Harnack inequalities for stochastic (functional) differential equations with non-Lipschitzian coefficients[*]

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Abstract

By using coupling arguments, Harnack type inequalities are established for a class of stochastic (functional) differential equations with multiplicative noises and non-Lipschitzian coefficients. To construct the required couplings, two results on existence and uniqueness of solutions on an open domain are presented.

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1 Introduction

Consider the following stochastic differential equation (SDE):

$$\mathrm{d}X(t) = \sigma(t, X(t))\mathrm{d}B(t) + b(t, X(t))\mathrm{d}t, \tag{1.1}$$

where $(B(t))_{t \geq 0}$ is the d-dimensional Brownian motion on a complete filtered probability space $(\Omega, (\mathscr{F}_t)_{t \geq 0}, \mathscr{F}, \mathbb{P})$, $\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable, locally bounded in the first variable and continuous in the second variable. This time-dependent stochastic differential equation has intrinsic links to non-linear PDEs (cf. [20]) as well as geometry with time-dependent metric (cf. [8]). When the equation has a unique solution for any initial data x, we denote the solution by $X^x(t)$. In this paper we aim to investigate Harnack inequalities for the associated family of Markov operators $(P(t))_{t \geq 0}$:

$$P(t)f(x) := \mathbb{E}f(X^x(t)), \quad t \geq 0, x \in \mathbb{R}^d, f \in \mathscr{B}_b(\mathbb{R}^d),$$

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where $\mathscr{B}_b(\mathbb{R}^d)$ is the set of all bounded measurable functions on $\mathbb{R}^d$.

In the recent work [24] the second named author established some Harnack-type inequalities for $P(t)$ under certain ellipticity and semi-Lipschitz conditions. Precisely, if there exists an increasing function $K : [0, \infty) \to \mathbb{R}$ such that

$$\|\sigma(t, x) - \sigma(t, y)\|_{\mathrm{HS}}^2 + 2\langle b(t, x) - b(t, y), x - y \rangle \leq K(t)|x - y|^2, \ x, y \in \mathbb{R}^d, \ t \geq 0,$$

and there exists a decreasing function $\lambda : [0, \infty) \to (0, \infty)$ such that

$$\|\sigma(t, x)\xi\| \geq \lambda(t)|\xi|, \ \ t \geq 0, \xi, x \in \mathbb{R}^d,$$

then for each $T > 0$, the log-Harnack inequality

$$P(T)\log f(y) \leq \log P(T)f(x) + \frac{K(T)|x - y|^2}{2\lambda(T)^2(1 - e^{-K(T)T})}, x, y \in \mathbb{R}^d \tag{1.2}$$

holds for all strictly positive $f \in \mathscr{B}_b(\mathbb{R}^d)$. If, in addition, there exists an increasing function $\delta : [0, \infty) \to (0, \infty)$ such that almost surely

$$\left|\big(\sigma(t, x) - \sigma(t, y)\big)^*(x - y)\right| \leq \delta(t)|x - y|, \ x, y \in \mathbb{R}^d, t \geq 0,$$

then for $p > (1 + \frac{\delta(T)}{\lambda(T)})^2$ there exists a positive constant $C(T)$ (see [24, Theorem 1.1(2)] for expression of this constant) such that the following Harnack inequality with power p holds:

$$\big(P(T)f(y)\big)^p \leq \big(P(T)f^p(x)\big)e^{C(T)|x - y|^2}, \ x, y \in \mathbb{R}^d, \ f \in \mathscr{B}_b(\mathbb{R}^d). \tag{1.3}$$

This type Harnack inequality is first introduced in [21] for diffusions on Riemannian manifolds, while the log-Harnack inequality is firstly studied in [15, 23] for semi-linear SPDEs and reflecting diffusion process on Riemannian manifolds respectively. Both inequalities have been extended and applied in the study of various finite- and infinite-dimensional models, see [1, 2, 4, 5, 7, 12, 14, 22, 24] and references within. In particular, these inequalities have been studied in [25] for the stochastic functional differential equations (SFDE)

$$\mathrm{d}X(t) = \big\{Z(t, X(t)) + a(t, X_t)\big\}\mathrm{d}t + \sigma(t, X(t))\mathrm{d}B(t), \ \ X_0 \in \mathscr{C}, \tag{1.4}$$

where $\mathscr{C} = C([-r_0, 0]; \mathbb{R}^d)$ for a fixed constant $r_0 > 0$ is equipped with the uniform norm $\|\cdot\|_\infty$; $X_t \in \mathscr{C}$ is given by $X_t(u) = X(t + u), u \in [-r_0, 0]$; $\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, $Z : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$, and $a : [0, \infty) \times \mathscr{C} \to \mathbb{R}^d$ are measurable, locally bounded in the first variable and continuous in the second variable. Let X_t^ϕ be the solution to this equation with $X_0 = \phi \in \mathscr{C}$. In [25] the log-Harnack inequality of type (1.2) and the Harnack inequality of type (1.3) were established for

$$P_t F(\phi) := \mathbb{E}F(X_t^\phi), \ \ t > 0, F \in \mathscr{B}_b(\mathscr{C})$$

provided σ is invertible and for any $T > 0$ there exist constants $K_1, K_2 \geq 0, K_3 > 0$ and $K_4 \in \mathbb{R}$ such that

(1) $\left|\sigma(t, \eta(0))^{-1}\{a(t, \xi) - a(t, \eta)\}\right| \leq K_1\|\xi - \eta\|_\infty, \ t \in [0, T], \xi, \eta \in \mathscr{C}$;

(2) $\left|\big(\sigma(t, x) - \sigma(t, y)\big)\right| \leq K_2(1 \wedge |x - y|), \ t \in [0, T], x, y \in \mathbb{R}^d$;

(3) $\left|\sigma(t, x)^{-1}\right| \leq K_3, \ t \geq 0, x \in \mathbb{R}^d$;

(4) $\|\sigma(t, x) - \sigma(t, y)\|_{\mathrm{HS}}^2 + 2\langle x - y, Z(t, x) - Z(t, y)\rangle \leq K_4|x - y|^2, \ t \in [0, T], x, y \in \mathbb{R}^d$.

The aim of this paper is to extend the above mentioned results to SDEs and SFDEs with less regular coefficients as considered in Fang and Zhang [6] (see also [11]), where the existence and uniqueness of solutions were investigated. In section 2, we consider the SDE case; and in section 3, we consider the SFDE case. Finally, in section 4 we present two results for the existence and uniqueness of solutions on open domains of SDEs and SFDEs with non-Lipschitz coefficients, which are crucial for constructions of couplings in the proof of Harnack-type inequalities.

2 SDE with non-Lipschitzian coefficients

To characterize the non-Lipschitz regularity of coefficients, we introduce the class

$$\mathscr{U} := \left\{ u \in C^1((0,\infty); [1,\infty)) : \int_0^1 \frac{\mathrm{d}s}{su(s)} = \infty, \ \liminf_{r\downarrow 0} \left\{ u(r) + ru'(r) \right\} > 0 \right\}. \quad (2.1)$$

Here, the restriction that $u \geq 1$ is more technical than essential, since in applications one may usually replace u by $u \vee 1$ (see condition **(H1)** below).

To ensure the existence and uniqueness of the solution and to establish the log-Harnack inequality, we shall need the following assumptions:

(H1) There exist $u, \tilde{u} \in \mathscr{U}$ with $u' \leq 0$ and increasing functions $K, \tilde{K} \in C([0,\infty); (0,\infty))$ such that for all $t \geq 0$ and $x, y \in \mathbb{R}^d$,

$$\langle b(t,x) - b(t,y), x - y \rangle + \frac{1}{2}\|\sigma(t,x) - \sigma(t,y)\|_{\mathrm{HS}}^2 \leq K(t)|x-y|^2 u(|x-y|^2)$$

$$\|\sigma(t,x) - \sigma(t,y)\|_{\mathrm{HS}}^2 \leq \tilde{K}(t)|x-y|^2 \tilde{u}(|x-y|^2).$$

(H2) There exists a decreasing function $\lambda \in C([0,\infty); (0,\infty))$ such that

$$|\sigma(t,x)y| \geq \lambda(t)|y|, \ \ t \geq 0, x, y \in \mathbb{R}^d.$$

The log-Harnack inequality we are establishing depends only on functions u, K and λ, $\tilde{K}$ and $\tilde{u}$ will be only used to ensure the existence of coupling constructed in the proof. As in [24], in order to derive the Harnack inequality with a power, we need the following additional assumption:

(H3) There exists an increasing function $\delta \in C([0,\infty); [0,\infty))$ such that

$$|(\sigma(t,x) - \sigma(t,y))^*(x-y)| \leq \delta(t)|x-y|, \ \ x, y \in \mathbb{R}^d, t > 0.$$

Theorem 2.1. *Assume that* **(H1)** *holds.*

(1) *For any initial data $X(0)$, the equation* (1.1) *has a unique solution, and the solution is non-explosive.*

(2) *If moreover* **(H2)** *holds and*

$$\varphi(s) := \int_0^s u(r)\mathrm{d}r \leq \gamma s u(s)^2, \ \ s > 0 \quad (2.2)$$

for some constant $\gamma > 0$, then for each $T > 0$ and strictly positive $f \in \mathscr{B}_b(\mathbb{R}^d)$,

$$P(T)\log f(y) \leq \log P(T)f(x) + \frac{K(T)\varphi(|x-y|^2)}{\lambda(T)(1 - \exp[-2K(T)T/\gamma])}, \ \ f \geq 1, \ x, y \in \mathbb{R}^d.$$

(3) *If, additional to conditions in* (2), **(H3)** *holds, then*

$$\big(P(T)f(y)\big)^q \le P(T)f^q(x)\cdot\exp\left[\frac{K(T)\sqrt{q}(\sqrt{q}-1)\varphi(|x-y|^2)}{2\delta(T)\big((\sqrt{q}-1)\lambda(T)-\delta(T)\big)\big(1-\exp[-2K(T)T/\gamma]\big)}\right]$$

holds for $T > 0$, *for* $q > 1 + \frac{\delta(T)+2\lambda(T)\sqrt{\delta(T)}}{\lambda(T)^2}$, $x, y \in \mathbb{R}^d$, *and* $f \in \mathscr{B}_b^+(\mathbb{R}^d)$, *the set of all non-negative elements in* $\mathscr{B}_b(\mathbb{R}^d)$.

Typical examples for $u \in \mathscr{U}$ satisfying $u' \le 0$ and (2.2) contain $u(s) = \log(\mathrm{e} \vee s^{-1})$, $u(s) = \{\log(\mathrm{e} \vee s^{-1})\}\log\log(\mathrm{e}^{\mathrm{e}} \vee s^{-1}), \cdots$.

Although the main idea of the proof is based on [24], due to the non-Lipschitzian coefficients we have to overcome additional difficulties for the construction of coupling. In fact, to show that the coupling we are going to construct is well defined, a new result concerning existence and uniqueness of solutions to SDEs on a domain is addressed in section 4.

2.1 Construction of the coupling and some estimates

It is easy to see from Theorem 4.1 that the equation (1.1) has a unique strong solution which is non-explosive (see the beginning of the next subsection). To establish the desired log-Harnack inequality, we modify the coupling constructed in [24]. For fixed $T > 0$ and $\theta \in (0, 2)$, let

$$\xi(t) = \frac{2-\theta}{2K(T)}\left[1 - \mathrm{e}^{\frac{2K(T)}{\gamma}(t-T)}\right], \quad t \in [0, T],$$

then ξ is a smooth and strictly positive function on $[0, T)$ so that

$$2 - 2K(T)\xi(t) + \gamma\xi'(t) = \theta, \ t \in [0, T). \tag{2.3}$$

For any $x, y \in \mathbb{R}^d$, we construct the coupling processes $(X(t), Y(t))_{t\ge 0}$ as follows:

$$\begin{cases} \mathrm{d}X(t) = \sigma(t, X(t))\mathrm{d}B(t) + b(t, X(t))\mathrm{d}t, \ \ X_0 = x, \\ \mathrm{d}Y(t) = \sigma(t, Y(t))\mathrm{d}B(t) + b(t, Y(t))\mathrm{d}t \\ \qquad + \frac{1}{\xi(t)}\sigma(t, Y(t))\sigma(t, X(t))^{-1}(X(t)-Y(t))u(|X(t)-Y(t)|^2)\mathrm{d}t, \ \ Y_0 = y. \end{cases} \tag{2.4}$$

We intend to show that the $Y(t)$ (hence, the coupling process) is well defined up to time τ and $\tau \le T$, where

$$\tau := \inf\{t \ge 0 : X(t) = Y(t)\}$$

is the coupling time. To this end, we apply Theorem 4.1 to

$$D = \{(x', y') \in \mathbb{R}^d \times \mathbb{R}^d : \ x' \ne y'\}.$$

It is easy to verify (4.2) from **(H1)**. Then $Y(t)$ is well defined up to time $\zeta \wedge \tau$, where

$$\zeta := \lim_{n\to\infty}\zeta_n, \ \text{and} \ \zeta_n := \inf\{t \in [0, T); \ |Y(t)| \ge n\}.$$

Here and in what follows, we set $\inf \emptyset = \infty$.

As in [24], to derive Harnack-type inequalities, we need to prove that the coupling is successful before $\zeta \wedge T$ under the weighted probability $\mathbb{Q} := R(T\wedge\tau\wedge\zeta)\mathbb{P}$, where

$$\begin{aligned} R(s) := \exp\bigg[&-\int_0^s \frac{1}{\xi(t)}\big\langle\sigma(t, X(t))^{-1}(X(t)-Y(t))u(|X(t)-Y(t)|^2), \mathrm{d}B(t)\big\rangle \\ &-\frac{1}{2}\int_0^s \frac{1}{\xi(t)^2}|\sigma(t, X(t))^{-1}(X(t)-Y(t))|^2 u^2(|X(t)-Y(t)|^2)\mathrm{d}t\bigg], \end{aligned} \tag{2.5}$$

for $s \in [0, T \wedge \zeta \wedge \tau)$. To ensure the existence of the density $R(T \wedge \tau \wedge \zeta)$, letting

$$\tau_n = \inf\{t \in [0, \zeta) : |X(t) - Y(t)| \geq n^{-1}\}, \quad n \geq 1,$$

we verify that $(R(s \wedge \zeta_n \wedge \tau_n))_{s \in [0,T), n \geq 1}$ is uniformly integrable, so that

$$R(T \wedge \tau \wedge \zeta) := \lim_{n \to \infty} R((T - n^{-1}) \wedge \tau_n \wedge \zeta_n)$$

is a well defined probability density due to the martingale convergence theorem. Then we prove that $\zeta \wedge T \geq \tau$ a.s.-$\mathbb{Q}$, so that $\mathbb{Q} = R(\tau)\mathbb{P}$. Both assertions are ensured by the following lemma.

Lemma 2.2. *Assume that the conditions* **(H1)** *and* **(H2)** *hold for some u satisfying* (2.2). *Then*

(1) *For any $s \in [0, T)$ and $n \geq 1$,*

$$\mathbb{E}[R(s \wedge \tau_n \wedge \zeta_n) \log R(s \wedge \tau_n \wedge \zeta_n)] \leq \frac{K(T)\varphi(|x-y|^2)}{\lambda(T)^2 \, \theta(2-\theta)(1 - \exp[-2K(T)T/\gamma])}.$$

Consequently, $R(T \wedge \zeta \wedge \tau) := \lim_{n \to \infty} R((T - n^{-1}) \wedge \tau_n \wedge \zeta_n)$ exists as a probability density function of $\mathbb{P}$, and

$$\mathbb{E}\{R(T \wedge \zeta \wedge \tau) \log R(T \wedge \zeta \wedge \tau)\} \leq \frac{K(T)\varphi(|x-y|^2)}{\lambda(T)^2 \, \theta(2-\theta)(1 - \exp[-2K(T)T/\gamma])}.$$

(2) *Let $\mathbb{Q} = R(T \wedge \zeta \wedge \tau)\mathbb{P}$, then $\mathbb{Q}(\zeta \wedge T \geq \tau) = 1$. Thus, $\mathbb{Q} = R(\tau)\mathbb{P}$ and*

$$\mathbb{E}\{R(\tau) \log R(\tau)\} \leq \frac{K(T)\varphi(|x-y|^2)}{\lambda(T)^2 \, \theta(2-\theta)(1 - \exp[-2K(T)T/\gamma])}.$$

Proof. (1) Let

$$\widetilde{B}(t) = B(t) + \int_0^t \frac{1}{\xi(s)}\sigma(s, X(s))^{-1}(X(s) - Y(s))u(|X(s) - Y(s)|^2)\mathrm{d}s, \quad t < T \wedge \tau \wedge \zeta. \quad (2.6)$$

Then, before time $T \wedge \tau \wedge \zeta$, (2.4) can be reformulated as

$$\begin{cases} \mathrm{d}X(t) = \sigma(t, X(t))\mathrm{d}\widetilde{B}(t) + b(t, X(t))\mathrm{d}t - \frac{X(t) - Y(t)}{\xi(t)}u(|X(t) - Y(t)|^2)\mathrm{d}t, \ X_0 = x, \\ \mathrm{d}Y(t) = \sigma(t, Y(t))\mathrm{d}\widetilde{B}(t) + b(t, Y(t))\mathrm{d}t, \ Y_0 = y. \end{cases} \quad (2.7)$$

For fixed $s \in [0, T)$ and $n \geq 1$, let $\vartheta_{n,s} = s \wedge \tau_n \wedge \zeta_n$ and $\mathbb{Q}_{n,s} = R(\vartheta_{n,s})\mathbb{P}$. Then by the Girsanov theorem, $(\widetilde{B}(t))_{t \in [0, \vartheta_{n,s}]}$ is a d-dimensional Brownian motion under the probability measure $\mathbb{Q}_{n,s}$. Let $Z(t) = X(t) - Y(t)$. By the Itô formula and condition **(H1)**, we obtain

$$\mathrm{d}|Z(t)|^2 = 2\Big\langle Z(t), b(t, X(t)) - b(t, Y(t)) - \frac{Z(t)u(|Z(t)|^2)}{\xi(t)} \Big\rangle \mathrm{d}t + \|\sigma(t, X(t)) - \sigma(t, Y(t))\|_{\mathrm{HS}}^2 \mathrm{d}t$$

$$+ 2\Big\langle Z(t), (\sigma(t, X(t)) - \sigma(t, Y(t)))\mathrm{d}\widetilde{B}(t) \Big\rangle$$

$$\leq 2\Big(K(T) - \frac{1}{\xi(t)}\Big)|Z(t)|^2 u(|Z(t)|^2)\mathrm{d}t$$

$$+ 2\Big\langle Z(t), (\sigma(t, X(t)) - \sigma(t, Y(t)))\mathrm{d}\widetilde{B}(t) \Big\rangle, \ t \leq \vartheta_{n,s}.$$

Applying the Itô formula to $\varphi(|Z(t)|^2)$ and noting that $\varphi'' = u' \leq 0$, we derive

$$\mathrm{d}\varphi(|Z(t)|^2) \leq \mathrm{d}M(t) + 2\Big(K(T) - \frac{1}{\xi(t)}\Big)|Z(t)|^2 u^2(|Z(t)|^2)\mathrm{d}t, \ t \leq \vartheta_{n,s},$$

where

$$M(t) := \int_0^t 2u(|Z_s|^2)\langle Z(s), (\sigma(s, X(s)) - \sigma(s, Y(s)))\mathrm{d}\widetilde{B}(s)\rangle, \quad t \le \vartheta_{n,s}$$

is a $\mathbb{Q}_{n,s}$-martingale. Thus, by (2.2) and (2.3),

$$
\begin{aligned}
\mathrm{d}\frac{\varphi(|Z(t)|^2)}{\xi(t)} &\le \frac{1}{\xi(t)}\mathrm{d}M(t) + \frac{2K(T)\xi(t) - 2}{\xi(t)^2}|Z(t)|^2 u^2(|Z(t)|^2)\mathrm{d}t - \frac{\xi'(t)}{\xi(t)^2}\varphi(|Z(t)|^2)\mathrm{d}t \\
&\le \frac{1}{\xi(t)}\mathrm{d}M(t) + \frac{|Z(t)|^2 u^2(|Z(t)|^2)}{\xi(t)^2}(-2 + 2K(T)\xi(t) - \gamma\xi'(t))\mathrm{d}t \qquad (2.8) \\
&= \frac{1}{\xi(t)}\mathrm{d}M(t) - \theta\frac{|Z(t)|^2 u^2(|Z(t)|^2)}{\xi(t)^2}\mathrm{d}t, \quad t \le \vartheta_{n,s}.
\end{aligned}
$$

Taking the expectation w.r.t. the probability measure $\mathbb{Q}_{n,s}$ and noting $(\widetilde{B}(t))_{t\in[0,\vartheta_{n,s}]}$ is a Brownian motion under $\mathbb{Q}_{n,s}$, we get

$$\mathbb{E}_{\mathbb{Q}_{n,s}}\left[\int_0^{\vartheta_{n,s}} \frac{|Z(t)|^2 u^2(|Z(t)|^2)}{\xi(t)^2}\mathrm{d}t\right] \le \frac{\varphi(|x-y|^2)}{\theta\,\xi(0)}. \qquad (2.9)$$

On the other hand, it follows from **(H2)** that

$$
\begin{aligned}
\log R(\vartheta_{n,s}) =& -\int_0^{\vartheta_{n,s}} \frac{1}{\xi(t)}\langle\sigma(t, X(t))^{-1}Z(t)u(|Z(t)|^2), \mathrm{d}\widetilde{B}(t)\rangle \\
&+ \frac{1}{2}\int_0^{\vartheta_{n,s}} \frac{|\sigma(t, X(t))^{-1}Z(t)|^2 u^2(|Z(t)|^2)}{\xi(t)^2}\mathrm{d}t \\
\le& -\int_0^{\vartheta_{n,s}} \frac{1}{\xi(t)}\langle\sigma(t, X(t))^{-1}Z(t)u(|Z(t)|^2), \mathrm{d}\widetilde{B}(t)\rangle \\
&+ \frac{1}{2\lambda(T)^2}\int_0^{\vartheta_{n,s}} \frac{|Z(t)|^2 u^2(|Z(t)|^2)}{\xi(t)^2}\mathrm{d}t.
\end{aligned}
$$

Combining with (2.9), we arrive at

$$\mathbb{E}\big[R(\vartheta_{n,s})\log R(\vartheta_{n,s})\big] = \mathbb{E}_{\mathbb{Q}_{n,s}}\big[\log R(\vartheta_{n,s})\big] \le \frac{\varphi(|x-y|^2)}{2\lambda(T)^2\theta\,\xi(0)}, \quad s \in [0, T), \, n \ge 1. \qquad (2.10)$$

This implies the desired inequality in (1), and the consequence then follows from the martingale convergence theorem.

(2) Let $\zeta_n^X = \inf\{t \ge 0; |X(t)| \ge n\}$. Since $X(t)$ is non-explosive as mentioned above, $\zeta_n^X \uparrow \infty$ $\mathbb{P}$-a.s. and hence, also $\mathbb{Q}$-a.s. For $n > m > 1$, it follows from (2.8) that

$$\frac{\mathbb{Q}(\zeta_m^X > s \wedge \tau_m > \zeta_n)}{\xi(0)}\int_0^{(n-m)^2} u(s)\mathrm{d}s \le \mathbb{E}_{\mathbb{Q}}\left[\frac{\varphi(|Z(\vartheta_{n,s})|^2)}{\xi_{\vartheta_{n,s}}}\right] \le \frac{\varphi(|x-y|^2)}{\xi(0)}. \qquad (2.11)$$

Letting first $n \to \infty$, then $m \to \infty$, and noting that $u \ge 1$, we obtain $\mathbb{Q}(\zeta < s \wedge \tau) = 0$ for all $s \in [0, T)$. Therefore, $\mathbb{Q}(\zeta \ge T \wedge \tau) = 1$. So, it remains to show that $\mathbb{Q}(\tau \le T) = 1$ and according to (1) and (2.9),

$$\mathbb{E}_{\mathbb{Q}}\int_0^{T\wedge\tau} \frac{|Z(t)|^2 u(|Z(t)|^2)^2}{\xi(t)^2}\mathrm{d}t \le \frac{K(T)\varphi(|x-y|^2)}{\lambda(T)^2\,\theta(2-\theta)(1 - \exp[-2K(T)T/\gamma])}.$$

Since $\int_0^T \frac{1}{\xi(t)^2}\mathrm{d}t = \infty$, $\tau > T$ implies that

$$\inf_{t\in[0,T)} |Z(t \wedge \tau)|^2 u(|Z(t \wedge \tau)|^2)^2 > 0,$$

which yields that

$$\mathbb{Q}(T < \tau) \leq \mathbb{Q}\left(\int_0^{T \wedge \tau} \frac{|Z(t)|^2 u(|Z(t)|^2)^2}{\xi(t)^2} \mathrm{d}t = \infty \right) = 0.$$

Combining this with $\mathbb{Q}(\zeta \geq T \wedge \tau) = 1$, we prove (2). $\qquad\square$

If moreover **(H3)** holds, then we have the following moment estimate on $R(\tau)$, which will be used to prove the Harnack inequality with power.

Lemma 2.3. *Assume* **(H1)**, **(H2)** *and* **(H3)** *hold. Then for* $p := \frac{c^2 \theta^2}{4\delta(T)^2 + 4\theta\lambda(T)\delta(T)} > 0$,

$$\mathbb{E}R(\tau)^{1+p} \leq \exp\left[\frac{(2\delta(T) + \lambda(T)\theta)\theta\varphi(|x-y|^2)}{4\delta(T)\xi(0)(2\delta(T) + 2\lambda(T)\theta)} \right]. \tag{2.12}$$

Proof. By (2.8) and **(H3)**, for any $r > 0$ we have

$$\mathbb{E}_{\mathbb{Q}_{n,s}} \exp\left[r \int_0^{\vartheta_{n,s}} \frac{|Z(t)|^2 u^2(|Z(t)|^2)}{\xi(t)^2} \mathrm{d}t \right]$$

$$\leq \exp\left[\frac{r\varphi(|x-y|^2)}{\theta\,\xi(0)} \right] \mathbb{E}_{\mathbb{Q}_{n,s}} \exp\left[\frac{2r}{\theta} \int_0^{s \wedge \tau_n} \frac{u(|Z(t)|^2)}{\xi(t)} \langle Z(t), \left(\sigma(t,X(t)) - \sigma(t,Y(t))\right)\mathrm{d}\widetilde{B}(t) \rangle \right]$$

$$\leq \exp\left[\frac{r\varphi(|x-y|^2)}{\theta\,\xi(0)} \right] \left(\mathbb{E}_{\mathbb{Q}_{n,s}} \exp\left[\frac{8\delta(T)^2 r^2}{\theta^2} \int_0^{\vartheta_{n,s}} \frac{|Z(t)|^2 u^2(|Z(t)|^2)}{\xi(t)^2} \mathrm{d}t \right] \right)^{1/2},$$

where in the last step we use the inequality

$$\mathbb{E}e^{M(t)} \leq \left(\mathbb{E}e^{2\langle M\rangle(t)} \right)^{1/2},$$

for a continuous exponentially integrable martingale $M(t)$, and $\langle M\rangle(t)$ denotes the quadratic variational process corresponding to $M(t)$. Putting $r = \frac{\theta^2}{8\delta(T)^2}$ such that $r = \frac{8r^2\delta(T)^2}{\theta^2}$, we get

$$\mathbb{E}_{\mathbb{Q}_{n,s}} \exp\left[\frac{\theta^2}{8\delta(T)^2} \int_0^{\vartheta_{n,s}} \frac{|Z(t)|^2 u^2(|Z(t)|^2)}{\xi(t)^2} \mathrm{d}t \right] \leq \exp\left[\frac{\theta\varphi(|x-y|^2)}{4\delta(T)^2\xi(0)} \right].$$

Due to Lemma 2.2, we have $\tau \leq T \wedge \zeta, \mathbb{Q}$-a.s. By taking $s = T - n^{-1}$ and letting $n \to \infty$ in the above inequality, we arrive at

$$\mathbb{E}_{\mathbb{Q}} \exp\left[\frac{\theta^2}{8\delta(T)^2} \int_0^{\tau} \frac{|Z(t)|^2 u^2(|Z(t)|^2)}{\xi(t)^2} \mathrm{d}t \right] \leq \exp\left[\frac{\theta\varphi(|x-y|^2)}{4\delta(T)^2\xi(0)} \right]. \tag{2.13}$$

Since for any continuous $\mathbb{Q}$-martingale $M(t)$

$$\mathbb{E}_{\mathbb{Q}} \exp\left[pM(t) + \frac{p}{2}\langle M\rangle(t) \right]$$

$$\leq \left(\mathbb{E}_{\mathbb{Q}} \exp\left[pqM(t) - p^2q^2\langle M\rangle(t)/2 \right] \right)^{1/q} \left(\mathbb{E}_{\mathbb{Q}} \exp\left[\frac{pq(pq+1)}{2(q-1)}\langle M\rangle(t) \right] \right)^{(q-1)/q}$$

$$\leq \left(\mathbb{E}_{\mathbb{Q}} \exp\left[\frac{pq(pq+1)}{2(q-1)}\langle M\rangle(t) \right] \right)^{(q-1)/q}, \quad q > 1,$$

we obtain from **(H2)** that

$$
\mathbb{E}R(\tau)^{1+p} = \mathbb{E}_{\mathbb{Q}} \exp\left[-p \int_0^\tau \frac{1}{\xi(t)} \langle \sigma(t, X(t))^{-1} Z(t) u(|Z(t)|^2), \mathrm{d}\widetilde{B}(t) \rangle \right.
$$
$$
\left. + \frac{p}{2} \int_0^\tau \frac{1}{\xi(t)^2} |\sigma(t, X(t))^{-1} Z(t) u(|Z(t)|^2)|^2 \mathrm{d}t \right]
$$
$$
\leq \left(\mathbb{E}_{\mathbb{Q}} \exp\left[\frac{pq(pq+1)}{2\lambda(T)^2(q-1)} \int_0^\tau \frac{|Z(t)|^2 u^2(|Z(t)|^2)}{\xi(t)^2} \mathrm{d}t \right] \right)^{(q-1)/q}.
$$

Taking $q = 1 + \sqrt{1 + p^{-1}}$ which minimizes $q(pq+1)/(q-1)$, and using the definition of p, we have

$$
\frac{pq(pq+1)}{2\lambda(T)^2(q-1)} = \frac{(p + \sqrt{p^2+p})^2}{2\lambda(T)^2} = \frac{\theta^2}{8\delta(T)^2}, \quad \frac{q-1}{q} = \frac{2\delta(T) + \lambda(T)\theta}{2\delta(T) + 2\lambda(T)\theta}.
$$

Combining this with (2.13), we complete the proof. □

2.2 Proof of Theorem 2.1

According to Theorem 4.1 below for $D = \mathbb{R}^d$, **(H1)** implies that (1.1) has a unique solution. Since u is decreasing, the first inequality in **(H1)** with $y = 0$ implies that for $|x| \geq 1$,

$$
2\langle b(t,x), x \rangle + \|\sigma(t,x)\|_{\mathrm{HS}}^2 \leq 2\langle b(t,0), x \rangle + \|\sigma(t,0)\|_{\mathrm{HS}}^2 + 2\|\sigma(t,0)\|_{\mathrm{HS}} \|\sigma(t,x)\|_{\mathrm{HS}} + K(t)|x|^2 u(1). \tag{2.14}
$$

Moreover, the second inequality in **(H1)** with $y = 0$ implies that for $|x| \geq 1$,

$$
\|\sigma(t,x)\|_{\mathrm{HS}} \leq \|\sigma(t,0)\|_{\mathrm{HS}} + \sum_{k=1}^{[|x|]} \left\| \sigma\left(t, \frac{kx}{[|x|]}\right) - \sigma\left(t, \frac{(k-1)x}{[|x|]}\right) \right\|_{\mathrm{HS}}
$$
$$
\leq \|\sigma(t,0)\|_{\mathrm{HS}} + 2|x|\sqrt{\tilde{K}(t)}\sqrt{u(1)}
$$

where $[|x|]$ stands for the integer part of $|x|$. Combining this with (2.14) we may find a function $h \in C([0,\infty); (0,\infty))$ such that

$$
2\langle b(t,x), x \rangle + \|\sigma(t,x)\|_{\mathrm{HS}}^2 \leq h(t)(1 + |x|^2),
$$

which implies the non-explosion of $X(t)$ as is well known. Thus, the proof of (1) is finished.

Next, by Lemma 2.2 and the Girsanov theorem,

$$
\tilde{B}(t) := B(t) + \int_0^{t \wedge \tau} \frac{\sigma(s, X(s))^{-1}(X(s) - Y(s))}{\xi(s)} u(|X(s) - Y(s)|^2)\mathrm{d}s, \quad t \geq 0
$$

is a d-dimensional Brownian motion under the probability measure $\mathbb{Q}$. Then, according to Theorem 2.1(1), the equation

$$
\mathrm{d}Y(t) = \sigma(t, Y(t))\mathrm{d}\tilde{B}(t) + b(t, Y(t))\mathrm{d}t, \quad Y(0) = y \tag{2.15}
$$

has a unique solution for all $t \geq 0$. Moreover, it is easy to see that $(X(t))_{t \geq 0}$ solves the equation

$$
\mathrm{d}X(t) = \sigma(t, X(t))\mathrm{d}\tilde{B}(t) + b(t, X(t))\mathrm{d}t - \frac{X(t) - Y(t)}{\xi(t)} 1_{\{t < \tau\}}\mathrm{d}t, \quad X(0) = x. \tag{2.16}
$$

Thus, we have extended equation (2.7) to all $t \geq 0$, which has a global solution $(X(t), Y(t))_{t \geq 0}$ under the probability measure $\mathbb{Q}$, and

$$
\tau := \inf\{t \geq 0 : X(t) = Y(t)\} \leq T, \quad \mathbb{Q}\text{-a.s.}
$$

Moreover, since the equations (2.15) and (2.16) coincide for $t \geq \tau$, by the uniqueness of the solution and $X(\tau) = Y(\tau)$, we conclude that $X(T) = Y(T)$, $\mathbb{Q}$-a.s.

Now, by Lemma 2.2 and the Young inequality we obtain

$$
\begin{aligned}
P(T) \log f(y) = \mathbb{E}_{\mathbb{Q}} \big[\log f(Y(T)) \big] &= \mathbb{E} \big[R(\tau) \log f(Y(T)) \big] \\
&\leq \log \mathbb{E}[f(X(T))] + \mathbb{E} \big[R(\tau) \log R(\tau) \big] \\
&\leq \log P(T) f(x) + \frac{K(T) \varphi(|x-y|^2)}{\lambda(T) \theta (2-\theta) \big(1 - \exp[-2K(T)T/\gamma] \big)}.
\end{aligned}
$$

Taking $\theta = 1$, we derive the desired log-Harnack inequality.

Moreover, by the Hölder inequality, for any $q > 1$ we have

$$
\big(P(T) f(y) \big)^q = \big(\mathbb{E}_{\mathbb{Q}} \big[f(Y(T)) \big] \big)^q = \big(\mathbb{E} \big[R_\tau f(X(T)) \big] \big)^q \leq \big(P(T) f^q(x) \big) \big(\mathbb{E} \big[R_\tau^{q/(q-1)} \big] \big)^{q-1}.
$$

Setting $q = 1 + \dfrac{4\delta(T)^2 + 4\theta\lambda(T)\delta(T)}{\lambda(T)^2 \theta^2}$ such that

$$
\frac{q}{q-1} = 1 + p = 1 + \frac{\lambda(T)^2 \theta^2}{4\delta(T)^2 + 4\theta\lambda(T)\delta(T)}, \tag{2.17}
$$

it then follows from Lemma 2.3 that

$$
\big(P(T) f(y) \big)^q \leq P(T) f^q(x) \cdot \exp \left[\frac{2\delta(T) + \lambda(T)\theta}{2\lambda(T)^2 \delta(T) \theta \, \xi(0)} \varphi(|x-y|^2) \right].
$$

It is easy to see that for any $q > 1 + \frac{\delta(T)^2 + 2\lambda(T)\delta(T)}{\lambda(T)^2}$, (2.17) holds for $\theta = \dfrac{2\delta(T)}{\lambda(T)(\sqrt{q}-1)}$. Therefore, the desired Harnack inequality with power q follows.

3 SFDEs with non-Lipschitzian coefficients

For a fixed $r_0 > 0$, let $\mathscr{C} := C([-r_0, 0]; \mathbb{R}^d)$ denote all continuous functions from $[-r_0, 0]$ to $\mathbb{R}^d$ endowed with the uniform norm, i.e.

$$
\|\phi\|_\infty := \max_{-r_0 \leq s \leq 0} |\phi(s)|, \quad \text{for } \phi \in \mathscr{C}.
$$

Let $T > r_0$ be fixed, for any $h \in C([-r_0, T]; \mathbb{R}^d)$ and $t \geq 0$, let $h_t \in \mathscr{C}$ such that

$$
h_t(s) := h(t+s), \ s \in [-r_0, 0].
$$

Consider the following type of stochastic functional differential equation

$$
\mathrm{d}X(t) = \{b(t, X(t)) + a(t, X_t)\} \mathrm{d}t + \bar{\sigma}(t, X_t) \mathrm{d}B(t), \quad X_0 \in \mathscr{C}, \tag{3.1}
$$

where $a : [0, \infty) \times \mathscr{C} \to \mathbb{R}^d$, $\bar{\sigma} : [0, \infty) \times \mathscr{C} \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : [0, \infty) \to \mathbb{R}^d$ are measurable, locally bounded in the first variable and continuous in the second variable.

According to the proof of Theorem 4.2 below, we introduce the following class of functions to characterize the non-Lipschitz regularity of the coefficients:

$$
\bar{\mathscr{U}} := \left\{ u \in C^1((0, \infty), [1, \infty)) : \int_0^1 \frac{\mathrm{d}s}{su(s)} = \infty, \ s \mapsto su(s) \text{ is increasing and concave} \right\}.
$$

According to Theorem 4.2 with $D = \mathbb{R}^d$, the equation (3.1) has a unique strong solution provided there exist a locally bounded function $K : [0, \infty) \to (0, \infty)$ and $u \in \bar{\mathscr{U}}$ such that

$$
\begin{aligned}
2\langle b(t, \phi(0)) - b(t, \psi(0)) &+ a(t, \phi) - a(t, \psi), \phi(0) - \psi(0) \rangle + \|\bar{\sigma}(t, \phi) - \bar{\sigma}(t, \psi)\|_{\mathrm{HS}}^2 \\
&\leq K(t) \|\phi - \psi\|_\infty^2 u(\|\phi - \psi\|_\infty^2), \\
\|\bar{\sigma}(t, \phi) - \bar{\sigma}(t, \psi)\|_{\mathrm{HS}}^2 &\leq K(t) \|\phi - \psi\|_\infty^2 u(\|\phi - \psi\|_\infty^2)
\end{aligned} \tag{3.2}
$$

holds for all $t \geq 0$ and $\phi, \psi \in \mathscr{C}$. Since $su(s)$ is increasing and concave in s, we have $su(s) \leq c(1+s)$ for some constant $c > 0$. Therefore, it is easy to see that the above conditions also imply the non-explosion of the solution.

Let X_t^ϕ be the segment solution to (3.1) for $X_0 = \phi$. We aim to establish the Harnack inequality for the associated Markov operators $(P_t)_{t \geq 0}$:

$$P_t f(\phi) := \mathbb{E} f(X_t^\phi), \quad f \in \mathscr{B}_b(\mathscr{C}), \phi \in \mathscr{C}.$$

As already known in [5, 25], to establish a Harnack inequality using coupling method, one has to assume that $\bar{\sigma}(\cdot, \phi)$ depends only on $\phi(0)$; that is, $\bar{\sigma}(t, \phi) = \sigma(t, \phi(0))$ holds for some $\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$. Therefore, below we will consider the equation

$$\mathrm{d}X(t) = \{b(t, X(t))\} + a(t, X_t)\}\mathrm{d}t + \sigma(t, X(t))\mathrm{d}B(t), \quad X_0 \in \mathscr{C}, \tag{3.3}$$

where $a : [0, \infty) \times \mathscr{C} \to \mathbb{R}^d$, $\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ are measurable, locally bounded in the first variable and continuous in the second variable. We shall make use of the following assumption, which is weaker than (1)-(4) introduced in the end of Section 1 since u might be unbounded.

(A) There exist $u \in \bar{\mathscr{U}}$ and increasing function $K, K_1, K_2, K_3, K_4 \in C([0, \infty); (0, \infty))$ such that for all $t \geq 0$,

(i) $\langle b(t, x) - b(t, y), x - y \rangle + \frac{1}{2} \|\sigma(t, x) - \sigma(t, y)\|_{\mathrm{HS}}^2 \leq K_1(t)|x - y|^2 u(|x - y|^2)$, $x, y \in \mathbb{R}^d$;

(ii) $\|\sigma(t, x) - \sigma(t, y)\|_{\mathrm{HS}}^2 \leq K(t)|x - y|^2 u(|x - y|^2)$, $x, y \in \mathbb{R}^d$;

(iii) $|a(t, \phi) - a(t, \psi)|^2 \leq K_2(t)\|\phi - \psi\|_\infty^2 u(\|\phi - \psi\|_\infty^2)$, $\phi, \psi \in \mathscr{C}$;

(iv) $\|(\sigma(t, x) - \sigma(t, y))\sigma(t, y)^{-1}\|^2 \leq K_3(t)$, $\|\sigma(t, x)^{-1}\|^2 \leq K_4(t)$, $x, y \in \mathbb{R}^d$.

Obviously, **(A)** implies (3.2) so that the equation (3.3) has a unique strong solution and the solution is non-explosive. Let $G(s) = \int_1^s \frac{1}{ru(r)} \mathrm{d}r, s > 0$. It is easy to see that G is strictly increasing with full range $\mathbb{R}$. Let

$$C(T, r) = G^{-1}\Big(G(2r^2) + G\big(4\{K_1(T) + 2K_2(T)K_3(T) + 32K(T)\}\big)\Big),$$
$$\Phi(T, r) = C(T, r)u(C(T, r)), \quad T > 0.$$

Since $G(0) := \lim_{s \downarrow 0} G(s) = -\infty$, we have $C(T, 0) = 0$ for any $T > 0$. So, if $\lim_{s \downarrow 0} su(s) = 0$ then $\Phi(T, 0) = 0$. The main result in this section is the following.

Theorem 3.1. *Assume* **(A)**. *If* (2.2) *holds for some constant $\gamma > 0$, then for $T > 0$*

$$P_{T+r_0} \log f(\psi) - \log P_{T+r_0} f(\phi)$$
$$\leq K_4(T)\Big(\frac{2\gamma\varphi(|\phi(0) - \psi(0)|^2)}{T} + T\{8K_1(T)^2 + 8K_2(T)K_3(T) + K_2(T)\}\Phi(T, \|\phi - \psi\|_\infty)\Big),$$

holds for all strictly positive $f \in \mathscr{B}_b(\mathscr{C})$ and $\phi, \psi \in \mathscr{C}$.

The proof is modified from Section 2. But in the present setting we are not able to derive the Harnack inequality with power as in Theorem 2.1(3). The reason is that according to the proof of Lemma 3.3 below, to estimate $\mathbb{E}R(\tilde{\tau})^q$ for $q > 0$ one needs upper bounds of the exponential moments of $\|Z_t\|_\infty^2 u(\|Z_t\|_\infty^2)$, which is however not available.

Let $T > 0$ and $\phi, \psi \in \mathscr{C}$ be fixed. Combining the construction of coupling in Section 2 for the SDE case with non-Lipschitz coefficients and that in [25] for the SFDE case with Lipschitz coefficients, we construct the coupling process $(X(t), Y(t))$ as follows:

$$
\begin{cases}
\mathrm{d}X(t) = \{b(t, X(t)) + a(t, X_t)\}\mathrm{d}t + \sigma(t, X(t))\mathrm{d}B(t), \ X_0 = \phi, \\
\mathrm{d}Y(t) = \{b(t, Y(t)) + a(t, X_t)\}\mathrm{d}t + \sigma(t, Y(t))\mathrm{d}B(t) \\
\qquad + \frac{\sigma(t, Y(t))\sigma(t, X(t))^{-1}(X(t) - Y(t))}{\tilde{\xi}(t)} 1_{[0,T)}(t) u(|X(t) - Y(t)|^2)\mathrm{d}t, \ Y_0 = \psi,
\end{cases}
\tag{3.4}
$$

where

$$
\tilde{\xi}(t) = \frac{T - t}{2\gamma}, \ \ t \in [0, T].
$$

As explained in Subsection 2.1 for the existence of solution to (2.4) using Theorem 4.1, due to Theorem 4.2 and (i) in **(A)**, the equation (3.4) has a unique solution up to the time $T \wedge \tilde{\zeta} \wedge \tilde{\tau}$, where

$$
\tilde{\tau} := \inf\{t > 0: \ X(t) = Y(t)\}, \ \ \tilde{\zeta} := \lim_{n \to \infty} \tilde{\zeta}_n; \ \ \tilde{\zeta}_n := \inf\{t \in [0, \tilde{T}): \ |Y(t)| \geq n\}.
$$

From **(A)** it is easy to see that $\tilde{\zeta} \geq T$. If $\tilde{\tau} \leq T$, we set $Y(t) = X(t)$ for $t \geq \tilde{\tau}$ so that $(X(t), Y(t))$ solves (3.4) for all $t \geq 0$ (this is not true if $\sigma(t, Y(t))$ is replaced by $\bar{\sigma}(t, Y_t)$ depending on $Y(t + s), s \in [-r_0, 0]$). In particular, $\tilde{\tau} \leq T$ implies that $X_{T+r_0} = X_{T+r_0}$. To show that $\tilde{\tau} \leq T$, we make use of the Girsanov theorem as in Section 2. Let $Z(t) = X(t) - Y(t)$ and

$$
\Lambda(t) := \frac{u(|Z(t)|^2)\sigma(t, X(t))^{-1}Z(t)}{\tilde{\xi}(t)} + \sigma(t, Y(t))^{-1}\big(a(t, X_t) - a(t, Y_t)\big).
$$

We intend to show that

$$
R(s) := \exp\left[-\int_0^s \langle \Lambda(t), \mathrm{d}B(t) \rangle - \frac{1}{2}\int_0^s |\Lambda(t)|^2 \mathrm{d}t \right]
\tag{3.5}
$$

is a uniformly integrable martingale for $s \in [0, T \wedge \tilde{\tau})$, so that due to the Girsanov theorem,

$$
\tilde{B}(s) := B(s) + \int_0^s \Lambda(t)\mathrm{d}t, \ \ t < T \wedge \tilde{\tau}
\tag{3.6}
$$

is a d-dimensional Brownian motion under the probability $\mathbb{Q} := R(\tilde{\tau} \wedge \tilde{\zeta} \wedge T)\mathbb{P}$. To this end, we make use of the approximation argument as in Section 2.

Define

$$
\tilde{\tau}_n = \inf\{t \in [0, \tilde{T}); \ |X(t) - Y(t)| \geq n^{-1}\}, \ \ n \geq 1.
$$

By the Girsanov theorem, for any $s \in (0, T)$ and $n \geq 1$, $\{R(t)\}_{t \in [0, s \wedge \tilde{\tau}_n \wedge \tilde{\zeta}_n]}$ is a martingale and $\{\tilde{B}(t)\}_{t \in [0, s \wedge \tilde{\tau}_n \wedge \tilde{\zeta}_n]}$ is a d-dimensional Brownian motion under the probability $\mathbb{Q}_{s,n} := R(s \wedge \tilde{\zeta}_n \wedge \tilde{\tau}_n)\mathbb{P}$.

For $t < T \wedge \tilde{\zeta}_n \wedge \tilde{\tau}_n$, rewrite (3.4) as

$$
\begin{cases}
\mathrm{d}X(t) = \{b(t, X(t)) + a(t, X_t)\}\mathrm{d}t + \sigma(t, X(t))\mathrm{d}\tilde{B}(t) - \frac{Z(t)}{\tilde{\xi}(t)}u(|Z(t)|^2)\mathrm{d}t \\
\qquad - \sigma(t, X(t))\sigma(t, Y(t))^{-1}\big(a(t, X_t) - a(t, Y_t)\big)\mathrm{d}t, \ \ X_0 = \phi, \\
\mathrm{d}Y(t) = \{b(t, Y(t)) + a(t, Y_t)\}\mathrm{d}t + \sigma(t, Y(t))\mathrm{d}\tilde{B}(t), \ \ Y_0 = \psi.
\end{cases}
$$

We have $Z_0 = \phi - \psi$ and

$$
\mathrm{d}Z(t) = \big(\sigma(t, X(t)) - \sigma(t, Y(t))\big)\mathrm{d}\tilde{B}(t) + \Big(b(t, X(t)) - b(t, Y(t)) - \frac{u(|Z(t)|^2)Z(t)}{\tilde{\xi}(t)}\Big)\mathrm{d}t
$$
$$
+ \{\sigma(t, Y(t)) - \sigma(t, X(t))\}\sigma(t, Y(t))^{-1}(a(t, X_t) - a(t, Y_t))\mathrm{d}t
\tag{3.7}
$$

for $t < T \wedge \tilde{\tau}_n \wedge \tilde{\zeta}_n$.

Lemma 3.2. *Assume (i), (ii) and (iii) in* **(A)**. *Let* $\mathbb{E}_{s,n}$ *stands for taking the expectation w.r.t. the probability measure* $\mathbb{Q}_{s,n} := R(s \wedge \tilde{\zeta}_n \wedge \tilde{\tau}_n)\mathbb{P}$. *Then*

$$\sup_{n \geq 1, s \in [0,T)} \mathbb{E}_{s,n} \left(\sup_{-r_0 \leq t \leq s \wedge \tilde{\zeta}_n \tilde{\wedge} \tilde{\tau}_n} |Z(t)|^2 \right) \leq C(T, \|Z_0\|_\infty).$$

Proof. Let $\ell_n(t) = \sup_{-r_0 \leq r \leq t \wedge \tilde{\tau}_n \wedge \tilde{\zeta}_n} |Z(r)|^2$. By the first inequality (i) and (iii) in **(A)**, (3.7) and using the Itô formula, we get

$$\begin{aligned}
\mathrm{d}|Z(t)|^2 \leq &2\big\langle Z(t), (\sigma(t,X(t)) - \sigma(t,Y(t)))\mathrm{d}\tilde{B}(t)\big\rangle \\
&+ 2\Big(K_1(t)|Z(t)|^2 u(|Z(t)|^2) + |Z(t)|\sqrt{K_2(t)K_3(t)\|Z_t\|_\infty^2 u(\|Z_t\|_\infty^2)}\Big)\mathrm{d}t
\end{aligned} \tag{3.8}$$

for $t \leq s \wedge \tilde{\tau}_n \wedge \tilde{\zeta}_n$. Moreover, according to the Burkholder-Davis-Gundy inequality, for any continuous martingale $M(t)$ one has

$$\mathbb{E} \sup_{s \in [0,t]} M(s) \leq 2\sqrt{2}\mathbb{E}\sqrt{\langle M\rangle(t)}, \;\; t \geq 0.$$

Combining this with (3.8) and (ii) in **(A)**, and noting that $su(s)$ is increasing in s so that

$$|Z(t)|^2 u(|Z(t)|^2) \leq \|Z_t\|_\infty^2 u(\|Z_t\|_\infty^2) \leq \ell_n(t)u(\ell_n(t)), \;\; t \leq s \wedge \tilde{\tau}_n \wedge \tilde{\zeta}_n,$$

we obtain

$$\begin{aligned}
\mathbb{E}_{s,n}\ell_n(t) \leq &\|Z_0\|_\infty^2 + 8\mathbb{E}_{s,n}\sqrt{K(T)}\bigg(\int_0^t \ell_n(r)^2 u(\ell_n(r))\mathrm{d}r \bigg)^{1/2} + \frac{1}{4}\mathbb{E}_{s,n}\ell_n(t) \\
&+ \{2K_1(T) + 4K_2(T)K_3(T)\}\int_0^t \mathbb{E}_{s,n}\ell_n(r)u(\ell_n(r))\mathrm{d}r \\
\leq &\|Z_0\|_\infty^2 + \frac{1}{2}\mathbb{E}_{s,n}\ell_n(t) + 2\{K_1(T) + 2K_2(T)K_3(T) + 32K(T)\}\int_0^t \mathbb{E}_{s,n}\big[\ell_n(r)u(\ell_n(r))\big]\mathrm{d}r.
\end{aligned}$$

Since $su(s)$ is concave in s so that $\mathbb{E}_{s,n}[\ell_n(r)u(\ell_n(r))] \leq \mathbb{E}_{s,n}\ell_n(r)u(\mathbb{E}_{s,n}\ell_n(r))$, this implies that

$$\mathbb{E}_{s,n}\ell_n(t) \leq 2\|Z_0\|_\infty^2 + 4\{K_1(T) + 2K_2(T)K_3(T) + 32K(T)\}\int_0^t \mathbb{E}_{s,n}\ell_n(r)u(\mathbb{E}_{s,n}\ell_n(r))\mathrm{d}r, \;\; t \leq s.$$

Therefore, the desired estimate follows from the Bihari's inequality (see, for example, [13, Theorem 1.8.2]). $\qquad\square$

Lemma 3.3. *Assume* **(A)**. *If* (2.2) *holds for some constant* $\gamma > 0$, *then*

$$\begin{aligned}
\sup_{s \in [0,\tilde{T}), n \geq 1} &\mathbb{E}\big[R(s \wedge \tilde{\zeta}_n \wedge \tilde{\tau}_n) \log R(s \wedge \tilde{\zeta}_n \wedge \tilde{\tau}_n)\big] \\
&\leq K_4(T)\Big(\frac{2\varphi(|Z(0)|^2)}{T} + T\{8K_1(T)^2 + 8K_2(T)K_3(T) + K_2(T)\}\Phi(T, \|Z_0\|_\infty)\Big)
\end{aligned}$$

Proof. By the first inequality in **(A2)**, (3.7) and using the Itô formula, we obtain

$$\begin{aligned}
\mathrm{d}|Z(t)|^2 \leq &2\big\langle Z(t), (\sigma(t,X(t)) - \sigma(t,Y(t)))\mathrm{d}\tilde{B}(t)\big\rangle - \frac{2|Z(t)|^2 u(|Z(t)|^2)}{\tilde{\xi}(t)}\mathrm{d}t \\
&+ 2\Big(K_1(t)|Z(t)|^2 u(|Z(t)|^2) + |Z(t)|\sqrt{K_2(t)K_3(t)\|Z_t\|_\infty^2 u(\|Z_t\|_\infty^2)}\Big)\mathrm{d}t
\end{aligned}$$

for $t \leq s \wedge \tilde{\tau}_n \wedge \tilde{\zeta}_n$. So, as in the proof of Lemma 2.2, there exists a $\mathbb{Q}_{s,n}$-martingale $M(t)$ such that for $t \leq s \wedge \tilde{\tau}_n \wedge \tilde{\zeta}_n$,

$$
\begin{aligned}
\mathrm{d}\frac{\varphi(|Z(t)|^2)}{\tilde{\xi}(t)} \leq & \mathrm{d}M(t) - \frac{|Z(t)|^2 u^2(|Z(t)|^2)}{\tilde{\xi}(t)^2}\big(2 + \gamma\xi'(t)\big)\,\mathrm{d}t \\
& + \frac{2}{\tilde{\xi}(t)}\Big(K_1(t)|Z(t)|^2 u(|(Z(t)|^2)) + |Z(t)|\sqrt{K_2(t)K_3(t)}\|Z_t\|_\infty^2 u(\|Z_t\|_\infty^2)\Big)\mathrm{d}t \\
\leq & \mathrm{d}M(t) + \Big(4\{K_1(t)^2 + K_2(t)K_3(t)\}\|Z_t\|_\infty^2 u(\|Z_t\|_\infty^2)\mathrm{d}t - \frac{|Z(t)|^2 u^2(|Z(t)|^2)}{2\tilde{\xi}(t)^2}\Big)\mathrm{d}t,
\end{aligned}
$$

where in the last step we have used $u \geq 1$ and $\tilde{\xi}'(t) = -\frac{1}{2\gamma}$. Therefore,

$$
\begin{aligned}
& \mathbb{E}_{s,n}\int_0^{s\wedge\tilde{\tau}_n\wedge\tilde{\zeta}_n} \frac{|Z(t)|^2 u^2(|Z(t)|^2)}{\tilde{\xi}(t)^2}\mathrm{d}t \\
& \leq \frac{2\varphi(|Z(0)|^2)}{\tilde{\xi}(0)} + 8T\{K_1(T)^2 + K_2(T)K_3(T)\}\mathbb{E}_{s,n}\ell_n(T)u(\ell_n(T)).
\end{aligned}
\tag{3.9}
$$

Since by Lemma 3.2 and the concavity of $r \mapsto ru(r)$

$$
\mathbb{E}_{s,n}\ell_n(T)u(\ell_n(T)) \leq C(T, \|Z_0\|_\infty)u(C(T, \|Z_0\|_\infty)) = \Phi(T, \|Z_0\|_\infty),
$$

combining (3.9) with Lemma 3.2 and (iv) in **(A)** we arrive at that

$$
\begin{aligned}
& \mathbb{E}\big[R(s \wedge \tilde{\tau}_n \wedge \tilde{\zeta}_n)\log R(s \wedge \tilde{\tau}_n \wedge \tilde{\zeta}_n)\big] = \frac{1}{2}\mathbb{E}_{s,n}\int_0^{s\wedge\tilde{\tau}_n\wedge\tilde{\zeta}_n}|\Lambda(t)|^2\mathrm{d}t \\
& = K_4(T)\mathbb{E}_{s,n}\int_0^{s\wedge\tilde{\tau}_n\wedge\tilde{\zeta}_n}\Big(\frac{|Z(t)|^2 u^2(|Z(t)|^2)}{\tilde{\xi}(t)^2} + K_2(T)\|Z_t\|_\infty^2 u(\|Z_t\|_\infty^2)\Big)\mathrm{d}t \\
& \leq K_4(T)\Big(\frac{2\gamma\varphi(|Z(0)|^2)}{T} + T\{8K_1(T)^2 + 8K_2(T)K_3(T) + K_2(T)\}\Phi(T, \|Z_0\|_\infty)\Big).
\end{aligned}
$$

$\square$

Proof of Theorem 3.1. As discussed in Section 2 that Lemma 3.3 and (3.9) imply that $\tilde{\tau} \leq T \wedge \tilde{\zeta}$ $\mathbb{Q}$-a.s., where $\mathbb{Q} := R(\tilde{\tau} \wedge T \wedge \tilde{\zeta})\mathbb{P} = R(\tilde{\tau})\mathbb{P}$. Since by the construction we have $X(t) = Y(t)$ for $t \geq \tilde{\tau}$, this implies that $X_{T+r_0} = Y_{T+r_0}$. Applying the Young inequality and Lemma 3.3, we obtain

$$
\begin{aligned}
& P_{T+r_0}\log f(\psi) - \log P_{T+r_0}f(\phi) = \mathbb{E}_\mathbb{Q}\big[\log f(Y_{T+r_0})\big] - \log P_{T+r_0}f(\phi) \\
& = \mathbb{E}\big[R(\tilde{\tau})\log f(X_{T+r_0})\big] - \log\mathbb{E}\big[f(X_{T+r_0})\big] \leq \mathbb{E}\big[R(\tilde{\tau})\log R(\tilde{\tau})\big] \\
& \leq K_4(T)\Big(\frac{2\gamma\varphi(|Z(0)|^2)}{T} + T\{8K_1(T)^2 + 8K_2(T)K_3(T) + K_2(T)\}\Phi(T, \|Z_0\|_\infty)\Big).
\end{aligned}
$$

$\square$

4 Existence and uniqueness of solutions

There is a lot of literature on the existence and uniqueness of SDEs and SFDEs under non-Lipschitz condition, see e.g. Taniguchi [18, 19] and references therein. In the following two subsections, for the construction of couplings given in the previous sections, we present below two results in this direction for SDEs and SFDEs on open domains respectively.

4.1 Stochastic differential equations

Let D be a non-empty open domain in $\mathbb{R}^d$, and let $T > 0$ be fixed. Consider the following SDE:

$$\mathrm{d}X(t) = \sigma(t, X(t))\mathrm{d}B(t) + b(t, X(t))\mathrm{d}t, \tag{4.1}$$

where $(B(t))_{t \geq 0}$ is the m-dimensional Brownian motion on a complete filtered probability space $(\Omega, (\mathscr{F}_t)_{t \geq 0}, \mathscr{F}, \mathbb{P})$, $\sigma : [0, T] \times D \to \mathbb{R}^d \otimes \mathbb{R}^m$ and $b : [0, T] \times D \to \mathbb{R}^d$ are measurable, locally bounded in the first variable and continuous in the second variable.

Theorem 4.1. *If there exist $u \in \mathscr{U}$, a sequence of compact sets $\mathbf{K}_n \uparrow D$ and functions $\{\Theta_n\}_{n \geq 1} \in C([0, T); (0, \infty))$ such that for every $n \geq 1$,*

$$\begin{aligned}
&2\langle b(t, x) - b(t, y), x - y \rangle + \|\sigma(t, x) - \sigma(t, y)\|_{\mathrm{HS}}^2 \\
&\leq \Theta_n(t)|x - y|^2 u(|x - y|^2), \quad |x - y| \leq 1, x, y \in \mathbf{K}_n, t \in [0, T).
\end{aligned} \tag{4.2}$$

Then for any initial data $X(0) \in D$, the equation (4.1) *has a unique solution $X(t)$ up to life time*

$$\zeta := T \wedge \lim_{n \to \infty} \inf \{ t \in [0, T) : X(t) \notin \mathbf{K}_n \},$$

where $\inf \emptyset := \infty$.

Proof. For each $n \geq 1$, we may find $h_n \in C^\infty(\mathbb{R}^d)$ with compact support contained in D such that $h_n|_{\mathbf{K}_n} = 1$. Let

$$b_n(t, x) = h_n(x)b(t, x), \quad \sigma_n(t, x) = h_n(x)\sigma(t, x).$$

Then for any $n \geq 1$, b_n and σ_n are bounded on $[0, \frac{nT}{n+1}] \times \mathbb{R}^d$ and continuous in the second variable. According to the Skorokhod theorem [16] (see also [9, Theorem 0.1]), the equation

$$\mathrm{d}X_n(t) = \sigma_n(t, X_n(t))\mathrm{d}B(t) + b_n(t, X_n(t))\mathrm{d}t, \quad X_n(0) = X_0 \tag{4.3}$$

has a weak solution for $t \in [0, \frac{nT}{n+1}]$. So, by Yamada-Watanabe principle [26], to prove the existence and uniqueness of the (strong) solution, we only need to verify the pathwise uniqueness.

Let $X_n(t), \tilde{X}_n(t)$ be two solutions to (4.3) for $t \in [0, \frac{nT}{n+1}]$. Since the support of h_n is a compact subset of D and since $K_m \uparrow D$, there exists $m > n$ such that $K_m \supset \operatorname{supp} h_n$. Then (4.2) yields that

$$2\langle b_n(t, x) - b_n(t, y), x - y \rangle + \|\sigma_n(t, x) - \sigma_n(t, y)\|_{\mathrm{HS}}^2 \leq C_n|x - y|^2 u(|x - y|^2)$$

holds for some constant $C_n > 0$, all $t \in [0, \frac{nT}{n+1}]$ and $x, y \in \mathbb{R}^d$ with $|x - y| \leq 1$. By the Itô formula, this implies

$$\begin{aligned}
\mathrm{d}|X_n(t) - \tilde{X}_n(t)|^2 \leq{}& C_n|X_n(t) - \tilde{X}_n(t)|^2 u(|X_n(t) - \tilde{X}_n(t)|^2)\mathrm{d}t \\
&+ 2\langle X_n(t) - \tilde{X}_n(t), \{\sigma_n(t, X_n(t)) - \sigma_n(t, \tilde{X}_n(t))\}\mathrm{d}B(t)\rangle
\end{aligned} \tag{4.4}$$

for $t \in [0, \frac{nT}{n+1}]$. On the other hand, $u \in \mathscr{U}$ implies that

$$u(r) + ru'(r) \geq \lambda, \quad r \in [0, \rho_0]$$

holds for some constants $\lambda, \rho_0 > 0$. Let

$$\Psi_\varepsilon(r) = \exp\left[\lambda \int_1^r \frac{\mathrm{d}s}{\varepsilon + su(s)}\right], \quad r, \varepsilon \geq 0.$$

Then, for any $\varepsilon > 0$, we have $\Psi_\varepsilon \in C^2([0, \infty))$ and

$$ru(r)\Psi'_\varepsilon(r) = \frac{\lambda r u(r)}{\varepsilon + ru(r)}\Psi_\varepsilon(r) \leq \lambda\Psi_\varepsilon(r),$$

$$\Psi''_\varepsilon(r) = \frac{\lambda^2 - \lambda\{u(r) + ru'(r)\}}{(\varepsilon + ru(r))^2} \leq 0, \quad r \in [0, \rho_0].$$

Therefore, letting

$$\tau_0 = \inf\left\{t \in \left[0, \frac{nT}{n+1}\right] : \ |X_n(t) - \tilde{X}_n(t)|^2 \geq \rho_0\right\},$$

it follows from (4.4) and the Itô formula that

$$\mathrm{d}\Psi_\varepsilon(|X_n(t) - \tilde{X}_n(t)|^2) \leq \lambda C_n \Psi_\varepsilon(|X_n(t) - \tilde{X}_n(t)|^2)\mathrm{d}t$$
$$+ 2\Psi'_\varepsilon(|X_n(t) - \tilde{X}_n(t)|^2)\langle X_n(t) - \tilde{X}_n(t), \{\sigma_n(t, X_n(t)) - \sigma_n(t, \tilde{X}_n(t))\}\mathrm{d}B(t)\rangle$$

holds for $t \leq \tau_0 \wedge \frac{nT}{n+1}$. Hence,

$$\mathbb{E}\Psi_\varepsilon(|X_n(t \wedge \tau_0) - \tilde{X}_n(t \wedge \tau_0)|^2) \leq \mathrm{e}^{\lambda C_n t}\Psi_\varepsilon(0), \quad t \leq \frac{nT}{n+1}.$$

Letting $\varepsilon \downarrow 0$ and noting that $\Psi_0(0) = 0$, we arrive at

$$\mathbb{E}\Psi_0(|X_n(t \wedge \tau_0) - \tilde{X}_n(t \wedge \tau_0)|^2) = 0.$$

Thus, $X_n(t \wedge \tau_0) - \tilde{X}_n(t \wedge \tau_0)$ holds for all $t \in [0, \frac{nT}{n+1}]$. Therefore, $\tau_0 = \infty$ and $X_n(t) = \tilde{X}_n(t)$ holds for all $t \in [0, \frac{nT}{n+1}]$. In conclusion, for every $n \geq 1$, the equation (4.3) has a unique solution up to time $\frac{nT}{n+1}$.

Since $h_n = 1$ on $\mathbf{K}_n$ so that (4.3) coincides with (4.1) before the solution leaves $\mathbf{K}_n$, the equation (4.1) has a unique solution $X(t)$ up to the time

$$\zeta_n := \frac{nT}{n+1} \wedge \inf\{t \geq 0 : X(t) \notin \mathbf{K}_n\}.$$

Therefore, (4.1) has a unique solution up to the life time $\zeta = T \wedge \lim_{n\to\infty} \zeta_n$. $\qquad\square$

4.2 Stochastic functional differential equations

Let $\mathscr{C} := \mathscr{C}([-r_0, 0]; \mathbb{R}^d)$ for a fixed number $r_0 > 0$, and for any set $A \subset \mathbb{R}^d$ let $A^\mathscr{C} = \{\phi \in \mathscr{C} : \phi([-r_0, 0]) \subset A\}$. For fixed $T > 0$ and a non-empty open domain D in $\mathbb{R}^d$, we consider the SFDE

$$\mathrm{d}X(t) = \bar{b}(t, X_t)\mathrm{d}t + \bar{\sigma}(t, X_t)\mathrm{d}B(t), \quad X_0 \in D^\mathscr{C}, \tag{4.5}$$

where $B(t)$ is the m-dimensional Brownian motion, $\bar{b} : [0, T) \times D^\mathscr{C} \to \mathbb{R}^d$ and $\bar{\sigma} : [0, T) \times D^\mathscr{C} \to \mathbb{R}^d \otimes \mathbb{R}^m$ are measurable, bounded on $[0, t] \times K^\mathscr{C}$ for $t \in [0, T)$ and compact set $K \subset D$, and continuous in the second variable.

Theorem 4.2. *Assume that there exists a sequence of compact sets $\mathbf{K}_n \uparrow D$ such that for every $n \geq 1$,*

$$2\langle\bar{b}(t, \phi) - \bar{b}(t, \psi), \phi(0) - \psi(0)\rangle + \|\bar{\sigma}(t, \phi) - \bar{\sigma}(t, \psi)\|^2_{\mathrm{HS}} \leq \|\phi - \psi\|^2_\infty u_n(\|\phi - \psi\|^2_\infty) \tag{4.6}$$

and

$$\|\bar{\sigma}(t, \phi) - \bar{\sigma}(t, \psi)\|^2_{\mathrm{HS}} \leq \|\phi - \psi\|^2_\infty u_n(\|\phi - \psi\|^2_\infty) \tag{4.7}$$

hold for some $u_n \in \mathscr{U}$ and all $\phi, \psi \in \mathbf{K}_n^\mathscr{C}, t \leq \frac{nT}{n+1}$. Then for any initial data $X_0 \in D^\mathscr{C}$, the equation (4.5) has a unique solution $X(t)$ up to life time

$$\zeta := T \wedge \lim_{n\to\infty} \inf\left\{t \in [0, T) : \ X(t) \notin \mathbf{K}_n\right\}.$$

Proof. Using the approximation argument in the proof of Theorem 4.1, we may and do assume that $D = \mathbb{R}^d$ and $\bar{b}$ and $\bar{\sigma}$ are bounded and continuous in the second variable and prove the existence and uniqueness of solution up to any time $T' < T$. According to the Yamada-Watanabe principle, we shall verify below the existence of a weak solution and the pathwise uniqueness of the strong solution respectively.

(1) The proof of the existence of a weak solution is standard up to an approximation argument. Let $\mathbf{B}(s) = B(r_0 + 1 + s), s \in [-r_0, 0]$, where $B(s)$ is a d-dimensional Brownian motion. Define

$$\bar{\sigma}_n(t, \phi) = \mathbb{E}\bar{\sigma}(t, \phi + n^{-1}\mathbf{B}), \bar{b}_n(t, \phi) = \mathbb{E}\bar{b}(t, \phi + n^{-1}\mathbf{B}), \quad n \geq 1.$$

Applying [3, Corollary 1.3] for $\sigma = \frac{1}{n}I_{d \times d}, m = 0, Z = b = 0$ and $T = 1 + r_0$, we see that for every $n \neq 1$, $\bar{\sigma}_n$ and $\bar{b}_n$ are Lipschitz continuous in the second variable uniformly in the first variable. Therefore, the equation

$$\mathrm{d}X^{(n)}(t) = \bar{b}_n(t, X_t^{(n)})\mathrm{d}t + \bar{\sigma}_n(t, X_t^{(n)})\mathrm{d}B(t), \quad X_0^{(n)} = X_0$$

has a unique strong solution up to time T': $X^{(n)} \in C([0, T']; \mathbb{R}^d)$. To see that $X^{(n)}$ converges weakly as $n \to \infty$, we take the reference function

$$g_\varepsilon(h) := \sup_{t \in [0,T)} \sup_{s \in (0,(T-t)\wedge 1)} \frac{|h(t+s)) - h(t)|}{s^\varepsilon}$$

for a fixed number $\varepsilon \in (0, \frac{1}{2})$. It is well known that g_ε is a compact function on $C([0, T']; \mathbb{R}^d)$, i.e. $\{g_\varepsilon \leq r\}$ is compact under the uniform norm for any $r > 0$. Since $\bar{b}_n$ and $\bar{\sigma}_n$ are uniformly bounded and $\varepsilon \in (0, \frac{1}{2})$, we have

$$\sup_{n \geq 1} \mathbb{E}g_\varepsilon(X^{(n)}) < \infty.$$

Let $\mathbb{P}^{(n)}$ be the distribution of $X^{(n)}$. Then the family $\{\mathbb{P}^{(n)}\}_{n \geq 1}$ is tight, and hence (up to a sub-sequence) converges weakly to a probability measure $\mathbb{P}$ on $\Omega := C([0, T;]; \mathbb{R}^d)$. Let $\mathscr{F}_t = \sigma(\omega \mapsto \omega(s) : s \leq t)$ for $t \in [0, T']$. Then the coordinate process

$$X(t)(\omega) := \omega(t), \quad t \in [0, T'], \omega \in \Omega$$

is $\mathscr{F}_t$-adapted. Since $\mathbb{P}^{(n)}$ is the distribution of $X^{(n)}$, we see that

$$M^{(n)}(t) := X(t) - \int_0^t \bar{b}_n(s, X_s)\mathrm{d}s, \quad t \in [0, T']$$

is a $\mathbb{P}^{(n)}$-martingale with

$$\langle M_i^{(n)}, M_j^{(n)}\rangle(t) = \sum_{k=1}^m \int_0^t \{(\bar{\sigma}_n)_{ik}(\bar{\sigma}_n)_{jk}\}(s, X_s)\mathrm{d}s, \quad 1 \leq i, j \leq d.$$

Since $\bar{\sigma}_n \to \bar{\sigma}$ and $\bar{b}_n \to \bar{b}$ uniformly and $\mathbb{P}^{(n)} \to \mathbb{P}$ weakly, by letting $n \to \infty$ we conclude that

$$M(t) := X(t) - \int_0^t \bar{b}(s, X_s)\mathrm{d}s, \quad s \in [0, T']$$

is a $\mathbb{P}$-martingale with

$$\langle M_i, M_j\rangle(t) = \sum_{k=1}^m \int_0^t \{\bar{\sigma}_{ik}\bar{\sigma}_{jk}\}(s, X_s)\mathrm{d}s, \quad 1 \leq i, j \leq d.$$

According to [10, Theorem II.7.1′], this implies

$$M(t) = \int_0^t \bar{\sigma}(s, X_s) \mathrm{d}B(s), \ \ t \in [0, T']$$

for some m-dimensional Brownian motion B on the filtered probability space $(\Omega, \mathscr{F}_t, \mathbb{P})$. Therefore, the equation has a weak solution up to time T'.

(2) The pathwise uniqueness. Let $X(t)$ and $Y(t)$ for $t \in [0, T']$ be two strong solutions with $X_0 = Y_0$. Let $Z = X - Y$ and

$$\tau_n = T' \wedge \inf \big\{ t \in [0, T) : \ |X(t)| + |Y(t)| \geq n \big\}.$$

By the Itô formula and (4.6), we have

$$\mathrm{d}|Z(t)|^2 \leq 2\langle (\bar{\sigma}(t, X_t) - \bar{\sigma}(t, Y_t))\mathrm{d}B(t), Z_t \rangle + \|Z_t\|_\infty^2 u_n(\|Z_t\|_\infty^2), \ \ t \leq \tau_n. \tag{4.8}$$

Let

$$\ell_n(t) := \sup_{s \leq t \wedge \tau_n} |Z_s|^2, \ \ t \geq 0.$$

Noting that $su_n(s)$ is increasing in s, we have

$$\|Z_t\|_\infty^2 u_n(\|Z_t\|_\infty^2) \leq \ell_n(t) u_n(\ell_n(t)), \ \ t \geq 0.$$

So, by (4.7), (4.8) and using the Burkholder-Davis-Gundy inequality, there exist constants $C_1, C_2 > 0$ such that

$$\mathbb{E}\ell_n(t) \leq \int_0^t \mathbb{E}\ell_n(s) u_n(\ell_n(s)) \mathrm{d}s + C_1 \mathbb{E}\left(\ell_n(t) \int_0^t \ell_n(s) u_n(\ell_n(s)) \mathrm{d}s \right)^{1/2}$$

$$\leq \frac{1}{2} \mathbb{E}\ell_n(t) + C_2 \int_0^t \mathbb{E}\ell_n(s) u_n(\ell_n(s)) \mathrm{d}s.$$

Since $s \mapsto su_n(s)$ is concave, due to Jensen's inequality this implies that

$$\mathbb{E}\ell_n(t) \leq 2C_2 \int_0^t \mathbb{E}\ell_n(s) u_n\big(\mathbb{E}\ell_n(s)\big) \mathrm{d}s.$$

Let $G(s) = \int_1^s \frac{1}{su_n(s)} \mathrm{d}s$, $s > 0$, and let G^{-1} be the inverse of G. Since $\int_0^1 \frac{1}{su_n(s)} \mathrm{d}s = \infty$, we have $[-\infty, 0] \subset \mathrm{Dom}(G^{-1})$ with $G^{-1}(-\infty) = 0$. Then, by Bihari's inequality, we obtain

$$\mathbb{E}\ell_n(t) \leq G^{-1}\big(G(0) + G(2C_2 t)\big) = G^{-1}(-\infty) = 0.$$

This implies that $X(t) = Y(t)$ for $t \leq \tau_n$ for any $n \geq 1$. Since $\bar{b}$ and $\bar{\sigma}$ are bounded, we have $\tau_n \uparrow T'$. Therefore, $X(t) = Y(t)$ for $t \in [0, T']$. $\qquad \square$

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