the results of Section 1 and their specialization to asymmetric trap model on the complete graph. Section 3 begins the investigation of Bouchaud's asymmetric trap model on the complete graph proper: there we define the model and state the results. Their proofs occupy the rest of the paper (Section 4-6) up to a short appendix on regular variations and renewal theory.

[3] Equivalently, if S is a stable subordinator with index $\alpha\in(0,1)$.

2 Convergence of the clock process and related results

This section is divided in four parts. In Subsection 2.1 we state a result by Durrett and Resnick [21] that is central to the proofs of Theorem 1.1 and Theorem 1.3. The latter are done in Subsection 2.2, and the proof of Theorem 1.6 is done in Subsection 2.3. In Subsection 2.4 we specialize Theorem 1.1 and Theorem 1.3 to the asymmetric trap model on the complete graph. We also give sufficient conditions for convergence of the re-scaled clock process to a partial-sum process in the case, not covered by the theorems of Section 1, where the auxiliary time scale a_n is a constant (see Theorem 2.4 and Theorem 2.5).

2.1 A result by Durrett and Resnick.

In [21] a method is developed for proving convergence of partial sums processes with dependent increments to Lévy processes. This method consists of two steps. In the first step, one shows that a sequence of point processes associated with the increments converges weakly to a two dimensional Poisson process. Then, applying appropriate functionals (to 'sum up the points') and continuity arguments, one obtains weak convergence of the sum to a limiting Lévy process.

In this section we specialize this result, namely Theorem 4.1 of [21], to the case of processes with non-negative increments. Our framework is the following. Let $\{Z_{n,i}, n \geq 1, i \geq 1\}$, $Z_{n,i} \geq 0$, be an array of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and let $\{\mathcal{F}_{n,i}, n \geq 1, i \geq 0\}$ be an array of sub-sigma fields of $\mathcal{F}$ such that for each n and $i \geq 1$, $Z_{n,i}$ is $\mathcal{F}_{n,i}$ measurable and $\mathcal{F}_{n,i-1} \subset \mathcal{F}_{n,i}$. Let $k_n(t)$ be a nondecreasing right continuous function with range $\{0, 1, 2, \dots\}$ and assume that for each $t > 0$ $k_n(t)$ is a stopping time. Set

$$\widetilde{S}_{n,k} = \sum_{i=1}^{k} Z_{n,i}\,, \tag{2.1}$$

for $k \geq 1$, $\widetilde{S}_{n,0} = 0$, and define

$$S_n(t) = \widetilde{S}_{n,k_n(t)}\,. \tag{2.2}$$

The next theorem gives conditions for S_n to converge to a subordinator. To state it we will need the following extra notation: for $\delta \geq 0$ set $Z^{\delta}_{n,i} = Z_{n,i} \mathbb{1}_{\{Z_{n,i} \leq \delta\}}$; further set

$$\widetilde{S}^{\delta}_{n,k} = \sum_{i=1}^{k} Z^{\delta}_{n,i} \tag{2.3}$$

for $k \geq 1$, $\widetilde{S}^{\delta}_{n,0} = 0$, and define

$$S^{\delta}_n(t) = \widetilde{S}^{\delta}_{n,k_n(t)}\,. \tag{2.4}$$

Theorem 2.1. *(Durrett and Resnick) Let ν be a σ-finite measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge x) \nu(dx) < \infty$, and let $\{S(t), t \geq 0\}$ be the subordinator of Laplace exponent $\Phi(\theta) = \int_{(0,\infty)} \left(1 - e^{-\theta x}\right) \nu(dx)$, $\theta \geq 0$. If, as $n \to \infty$,*

(D1) For all $t > 0$ and for $x > 0$ such that $\nu(\{x\}) = 0$,

$$\sum_{i=1}^{k_n(t)} \mathcal{P}\left(Z_{n,i} > x \mid \mathcal{F}_{n,i-1}\right) \stackrel{proba}{\longrightarrow} t\nu(x, \infty)\,, \tag{2.5}$$

(D2) For all $t > 0$ and and all $\varepsilon > 0$,

$$\sum_{i=1}^{k_n(t)} \left[\mathcal{P}\left(Z_{n,i} > \varepsilon \mid \mathcal{F}_{n,i-1}\right)\right]^2 \stackrel{proba}{\longrightarrow} 0, \tag{2.6}$$

and

(D3) For all $t > 0$ and all $\varepsilon > 0$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathcal{P}\left(S_n^\delta(t) > \varepsilon\right) = 0\,, \tag{2.7}$$

then $S_n \Rightarrow S$ in the space $D([0,\infty))$ of càdlàg functions on $[0,\infty)$ equipped with the Skorohod topology.

Remark 2.2. *In [21], Conditions (D2) and (D3) are stated for t fixed and equal to one. This does not seem to be correct.*

2.2 Convergence to subordinators.

In this subsection we prove Theorem 1.1 and the first assertion of Theorem 1.3, and give an alternative to Condition (A3).

Proof of Theorem 1.1. Our aim is to apply Theorem 2.1 to the sum

$$S_n(t) = c_n^{-1}\sum_{i=1}^{\lfloor a_n t\rfloor}\lambda_n^{-1}(J_n(i))e_{n,i}\,. \tag{2.8}$$

Let us first do this for a fixed realization $\omega\in\Omega^\tau$ of the environment. Set $k_n(t) = \lfloor a_n t\rfloor$, $Z_{n,i} = (c_n\lambda_n(J_n(i)))^{-1}e_{n,i}$, and define $\mathcal{F}_{n,i} = \mathcal{B}(J_n(0),\dots,J_n(i),e_{n,0},\dots,e_{n,i})$. Clearly, for each n and $i\geq 1$, $Z_{n,i}$ is $\mathcal{F}_{n,i}$ measurable and $\mathcal{F}_{n,i-1}\subset\mathcal{F}_{n,i}$. Next observe that

$$\begin{aligned}\mathcal{P}_{\mu_n}\left(J_n(i)=x, Z_{n,i}>z\mid\mathcal{F}_{n,i-1}\right) &= \mathcal{P}_{\mu_n}\left(J_n(i)=x, Z_{n,i}>z\mid J_n(i-1)\right)\\ &= p_n(J_n(i-1),x)\mathcal{P}_{\mu_n}\left((\lambda_n(x))^{-1}e_{n,i}>z\right)\\ &= p_n(J_n(i-1),x)\exp\{-zc_n\lambda_n(x)\}\,. \end{aligned}\tag{2.9}$$

From this it follows that

$$\sum_{i=1}^{k_n(t)}\mathcal{P}_{\mu_n}\left(Z_{n,i}>z\mid\mathcal{F}_{n,i-1}\right) = \sum_{i=1}^{\lfloor a_n t\rfloor}\sum_{x\in\mathcal{V}_n}p_n(J_n(i-1),x)\exp\{-zc_n\lambda_n(x)\}\,, \tag{2.10}$$

and

$$\sum_{i=1}^{k_n(t)}\left[\mathcal{P}_{\mu_n}\left(Z_{n,i}>\varepsilon\mid\mathcal{F}_{n,i-1}\right)\right]^2 = \sum_{i=1}^{\lfloor a_n t\rfloor}\left[\sum_{x\in\mathcal{V}_n}p_n(J_n(i-1),x)\exp\{-uc_n\lambda_n(x)\}\right]^2\,, \tag{2.11}$$

so Condition (A2) and (A1) of Theorem 1.1 are, respectively, Conditions (D2) and condition (D1) of Theorem 2.1.

We will now show that Condition (A1) together with Condition (A3) imply Condition (D3). To simplify the notation in Conditions (A1)-(A3) we write $\bar\nu(u)\equiv\nu(u,\infty)$, and set

$$\bar\nu_n^{J,t}(u) = \sum_{j=1}^{\lfloor a_n t\rfloor}\sum_{x\in\mathcal{V}_n}p_n(J_n(j-1),x)\exp\{-uc_n\lambda_n(x)\}\,. \tag{2.12}$$

Consider now Condition (D3). By Tchebychev inequality $\mathcal{P}_{\mu_n}\left(S_n^\delta(t)>\varepsilon\right)\leq\varepsilon^{-1}\mathcal{E}_{\mu_n}S_n^\delta(t)$. Expressed in terms of the truncated variables $Z_{n,i}^\delta = Z_{n,i}\mathbb{1}_{\{Z_{n,i}<\delta\}}$, $\delta\geq 0$, the latter expectation becomes,

$$\mathcal{E}_{\mu_n}S_n^\delta(t) = \mathcal{E}_{\mu_n}\sum_{j=1}^{\lfloor a_n t\rfloor}Z_{n,i}^\delta = E_{\mu_n}\sum_{j=1}^{\lfloor a_n t\rfloor}\mathcal{E}_{\mu_n}\left(Z_{n,j}^\delta\;\middle|\;J_n(j-1)\right)\,. \tag{2.13}$$

Integrating by parts,

$$\begin{aligned}\mathcal{E}_{\mu_n}\left(Z^\delta_{n,i}\,\middle|\,J_n(i-1)\right) &= \int_0^\infty \mathcal{P}_{\mu_n}\left(Z^\delta_{n,i}(J_n(i))>y \mid J_n(i-1)\right)dy \\ &= \int_0^\delta \mathcal{P}_{\mu_n}\left(Z_{n,i}\geq z \mid J_n(i-1)\right)dz - \delta\mathcal{P}_{\mu_n}\left(Z_{n,i}>\delta \mid J_n(i-1)\right)\,,\end{aligned}$$

and since $\sum_{i=1}^{\lfloor a_n t\rfloor}\mathcal{P}\left(Z_{n,i}>z \mid J_n(i-1)\right)=\bar\nu_n^{J,t}(u)$, as follows from (2.10) and (2.12), we arrive at

$$\mathcal{E}_{\mu_n}S^\delta_n(t) = E_{\mu_n}\left(\int_0^\delta du\bar\nu_n^{J,t}(u) - \delta\bar\nu_n^{J,t}(\delta)\right)\,. \tag{2.14}$$

Now by Condition (A1), $E_{\mu_n}\delta\bar\nu_n^{J,t}(\delta)\leq t\delta\bar\nu(\delta)+o(1)$ and $\lim_{\delta\to 0}\delta\bar\nu(\delta)=0$, whereas Condition (A3) states that $E_{\mu_n}\left(\int_0^\delta du\bar\nu_n^{J,t}(u)\right)\leq t\varepsilon_n(\delta)$, where $\lim_{\delta\to0}\limsup_{n\to\infty}\varepsilon_n(\delta)=0$. Hence, if both these conditions are satisfied, $\lim_{\delta\to0}\limsup_{n\to\infty}\mathcal{E}_{\mu_n}S^\delta_n(t)=0$, so that Condition (D3) also is satisfied.

We may now conclude the proof of Theorem 1.1. We established that (A1) $\Rightarrow$ (D1), (D2) $\Leftrightarrow$ (A2), and (A1) & (A3) $\Rightarrow$ (D3). Therefore, by Theorem 2.1, $S_n\Rightarrow S$ in $D([0,\infty))$ where S is the subordinator (1.13).

So far we kept $\omega\in\Omega^\tau$ fixed, i.e. we worked with a fixed realization of the environment. Let us now introduce the subsets $\Omega^\tau_{n,1},\Omega^\tau_{n,2}\subset\Omega^\tau$ (with the notation of (2.12))

$$\begin{aligned}\Omega^\tau_{n,1} &= \left\{\forall t>0,\forall u>0,\forall\varepsilon>0, P\left(\left|\bar\nu_n^{J,t,\omega}(u)-t\bar\nu(u)\right|<\varepsilon\right)=1-o(1)\right\}, \qquad (2.15)\\ \Omega^\tau_{n,2} &= \left\{\forall t>0,\forall\varepsilon>0, P\left(\sum_{j=1}^{\lfloor a_n t\rfloor}\left[\sum_{x\in\mathcal{V}_n}p^\omega_n(J^\omega_n(j-1),x)e^{-uc_n\lambda^\omega_n(x)}\right]^2<\varepsilon\right)=1-o(1)\right\},\end{aligned}$$

and set $\widetilde\Omega^\tau_n=\Omega^\tau_{n,1}\cap\Omega^\tau_{n,2}$. By definition of weak convergence what we have just established is that for each $\omega\in\widetilde\Omega^\tau_n$, and large enough n,

$$\left|\mathcal{E}\left(f(S_n)\right)-\mathcal{E}\left(f(S)\right)\right|=o(1)\,, \tag{2.16}$$

for each continuous bounded function f on the space $D([0,\infty))$ equipped with Skorohod metric ρ_∞. If it holds true that $\mathbb{P}\left(\bigcup_m\bigcap_{n>m}\widetilde\Omega^\tau_n\right)=1$, then $S_n\Rightarrow S$ $\mathbb{P}$-almost surely. If instead we have $\lim_{n\to\infty}\mathbb{P}(\widetilde\Omega^\tau_n)=1$, then $S_n\Rightarrow S$ in $\mathbb{P}$-probability. Theorem 1.1 is thus proved. □

Proof of Theorem 1.3. As in the proof of Theorem 1.1 we first establish (1.15) for a fixed realization $\omega\in\Omega^\tau$ of the environment. Note that the additional Condition (A0) is designed to guarantee that σ_n converges in distribution to σ. Indeed, since $\sigma_n=c_n^{-1}\widetilde S_n(0)=c_n^{-1}\lambda_n^{-1}(J_n(0))e_{n,0}$, we have $1-\mathcal{P}_{\mu_n}(\sigma_n<v)=\sum_{x\in\mathcal{V}_n}\mu_n(x)e^{-vc_n\lambda_n(x)}$, so that (1.14) becomes $|\mathcal{P}_{\mu_n}(\sigma_n<v)-F(v)|=o(1)$. Thus, supplementing Conditions (A1) and (A2) with Condition (A0), it follows from Theorem 1.1 that, viewing σ_n as a constant function in $D([0,\infty))$, the pairs (σ_n,S_n) jointly converge, weakly, to the pair (σ,S), in $D^2([0,\infty))$. It next follows from the continuous mapping theorem, upon adding σ_n and S_n, that $\sigma_n+S_n\Rightarrow\widehat S=\sigma+S$ in $D([0,\infty))$ (see [36], p. 84, last paragraph of Section 3.3, for the continuity of the addition of an arbitrary element of $D([0,\infty))$ and the constant function). Eq. (1.15) being established for a fixed realization $\omega\in\Omega^\tau$, we conclude the proof proceeding exactly as in the proof of Theorem 1.1 [4], introducing the extra subsets $\Omega^\tau_{n,3}=\left\{\left|\sum_{x\in\mathcal{V}_n}\mu_n(x)e^{-vc_n\lambda_n(x)}-(1-F(v))\right|=o(1)\right\}$, and setting $\widetilde\Omega^\tau_n=\Omega^\tau_{n,1}\cap\Omega^\tau_{n,2}\cap\Omega^\tau_{n,3}$. □

[4] see the paragraph beginning above (2.15).

Condition (A3) may not always be easy to handle. Here is an alternative:
Condition (A3'). There exists a sequence of functions $\varepsilon_n \geq 0$ satisfying $\lim_{\delta\to 0}\limsup_{n\to\infty}\varepsilon_n(\delta) = 0$ such that, for some $0<\delta_0\leq 1$, for all $0<\delta\leq\delta_0$ and all $t>0$,

$$E_{\mu_n}\left(\sum_{j=1}^{\lfloor a_n t\rfloor}\sum_{x\in\mathcal{V}_n} p_n^\omega(J_n^\omega(j-1),x)\frac{\mathbb{1}_{\{(c_n\lambda_n^\omega(x))^{-1}\leq\delta\}}}{c_n\lambda_n^\omega(x)}\right)\leq t\varepsilon_n(\delta)\,. \tag{2.17}$$

Lemma 2.3. *A sufficient condition for (A3) is (A3').*

Proof. We will show that if Condition (A1) and Condition (A3') then so is Condition (A3). As in the proof of Theorem 1.1 we write $\bar\nu(u)\equiv\nu(u,\infty)$ and let $\bar\nu_n^{J,t}(u)$ be defined through (2.12). Then (1.12) of Condition (A3) becomes $E_{\mu_n}\big(\int_0^\delta du\bar\nu_n^{J,t}(u)\big)\leq t\varepsilon_n(\delta)$. Clearly,

$$\textstyle\int_0^\delta du\bar\nu_n^{J,t}(u)=\sum_{j=1}^{\lfloor a_n t\rfloor}\sum_{x\in\mathcal{V}_n}p_n^\omega(J_n^\omega(j-1),x)\frac{1-e^{-\delta c_n\lambda_n^\omega(x)}}{c_n\lambda_n^\omega(x)}\,. \tag{2.18}$$

Now on the one hand, since $\frac{1-e^{-y}}{y}\leq e^\rho e^{-y}$, $0\leq y\leq\rho$,

$$\tfrac{1-e^{-\delta c_n\lambda_n^\omega(x)}}{c_n\lambda_n^\omega(x)}\mathbb{1}_{\{c_n\lambda_n^\omega(x)\leq\delta\}}\leq\delta e^\rho e^{-\delta c_n\lambda_n^\omega(x)}\mathbb{1}_{\{\delta c_n\lambda_n^\omega(x)\leq\rho\}}\leq\delta e^\rho e^{-\delta c_n\lambda_n^\omega(x)}\,, \tag{2.19}$$

for all $\rho>0$, while on the other hand $\frac{1-e^{-\delta c_n\lambda_n^\omega(x)}}{c_n\lambda_n^\omega(x)}\mathbb{1}_{\{\delta c_n\lambda_n^\omega(x)\geq\rho\}}\leq\frac{\mathbb{1}_{\{(c_n\lambda_n^\omega(x))^{-1}\leq\delta/\rho\}}}{c_n\lambda_n^\omega(x)}$. Inserting these two bounds in (2.18) yields

$$\textstyle\int_0^\delta du\bar\nu_n^{J,t}(u)\leq\delta e^\rho\bar\nu_n^{J,t}(\delta)+\sum_{j=1}^{\lfloor a_n t\rfloor}\sum_{x\in\mathcal{V}_n}p_n^\omega(J_n^\omega(j-1),x)\frac{\mathbb{1}_{\{(c_n\lambda_n^\omega(x))^{-1}\leq\delta/\rho\}}}{c_n\lambda_n^\omega(x)}\,. \tag{2.20}$$

Recall that by Condition (A1), $E_{\mu_n}\delta\bar\nu_n^{J,t}(\delta)\leq t\delta\bar\nu(\delta)+o(1)$ where $\lim_{\delta\to 0}\delta\bar\nu(\delta)=0$. Thus, averaging out (2.20) and using Condition (A1) together with (2.17) of Condition (A3') to bound the resulting right hand side, we get that, for all $\rho>0$, $E_{\mu_n}\big(\int_0^\delta du\bar\nu_n^{J,t}(u)\big)\leq t\varepsilon_n(\delta/\rho)+\delta e^\rho(t\delta\bar\nu(\delta)+o(1))$. Finally, taking e.g. $\rho=\sqrt{\delta}$, $\lim_{\delta\to 0}\limsup_{n\to\infty}\{t\varepsilon_n(\delta/\rho)+\delta e^\rho(t\delta\bar\nu(\delta)+o(1))\}=0$. Condition (A3) is therefore satisfied. □

2.3 Convergence of the time-time correlation function.

We will now exploit the convergence of $\widehat{S}_n$ established above to prove convergence of the time-time correlation function, using the continuous-mapping theorem.

Proof of Theorem 1.6. This pattern of proof is classical (see [36] section 9.7.2) and relies on the continuity property of a certain function of the inverse mapping on $D([0,\infty))$, the so-called overshoot, which we now define. Let $\eta\in D([0,\infty))$. For $t>0$ let $\mathcal{L}_t$ be the time of the first passage to a level beyond t; *i.e.*,

$$\mathcal{L}_t(\eta)\equiv\eta^{-1}(t)\equiv\{\inf u\geq 0\mid\eta(u)>t\} \tag{2.21}$$

(with $\mathcal{L}_t(\eta)=\infty$ if $\eta(u)\leq t$ for all u). Let $D_t(\eta)=\eta(\mathcal{L}_t(\eta))$ be the first visit to the set $\{\eta(u)\,,u>0\}$ after time t. The associated overshoot is the function $\theta_t(\eta)$ defined through

$$\theta_t(\eta)=D_t(\eta)-t\,. \tag{2.22}$$

With this definition the time-time correlation function (1.21) may be rewritten as

$$\mathcal{C}_n(t,s)=\mathcal{P}_{\mu_n}\left(\left\{\widehat{S}_n(u)\,,u>0\right\}\cap(t,t+s)=\emptyset\right)=\mathcal{P}_{\mu_n}\left(\theta_t(\widehat{S}_n)\geq s\right). \tag{2.23}$$

Similarly, (1.22) can be rewritten as

$$\mathcal{C}_\infty(t,s)=\mathcal{P}\left(\{S(u)\,,u>0\}\cap(t,t+s)=\emptyset\right)=\mathcal{P}\left(\theta_t(S)\geq s\right). \tag{2.24}$$

The point of doing this is that the overshoot function is an almost surely continuous function on $D([0,\infty))$ with respect to Lévy motions having almost surely diverging paths (see [36], Theorem 13.6.5 p.447). Hence, if (1.15) holds true $\mathbb{P}$-almost surely, the continuous mapping theorem (applied for each fixed ω that belongs to the set of full measure for which $S_n \Rightarrow S$ obtains) yields that $\mathbb{P}$-almost surely, uniformly in $0 \le t < t+s$, $\lim_{n\to\infty} \mathcal{P}_{\mu_n}\big(\theta_t(\widehat{S}_n) \ge s\big) = \mathcal{P}\big(\theta_t(\widehat{S}) \ge s\big)$. Assume now that $\widehat{S}_n \Rightarrow \widehat{S}$ in $\mathbb{P}$-probability. Note that for each continuous bounded function g on $[0,\infty)$ the function $g\circ\theta_t$ is a continuous bounded function on $D([0,\infty))$. Thus, by (2.16), for each $\omega \in \widetilde{\Omega}^\tau_n$ and large enough n,

$$\big|\mathcal{E}\big(g\circ\theta_t(\widehat{S}_n)\big) - \mathcal{E}\big(g\circ\theta_t(\widehat{S})\big)\big| = o(1)\,. \tag{2.25}$$

From this and the definition of weak convergence it follows that $\lim_{n\to\infty} \mathcal{P}_{\mu_n}\big(\theta_t(\widehat{S}_n) \ge s\big) = \mathcal{P}\big(\theta_t(\widehat{S}) \ge s\big)$ in $\mathbb{P}$-probability. Since the sequence of subsets $\widetilde{\Omega}^\tau_n$ does not depend on t and s, convergence holds uniformly in $0 \le t < t+s$, in $\mathbb{P}$-probability.

It remains to express $\mathcal{P}\big(\theta_t(\widehat{S}) \ge s\big)$ in terms of $\mathcal{C}_\infty(t,s)$ and F. If $\sigma = 0$ then $\widehat{S} = S$, and by (2.24), $\mathcal{P}\big(\theta_t(\widehat{S}) \ge s\big) = \mathcal{C}_\infty(t,s)$, which proves (1.24). Otherwise, from the assumption that σ and S in (1.15) are independent r.v.'s on $(\Omega^X, \mathcal{F}^X, \mathcal{P})$ for each fixed $\omega \in \Omega^\tau$, we get, conditioning on σ, that

$$\mathcal{P}\big(\theta_t(\widehat{S}) \ge s\big) = 1-F(t+s)+\int_0^t \mathcal{P}\big(\theta_{t-v}(S) \ge s\big)dF(v) = 1-F(t+s)+\int_0^t \mathcal{C}_\infty(t-v,s)dF(v)\,. \tag{2.26}$$

Since (2.26) holds true for each $\omega \in \Omega^\tau$ uniformly in $0 \le t < t+s$, (1.23) obtains uniformly in $0 \le t < t+s$, and inherits the convergence mode of $\mathcal{P}_{\mu_n}\big(\theta_t(\widehat{S}_n) \ge s\big)$, that is to say, the convergence mode of $\widehat{S}_n$. The proof of Theorem 1.6 is now complete. ☐

2.4 The special case of the asymmetric trap model on the complete graph (Convergence to renewal processes).

Our aim in this section is twofold: specialize the results of Theorems 1.1, 1.3 and 1.6 to the asymmetric trap model on the complete graph defined through (3.2)-(3.3) (we do not however specify the distribution of the landscape variables, i.e. we do not assume (3.1)), and give sufficient conditions for convergence of the re-scaled clock processes to a renewal process in the case where the auxiliary time scale a_n of the re-scaled clock process (1.8) is a constant. For such time scales the sample paths of S_n are increasing functions on $[0,\infty)$ that have discontinuities at all integer time points. The natural topological space in which to interpret weak convergence of S_n is, here, the space $\mathbb{R}^\infty$ of infinite sequences equipped with the usual Euclidean topology (see *e.g.* [13] section 3). We will use the arrow $\Rrightarrow$ to denote weak convergence in that space. As in Theorem 1.1, weak convergence in Skorohod topology on $D([0,\infty))$ will be denoted by $\Rightarrow$. Set $r_n = c_n^{1/(1-a)}$ and define

$$\nu_n(u,\infty) = a_n \frac{\sum_{x\in\mathcal{V}_n} \tau^a(x)\exp\{-u(r_n/\tau(x))^{(1-a)}\}}{\sum_{x\in\mathcal{V}_n}\tau^a(x)}\,, \quad u \ge 0\,. \tag{2.27}$$

Theorem 2.4. *Consider the asymmetric trap model on the complete graph on time scale c_n. The following holds for any choice of the initial distribution μ_n.*

(i) If there exists a sequence a_n satisfying $a_n \uparrow \infty$ as $n \uparrow \infty$, a σ-finite measure ν on $(0,\infty)$ satisfying $\int_{(0,\infty)}(1\wedge u)\nu(du) < \infty$, and a function $\varepsilon \ge 0$ satisfying $\lim_{\delta\to 0}\varepsilon(\delta) = 0$, such that, either $\mathbb{P}$-almost surely or in $\mathbb{P}$-probability,

$$\lim_{n\to\infty} \nu_n(u,\infty) = \nu(u,\infty) \quad \forall u > 0\,, \tag{2.28}$$

and, for all $0 < \delta \le \delta_0$*, for some* $0 < \delta_0 \le 1$*,*

$$\limsup_{n\to\infty} \int_0^\delta \nu_n(u,\infty)du \le \varepsilon(\delta)\,, \tag{2.29}$$

then, w.r.t. the same convergence mode,

$$S_n(\cdot) \Rightarrow S(\cdot) = \sum_{t_k \le \cdot} \xi_k\,, \tag{2.30}$$

where $\{(t_k,\xi_k)\}$ *are the marks of a Poisson process on* $[0,\infty)\times(0,\infty)$ *with mean measure* $dt \times d\nu$*.*

(ii) If, taking $a_n = 1$*, there exists a probability distribution* ν *on* $(0,\infty)$ *such that, either* $\mathbb{P}$*-almost surely or in* $\mathbb{P}$*-probability, (2.28) is verified for all* $u \ge 0$*, then, w.r.t. the same convergence mode,*

$$S_n(\cdot) \Rrightarrow R(\cdot) = \sum_{k \le \cdot} \xi_k\,, \tag{2.31}$$

where $\{\xi_k, k \ge 1\}$ *are independent r.v.'s with identical distribution* ν*.*

In the sequel we will adopt the terminology used in [24] and call the sequence $\{R(k)\,, k \in \mathbb{N}\}$ a *renewal process of inter-arrival distribution* ν (equivalently, of *inter-arrival times* ξ_k). As in Theorem 1.3 the extra Condition (A0) on the convergence of the initial increment σ_n enables us to deduce convergence of the full clock process $\widehat{S}_n$ from that of S_n.

Theorem 2.5.

(i) If, in addition to the assumptions of assertion (i) of Theorem 2.4, Condition (A0) is satisfied w.r.t. the same convergence mode as in (2.28), then, in this convergence mode, denoting by σ *the random variable of (possibly random) distribution function* F*, the following holds: For* S *defined in (2.30),*

$$\widehat{S}_n \Rightarrow \widehat{S} = \sigma + S\,, \tag{2.32}$$

where σ *and* S *are independent. Moreover for* $\mathcal{C}_\infty(t,s)$ *defined in (1.22), for all* $0 \le t < t+s$*,*

$$\lim_{n\to\infty} \mathcal{C}_n(t,s) = 1 - F(t+s) + \int_0^t \mathcal{C}_\infty(t-v,s)dF(v)\,. \tag{2.33}$$

In particular, if $\sigma = 0$*,*

$$\lim_{n\to\infty} \mathcal{C}_n(t,s) = \mathcal{C}_\infty(t,s)\,. \tag{2.34}$$

(ii) Substituting the assumptions of assertion (ii) of Theorem 2.4 to those of assertion (i) in the statement of assertion (i') above, and leaving the definition of σ *unchanged, the following holds: For* R *defined in (2.31),*

$$\widehat{S}_n \Rrightarrow \widehat{R} = \sigma + R\,, \tag{2.35}$$

where σ *and* R *are independent. Moreover, (2.33)-(2.34) hold true with* $\mathcal{C}_\infty(t,s)$ *defined through*

$$\mathcal{C}_\infty(t,s) = \mathcal{P}\left(\{R(k)\,, k \in \mathbb{N}\} \cap (t,t+s) = \emptyset\right)\,, \quad 0 \le t < t+s\,. \tag{2.36}$$

Thus, when a_n diverges, $\widehat{S}_n$ converges to a delayed subordinator, and it converges to a delayed renewal process otherwise.

Remark 2.6. *As in Theorem 1.6, the statement that σ and S are independent in (2.32) has the precise meaning that for each fixed $\omega \in \Omega^\tau$, σ and S are independent random variables on the probability space $(\Omega^X, \mathcal{F}^X, \mathcal{P})$. The same remark applies to the statement that σ and R in (2.35) are independent.*

Specializing the previous theorem to the case where the initial distribution μ_n is the invariant measure π_n of the jump chain (see (3.5)) yields the following:

Corollary 2.7. *Let $\mu_n = \pi_n$. Under the assumptions of assertion (i) (respectively, assertion (ii)) of Theorem 2.4, w.r.t. the same convergence mode as in (2.28) (equivalently, (2.30), respectively, (2.31)),*

$$\lim_{n\to\infty} \mathcal{C}_n(t,s) = \mathcal{C}_\infty(t,s)\,, \quad 0 \le t < t+s\,, \tag{2.37}$$

where $\mathcal{C}_\infty(t,s)$ is defined in (1.22) (respectively, (2.36)).

Proof of Theorem 2.4. The first assertion of Theorem 2.4 is an elementary specialization of Theorem 1.1 to the asymmetric trap model on the complete graph. Simply note that

$$\sum_{j=1}^{\lfloor a_n t\rfloor} \sum_{x\in\mathcal{V}_n} p_n(J_n(j-1),x)\exp\{-uc_n\lambda_n(x)\} = \frac{\lfloor a_n t\rfloor}{a_n}\nu_n(u,\infty)\,, \tag{2.38}$$

where the r.h.s. is chain independent. Thus, if a_n is a diverging sequence, (1.10) and (1.11) of Conditions (A1) and (A2) of Theorem 1.1 reduce, respectively, to

$$\nu_n(u,\infty) \to \nu(u,\infty)\,, \tag{2.39}$$

$$\frac{1}{a_n}\left[\nu_n(u,\infty)\right]^2 \to 0\,, \tag{2.40}$$

as $n\to\infty$, and, clearly, (2.39) implies (2.40). Similarly, (1.10) of Condition (A3) becomes (2.29).

Remark 2.8. *Note that, setting $h_n(v) = \sum_{x\in\mathcal{V}_n} p_n(v,x)e^{-uc_n\lambda_n(x)}$, (2.38) can be written as*

$$\lfloor a_n t\rfloor^{-1} \sum_{j=1}^{\lfloor a_n t\rfloor} h_n(J_n(j-1)) = \sum_{y\in\mathcal{V}_n} \pi_n(y)h_n(y) = E_{\pi_n} h_n(J_n(j-1))\,. \tag{2.41}$$

In other words the sum appearing in Condition (A1) of Theorem 1.1 is ‘ergodic’. A similar observation holds for Condition (A2).

The new part of Theorem 2.4 is assertion (ii), whose elementary proof we now give. Assume first that there exists a probability distribution ν such that, for all $u\ge 0$, (2.28) holds in $\mathbb{P}$-probability. Set $\xi_{n,i} = c_n^{-1}\lambda_n^{-1}(J_n(i))e_{n,i}$, $i\ge 0$. Putting $a_n = 1$ in (1.8), $S_n(t) = \overline{S}_n(\lfloor t\rfloor) = \sum_{i=1}^{\lfloor t\rfloor}\xi_{n,i}$. Notice that for each $\omega\in\Omega^\tau$, $\{\xi_{n,i}, i\ge 1\}$ is an i.i.d. sequence on the probability space $(\Omega^X, \mathcal{F}^X, \mathcal{P})$ since, by (3.8), the chain variables $(J_n(i), i\in\mathbb{N})$ form an i.i.d. sequence, and since $\mathcal{P}_{\mu_n}(\xi_{n,i} > u) = \nu_n(u,\infty)$ does not depend on i. This means that $\overline{S}_n$ has stationary positive increments. To prove (2.31) it thus suffices to prove that, in $\mathbb{P}$-probability, for each integer k (finite and independent of n), $\overline{S}_n(k) \stackrel{d}{\to} R(k)$ (see e.g. [13] p. 30). To this end consider the Laplace transforms $\Lambda_n(k,\theta) = \mathcal{E}_{\mu_n} e^{-\theta\overline{S}_n(k)}$ and $\Lambda(k,\theta) = \mathcal{E}e^{-\theta R(k)}$, $\theta > 0$. From the assumption that, for all $u\ge 0$, (2.28) holds in $\mathbb{P}$-probability, it follows that there exists a sequence $\widetilde{\Omega}_n^\tau \subset \Omega^\tau$ satisfying $\lim_{n\to\infty}\mathbb{P}(\widetilde{\Omega}_n^\tau) = 1$, and such that, for all large enough n,

$$\sup_{u\ge 0} |\mathcal{P}_{\mu_n}(\xi_{n,i} > u) - \nu(u,\infty)| = o(1)\,, \quad 1\le i\le n\,, \tag{2.42}$$

for all $\omega \in \widetilde{\Omega}^{\tau}_n$. Let now $\omega \in \widetilde{\Omega}^{\tau}_n$ be fixed, where n will be taken as large as needed. By independence, $\Lambda_n(k,\theta) = \left(\mathcal{E}e^{-\theta\xi_{n,i}}\right)^k$. From the integration by parts formula $\mathcal{E}_{\mu_n} e^{-\theta\xi_{n,i}} = 1 - \theta\int_0^\infty e^{-\theta u}\mathcal{P}_{\mu_n}(\xi_{n,i} > u)du$, it follows that

$$\left|\mathcal{E}_{\mu_n} e^{-\theta\xi_{n,i}} - \mathcal{E}e^{-\theta\xi_i}\right| \leq \sup_{u\geq 0}\left|\mathcal{P}_{\mu_n}(\xi_{n,i} > u) - \nu(u,\infty)\right| . \tag{2.43}$$

Thus, by (2.42), for all n large enough, for each k, $\sup_{\theta>0}|\Lambda_n(k,\theta) - \Lambda(k,\theta)| = o(1)$. Now, by Feller's continuity theorem (see e.g. [24], XIII.1, Theorem 2a), this implies that, for all n large enough, for each k, $\sup_{u>0}|\mathcal{P}_{\mu_n}(\xi_{n,i} > u) - \mathcal{P}(\xi_i > u)| = o(1)$. Since this holds true for each fixed $\omega \in \widetilde{\Omega}^{\tau}_n$, it is tantamount to the statement that, for each k, $\overline{S}_n(k) \overset{d}{\to} R(k)$ in $\mathbb{P}$-probability. The proof of assertion (ii) when (2.28) holds in $\mathbb{P}$-probability is now complete. The proof in the case of $\mathbb{P}$-almost sure convergence is an elementary adaptation whose details we skip. The proof of Theorem 2.4 is now done. □

Proof of Theorem 2.5. We first deal with assertion (i'). Eq. (2.32) is proved just as (1.15) of Theorem 1.3. Assuming that for each $\omega \in \Omega^\tau$, σ and S in (2.32) are independent random variables on the probability space $(\Omega^X, \mathcal{F}^X, \mathcal{P})$, (2.33) is proved in the same way as (1.23) of Theorem 1.6, and the special case $\sigma = 0$ of (2.34) is nothing but (1.24).

Let us show that the above independence assumption is verified. For this let $\omega \in \Omega^\tau$ be fixed. Note that by (3.8) the jump chain $(J_n(i), i \in \mathbb{N})$ becomes stationary in exactly one step. Namely, for any initial distribution μ_n, for all $i \geq 1$, $P_{\mu_n}(J_n(i) = x) = \pi_n(x)$, $x \in \mathcal{V}_n$. Thus, for each n, σ_n and $\{\overline{S}_n(k), k \geq 1\}$ in (1.7) are independent r.v.'s on $(\Omega^X, \mathcal{F}^X, \mathcal{P})$. This in turn implies that, for each n, σ_n and $\{S_n(t), t > 0\}$ in the r.h.s. of (1.9) are independent r.v.'s on $(\Omega^X, \mathcal{F}^X, \mathcal{P})$. Thus σ and $S(\cdot)$ are independent, and since this is true for each $\omega \in \Omega^\tau$, the claim follows.

We skip the proof of assertion (i”), which is a re-run of the proof of assertion (i') (and, upstream from it, of Theorems 1.3 and 1.6) in the simpler setting of discrete time process. □

Proof of Corollary 2.7. Since $1 - \mathcal{P}_{\mu_n}(\sigma_n < v) = \sum_{x\in\mathcal{V}_n}\mu_n(x)e^{-vc_n\lambda_n(x)}$ (see e.g. the proof of assertion (i) of Theorem 1.3) it follows from (2.27) and the choice $\mu_n = \pi_n$ that

$$1 - \mathcal{P}_{\mu_n}(\sigma_n < v) = \frac{1}{a_n}\nu_n(u,\infty) . \tag{2.44}$$

Suppose first that the assumptions of assertion (i) of Theorem 2.4 are verified. In view of (2.28) and (2.44), $1 - \mathcal{P}_{\mu_n}(\sigma_n < v) \to 0$ for all $v \geq 0$, so that Condition (A0) is satisfied with $F(v) = 1$, $v \geq 0$, w.r.t. the same convergence mode as in (2.28). Eq. (2.37) then follows from (2.34). Suppose next that the assumptions of assertion (ii) of Theorem 2.4 are verified. Reasoning as above we readily see that Condition (A0) is satisfied with $F(v) = \nu(u,\infty)$, $v \geq 0$, w.r.t. the same convergence mode as in (2.28). Thus, by (2.35), the first increment σ of the limiting renewal process $\widehat{R}$ has the same distribution as the inter-arrival times ξ_k of R. Hence, for all $0 \leq t < t+s$,

$$\mathcal{P}\left(\{\sigma + R(k), k \in \mathbb{N}\} \cap (t,t+s) = \emptyset\right) = \mathcal{P}\left(\{R(k), k \in \mathbb{N}\} \cap (t,t+s) = \emptyset\right) = \mathcal{C}_\infty(t,s) , \tag{2.45}$$

where the last equality is (2.36). Since σ and R in (2.35) are independent, we also have, conditioning on σ and using (2.33), that, for all $0 \leq t < t+s$,

$$\mathcal{P}\left(\{\sigma + R(k), k \in \mathbb{N}\} \cap (t,t+s) = \emptyset\right) = 1 - F(t+s) + \int_0^t \mathcal{C}_\infty(t-v,s)dF(v) = \lim_{n\to\infty}\mathcal{C}_n(t,s) . \tag{2.46}$$

Equating the r.h.s. of (2.44) with the r.h.s. of (2.45) gives (2.37). The proof of Corollary 2.7 is done. The proof of Corollary 2.7 is done. □

3 Bouchaud's asymmetric trap model on the complete graph.

We now begin the investigation of Bouchaud's asymmetric trap model on the complete graph, which will occupy the rest of the paper. The results we present are the first aging results for a trap model of mean field type which is not a time change of a simple random walk.

This section is organized as follows. In Subsection 3.1 we describe the model and some of its static properties. We then state our main results, first, on the convergence of the time correlation function (Subsection 3.2), and next, on the clock process (Subsection 3.3).

3.1 The model.

This model appeared in [16] where it was proposed and studied on various graphs $G_n(\mathcal{V}_n, \mathcal{E}_n)$. The random landscape $(\tau(x), x \in \mathcal{V}_n)$ is a sequence of i.i.d. random variables that represent the depths of traps, and whose distribution belongs to the domain of attraction of a positive stable law with parameter $\alpha \in (0,1)$. This means that there exists a function L, slowly varying at infinity, such that

$$\mathbb{P}(\tau(x) > u) = u^{-\alpha} L(u)\,, \quad u \geq 0\,. \tag{3.1}$$

Given a parameter $0 \leq a < 1$, the Markov jump process X_n has holding time parameters

$$\lambda_n(x) = (\tau_n(x))^{-(1-a)}\,, \quad \forall x \in \mathcal{V}_n\,, \tag{3.2}$$

and its jump chain, J_n, has one-step transition probabilities

$$p_n(x,y) = \frac{\tau_n^a(y)}{\sum_{y:(x,y)\in\mathcal{E}_n} \tau_n^a(y)}\,, \text{if}(x,y) \in \mathcal{E}_n\,, \tag{3.3}$$

$p_n(x,y) = 0$ otherwise. When $a = 0$, J_n simply is the homogeneous random walk on G_n, whereas when $a > 0$, J_n favors jumps to the neighboring traps of largest depths. Models with $a > 0$ are called *asymmetric* as opposed to the *symmetric* ones where $a = 0$.

The first rigorous results for the asymmetric trap model were obtained for the graph $\mathbb{Z}$ in [**?**]. There, it is shown that the time-time correlation function (1.21) does not exhibit an arcsine aging regime but is sub-aging, and has the same (a-dependent) aging regime for all $a \in [0,1]$. The recent work [1] suggests that on the contrary, on the graphs $\mathbb{Z}^d$, $d \geq 3$, the asymmetry parameter, a, has no relevance on the aging phenomenon. These results contrast with the case of the complete graph where the asymmetry parameter triggers a dynamical phase transition. More precisely, we show that there exists a positive threshold value in a below which the model exhibits an (a-dependent) arcsine aging regime, whereas above it arcsine aging is interrupted. This phenomenon occurs "on all time scales", i.e. from time scale one up to, and including, the time scale of stationarity. We also show how, on the time scale of stationarity, the model can be driven from an arcsine aging regime to its stationary regime.

From now on we focus on the model where $G_n(\mathcal{V}_n, \mathcal{E}_n)$ is the complete graph on $\mathcal{V}_n \equiv \{1, \ldots, n\}$ that has a loop at each vertex. Clearly X_n has a unique reversible invariant measure, denoted by $\mathcal{G}_{\alpha,n}$, which is the Gibbs measure of the model, i.e.

$$\mathcal{G}_{\alpha,n}(x) = \frac{\tau(x)}{\sum_{x\in\mathcal{V}_n} \tau(x)}\,, \quad x \in \mathcal{V}_n\,. \tag{3.4}$$

Clearly also, the jump chain J_n has a unique reversible invariant measure, π_n, given by

$$\pi_n(x) = \frac{\tau^a(x)}{\sum_{x\in\mathcal{V}_n} \tau^a(x)}\,, \quad x \in \mathcal{V}_n\,. \tag{3.5}$$

Thus π_n is nothing but the Gibbs measure with parameter α/a; more precisely, $\pi_n =^d \mathcal{G}_{\beta,n}$, $\beta \equiv \alpha/a \in (0,\infty)$, where $=^d$ denotes equality in distribution.

Let us thus take a closer look at the asymptotics of the Gibbs measure. Its behavior changes at the critical value $\alpha = 1$. When $\alpha < 1$ the order statistics of the Gibbs weights converges in distribution to Poisson-Dirichlet distribution with parameter α. Namely, let $\mathrm{PRM}(\mu)$ be Poisson random measure with intensity measure

$$\mu(x,\infty) = x^{-\alpha}\,, \quad x > 0\,, \tag{3.6}$$

and denote by $\{\gamma_k\}$ its marks. Next denote by $\bar{\gamma}_1 \geq \bar{\gamma}_2 \geq \dots$ the ranked Poisson marks. Then Poisson-Dirichlet distribution with parameter α can be represented as the distribution of the sequence

$$\bar{w}_1 \geq \bar{w}_2 \geq \dots \quad \text{where} \quad \bar{w}_k = \frac{\bar{\gamma}_k}{\sum_l \bar{\gamma}_l}\,. \tag{3.7}$$

If we now label $\tau_n(x(1)) \geq \cdots \geq \tau_n(x(n))$ the landscape variables arranged in decreasing order of magnitude, then $\left(\mathcal{G}_{\alpha,n}(x(k))\right)_{k\geq 1} \xrightarrow{d} (\bar{w}_k)_{k\geq 1}$, as $n \to \infty$, (see [34], Proposition 10). This readily implies that most of the mass of the Gibbs measure is supported by the points $x(k)$ with largest weights (i.e. with deepest traps). In contrast, when $\alpha > 1$, no single point carries a positive mass asymptotically. In particular, it is not hard to show that $\lim_{n\to\infty}\sup_{x\in\mathcal{V}_n}\mathcal{G}_{\alpha,n}(x) = 0$ in $\mathbb{P}$-probability. Here the Gibbs measure "resembles a uniform measure".

It is now easy to see why the chain X_n should undergo a dynamical phase transition at the value $a = \alpha$. By (3.3) and (3.5),

$$p_n(x,y) = \pi_n(y)\,, \quad \forall (x,y) \in \mathcal{E}_n\,, \tag{3.8}$$

where, because $\pi_n =^d \mathcal{G}_{\alpha/a,n}$, π_n undergoes a transition at the value $a = \alpha$. Thus, when $a > \alpha$ the jump chain should resemble a symmetric random walk, and may explore the entire landscape. In contrast, when $a < \alpha$ the jump chain will quickly go and visit a trap of extreme depth from which it will not be able to escape, unless time is measured on the scale of stationarity.

3.2 Aging of $\mathcal{C}_n(t,s)$.

We now state our main results on the asymptotic behavior of the time-time correlation function $\mathcal{C}_n(t,s)$ of (1.21). We cover all choices of a and α with $0 < \alpha < 1$, $0 \leq a < 1$, and $a \neq \alpha$, and any choice of the time scale c_n up to and including the time scale of stationarity. All these results are obtained for a special choice of the initial distribution μ_n, namely, $\mu_n = \pi_n$.

By (3.2), $(c_n\lambda_n(x))^{-1/(1-a)} = \tau(x)/{c_n}^{1/(1-a)}$. This relation prompts us to call $r_n \equiv c_n^{1/(1-a)}$ a *space scale*. We will distinguish three types of space scales: the *constant* scales (which simply are constant sequences), the *intermediate*, and the *extreme* scales.

Definition 3.1. *We say that a positive and diverging sequence r_n is:*

(i) an intermediate space scale if there exists an increasing and diverging sequence $b_n > 0$ such that

$$\frac{b_n}{n} = o(1) \quad \textit{and} \quad b_n\mathbb{P}(\tau(x) \geq r_n) \sim 1\,, \tag{3.9}$$

(ii) an extreme space scale if there exists an increasing and diverging sequence $0 < b_n \leq n$ such that

$$\frac{b_n}{n} \sim 1 \quad \textit{and} \quad b_n\mathbb{P}(\tau(x) \geq r_n) \sim 1\,. \tag{3.10}$$

Remark 3.2. *These scales are well separated. Namely, if r_n^{cst}, r_n^{int} and r_n^{ext} denote, respectively, a constant, an intermediate and an extreme space scale, then $r_n^{cst} \ll r_n^{int} \ll r_n^{ext}$.*

Our first theorem establishes that if $a < \alpha$ then $\mathcal{C}_n(t, \rho t)$ ages on all time scales.

Theorem 3.3 (Arcsine aging regime)**.** *Assume that $a < \alpha$ and take $\mu_n = \pi_n$.*

(i) If r_n is a constant space scale then, $\mathbb{P}$-almost surely, for all $\rho > 0$,

$$\lim_{t\to\infty}\lim_{n\to\infty} \mathcal{C}_n(t, \rho t) = \text{Asl}_{\frac{\alpha-a}{1-a}}(1/1+\rho)\,. \tag{3.11}$$

(ii) If r_n is an intermediate space scale then for all $t \geq 0$ and all $\rho > 0$,

$$\lim_{n\to\infty} \mathcal{C}_n(t, \rho t) = \text{Asl}_{\frac{\alpha-a}{1-a}}(1/1+\rho)\,. \tag{3.12}$$

This holds $\mathbb{P}$-a.s. if b_n is regularly varying at infinity with index $\zeta < 1$, and in $\mathbb{P}$-probability otherwise.

(iii) If r_n is an extreme space scale then, for all $\rho > 0$, in $\mathbb{P}$-probability,

$$\lim_{t\to 0+}\lim_{n\to\infty} \mathcal{C}_n(t, \rho t) = \text{Asl}_{\frac{\alpha-a}{1-a}}(1/1+\rho)\,. \tag{3.13}$$

When $a > \alpha$, none of the time scale and limiting procedures of Theorem 3.3 yields aging:

Theorem 3.4 (Stranded in deep traps)**.** *Assume that $a > \alpha$ and take $\mu_n = \pi_n$.*

(i) If r_n is a constant or intermediate space scale then, for all $0 \leq t < t + s$,

$$\lim_{n\to\infty} \mathcal{C}_n(t, s) = 1 \quad \textit{in} \quad \mathbb{P}-probability. \tag{3.14}$$

(ii) If r_n is an extreme space scale then, for all $\rho > 0$,

$$\lim_{t\to 0}\lim_{n\to\infty} \mathcal{C}_n(t, \rho t) = 1 \quad \textit{in} \quad \mathbb{P}-probability. \tag{3.15}$$

At a heuristic level Theorem 3.4 is easy to understand. For $a > \alpha$ the initial distribution μ_n behaves like a "low temperature" Gibbs measure, namely $\mu_n \stackrel{d}{=} \mathcal{G}_{\beta,n}$, $\beta = \alpha/a < 1$. This means that almost all its mass is carried by traps whose size is of the order of extreme space scales. Now the mean waiting time in such deep traps diverges with n whenever time is measured on a scale which is small compared to extreme scales: the chain gets stranded.

The last theorem below is valid for all $0 \leq a < 1$. It states that, as expected, on extreme time scales, taking the infinite volume limit first, the process reaches stationarity as $t \to \infty$. As before let $\{\gamma_k\}$ denote the marks of PRM(μ) on $(0, \infty)$, and define

$$\mathcal{C}_\infty^{sta}(s) = \sum_k \frac{\gamma_k}{\sum_l \gamma_l} e^{-s\gamma_k^{-(1-a)}}\,, \quad s \geq 0\,. \tag{3.16}$$

Theorem 3.5 (Crossover to stationarity)**.** *Let $c_n = r_n^{1-a}$ where r_n is an extreme space scale. The following holds for all $0 \leq a < 1$, $a \neq \alpha$:*

(i) If $\mu_n = \mathcal{G}_{\alpha,n}$ then, for all $s \leq t < t + s$,

$$\lim_{n\to\infty} \mathcal{C}_n(t, s) \stackrel{d}{=} \mathcal{C}_\infty^{sta}(s)\,, \tag{3.17}$$

where $\stackrel{d}{=}$ denotes equality in distribution.

(ii) If $\mu_n = \pi_n$, for all $s > 0$,

$$\lim_{t\to\infty}\lim_{n\to\infty} \mathcal{C}_n(t,s) \stackrel{d}{=} \mathcal{C}_\infty^{sta}(s)\,. \tag{3.18}$$

Comparing (3.13) and (3.18) we see that when time goes from 0 to ∞, for $a < \alpha$, the chain moves out of an arcsine aging regime and crosses over to its stationary regime. Aging is then interrupted.

3.3 Convergence of the clock process.

This section contains the ingredients needed in the proofs of Theorem 3.3, 3.4 and 3.5. Consider the pure clock process S_n of (1.8). For $a < \alpha$ let a_n be any sequence such that $a_n r_n^a/(b_n\mathbb{E}\tau^a) \sim 1$, and for $a > \alpha$, take $a_n \sim 1$. As before $\mu_n = \pi_n$. . We show below that S_n has different limits depending on the choice of space scales (constant, intermediate, and extremal) and the value of a. The proofs of these results rely on Theorem 2.4 whose notations we now use.

Proposition 3.6. *Let r_n be a constant space scale. For $a < \alpha$ and $\tau \equiv \tau(1)$, let $\nu^{cst,-}$ be the measure defined through*

$$\nu^{cst,-}(u,\infty) = \mathbb{E}\tau^a e^{-u/\tau^{(1-a)}}\,, \quad u > 0\,. \tag{3.19}$$

(i) If $a < \alpha$ then $S_n \Rrightarrow R^{cst,-}$ $\mathbb{P}$-a.s., where $R^{cst,-}$ is the renewal process of inter-arrival distribution $\nu^{cst,-}$.

(ii) If $a > \alpha$ then, $S_n \Rrightarrow R^{cst,+}$ in $\mathbb{P}$-probability, where $R^{cst,+}$ is the degenerate renewal process of inter-arrival distribution $\nu^{cst,+} = \delta_\infty$.

The lemma below shows that $\nu^{cst,-}(u,\infty)$ is regularly varying at infinity with index $\frac{\alpha-a}{1-a}$.

Lemma 3.7. *As $u \to \infty$, $\nu^{cst,-}(u,\infty) \sim u^{-\frac{\alpha-a}{1-a}}\ell(u)\Gamma\left(\frac{\alpha-a}{1-a}\right)$ for some function $\ell(u)$ slowly varying at infinity.*

Proposition 3.8. *Let r_n be an intermediate space scale. For $a < \alpha$, let $\nu^{int,-}$ be the measure on $(0,\infty)$ defined through*

$$\nu^{int,-}(u,\infty) = u^{-\frac{\alpha-a}{1-a}}\frac{\alpha}{1-a}\Gamma\left(\frac{\alpha-a}{1-a}\right), \quad u > 0\,. \tag{3.20}$$

(i) If $a < \alpha$ then, $S_n \Rightarrow S^{int,-}$ where $S^{int,-}$ is the stable subordinator of Lévy measure $\nu^{int,-}$. Convergence holds $\mathbb{P}$-a.s. if $b_n = n^\zeta$ for some $0 < \zeta < 1$, and in $\mathbb{P}$-probability otherwise.

(ii) If $a > \alpha$ then, $S_n \Rrightarrow R^{int,+}$ in $\mathbb{P}$-probability, where $R^{int,+}$ is the degenerate renewal process of inter-arrival distribution $\nu^{int,+} = \delta_\infty$.

To formulate the results on extreme scales recall that for μ defined in (3.6), $\{\gamma_k\}$ denote the marks of $\text{PRM}(\mu)$ on $(0,\infty)$, and introduce the re-scaled landscape variables:

$$\gamma_n(x) = r_n^{-1}\tau(x) \quad x \in \mathcal{V}_n\,. \tag{3.21}$$

Proposition 3.9. *If r_n is an extreme space scale then both the sequence of re-scaled landscapes $(\gamma_n(x), x \in \mathcal{V}_n)$, $n \geq 1$, and the marks of $\text{PRM}(\mu)$ can be represented on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that, in this representation, denoting by $\mathbf{S}_n$*

the process (1.8), the following holds. For $a < \alpha$, resp. $a > \alpha$, let $\nu^{ext,-}$, resp. $\nu^{ext,+}$ be the random measures on $(0,\infty)$ defined on $(\Omega,\mathcal{F},\mathbf{P})$ through

$$\begin{aligned}\nu^{ext,-}(u,\infty) &= \sum_k \gamma_k^a e^{-u\gamma_k^{-(1-a)}}, \quad u>0\,,\\ \nu^{ext,+}(u,\infty) &= \sum_k \frac{\gamma_k^a}{\sum_l \gamma_l^a} e^{-u\gamma_k^{-(1-a)}}, \quad u>0\,.\end{aligned} \tag{3.22}$$

Then, $\mathbf{P}$-almost surely,

$$\begin{aligned}\mathbf{S}_n &\Rightarrow S^{ext,-} \quad \text{if} \quad a<\alpha\,,\\ \mathbf{S}_n &\Rrightarrow R^{ext,+} \quad \text{if} \quad a>\alpha\,,\end{aligned} \tag{3.23}$$

where $S^{ext,-}$ is the subordinator of Lévy measure $\nu^{ext,-}$, and $R^{ext,+}$ is the renewal process of inter-arrival distribution $\nu^{ext,+}$.

Here the subordinator $S^{ext,-}$ is not stable. However $\nu^{ext,-}(u,\infty)$ is regularly varying at 0^+ with index $\frac{\alpha-a}{1-a}$:

Lemma 3.10. *$\mathbf{P}$-a.s., $\nu^{ext,-}(u,\infty) \sim u^{-\frac{\alpha-a}{1-a}} \frac{\alpha}{1-a}\Gamma\left(\frac{\alpha-a}{1-a}\right)$ as $u \to 0^+$.*

By Corollary 2.7, Theorem 3.3, Theorem 3.4 and Theorem 3.5 are direct consequences of the above results and Theorem 1.8 (see also Theorem 7.2 of Appendix 7.1 on renewal theory for the discrete time version of Theorem 1.8; for the proof of Theorem 3.5, (i), see Theorem 7.3). Similarly, Proposition 3.6, (ii), and Proposition 3.8, (ii), are simple consequences of Theorem 2.4, (ii). We omit their proofs. Details can be found in [26].

The rest of the paper is devoted to the proofs of Proposition 3.6, (i), Proposition 3.8, (i), and Proposition 3.9. As they rely on very different tools, we give them in three separate sections (Section 4, 5, and 6 respectively).

4 Constant scales.

Proof of Proposition 3.6, (i). It suffices to check that the conditions of Theorem 2.4, (ii), are satisfied $\mathbb{P}$-almost surely. For all $a < \alpha$ $\mathbb{E}\tau^a < \infty$ so that $\mathbb{E}\tau^a e^{-u/\tau^{(1-a)}} \le \mathbb{E}\tau^a < \infty$ for all $u \ge 0$. Thus, for all $u \ge 0$, the strong law of large numbers applies to both the numerator and denominator of (2.27), yielding $\lim_{n\to\infty}\nu_n(u,\infty) = \nu^{cst,-}(u,\infty)$ $\mathbb{P}$-almost surely. Together with the monotonicity of $\nu_n(u,\infty)$ and the continuity of the limiting function $\nu^{cst,-}(u,\infty)$, this implies that there exists of a subset $\Omega_1^\tau \subset \Omega^\tau$ of the sample space Ω^τ of the τ's with the property that $\mathbb{P}(\Omega_1^\tau)=1$, and such that, on Ω_1^τ,

$$\lim_{n\to\infty} \nu_n(u,\infty) = \nu^{cst,-}(u,\infty)\,, \quad \forall\, u>0\,. \tag{4.1}$$

The proof of Proposition 3.6 is done. □

Proof of Lemma 3.7. Let $a<\alpha$. For $u \ge 0$ and $y \ge 0$ set $\varphi_u(y) = y^a e^{-u/y^{(1-a)}}$. Integrating by parts, $\mathbb{E}\varphi_u(\tau) = \int_0^\infty \varphi_u'(x)\mathbb{P}(\tau > x)dx$. By the change of variable $x = u^{1/(1-a)}y$, noting that $\varphi_u'\left(u^{1/(1-a)}y\right) = u^{-1}\varphi_1'(y)$, we get $\mathbb{E}\varphi_u(\tau) = u^{\frac{a}{1-a}}\int_0^\infty \varphi_1'(y)\mathbb{P}\big(\tau > u^{\frac{1}{1-a}}y\big)dy$. By (3.1) $u^{\frac{\alpha}{1-a}}\mathbb{P}\big(\tau > u^{\frac{1}{1-a}}\big) = \ell(u)$ for some function ℓ slowly varying at infinity. Thus

$$u^{\frac{\alpha-a}{1-a}}\mathbb{E}\varphi_u(\tau) = \ell(u)\int_0^\infty \varphi_1'(y)\big[\mathbb{P}\big(\tau > u^{\frac{1}{1-a}}y\big)/\mathbb{P}\big(\tau > u^{\frac{1}{1-a}}\big)\big]dy\,, \tag{4.2}$$

where $\mathbb{P}\big(\tau > u^{\frac{1}{1-a}}y\big)/\mathbb{P}\big(\tau > u^{\frac{1}{1-a}}\big) \to y^{-\alpha}$ as $u \to \infty$. Because of the monotonicity the approach is uniform, and so

$$\int_0^\infty \varphi_1'(y)\big[\mathbb{P}\big(\tau > u^{\frac{1}{1-a}}y\big)/\mathbb{P}\big(\tau > u^{\frac{1}{1-a}}\big)\big]dy \to \int_0^\infty \varphi_1'(y)y^{-\alpha}dy = \Gamma\big(\tfrac{\alpha-a}{1-a}\big)\,, \tag{4.3}$$

as $u \to \infty$. Combining (4.2) and (4.3) proves the lemma. □

5 Intermediate scales.

In this section we prove Proposition 3.8, (i), in the case where $b_n = n^\zeta$ for some $0 < \zeta < 1$. (The case where convergence holds in probability only can be found in [26].) The proof hinges on Proposition (5.1) below, that will enable us to establish control on the quantity $\nu_n(u, \infty)$ from (2.27).

For fixed $u \geq 0$ set $\varphi_u(y) = y^a e^{-u/y^{(1-a)}}$, $y \geq 0$. Consider the array $\{Z^u_{n,i}, i \geq 1, n \geq 1\}$, $Z^u_{n,i} \equiv b_n\varphi_u(\tau(i)/r_n)$, and set $S^u_n = \frac{1}{n}\sum_{i=1}^n Z^u_{n,i}$.

Proposition 5.1. *Assume that $a < \alpha$. Let r_n be an intermediate space scale and assume that $b_n = n^\zeta$ for some $0 < \zeta < 1$. Then, for all $u > 0$, $\lim_{n\to\infty} \frac{1}{n}S^u_n = \nu^{int,-}(u, \infty)$ $\mathbb{P}$-a.s..*

Proof of Proposition 5.1. The cases $a = 0$ and $a > 0$ are very different. When $a = 0$ the $Z^u_{n,i}$'s have finite moments of all order and Proposition 5.1 immediately follows a classical exponential inequality. When $a > 0$ they only have a finite first moment. Our proof here is inspired from the strong law of large number of Etemadi [23], and relies on the next three lemmata. Introduce the truncated variables $\overline{Z}^u_{n,i} = Z^u_{n,i}\mathbb{1}_{\{Z^u_{n,i}<i\}}$ and set $\frac{1}{n}\overline{S}^u_n = \frac{1}{n}\sum_{i=1}^n \overline{Z}^u_{n,i}$.

Lemma 5.2. *For each $u > 0$, $\mathbb{P}\big(\overline{Z}^u_{n,i} \neq Z^u_{n,i}, \text{ i.o.}\big) = 0$.*

To prove Proposition 5.1 it thus suffices to prove that $\lim_{n\to\infty} \frac{1}{n}\overline{S}^u_n = \nu^{int,-}(u,\infty)$ $\mathbb{P}$-a.s., for all $u > 0$. The next two lemmata will allow to prove this along subsequences of the form $k_n = \lfloor \beta^n \rfloor$, $\beta > 1$.

Lemma 5.3. *For each $u > 0$, $\sum_{n=1}^\infty \mathbb{P}\big(\big|\overline{S}^u_{k_n} - \mathbb{E}\overline{S}^u_{k_n}\big| \geq \varepsilon k_n\big) < \infty$.*

Lemma 5.4. *For each $u > 0$, $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\overline{S}^u_n = \nu^{int,-}(u, \infty)$.*

By Lemma 5.3, the Borel-Cantelli Lemma, and Lemma 5.4, $\lim_{n\to\infty} \frac{1}{k_n}\overline{S}^u_{k_n} = \nu^{int,-}(u, \infty)$ $\mathbb{P}$-a.s., for all $u > 0$. It will then only remain to handle the intermediate values $k_n \leq m \leq k_{n+1}$.

Before doing this let us prove Lemmata 5.3, 5.4 and 5.2. The information on the slow variation properties of the function φ_u and its inverse that is needed in their proofs is collected below. (We use the notations of Appendix 7.2 on regular variations.) Assume that $a > 0$. Clearly φ_u is strictly increasing and $\varphi_u \in R_a$. Thus φ_u^{-1} is well defined, strictly increasing, and, by Lemma 7.6, $\varphi_u^{-1} \in R_{1/a}$. Let the functions ϕ_u be defined through

$$\varphi_u^{-1}(y) = y^{1/a}\phi_u(y)\,. \tag{5.1}$$

Then $\phi_u \in R_0$. The following elementary lemma, stated without proof, gives its explicit form for small u.

Lemma 5.5. *Let $0 < a < 1$. Then, for all $u > 0$, (i) $\phi_u(y) \geq 1$ for all $y \geq 0$, and (ii) $\phi_u(y) \leq e^{1/a}$ for all $y \geq u^{1/(1-a)}$. Moreover, (iii) if $u \geq v$ then $\phi_u(y) \leq \phi_v(y)$ for all $y \geq 0$.*

In what follows, c_i, $i \geq 1$, are finite positive constants that may depend on the parameters a and α, but not on u, and whose value may change from line to line. We sometimes write φ, ϕ instead φ_u, ϕ_u if no confusion may arise.

Proof of Lemma 5.3. Let m_n be any increasing sequence such that $b_{k_n} \ll m_n \ll k_n$ and $\sum_{n=1}^{\infty}(m_n/k_n)^2 < \infty$. By Tchebychev inequality it is enough to prove that

$$\sum_{n=1}^{\infty} \frac{1}{k_n^2} \sum_{i=m_n}^{n} \mathbb{E}(\overline{Z}_{k_n,i}^u)^2 < \infty. \tag{5.2}$$

Set

$$h_n(z) = \frac{\mathbb{P}(\tau(x) > r_{k_n} z)}{\mathbb{P}(\tau(x) > r_{k_n})}. \tag{5.3}$$

By integration by parts,

$$\begin{aligned}\sum_{i=m_n}^{n} \mathbb{E}(\overline{Z}_{n,i}^u)^2 &\leq 2\sum_{i=m_n}^{n} b_{k_n}^2 \int_0^{i/b_{k_n}} y\mathbb{P}(\varphi_u(\tau(i)/r_{k_n}) > y)dy \\ &= 2b_{k_n}[b_{k_n}\mathbb{P}(\tau(x) > r_{k_n})] \sum_{i=m_n}^{n} \int_0^{\varphi_u^{-1}(i/b_{k_n})} \varphi_u(z)\varphi_u'(z)h_n(z)dz. \end{aligned}\tag{5.4}$$

Note that by definition of intermediate scales $b_{k_n}\mathbb{P}(\tau(x) \geq r_{k_n}) \sim 1$ and $b_{k_n}/k_n = o(1)$, whereas $i/b_{k_n} \geq m_n/b_{k_n} \uparrow \infty$ by assumption on m_n. To further express the last integral in (5.4) we decompose it into $I_n(u) = I_n'(u) + I_n''(u)$, where $I_n'(u) = 2\int_0^1 \varphi_u(z)\varphi_u'(z)h_n(z)dz$ and $I_n''(u) = 2\int_1^{\varphi_u^{-1}(i/b_{k_n})} \varphi_u(z)\varphi_u'(z)h_n(z)dz$.

To deal with $I_n'(u)$ we use that $h_n(z) \to z^{-\alpha}$, $n \to \infty$, where the convergence is uniform in $0 \leq z \leq 1$, since for each n, $h_n(z)$ is a monotone function, and since the limit, z^{α}, is continuous. Thus for all $\varepsilon > 0$ there exists $n(\varepsilon)$ such that for all $n \geq n(\varepsilon)$

$$\left| I_n'(u) - 2\int_0^1 \frac{\varphi_u(z)\varphi_u'(z)}{z^{\alpha}}dz \right| \leq \varepsilon 2\int_0^1 \varphi_u(z)\varphi_u'(z)dz \leq \frac{\varepsilon}{(2a+1)}, \quad \forall u > 0. \tag{5.5}$$

Integrating by parts, $2\int_0^1 \frac{\varphi_u(z)\varphi_u'(z)}{z^{\alpha}}dz = \varphi_u^2(1) + \alpha\int_0^1 \frac{\varphi_u^2(z)}{z^{1+\alpha-\varepsilon_n}}dz$. Performing the change of variable $z = y^{-1/(1-a)}$, the last integral may be written as, $j(u) = \frac{\alpha}{1-a}\int_{1/2}^{\infty} z^{\varrho-1}e^{-2uz}dz$ where we set $\varrho := \frac{\alpha-2a}{1-a}$. Now, if $\varrho > 0$, $j(u) \leq \frac{\alpha}{1-a}\int_0^{\infty} z^{\varrho-1}e^{-uz}dz = (2u)^{-\varrho}\frac{\alpha}{1-a}\Gamma(\varrho)$ where $-\varrho+1 > 0$, whereas if $\varrho \leq 0$, $j(u) \leq \frac{\alpha}{1-a}\int_{1/2}^{\infty} z^{-|\varrho|-1}dz = \frac{\alpha}{\varrho(1-a)}2^{|\varrho|}$. Combining these observations with (5.5) we conclude that for all $u > 0$ and all large enough n there exist constants $0 \leq c_0, c_4 < \infty$ and $0 < c_2 \leq 1$, that depend only on α and a, such that $I_n'(u) \leq c_0 + c_4 u^{-1+c_2}$.

To bound $I_n''(u)$ we note that $h_n(z) = x^{-\alpha}(L(r_{k_n}z)/L(r_{k_n}))$ and use that by Lemma 7.5, for each $x > 1$ and large enough n, $(1-\delta_n)z^{-\alpha-\varepsilon_n} \leq h_n(z) \leq (1+\delta_n)z^{-\alpha+\varepsilon_n}$, for some positive sequences ε_n and δ_n satisfying $\varepsilon_n \downarrow 0$, $\delta_n \downarrow 0$ as $n \uparrow \infty$. Thus

$$I_n''(u) \leq 2(1+\delta_n)\int_1^{\varphi_u^{-1}(i/b_{k_n})} \frac{\varphi_u(z)\varphi_u'(z)}{z^{\alpha-\varepsilon_n}}dz, \tag{5.6}$$

Integrating by parts $I_n''(u) \leq (1+\delta_n)\frac{\varphi_u^2(z)}{z^{\alpha-\varepsilon_n}}\Big|_1^{\varphi_u^{-1}(i/b_{k_n})} + (1+\delta_n)\int_1^{\varphi_u^{-1}(i/b_{k_n})}(\alpha-\varepsilon_n)\frac{\varphi_u^2(z)}{z^{1+\alpha-\varepsilon_n}}dz$.

Using Lemma 5.5, (i), we easily see that for all $u > 0$, $\frac{\varphi_u^2(z)}{x^{\alpha-\varepsilon_n}}\Big|_1^{\varphi_u^{-1}(i/b_{k_n})} \leq (\frac{i}{b_{k_n}})^{2-\frac{\alpha-\varepsilon_n}{a}}$. Next,

$$\begin{aligned}\int_1^{\varphi_u^{-1}(i/b_{k_n})} \frac{\varphi_u^2(x)}{x^{1+\alpha-\varepsilon_n}}dx &= \frac{1}{1-a}\int_{(\varphi_u^{-1}(i/b_{k_n}))^{-(1-a)}}^{1} x^{\frac{\alpha-2a-\varepsilon_n}{1-a}-1}e^{-2ux}dx \\ &\leq \tfrac{\alpha-\varepsilon_n}{|\alpha-2a-\varepsilon_n|}\left[1 + \left(\varphi_u^{-1}\left(\tfrac{i}{b_{k_n}}\right)\right)^{2a-\alpha+\varepsilon_n}\right] \leq 2\tfrac{\alpha-\varepsilon_n}{|\alpha-2a-\varepsilon_n|}\left(e\tfrac{i}{b_{k_n}}\right)^{2-\frac{\alpha-\varepsilon_n}{a}} \end{aligned}\tag{5.7}$$

which is valid for all $u > 0$. Indeed, if $u \leq v := (\frac{i}{b_{k_n}})^{1-a}$, then the last ineqality follows from Lemma 5.5, (ii); if on the contrary $u > v$, then, by Lemma 5.5, (iii), $\varphi_u^{-1}(i/b_{k_n}) \leq \varphi_v^{-1}(\frac{i}{b_{k_n}})$, whereas by Lemma 5.5, (ii), for all $y \geq \frac{i}{b_{k_n}}$, $\varphi_v^{-1}(y) \leq (e\frac{i}{b_{k_n}})^{1/a}$. Collecting our bounds we get that for all large enough n, $b_{k_n} I_n''(u) \leq c_5 b_{k_n} + c_6 b_{k_n}\left(\frac{i}{b_{k_n}}\right)^{2-\frac{\alpha-\varepsilon_n}{a}}$ for all $u > 0$.

Inserting our bounds on $I_n'(u)$ and $I_n''(u)$ in (5.4) and (5.2) successively yields

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{k_n^2}\sum_{i=m_n}^{n}\mathbb{E}(\overline{Z}^u_{k_n,i})^2 &\leq c_1\sum_{n=1}^{\infty}\frac{b_{k_n}}{k_n^2}\sum_{i=m_n}^{n}\left[(1+u^{-1+c_2})+\left(\frac{i}{b_{k_n}}\right)^{2-\frac{\alpha-\varepsilon_n}{a}}\right]\\ &\leq c_0(1+u^{-1+c_2})\sum_{n=1}^{\infty}\frac{b_{k_n}}{k_n}<\infty\,,\end{aligned}$$

where $0 < c_2 \leq 1$ and $0 \leq c_0, c_1 < \infty$. The proof of Lemma 5.3 is done. □

Proof of Lemma 5.4. It suffices to prove that $\frac{1}{n}\sum_{i=m_n}^{n}\mathbb{E}\overline{Z}^u_{n,i} = \nu^{int,-}(u,\infty)$ where m_n is an increasing sequence such that $b_n \ll m_n \ll n$. To this end write

$$\frac{1}{n}\mathbb{E}\overline{S}^u_n = \frac{1}{n}\sum_{i=m_n}^{n}\mathbb{E}Z^u_{n,i} - \frac{1}{n}\sum_{i=m_n}^{n}\mathbb{E}Z^u_{n,i}\mathbb{1}_{\{Z^u_{n,i}\geq i\}}\,. \tag{5.8}$$

Consider the first sum in the right hand side of (5.8). Integration by parts yields

$$\frac{1}{n}\sum_{i=m_n}^{n}\mathbb{E}Z^u_{n,i} = \frac{n-m_n+1}{n}[b_n\mathbb{P}(\tau(x)>r_nz)]\int_0^\infty\varphi_u'(z)h_n(z)dz := (1+o(1))J_n(u)\,, \tag{5.9}$$

where h_n is given in (5.3). Write $J_n(u) = J_n'(u) + J_n''(u)$, where $J_n'(u) = \int_0^1 \varphi_u'(x)h_n(x)dx$ and $J_n''(u) = \int_1^\infty \varphi_u'(x)h_n(x)dx$. On the one hand, proceeding as we did to establish (5.6), we obtain that $\lim_{n\to\infty} J_n'(u) = \int_0^1 \frac{\varphi_u'(x)}{x^\alpha}dx$ for all $u > 0$. On the other hand, using the bounds on h_n from the paragraph above (5.6),

$$(1-\delta_n)\int_1^\infty\frac{\varphi_u'(x)}{x^{\alpha+\varepsilon_n}}dx \leq J_n''(u) \leq (1+\delta_n)\int_1^\infty\frac{\varphi_u'(x)}{x^{\alpha-\varepsilon_n}}dx\,, \tag{5.10}$$

where $0 < \varepsilon_n, \delta_n \downarrow 0$ as $n \uparrow \infty$. Since $\int_0^\infty \frac{\varphi_u'(x)}{x^\alpha}dx = u^{-\frac{\alpha-a}{1-a}}\frac{\alpha}{1-a}\Gamma\left(\frac{\alpha-a}{1-a}\right) = \nu^{int,-}(u,\infty)$, which is finite for $u > 0$, dominated convergence applies and yields, $\lim_{n\to\infty} J_n''(u) = \int_1^\infty \frac{\varphi_u'(x)}{x^\alpha}dx$. Collecting our results we get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=m_n}^{n}\mathbb{E}Z^u_{n,i} = \int_0^\infty\frac{\varphi_u'(x)}{x^\alpha}dx = \nu^{int,-}(u,\infty)\,,\quad u>0\,. \tag{5.11}$$

It remains to prove that $\lim_{n\to\infty}\frac{1}{n}\sum_{i=m_n}^{n}\mathbb{E}Z^u_{n,i}\mathbb{1}_{\{Z^u_{n,i}\geq i\}} = 0$. Integration by parts and the change of variable $y = \varphi_u(z)$ yields

$$\mathbb{E}Z^u_{n,i}\mathbb{1}_{\{Z^u_{n,i}\geq i\}} = [b_n\mathbb{P}(\tau(x)>r_n)]J_n''\big(\varphi_u^{-1}(i/b_n)\big) + i\mathbb{P}\left(Z^u_{n,i}\geq i\right). \tag{5.12}$$

To bound $J_n''(\varphi_u^{-1}(i/b_n))$ we use the upper bound (5.10) (which is valid for all n large enough) and, proceeding as in the paragraph below (5.6) (but replacing φ_u^2 by φ_u), we readily obtain that $J_n''(\varphi_u^{-1}(i/b_n)) \leq (1+o(1))(i/b_n)^{1-\frac{\alpha-\varepsilon_n}{a}}$. Thus

$$\frac{1}{n}\sum_{i=m_n}^{n}J_n''(\varphi_u^{-1}(i/b_n)) \leq c_0\left(\frac{b_n}{n}\right)^{\frac{\alpha-\varepsilon_n}{a}-1}. \tag{5.13}$$

Turning to the last term in the right hand side of (5.12), we have

$$i\mathbb{P}(Z^u_{n,i} > i) = i\left(r_n(i/b_n)^{\frac{1}{a}}\phi(i/b_n)\right)^{-\alpha} L\left(r_n(i/b_n)^{\frac{1}{a}}\phi(i/b_n)\right), \tag{5.14}$$

where $\phi(y)$ is defined in (5.1). Using furthermore that $r_n^\alpha \mathbb{P}(\tau(x) > r_n) = L(r_n)$, we obtain

$$i\mathbb{P}(Z^u_{n,i} > i) = \left(\frac{b_n}{i}\right)^{\frac{\alpha}{a}-1} [b_n\mathbb{P}(\tau(x) > r_n)] \frac{L\left(r_n(i/b_n)^{\frac{1}{a}}\phi(i/b_n)\right)}{\phi(i/b_n)^\alpha L(r_n)}. \tag{5.15}$$

Consider the last quotient in the r.h.s. of (5.14). The bound $\phi_u\big(\frac{i}{b_n}\big) \geq 1$ of Lemma 5.5 (valid for all $u \geq 0$) together with Lemma 7.5 imply that there exist positive sequences ε_n and δ_n that verify $\varepsilon_n \downarrow 0$, $\delta_n \downarrow 0$ as $n \uparrow \infty$ and such that, for all n large enough, this quotient is bounded above by $(1+\delta_n)(i/b_n)^{\frac{\varepsilon_n}{a}}$. Inserting this in (5.15) we obtain,

$$\frac{1}{n}\sum_{i=m_n}^{n} i\mathbb{P}(Z^u_{n,i} > i) \leq \frac{2}{n}\sum_{i=m_n}^{n}\left(\frac{b_n}{i}\right)^{\frac{\alpha-\varepsilon_n}{a}-1} \leq 2\left(\frac{b_n}{m_n}\right)^{\frac{\alpha-\varepsilon_n}{a}-1}. \tag{5.16}$$

Finally, by (5.12), (5.13), and (5.16), since $\alpha/a > 1$ and $b_n \ll m_n \ll n$, we conclude that for all n large enough, $\lim_{n\to\infty}\frac{1}{n}\sum_{i=m_n}^{n}\mathbb{E}Z^u_{n,i}\mathbb{1}_{\{Z^u_{n,i}\geq i\}} = 0$. The proof of Lemma 5.4 is done. □

Proof of Lemma 5.2. It suffices to show that for some $n_0 < \infty$,

$$\lim_{i_0\to\infty}\mathbb{P}\big(\exists_{n\geq n_0}\exists_{i_0\leq i\leq n}\{\overline{Z}^u_{n,i} \neq Z^u_{n,i}\}\big) = 0. \tag{5.17}$$

Note that $\{\overline{Z}^u_{n,i} \neq Z^u_{n,i}\} = \{b_n\varphi_u(\tau(i)/r_n) > i\} = \{b_n\varphi_u(\tau(i)/r_n) > i, \tau(i) > 1\}$. Indeed if $\tau(i) \leq 1$ then there exists $n_0' < \infty$ such that for all $n \geq n_0'$, $b_n\varphi_u(\tau(i)/r_{k_n}) < 1$. Also note that for all fixed $y > 1$, $b_n\varphi_u(y/r_{k_n})$ is an increasing function of n for all $n \geq n_0''$ and some $n_0'' < \infty$. Therefore, taking $n_0 = \max(n_0', n_0'')$,

$$\begin{aligned}\mathbb{P}\big(\exists_{n\geq n_0}\exists_{i_0\leq i\leq n}\{\overline{Z}^u_{n,i} \neq Z^u_{n,i}\}\big) &\leq \mathbb{P}\big(\exists_{n\geq n_0}\exists_{i_0\leq i\leq n}\{b_n\varphi_u(\tau(i)/r_n) > i\}\big)\\ &\leq \mathbb{P}\big(\exists_{i\geq i_0}\{b_i\varphi_u(\tau(i)/r_i) > i\}\big)\\ &\leq \sum_{i\geq i_0}\mathbb{P}\big(b_i\varphi_u(\tau(i)/r_i) > i\big).\end{aligned} \tag{5.18}$$

Proceeding as in (5.14)-(5.16) to bound the last probability we get that, for some $\varepsilon_n \downarrow 0$,

$$\sum_{i\geq i_0}\mathbb{P}\big(b_i\varphi_u(\tau(i)/r_i) > i\big) \leq 2\sum_{i\geq i_0}\frac{1}{b_i}\left(\frac{b_i}{i}\right)^{\frac{\alpha-\varepsilon_n}{a}}, \tag{5.19}$$

where $\frac{\alpha-\varepsilon_n}{a} > 1$ for all n large enough. From this and the assumption that $b_n < n^\zeta$, $\zeta > 0$, it follows that the r.h.s. of (5.19) is bounded above by

$$2\sum_{i\geq i_0}(1/i)^{1+(1-\zeta)(\frac{\alpha-\varepsilon_n}{a}-1)} \leq c_0\,(1/i_0)^{(1-\zeta)(\frac{\alpha-\varepsilon_n}{a}-1)}, \tag{5.20}$$

which tends to zero as $i_0 \to \infty$, proving (5.17). The proof of Lemma 5.2 is done. □

To conclude the proof of Proposition 5.1 it remains to handle the intermediate values $k_n \leq m \leq k_{n+1}$. Observe that for such values of m, we have:

$$\begin{aligned}\frac{1}{m}\overline{S}^u_m &\leq \left(\frac{r_{k_{n+1}}}{r_{k_n}}\right)^a\left(\frac{k_{n+1}}{k_n}\right)\frac{1}{k_n}\overline{S}^{u_n}_{k_n}, \quad u_n \equiv u\frac{r_{k_n}}{r_{k_{n+1}}},\\ \frac{1}{m}\overline{S}^u_m &\geq \left(\frac{r_{k_n}}{r_{k_{n+1}}}\right)^a\left(\frac{k_n}{k_{n+1}}\right)\frac{1}{k_{n+1}}\overline{S}^{u_n}_{k_{n+1}}, \quad u_n \equiv u\frac{r_{k_{n+1}}}{r_{k_n}}.\end{aligned} \tag{5.21}$$

We now claim that,

$$\beta^{-(1-\varepsilon)(\zeta/\alpha)-1} \leq \frac{r_{k_{n+1}}}{r_{k_n}} \leq \beta^{(1-\varepsilon)(\zeta/\alpha)+1}, \tag{5.22}$$

for some $0 < \varepsilon < 1$, and $\beta^{-1} \leq \frac{k_{n+1}}{k_n} \leq \beta$. The latter bounds are immediate. To prove (5.22) note that G^{-1} in (6.1) belongs to $R_{-1/\alpha}(0+)$. It thus follows from Definition 3.1 that

$$r_{k_{n+1}}/r_{k_n} = \frac{G^{-1}\big(b_{k_{n+1}}^{-1}(1+o(1))\big)}{G^{-1}\big(b_{k_n}^{-1}(1+o(1))\big)} = (1+o(1))\left(\frac{b_{k_{n+1}}}{b_{k_n}}\right)^{1/\alpha} \frac{\ell\big(b_{k_{n+1}}^{-1}(1+o(1))\big)}{\ell\big(b_{k_n}^{-1}(1+o(1))\big)}, \tag{5.23}$$

for some function ℓ slowly varying at $0+$. Furthermore, by assumtion, b_n is regularly varying at infinity with index $\zeta < 1$. Hence, by Lemma 7.5, $(1-\delta_n)\left(b_{k_{n+1}}/b_{k_n}\right)^{\zeta/\alpha-\varepsilon_n} \leq r_{k_{n+1}}/r_{k_n} \leq (1+\delta_n)\left(b_{k_{n+1}}/b_{k_n}\right)^{\zeta/\alpha+\varepsilon_n}$, where $\varepsilon_n \downarrow 0$, $\delta_n \downarrow 0$ as $n \uparrow \infty$. Since $\beta > 1$, $\beta^{-1}(1-\delta_n) \leq (1-\delta_n) \leq (1+\delta_n) \leq \beta$, for all n large enough. Combining our bounds proves (5.22).

Therefore, there exist strictly positive constants ε^-, ε^+, δ^- and δ^+ such that, $\mathbb{P}$-a.s.,

$$\begin{aligned} \beta^{-\delta^-}\nu^{int,-}(u/\beta^{\varepsilon^-},\infty) &\leq \liminf_{m\to\infty}\frac{1}{m}\overline{S}_m^u \\ &\leq \limsup_{m\to\infty}\frac{1}{m}\overline{S}_m^u \leq \beta^{\delta^+}\nu^{int,-}(u\beta^{\varepsilon^+},\infty). \end{aligned} \tag{5.24}$$

Since this is true for every $\beta > 1$, the proof of Proposition 5.1 is done. □

We are now ready to prove Proposition 3.8.

Proof of Proposition 3.8, (i). With the notation of Proposition 5.1, (2.27) becomes

$$\nu_n(u,\infty) = \frac{a_n r_n^a}{b_n} \frac{\frac{1}{n}S_n^u}{\frac{1}{n}\sum_{i=1}^n \tau^a(i)}. \tag{5.25}$$

Reasoning as in the proof of (4.1) it follows from Proposition 5.1 that, $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty}\frac{1}{n}S_n^u = \nu^{int,-}(u,\infty), \quad \forall u > 0. \tag{5.26}$$

By the strong law of large numbers, $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \tau^a(i) = \mathbb{E}\tau^a$ $\mathbb{P}$-a.s., and by assumption on a_n, $a_n r_n^a/(b_n\mathbb{E}\tau^a) \sim 1$. Therefore Condition (2.28) of Theorem 2.4, (i), is satisfied $\mathbb{P}$-a.s. with $\nu = \nu^{int,-}$. To see that Condition (2.29) also is satisfied we use that since convergence in (5.26) is uniform then, for each $0 < \delta \leq 1$, $\mathbb{P}$-a.s., $\lim_{n\to\infty}\int_0^\delta \nu_n^{int,-}(u,\infty)du = \int_0^\delta \nu^{int,-}(u,\infty)du = \delta^{\frac{1-\alpha}{1-a}}\frac{\alpha}{1-\alpha}\Gamma\big(\frac{\alpha-a}{1-a}\big)$. Now $\int_0^\delta \nu_n^{int,-}(u,\infty)du$ is a monotone increasing sequence having a continuous limit so that $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty}\int_0^\delta \nu_n^{int,-}(u,\infty)du = \int_0^\delta \nu^{int}(u,\infty)du, \quad \forall\, 0 < \delta \leq 1. \tag{5.27}$$

Both conditions of Theorem 2.4, (i), being satisfied in $\mathbb{P}$-a.s., the proof of Proposition 3.8, (i), is done. □

6 Extreme scales.

Consider the re-scaled sequence $\gamma_n(x) = r_n^{-1}\tau(x)$, $x \in \mathcal{V}_n$. For each n form the point process $\Upsilon_n = \sum_{x\in\mathcal{V}_n}\mathbb{1}_{\gamma_n(x)}$, and let $\Upsilon = \sum_{k=1}^\infty \mathbb{1}_{\gamma_k}$ be PRM(μ) with μ given by (3.6). It is well known that when $(\tau(x), x \in \mathcal{V}_n)$ are i.i.d. r.v.'s equi-distributed with $\tau \in \mathcal{D}(\alpha)$, Υ_n converges weakly to Υ if and only if r_n is an extreme space scale. By the Continuous Mapping Theorem, appropriate continuous functionals of Υ_n converge to the corresponding functionals of Υ. This convergence however is in distribution only, and this is not enough for our purposes. The usual way out of this difficulty is to think of weak convergence from Skorohod's representation Theorem and replace the sequence $(\gamma_n(x), x \in \mathcal{V}_n)$ by a new sequence with identical distribution, but almost sure convergence properties. This strategy was first implemented in the context of an aging system by Fontes et *al.* [25], and often used since. We in turn adopt it using, however, an explicit representation of the re-scaled landscape. The latter is given in Subsection 6.1. In Subsection 6.2 we consider the model obtained by substituting the representation for the original landscape and prove Proposition 3.9. The final Subsection 6.3 contains the proof of Lemma 3.10.

6.1 A representation of the re-scaled landscape.

The representation we now introduce is due to Lepage *et al.* [32] and relies on an elementary property of order statistics. Let $\bar{\tau}_n(1) \geq \cdots \geq \bar{\tau}_n(n)$ and $\bar{\gamma}_n(1) \geq \cdots \geq \bar{\gamma}_n(n)$ denote, respectively, the landscape and re-scaled landscape variables, $(\tau(x), x \in \mathcal{V}_n)$ and $(\gamma_n(x), x \in \mathcal{V}_n)$, arranged in decreasing order of magnitude. For $u \geq 0$ set $G(u) = \mathbb{P}(\tau(x) > u)$ and

$$G^{-1}(u) := \inf\{y \geq 0 : G(y) \leq u\}\,. \tag{6.1}$$

Let $(E_i, i \geq 1)$ be a sequence of i.i.d. mean one exponential random variables defined on a common probability space $(\Omega^E, \mathcal{F}^E, \mathbf{P})$. We will now see that both the ordered landscape variables and the limiting point process Υ can be expressed in terms of this sequence. Set, for $k \geq 1$,

$$\begin{aligned} \Gamma_k &= \sum_{i=1}^{k} E_i\,, \\ \gamma_k &= \Gamma_k^{-1/\alpha}\,, \end{aligned} \tag{6.2}$$

and, for $1 \leq k \leq n$, $n \geq 1$,

$$\gamma_{nk} = r_n^{-1} G^{-1}(\Gamma_k / \Gamma_{n+1})\,. \tag{6.3}$$

Lemma 6.1. *For each* $n \geq 1$, $(\bar{\gamma}_n(1), \ldots, \bar{\gamma}_n(n)) \stackrel{d}{=} (\gamma_{n1}, \ldots, \gamma_{nn})\,.$

Proof. Note that G is non-increasing and right-continuous so that G^{-1} is non-increasing and right-continuous. It is well known that if the random variable U is a uniformly distributed on $[0,1]$ we may write $\tau(0) \stackrel{d}{=} G^{-1}(U)$. In turn it is well known (see [24], Section III.3) that if $(U(k), 1 \leq k \leq n)$ are independent random variables uniformly distributed on $[0,1]$ then, denoting by $\bar{U}_n(1) \leq \cdots \leq \bar{U}_n(n)$ their ordered statistics, $(\bar{U}_n(1), \ldots, \bar{U}_n(n)) \stackrel{d}{=} (\Gamma_1/\Gamma_{n+1}, \ldots, \Gamma_n/\Gamma_{n+1})$. Combining these two facts yields the claim of the lemma. □

Next, let Υ be the point process in $M_P(\mathbb{R}_+)$ which has counting function

$$\Upsilon([a,b]) = \sum_{i=1}^{\infty} \mathbb{1}_{\{\gamma_k \in [a,b]\}}\,. \tag{6.4}$$

Lemma 6.2. Υ *is a Poisson random measure on* $(0,\infty)$ *with mean measure* μ *given by (3.6).*

Proof. The point process $\Gamma = \sum_{i=1}^{\infty} \mathbb{1}_{\{\Gamma_k\}}$ defines a homogeneous Poisson random measure on $[0,\infty)$ and thus, by the mapping theorem ([35], Proposition 3.7), setting $T(x) = x^{-1/\alpha}$ for $x > 0$, $\Upsilon = \sum_{i=1}^{\infty} \mathbb{1}_{\{T(\Gamma_k)\}}$ is Poisson random measure on $(0,\infty)$ with mean measure $\mu(x,\infty) = T^{-1}(x)$. □

Then, on the fixed probability space $(\Omega^E, \mathcal{F}^E, \mathbf{P})$, all random variables of interest will have an almost sure limit.

Proposition 6.3. *Let r_n be an extreme space scale. Let $f : (0,\infty) \to [0,\infty)$ be a continuous function that obeys*

$$\int_{(0,\infty)} \min(f(u),1) d\mu(u) < \infty \,. \tag{6.5}$$

Then, $\mathbf{P}$-almost surely,

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(\gamma_{nk}) = \sum_{k=1}^{\infty} f(\gamma_k) < \infty \,. \tag{6.6}$$

The proof of Proposition 6.3 closely follows the proof of Proposition 3.1 of [25]. We leave the details to the interested reader.

6.2 Proof of Proposition 3.9.

In this subsection we consider the model obtained by substituting the new landscape $(\gamma_{nk}, 1 \le k \le n)$ for the original (re-scaled) landscape $(\gamma_n(x), x \in \mathcal{V}_n)$. We assume throughout that r_n is an extreme space scale. As for short and intermediate space scales, the proof of Proposition 3.9 relies on Theorem 2.4. To distinguish the quantity $\nu_n(u,\infty)$, expressed in (2.27) in the original landscape variable, from its expression in the new landscape variables, we call the latter $\mathbf{v}_n(u,\infty)$. Therefore

$$\mathbf{v}_n(u,\infty) = a_n \frac{\sum_{k=1}^{n} (r_n\gamma_{nk})^a e^{-u/\gamma_{nk}^{(1-a)}}}{\sum_{k=1}^{n} (r_n\gamma_{nk})^a} \,, \quad u \ge 0 \,. \tag{6.7}$$

We first treat the numerator in (6.7). For $u \ge 0$ set

$$\varphi_u(y) = y^a e^{-u/y^{(1-a)}} \,, \quad y \ge 0 \,. \tag{6.8}$$

We want to apply Proposition 6.3 to the sum $\sum_{k=1}^{n} \varphi_u(\gamma_{nk})$. For this let x^* be defined through $\varphi_u(x^*) = 1$. Noting that $0 < x^* \le 1$ for $0 \le a < 1$ and $u \ge 0$, a simple calculation yields $\int_{(0,\infty)} \min(\varphi_u(y),1) d\mu(y) = \frac{\alpha}{1-a} \int_{1/x^*}^{\infty} y^{-\frac{1-\alpha}{1-a}} e^{-uy} dy + (x^*)^{-\alpha}$, which is always finite if $u > 0$, regardless of the respective size of a and α. Thus, for all $u > 0$, $\mathbf{P}$-almost surely,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \varphi_u(\gamma_{nk}) = \sum_{k=1}^{\infty} \varphi_u(\gamma_k) < \infty \,. \tag{6.9}$$

In contrast, the behavior of the denominator in (6.7) will depend on whether a is larger or smaller than α.

The case $a > \alpha$. Here we have $\int_{(0,\infty)} \min(x^a,1) d\mu(x) < \infty$, so that $\mathbf{P}$-almost surely,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \gamma_{nk}^a = \sum_{k=1}^{\infty} \gamma_k^a < \infty \,. \tag{6.10}$$

In that case, choosing $a_n = 1$ in (6.7), we get, collecting (6.9) and (6.10), that for all $u > 0$, $\mathbf{P}$-almost surely,

$$\lim_{n\to\infty} \mathbf{v}_n(u,\infty) = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \varphi_u(\gamma_{nk})}{\sum_{k=1}^{n} \gamma_{nk}^a} = \frac{\sum_{k=1}^{\infty} \varphi_u(\gamma_{nk})}{\sum_{k=1}^{\infty} \gamma_{nk}^a} = \nu^{ext,+}(u,\infty) \,. \tag{6.11}$$

It is plain that $\nu^{ext,+}$ is a probability measure with continuous density: indeed it is an infinite mixture of exponential densities, the coefficients of the mixture being the weights

$\gamma_k^a/\sum_l \gamma_l^a$ of Poisson-Dirichlet random probability measure with parameter α/a. From the monotonicity of $\mathbf{v}_n(u,\infty)$ and the continuity of the limiting function $\nu^{ext,+}(u,\infty)$ we conclude that there exists a subset $\Omega_1^E \subset \Omega^E$ of the sample space Ω^E of the γ's with the property that $\mathbf{P}(\Omega_1^E)=1$, and such that, on Ω_1^E,

$$\lim_{n\to\infty} \mathbf{v}_n(u,\infty) = \nu^{ext,+}(u,\infty), \quad \forall\, u \geq 0\,. \tag{6.12}$$

Condition 2.28 of assertion (i) of Theorem 2.4 is thus satisfied $\mathbf{P}$-almost surely. To see that Condition 2.29 also is satisfied on a set of full measure we use that on Ω_1^E, by (6.12), for all $0<\delta\leq\delta_0$ and some $0<\delta_0\leq 1$, $\lim_{n\to\infty}\int_0^\delta \mathbf{v}_n(u,\infty)du = \int_0^\delta \nu^{ext,+}(u,\infty)du$. Again the monotonicity of $\int_0^\delta \mathbf{v}_n(u,\infty)du$ and the continuity of the limiting function allow us to conclude that there exists of a subset $\Omega_2^E \subset \Omega^E$ with the property that $\mathbf{P}(\Omega_2^E)=1$, and such that, on Ω_2^E, $\lim_{n\to\infty}\int_0^\delta \mathbf{v}_n(u,\infty)du = \int_0^\delta \nu^{ext,+}(u,\infty)du$ for all $0<\delta\leq\delta_0$. We may thus pass to the limit $\delta\to 0$ and write $\lim_{\delta\to 0}\lim_{n\to\infty}\int_0^\delta \mathbf{v}_n(u,\infty)du = \lim_{\delta\to 0}\int_0^\delta \nu^{ext,+}(u,\infty)du$. Now by (3.22),

$$\int_0^\delta \nu^{ext,+}(u,\infty)du = \sum_k \frac{\gamma_k}{\sum_l \gamma_l^a}\left(1-e^{-\delta\gamma_k^{-(1-a)}}\right) \leq \delta e^{\sqrt{\delta}}\nu^{ext,+}(\delta,\infty)+\sum_k \frac{\gamma_k}{\sum_l \gamma_l^a}\mathbb{1}_{\{\gamma_k\leq\delta^{1/2(1-a)}\}}\,, \tag{6.13}$$

where we proceeded as in (2.18)-(2.20) to derive the upper bound. Now from this bound and Lemma 3.10 it follows that $\lim_{\delta\to 0}\int_0^\delta \nu^{ext,+}(u,\infty)du = 0$ $\mathbf{P}$-a.s.. All the assumptions of assertion (i) of Theorem 2.4 are thus satisfied $\mathbf{P}$-a.s. The proof of Proposition 3.9 in the case $a>\alpha$ is complete.

The case $a<\alpha$. Here $\mathbb{E}(r_n\gamma_{nk})^a<\infty$ and $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n (r_n\gamma_{nk})^a = \mathbb{E}\tau^a<\infty$ $\mathbf{P}$-a.s.. Thus, choosing a_n such that $a_n r_n^a/(b_n\mathbb{E}\tau^a)\sim 1$, we get that for all $u>0$, $\mathbf{P}$-a.s,

$$\lim_{n\to\infty}\mathbf{v}_n(u,\infty) = \lim_{n\to\infty}\frac{\sum_{k=1}^n \varphi_u(\gamma_{nk})}{\frac{1}{n}\sum_{k=1}^n (r_n\gamma_{nk})^a} = \sum_{k=1}^\infty \varphi_u(\gamma_{nk}) = \nu^{ext,-}(u,\infty)<\infty\,. \tag{6.14}$$

Now using Lemma 3.10 one easily checks that $\int_{(0,\infty)}(1\wedge u)\nu^{ext,-}(du)<\infty$ and that $\nu^{ext,-}(u,\infty)$ is continuous on a subset of full measure. Again we conclude that there exists a subset $\Omega_2^E \subset \Omega^E$ of the sample space Ω^E of the γ's with the property that $\mathbf{P}(\Omega_2^E)=1$, and such that, on Ω_2^E,

$$\lim_{n\to\infty} \mathbf{v}_n(u,\infty) = \nu^{ext,-}(u,\infty), \quad \forall\, u \geq 0\,. \tag{6.15}$$

The conditions of assertion (ii) of Theorem 2.4 are thus satisfied $\mathbf{P}$-almost surely. Proposition 3.9 is proved in the case $a<\alpha$. Of course, taking the intersection $\Omega_1^E\cap\Omega_2^E$, the two convergence results of (3.23) can be stated simultaneous on a common full measure set.

6.3 Proof of Lemma 3.10.

Recall (6.8) and write $\varphi\equiv\varphi_1$. Set $u^{-\frac{\alpha-a}{1-a}}=m$. By (3.22) we may write

$$u^{\frac{\alpha-a}{1-a}}\nu^{ext,-}(u,\infty) = \frac{1}{m}\sum_{k=1}^\infty \varphi(m^{1/\alpha}\gamma_k)\,. \tag{6.16}$$

Assertion (i) of the lemma will thus be proven if we can prove that

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^\infty \varphi(m^{1/\alpha}\gamma_k) = \tfrac{\alpha}{1-a}\Gamma\left(\tfrac{\alpha-a}{1-a}\right)\mathbf{P}\text{-almost surely.} \tag{6.17}$$

Note that for this it is enough to take the limit along the integers since, $\varphi(m^{1/\alpha}\gamma_k)$ being a strictly increasing function of m,

$$\frac{\lfloor m\rfloor}{m}\frac{1}{\lfloor m\rfloor}\sum_{k=1}^{\infty}\varphi(\lfloor m\rfloor^{1/\alpha}\gamma_k)\leq\frac{1}{m}\sum_{k=1}^{\infty}\varphi(m^{1/\alpha}\gamma_k)\leq\frac{\lceil m\rceil}{m}\frac{1}{\lceil m\rceil}\sum_{k=1}^{\infty}\varphi(\lceil m\rceil^{1/\alpha}\gamma_k)\,. \tag{6.18}$$

The proof now proceeds as follows. Given a threshold function $M\equiv M(m)$ (to be chosen later) let $\mathrm{PRM}(\mu_M^+)$ and $\mathrm{PRM}(\mu_M^-)$ be the Poisson point processes with points $\{\gamma_k^\pm\}$ whose intensity measures are defined through

$$\mu_M^-(A)=\mu(A\cap(0,M/m^{1/\alpha}))\text{and}\mu_M^+(A)=\mu(A\cap[M/m^{1/\alpha},\infty)) \tag{6.19}$$

for any Borel set $A\subseteq(0,\infty)$. (In other words $\mathrm{PRM}(\mu_M^+)$ and $\mathrm{PRM}(\mu_M^-)$ are $\mathrm{PRM}(\mu)$ restricted to the sets $(0,M/m^{1/\alpha})$ and $[M/m^{1/\alpha},\infty)$ respectively). Using these two processes we break the middle sum in (6.18) into $\frac{1}{m}\sum_{k=1}^{\infty}\varphi(m^{1/\alpha}\gamma_k^-)+\frac{1}{m}\sum_{k=1}^{\infty}\varphi(m^{1/\alpha}\gamma_k^+)$. We will show that if M is of the form $M=\varepsilon\big(\frac{m}{\log m}\big)^{\frac{1}{\alpha}}$, for some small enough $0<\varepsilon<1$, then, $\mathbf{P}$-almost surely,

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{\infty}\varphi(m^{1/\alpha}\gamma_k^-)=\tfrac{\alpha}{1-a}\Gamma\big(\tfrac{\alpha-a}{1-a}\big)\,, \tag{6.20}$$

$$\text{and}\quad\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{\infty}\varphi(m^{1/\alpha}\gamma_k^+)=0\,. \tag{6.21}$$

We first prove (6.20). The boundedness of the Poisson points γ_k^- enables us to use a classical large deviation upper bound. Set

$$A_m=\left\{\left|\frac{1}{m}\sum_{k=1}^{\infty}\varphi(m^{1/\alpha}\gamma_k^-)-\mathbf{E}\frac{1}{m}\sum_{k=1}^{\infty}\varphi(m^{1/\alpha}\gamma_k^-)\right|\geq\delta_m\right\}\,, \tag{6.22}$$

where $\delta_m=2\big(\frac{\log m}{m}\big)^{1-\frac{a}{\alpha}}$. By Tchebychev exponential inequality, for all $\lambda>0$,

$$\mathbf{P}\left(A_m\right)\leq 2\exp\left\{-\lambda\delta_m-\mathbf{E}(\lambda/m)\sum_{k=1}^{\infty}\varphi(m^{1/\alpha}\gamma_k^-)+\log\mathbf{E}\exp\left\{(\lambda/m)\sum_{k=1}^{\infty}\varphi(m^{1/\alpha}\gamma_k^-)\right\}\right\}\,. \tag{6.23}$$

Simple Poisson point process calculations yield $\mathbf{E}\frac{1}{m}\sum_{k=1}^{\infty}\varphi(m^{1/\alpha}\gamma_k^-)=\sigma_M^{(1)}$, where

$$\sigma_M^{(1)}=\tfrac{\alpha}{1-a}\int_{1/M^{1-a}}^{\infty}y^{\frac{\alpha-a}{1-a}-1}e^{-y}dy=(1-o(1))\tfrac{\alpha}{1-a}\Gamma\big(\tfrac{\alpha-a}{1-a}\big)\,, \tag{6.24}$$

and

$$\log\mathbf{E}\exp\left\{(\lambda/m)\sum_{k=1}^{\infty}\varphi(m^{1/\alpha}\gamma_k^-)\right\}=-\int_0^{\infty}(1-e^{\frac{\lambda}{m}\varphi(m^{1/\alpha}x)})d\mu_M^-(x)\,. \tag{6.25}$$

Furthermore, for all $l>1$, $\int_0^\infty\varphi^k(m^{1/\alpha}x))d\mu_M^-(x):=m\sigma_M^{(k)}$, where

$$\sigma_M^{(l)}=\tfrac{\alpha}{1-a}\int_{1/M^{1-a}}^{\infty}y^{\frac{\alpha-la}{1-a}-1}e^{-y}dy\,. \tag{6.26}$$

In the worst situation $\alpha<la$ for all $l>1$ (indeed if $\alpha\geq la$, then $\sigma_M^{(l)}\leq\frac{\alpha}{1-a}\Gamma\big(\frac{\alpha-la}{1-a}\big)<\infty$). Let us thus assume that $\alpha<la$ for all $l>1$. In this case, $\sigma_M^{(l)}\leq\bar{\sigma}_M^{(l)}:=\frac{\alpha}{(1-a)(2a-\alpha)}M^{al-\alpha}$, and so,

$$-\int_0^{\infty}(1-e^{\frac{\lambda}{m}f(m^{1/\alpha}x)})d\mu(x)\leq\sigma_M^{(1)}\lambda+\bar{\sigma}_M^{(2)}\frac{\lambda^2}{4m}e^{\frac{\lambda M^a}{2m}}\,. \tag{6.27}$$

Inserting this bound in (6.25), plugging the result in (6.23), and choosing $\lambda = \delta_m 2m/\bar{\sigma}_M^{(2)}$, we obtain

$$\mathbf{P}\left(A_m\right) \leq 2\exp\left\{-\delta_m^2 m/\bar{\sigma}_M^{(2)}\left(2 - e^{2\delta_m M^a/\bar{\sigma}_M^{(2)}}\right)\right\}. \tag{6.28}$$

If we now take $\delta_m^2 = 4\big(\frac{\log m}{m}\big)^{2(1-\frac{a}{\alpha})}$ and $M = \varepsilon\big(\frac{m}{\log m}\big)^{\frac{1}{\alpha}}$, $0 < \varepsilon < 1$, then

$$\begin{aligned} \delta_m^2 m/\bar{\sigma}_M^{(2)} &= 4^{\frac{(1-a)(2a-\alpha)}{\alpha}}\left(1/\varepsilon\right)^{2a-\alpha}\log m\,, \\ 2\delta_m M^a/\bar{\sigma}_M^{(2)} &= 4^{\frac{(1-a)(2a-\alpha)}{\alpha}}\varepsilon^{\alpha-a}\,, \end{aligned} \tag{6.29}$$

(recall that by assumption $2a > \alpha$ and $a < \alpha$). Choosing ε sufficiently small so as to guarantee that

$$\delta_m^2 m/\bar{\sigma}_M^{(2)} \geq 6 \text{and} 2\delta_m M^a/\bar{\sigma}_M^{(2)} \leq \log(4/3)\,, \tag{6.30}$$

the bound (6.28) becomes $\mathbf{P}\left(A_m\right) \leq \frac{2}{m^2}$. Thus $\sum_m \mathbf{P}\left(A_m\right) \leq \infty$ which, invoking the first Borel-Cantelli Lemma, proves (6.20).

From now on we take $M = \varepsilon\big(\frac{m}{\log m}\big)^{1/\alpha}$ and assume that ε satisfies (6.30). It remains to prove (6.21). Using that $\varphi(x) \leq x^a$, $x \geq 0$, we have

$$\frac{1}{m}\sum_{k=1}^{\infty}\varphi(m^{1/\alpha}\gamma_k^+) = \frac{1}{m}\sum_{k=1}^{\infty}\varphi(m^{1/\alpha}\gamma_k)\mathbb{1}_{\{\gamma_k > M/m^{1/\alpha}\}} \leq m^{-(1-a/\alpha)}\sum_{k=1}^{\infty}\gamma_k^a\mathbb{1}_{\{\gamma_k > \varepsilon/(\log m)^{\frac{1}{\alpha}}\}}\,. \tag{6.31}$$

We further decompose the last sum in the r.h.s. above into $\mathcal{S}^-(m) + \mathcal{S}^+(m)$, where

$$\begin{aligned} \mathcal{S}^-(m) &= m^{-(1-a/\alpha)}\sum_{k=1}^{\infty}\gamma_k^a\mathbb{1}_{\{\varepsilon/(\log m)^{1/\alpha} < \gamma_k \leq 1\}}\,, \\ \mathcal{S}^+(m) &= m^{-(1-a/\alpha)}\sum_{k=1}^{\infty}\gamma_k^a\mathbb{1}_{\{\gamma_k > 1\}}\,. \end{aligned} \tag{6.32}$$

To deal with $\mathcal{S}^-(m)$ we write

$$\mathcal{S}^-(m) \leq \frac{m^{a/\alpha}}{m}\sum_{k=1}^{\infty}\mathbb{1}_{\{\varepsilon/(\log m)^{1/\alpha} < \gamma_k \leq 1\}} = \frac{m^{a/\alpha}\mu((\log m)^{1/\alpha}, 1]}{m}\frac{\sum_{k=1}^{\infty}\mathbb{1}_{\{\varepsilon/(\log m)^{1/\alpha} < \gamma_k \leq 1\}}}{\mu((\log m)^{1/\alpha}, 1]}\,. \tag{6.33}$$

Since $\mu((\log m)^{1/\alpha}, 1] = \log m/\varepsilon^\alpha - 1 \uparrow \infty$ as $m \uparrow \infty$, it follows from the strong law of large numbers for non-homogeneous Poisson processes (see [30] p. 51) that

$$\lim_{m\to\infty}\frac{\sum_{k=1}^{\infty}\mathbb{1}_{\{\varepsilon/(\log m)^{1/\alpha} < \gamma_k \leq 1\}}}{\mu((\log m)^{1/\alpha}, 1]} = 1 \mathbf{P}\text{-almost surely.} \tag{6.34}$$

and since $m^{a/\alpha-1}\mu((\log m)^{1/\alpha}, 1] = o(1)$, as follows from the assumption that $a < \alpha$, we get that $\lim_{m\to\infty}\mathcal{S}^-(m)$ $\mathbf{P}$-a.s.. To treat $\mathcal{S}^+(m)$ note that $\int_{(0,\infty)}\min(u^a\mathbb{1}_{u>1}, 1)d\mu(u) < \infty$. Thus, by Campbell's Theorem, $\sum_{k=1}^{\infty}\gamma_k^a\mathbb{1}_{\{\gamma_k>1\}} < \infty$ $\mathbf{P}$-a.s.. From this and the fact that $m^{a/\alpha-1} = o(1)$, we get that $\lim_{m\to\infty}\mathcal{S}^+(m)$ $\mathbf{P}$-a.s.. Collecting our results yields that $\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{\infty}\varphi(m^{1/\alpha}\gamma_k^+) = 0$ $\mathbf{P}$-a.s., and establishes (6.21).

The proof of Lemma 3.10 is complete.

7 Appendix

7.1 Renewal theory.

We summarize here what we need to know about renewal theory for subordinators and renewal processes. Recall first that:

Definition 7.1. *A renewal process* $\{R(n)\,,n\in\mathbb{N}\}$ *is a partial sum process with identical and independent increments taking values in* $[0,\infty)$*.* $R(n)$ *is represented as*

$$R(n)=\textstyle\sum_{k\leq n}\xi_k\,, \tag{7.1}$$

where $\{\xi_k, k\geq 1\}$ *are independent r.v.'s with identical distribution* ν*. The* ξ_k*'s, which stand for the life-time of items, are called inter-arrival times; their law,* ν*, is called the inter-arrival distribution.*

7.1.1 The Dynkin-Lamperti Theorem.

The continuous time version of the Dynkin-Lamperti Theorem was stated as Theorem 1.8. We now give the “classical” discrete time version. Set

$$\mathcal{C}_\infty(t,s)=\mathcal{P}\left(\{R(k)\,,k\in\mathbb{N}\}\cap(t,t+s)=\emptyset\right)\,,\quad 0\leq t<t+s\,, \tag{7.2}$$

where R is a renewal process of inter-arrival distribution ν. Let $\theta_t(\cdot)$ denote the overshoot function (2.22) in discrete time. In this setting $\theta_t(R)$ is usually called the residual waiting time. Clearly, $\mathcal{C}_\infty(t,s)=\mathcal{P}\left(\theta_t(R)\geq s\right)$. One has (see [22] or [14], section 8.6):

Theorem 7.2 (Dynkin, 55(61) and Lamperti, 58)**.**

(i) [Arcsine law.] A necessary an sufficient condition for $\theta_t(R)/t$ *to have a non-degenerate limit law is that* ν *is regularly varying at infinity with index* $0<\alpha<1$*. In that case,*

$$\lim_{t\to\infty}\mathcal{C}_\infty(t,\rho t)=\mathrm{Asl}_\alpha(1/1+\rho)\,. \tag{7.3}$$

(ii) [Finite mean life time renewal.] If $\int_0^\infty\nu(x,\infty)dx=m<\infty$ *and if* ν *is non-latticed then, for each fixed* $s>0$*,*

$$\lim_{t\to\infty}\mathcal{C}_\infty(t,s)=\frac{1}{m}\int_s^\infty\nu(x,\infty)dx\,. \tag{7.4}$$

7.1.2 Stationarity of delayed processes with “finite mean life time”.

A *delayed* renewal process corresponding to a renewal process R is the process $\widehat{R}$ defined by $\widehat{R}=\sigma+R$ where σ is a nonnegative random variable independent of S. Similarly the *delayed* subordinator corresponding to a subordinator S is the process $\widehat{S}$ defined by $\widehat{S}=\sigma+S$ where σ is a nonnegative random variable independent of S. We will say that a renewal process or subordinator *pure* when we want to emphasize that $\sigma=0$.

It is well known (see e.g. [24]), and not difficult to prove, that when $\int_0^\infty\nu(x,\infty)dx=m<\infty$, the delayed renewal process $\widehat{R}=\sigma+R$, whose initial jump is sampled from the limit law of the residual waiting time $\theta_t(R)$, is stationary. A similar statement holds for the delayed subordinator $\widehat{S}:=\sigma+S$ (see [29]). These results are summarized in the theorem below.

Theorem 7.3. *Let* F *denote the distribution function of* σ*.*

(i) [Delayed renewal process] Under the assumptions and with the notations of Theorem 7.2, (ii), if $F(s)=\lim_{t\to\infty}\mathcal{C}_\infty(t,s)$*, then, denoting by* $\widehat{R}$ *the delayed renewal process* $\widehat{R}=\sigma+R$*,*

$$\widehat{R}\stackrel{d}{=}R\,. \tag{7.5}$$

(ii) [Delayed subordinator] Under the assumptions and with the notations of Theorem 1.8, (ii), if $F(s)\lim_{t\to\infty}\mathcal{C}_\infty(t,s)$*, then, denoting by* $\widehat{S}$ *the delayed subordinator* $\widehat{S}=\sigma+S$*,*

$$\widehat{S}\stackrel{d}{=}S\,. \tag{7.6}$$

7.2 Regular variations.

We assume as known the elementary properties of regularly and slowly varying functions as described in Section 1 of [14] and, in particular, the *Uniform Convergence Theorem* ([14], Theorem 1.2.1) for slowly varying functions ([14], Theorem 1.3.1). In the sequel we denote by R_0 the class of functions that are slowly varying at infinity, by R_ρ the class of functions that are regularly varying at infinity with index ρ, by $R_\rho(0+)$ the class of functions that are regularly varying at $0+$, and we set $R = \cup_{\rho\in\mathbb{R}} R_\rho$ ([14], Section 1.4.2). The results below are stated in the setting of slow variations at infinity. They can easily be adapted to that of slow variations at the origin by using that a function $f(x)$ is slowly (regularly) varying at zero if and only if $f(x^{-1})$ is slowly (regularly) varying at infinity. The next two lemmas contain bounds on slowly varying functions that will often be needed in Section 5 and 6.

Lemma 7.4 ([24],VIII.8, Lemma 2)**.** *If $\ell \in R_0$ then $x^{-\varepsilon} \le \ell(x) \le x^{\varepsilon}$ for any fixed $\varepsilon > 0$ and all x sufficiently large.*

We will also frequently use the following bounds of Potter's type.

Lemma 7.5. *Let $\ell \in R_0$ and let u_n and v_n be positive non decreasing sequences such that $v_n \uparrow \infty$, $u_n \uparrow \infty$ as $n \uparrow \infty$. For any given $x > 0$ there exist positive sequences ε_n and δ_n that verify $\varepsilon_n \downarrow 0$, $\delta_n \downarrow 0$ as $n \uparrow \infty$ and such that, for all n large enough,*

$$(1-\delta_n)\min\left\{\left(\tfrac{v_n}{u_n}x\right)^{\varepsilon_n}, \left(\tfrac{v_n}{u_n}x\right)^{-\varepsilon_n}\right\} \le \frac{\ell(v_n x)}{\ell(u_n)} \le (1+\delta_n)\max\left\{\left(\tfrac{v_n}{u_n}x\right)^{\varepsilon_n}, \left(\tfrac{v_n}{u_n}x\right)^{-\varepsilon_n}\right\}. \tag{7.7}$$

Both these lemmata are immediate consequences of the *Representation Theorem* for slowly varying functions (see e.g. [14], I.3.1, Theorem 1.3.1). Finally we state an important result about inverse of regularly varying functions. Let f be a function defined and locally bounded on $[0,\infty)$, and that tends to zero as $x \to \infty$. Its *generalized inverse*

$$f^{-1}(x) := \inf\{y \ge 0 : f(y) \le x\}, \tag{7.8}$$

is defined on $[f(0),\infty)$. The following result is an (easy) adaptation to the case of functions f in R_ρ with $\rho < 0$ of a theorem of [14] stated for $\rho > 0$.

Lemma 7.6 ([14], I.5.7, Theorem 1.5.12)**.** *If $f \in R_\rho$ with $\rho < 0$, there exists $g \in R_{1/\rho}(0+)$ with*

$$f(g(x)) \sim g(f(x)) \sim x\,, \quad x \to 0\,. \tag{7.9}$$

Here g (an 'asymptotic inverse' of f) is determined to within asymptotic equivalence, and one version of g is f^{-1}.

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