Electronic Journal of Probability

Electron. J. Probab. **17** (2012), no. 51, 1–21.
ISSN: 1083-6489 DOI: 10.1214/EJP.v17-2136

Taylor expansion for the solution of a stochastic differential equation driven by fractional Brownian motions

Fabrice Baudoin* Xuejing Zhang†

Abstract

We study the Taylor expansion for the solution of a differential equation driven by a multi-dimensional Hölder path with exponent $\beta > 1/2$. We derive a convergence criterion that enables us to write the solution as an infinite sum of iterated integrals on a nonempty interval. We apply our deterministic results to stochastic differential equations driven by fractional Brownian motions with Hurst parameter $H > 1/2$. We also study the convergence in L^2 of the stochastic Taylor expansion by using L^2 estimates of iterated integrals and Borel-Cantelli type arguments.

Keywords: fractional Brownian motion ; Taylor expansion.
AMS MSC 2010: NA.
Submitted to EJP on December 13, 2011, final version accepted on June 25, 2012.
Supersedes arXiv:1112.2760.

1 Introduction

This paper is divided into two parts. In the first part, we consider a deterministic differential equation of the following type:

$$\begin{cases} dX_t = \displaystyle\sum_{i=0}^{d} V_i\left(X_t\right) dy_t^i \\ X_0 = x_0 \end{cases} \tag{1.1}$$

where the V_i's are Lipschitz C^∞ vector fields on $\mathbb{R}^n$ with Lipschitz derivatives and where the driving signal $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^{d+1}$ is β-Hölder continuous with $\beta > 1/2$. Since $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^{d+1}$ is β-Hölder with $\beta > 1/2$, the integrals $\int_0^t V_i(X_s) dy_t^i$ are understood in Young's sense (see [17] and [18]) and it is known that the above equation has a unique solution that can be constructed as a fixed point of the Picard iteration in a convenient Banach space of functions (see [12], [18] and [15]).

*Purdue University, USA. E-mail: fbaudoin@math.purdue.edu
†Purdue University, USA. E-mail: zhang239@math.purdue.edu

Our goal is to express, under suitable conditions, the solution of the differential equation as an infinite series involving the iterated integrals of the Hölder path. To study the question of the convergence of the infinite series which, later on, we will call the Taylor expansion associated with the differential equation, we will develop some convergence criteria based on an explicit upper bound of the iterated integrals of the Hölder path. We can observe that these explicit upper bounds sharpen some recent results by Y. Hu [9], who also independently obtained upper bounds for iterated integrals of Hölder paths (see Proposition 2.7). Our improvement, with respect to Hu's paper is that we exhibit an explicit dependence of the constants with respect to the order of the integral and as a consequence we are able to study the convergence of the Taylor expansion even in the multidimensional case. The main tool we use in this part is the fractional calculus and the main assumption we need to perform the expansion is the analycity of the vector fields V_i's in a neighborhood of the initial condition x_0. To illustrate our results, we consider some examples of differential equations on a Lie group and in particular prove that the radius of convergence of the Magnus expansion is positive. This partly extends to the Hölder case, results by Magnus [10] and Strichartz [16].

The second part, which is of probabilistic nature, is concerned with a stochastic differential equation (in short SDE) of the following type:

$$\begin{cases} dX_t = \sum_{i=0}^{d} V_i\left(X_t\right) dB_t^i \\ X_0 = x_0 \end{cases}$$

where the V_i's are Lipschitz C^∞ vector fields on $\mathbb{R}^n$ with Lipschitz derivatives and where $(B_t)_{t\geq 0}$ is a d-dimensional fractional Brownian motion with Hurst parameter $H > 1/2$. Throughout the paper, we use the convention that $B_t^0 = t$. We will be interested in the convergence of Taylor expansion of the solution to the SDE. The deterministic estimates we obtained in the first part will automatically lead to convergence criteria for the stochastic Taylor expansion. The rate of convergence to 0 and L^p estimates of the remainder term are then studied. Afterwards, inspired by some ideas of Ben Arous [2] (see also Castell [6]), we will also use probabilistic methods to obtain a convergence criterion, which improves the convergence radius of the stochastic Taylor expansion in some cases. The key point to our probabilistic methods are the L_2 estimates of the iterated integrals of the fractional Brownian motion that were obtained by Baudoin-Coutin in [4] and Neuenkirch-Nourdin-Rößler-Tindel in [11]. Our results in this part widely generalize to the fractional Brownian case, results that were due in the Brownian motion case to Azencott [1] and Ben Arous [2].

2 Taylor expansion for differential equations driven by Hölder paths

In this section, we present our main theorem (Theorem 2.9). We derive a convergence criterion, under which, we can, on a nonempty time interval, write the solution of our differential equation (1.1) as an infinite series of iterated integrals of the Hölder path y. The materials in this section will be organized in the following way:

1. In Section 2.1, we will recall some basic facts about fractional calculus that will be needed in our analysis.

2. In Section 2.2, we will briefly discuss the existence and uniqueness of the solution to the differential equation (1.1). The main result is borrowed from [12]. We will also define, in apparently two different ways, the Taylor expansion of the solution of the differential equation (1.1). The two apparently different expansions are then shown to eventually be the same.
3. In Section 2.3, we will state our main theorem.
4. In Section 2.4 we will apply our results to the case of differential equations on Lie groups on which we study the convergence of the Magnus expansion (see Theorem 2.11).

2.1 Preliminaries : Fractional calculus

Let us first do some remainders about fractional calculus. For further details, we refer the reader to ([18]) or ([12]). Let $f \in L^1(a,b)$ and $\alpha > 0$. The left-sided and right-sided fractional integrals of f of order α are defined by:

$$I^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy$$

and

$$I^{\alpha}_{b-} f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) dy$$

respectively, where $(-1)^{-\alpha} = e^{-i\pi\alpha}$ and $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du$ is the Gamma function. Let us denote by $I^{\alpha}_{a+}(L^p)$ (respectively $I^{\alpha}_{b-}(L^p)$) the image of $L^p(a,b)$ by the operator I^{α}_{a+} (respectively I^{α}_{a+}). If $f \in I^{\alpha}_{a+}(L^p)$ (respectively $f \in I^{\alpha}_{b-}(L^p)$) and $0 < \alpha < 1$, we define for $x \in (a,b)$ the left and right Weyl derivatives by:

$$D^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy \right) \mathbf{1}_{(a,b)}(x)$$

and respectively,

$$D^{\alpha}_{b-} f(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^\alpha} + \alpha \int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha+1}} dy \right) \mathbf{1}_{(a,b)}(x)$$

Now recall that from ([12]), for a parameter $0 < \alpha < 1/2$, $W_T^{1-\alpha,\infty}(0,T)$ is defined as the space of measurable function $g : [0,T] \to \mathbb{R}$ such that:

$$\|g\|_{1-\alpha,\infty,T} = \sup_{0<s<t<T} \left(\frac{|g(t)-g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y)-g(s)|}{(y-s)^{2-\alpha}} dy \right) < \infty$$

Clearly, for every $\epsilon > 0$,

$$C^{1-\alpha+\epsilon}(0,T) \subset W_T^{1-\alpha,\infty}(0,T) \subset C^{1-\alpha}(0,T)$$

Moreover, if $g \in W_T^{1-\alpha,\infty}(0,T)$, its restriction to $(0,t)$ belongs to $I^{1-\alpha}_{t-}(L^\infty(0,t))$ for every t and

$$\begin{aligned}\Lambda_\alpha(g) :=& \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} |(D^{1-\alpha}_{t-} g_{t-})(s)|. \\ \le& \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|g\|_{1-\alpha,\infty,T} < \infty\end{aligned}$$

where $g_{t-}(s) = g(s) - g(t)$. We also denote by $W_0^{\alpha,1}(0,T)$ the space of measurable functions f on $[0,T]$ such that:

$$\|f\|_{\alpha,1} = \int_0^T \frac{f(s)}{s^\alpha} ds + \int_0^T \int_0^s \frac{|f(s) - f(y)|}{(s-y)^{\alpha+1}} dyds < \infty$$

The restriction of $f \in W_0^{\alpha,1}(0,T)$ to $(0,t)$ belongs to $I_{0+}^\alpha(L^1(0,t))$ for all t.

2.2 Taylor expansion of the solution

For the convenience of the reader, we first recall the main result concerning the existence and uniqueness results for solutions of the differential equation driven by Hölder path with Hölder exponent $\beta > 1/2$. Several authors made contributions to this existence and uniqueness result. It has been first proved by M.Zähle in [18]. D. Nualart and A.Răşcanu obtained the same result independently in [12]. Also, a different approach was introduced by A. A. Ruzimaikina to prove this existence and uniqueness theorem in [15]. The following result is due to Nualart and Răşcanu ([12])

Theorem 2.1. *Let $0 < \alpha < 1/2$ be fixed. Let $g \in W^{1-\alpha,\infty}(0,T;\mathbb{R}^d)$. Consider the deterministic differential equation on $\mathbb{R}^n$:*

$$x_t^i = x_0^i + \int_0^t b^i(s,x_s)ds + \sum_{j=1}^d \int_0^t \sigma^{i,j}(s,x_s)dg_s^j, \quad t \in [0,T] \tag{2.1}$$

$i = 1, \cdots, n$ where $x_0 \in \mathbb{R}^n$, and the coefficients $\sigma^{i,j}, b^i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ are measurable functions satisfying the following assumptions with $\rho = 1/\alpha$, $0 < \eta, \delta \le 1$ and

$$0 < \alpha < \alpha_0 = \min\left\{\frac{1}{2}, \eta, \frac{\delta}{1+\delta}\right\}$$

1. *$\sigma(t,x) = (\sigma^{i,j}(t,x))_{n\times d}$ is differentiable in x, and there exist some constants $0 < \eta, \delta \le 1$ and for every $N \ge 0$ there exists $M_N > 0$ such that the following properties hold:*

$$\begin{cases} \|\sigma(t,x) - \sigma(t,y)\| \le M_0\|x-y\|, \quad x \in \mathbb{R}^n, \forall t \in [0,T] \\ \\ \|\partial_{x_i}\sigma(t,x) - \partial_{y_i}\sigma(t,y)\| \le M_N\|x-y\|^\delta \quad \|x\|,\|y\| \le N, \forall t \in [0,T] \\ \\ \|\sigma(t,x) - \sigma(t,y)\| + \|\partial_{x_i}\sigma(t,x) - \partial_{y_i}\sigma(t,y)\| \le M_0\|t-s\|^\eta, \forall t,s \in [0,T] \end{cases} \tag{2.2}$$

2. *There exists $b_0 \in L^\rho(0,T;\mathbb{R}^n)$, where $\rho \ge 2$, and for every $N \ge 0$ there exists $L_N > 0$ such that the following properties hold:*

$$\begin{cases} \|b(t,x) - b(t,y)\| \le L_N\|x-y\|, \quad \forall \|x\|,\|y\| \le N, \forall t \in [0,T] \\ \\ \|b(t,x)\| \le L_0\|x\| + b_0(t), \quad \forall x \in \mathbb{R}^d, \forall t \in [0,T] \end{cases} \tag{2.3}$$

Then the differential equation (2.1) has a unique solution $x \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^n)$. Moreover, the solution x is $(1-\alpha)$-Hölder continuous.

With this existence and uniqueness result in hands, we now come back to the differential equation we are concerned with:

$$\begin{cases} dX_t = \sum_{i=0}^{d} V_i\,(X_t)\,dy_t^i \\ X_0 = x_0 \end{cases} \tag{2.4}$$

Throughout the section, we make the following assumptions:

1. The vector fields $V_i's$ are Lipschitz, C^∞, with Lipschitz derivatives, and analytic on the set $\{x : \|x - x_0\| \le C\}$ for some $C > 0$
2. The driving path $y : \mathbb{R}_{\ge 0} \to \mathbb{R}^{d+1}$ is β-Hölder continuous with $\beta > 1/2$.

Clearly, from by Theorem 2.1, under our assumptions, this differential equation admits a unique solution $(X_t)_{t\ge 0}$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function. By the change of variable formula, we have

$$f(X_t) = f(x_0) + \sum_{i=0}^{d} \int_0^t V_i f(X_s) dy_s^i.$$

Now, a new application of the change of variable formula to $V_i f(X_s)$ leads to

$$f(X_t) = f(x_0) + \sum_{i=0}^{d} V_i f(x_0) \int_0^t dy_s^i + \sum_{i,j=0}^{d} \int_0^t \int_0^s V_j V_i f(X_u) dy_u^j dy_s^i.$$

We can continue this procedure to get after N steps

$$f(X_t) = f(x_0) + \sum_{k=1}^{N} \sum_{I=(i_1,\cdots,i_k)} (V_{i_1} \cdots V_{i_k} f)(x_0) \int_{\triangle^k[0,t]} dy^I + R_N(t)$$

for some remainder term $R_N(t)$, where we used the notations:

1. $\triangle^k[0,t] = \{(t_1,\cdots,t_k) \in [0,t]^k, 0 \le t_1 \le t_2 \cdots \le t_k \le t\}$
2. If $I = (i_1,\cdots,i_k) \in \{0,1,\cdots,d\}^k$ is a word with length k,

$$\int_{\triangle^k[0,t]} dy^I = \int_{0\le t_1 \le t_2 \le \cdots \le t_k \le t} dy_{t_1}^{i_1} \cdots dy_{t_k}^{i_k}.$$

If we let $N \to +\infty$, we are led to the formal expansion formula:

$$f(X_t) = f(x_0) + \sum_{k=1}^{\infty} \sum_{I=(i_1,\cdots,i_k)} (V_{i_1} \cdots V_{i_k} f)(x_0) \int_{\triangle^k[0,t]} dy^I$$

Now let us denote $\pi^j(x) = x_j$, the j-th projection map. By using the previous expansion with $f = \pi^j$ we get

$$X_t^j = x_0^j + \sum_{k=1}^{\infty} \sum_{I=(i_1,\cdots,i_k)} (V_{i_1} \cdots V_{i_k} \pi^j)(x_0) \int_{\triangle^k[0,t]} dy^I$$

Therefore we have formally,

$$X_t = x_0 + \sum_{k=1}^{+\infty} g_k(t)$$

where

$$g_k^j(t) = \sum_{|I|=k} (V_{i_1} \cdots V_{i_k} \pi^j)(x_0) \int_{\triangle^k[0,t]} dy^I.$$

This leads to the following definition:

Definition 2.2. *The Taylor expansion associated with the differential equation (2.4) is defined as*

$$x_0 + \sum_{k=1}^{\infty} g_k(t)$$

where

$$g_k^j(t) = \sum_{|I|=k} P_I^j \int_{\triangle^k[0,t]} dy^I, \quad P_I^j = (V_{i_1} \cdots V_{i_k} \pi^j)(x_0).$$

Of course, at that point, the Taylor expansion is only a formal object in the sense that the convergence questions are not addressed yet. Our goal will be to provide natural assumptions ensuring the convergence of the expansion. Let us observe that the expansion can be written in a more compact way by using labeled rooted trees. For details on the combinatorics associated to these trees and the compact form of the expansion, we refer the interested reader to [11].

Another form of the expansion, which will be convenient for us, relies on the Taylor expansion of the vector fields and can be obtained as described below. We first introduce the multi-index notation

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

for $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. For a C^∞ function $f : \mathbb{R}^n \to \mathbb{R}$, we denote

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Now since the vector fields $V_i's$ in the differential equation (1.1) are C^∞, we can expand V_i at $x_0 \in \mathbb{R}^n$ as a formal Taylor series $\sum_{\alpha \in N^n} b_i^\alpha (x - x_0)^\alpha$, with

$$b_i^\alpha = \left(\frac{D^\alpha V_i^1(x_0)}{\alpha!}, \cdots \frac{D^\alpha V_i^n(x_0)}{\alpha!} \right) \in \mathbb{R}^n$$

Let now $C_i^m(h_1, \cdots, h_k)$ be the coefficient of z^m in $\sum_{\alpha \in \mathbb{N}^n} b_i^\alpha \left(h_1 z + \cdots + h_k z^k\right)^\alpha$.

Consider then the following rescaled differential equation, which depends on the parameter ϵ:

$$\begin{cases} dX_t^\epsilon = \sum_{i=0}^{d} \epsilon V_i(X_t^\epsilon) dy_t^i \\ X_0^\epsilon = x_0. \end{cases} \tag{2.5}$$

Assume now that X_t^ϵ admits an expansion in powers of ϵ that can be written $x_0 + \sum_{k=1}^{\infty} h_k(t)\epsilon^k$ for some h_k's. Heuristically, because of the analyticity of the $V_i's$, we can expand $V_i\left(x_0 + \sum_{k=1}^{\infty} h_k(t)\epsilon^k\right)$ at x_0. Therefore, on the right hand side of (2.5), we have:

$$\sum_{i=0}^{d} \epsilon V_i(X_t^\epsilon) dy_t^i = \sum_{i=0}^{d} \epsilon \left(V_i(x_0) + C_i^1(h_1)\epsilon + C_i^2\left(h_1, h_2\right)\epsilon^2 + \cdots \right) dy_t^i$$
$$= \left(\left(\sum_{i=0}^{d} V_i(x_0) dy_t^i \right) \epsilon + \left(\sum_{i=0}^{d} C_i^1(h_1) dy_t^i \right) \epsilon^2 + \cdots \right).$$

Thus, by identifying this expression with the left side of the equation we are let with the conclusion that the $h_k(t)'s$ need to satisfy the following inductive system of equations:

$$\begin{cases} dh_1(t) = \sum_{i=0}^{d} V_i(x_0) dy_t^i \\ dh_2(t) = \sum_{i=0}^{d} C_i^1\left(h_1(t)\right) dy_t^i \\ . \\ . \\ . \\ dh_k(t) = \sum_{i=0}^{d} C_i^{k-1}\left(h_1(t), \cdots, h_{k-1}(t)\right) dy_t^i. \end{cases} \tag{2.6}$$

If we let $\epsilon = 1$, we therefore see that $x_0 + \sum_{k=1}^{\infty} h_k(t)$ formally solves the equation (2.4) provided that the h_k's satisfy the above inductive system (2.6).

We have the following result:

Proposition 2.3. *Define h_k inductively by the system (2.6), then for every $k \in \mathbb{N}$ and $t \geq 0$ we have $h_k(t) = g_k(t)$.*

Proof. First, let us observe that for every $t \geq 0$, $\epsilon \to X_t^\epsilon$ is a smooth function, indeed the equation

$$\begin{cases} dX_t^\epsilon = \sum_{i=0}^{d} \epsilon V_i(X_t^\epsilon) dy_t^i \\ X_0^\epsilon = x_0. \end{cases}$$

can rewritten as

$$\begin{cases} dY_t = \sum_{i=0}^{d} \tilde{V}_i(Y_t) dy_t^i \\ Y_0 = (\epsilon, x_0), \end{cases}$$

here $Y_t = (\epsilon, X_t^\epsilon)$ is now valued in $\mathbb{R} \times \mathbb{R}^n$ and $\tilde{V}_i(\epsilon, x) = (0, \epsilon V_i(x))$. The problem of the smoothness of X_t^ϵ with respect to ϵ is thus reduced to the problem of the smoothness of the solution of an equation driven by Hölder paths with respect to its initial condition. Theorems that apply in this situation may be found in [7] and [14].

From the definition of $h_k(t)$ which is given by the system (2.6), it is then clear by induction that

$$h_k(t) = \frac{1}{k!} \frac{d^k X_t^\epsilon}{d\epsilon^k}(0).$$

On the other hand, as we have seen before, the repeated application of the change of variable formula to the smooth function $f(x) = \pi^j(x)$, gives

$$X_t^{\epsilon,j} = x_0^j + \sum_{k=1}^{N} \epsilon^k \sum_{I=(i_1,\cdots,i_k)} (V_{i_1}\cdots V_{i_k}\pi^j)(x_0) \int_{\triangle^k[0,t]} dy^I + \epsilon^{N+1} R_N^{\epsilon,j}(t),$$

where

$$R_N^{\epsilon,j}(t) = \sum_{I=(i_1,\cdots,i_{N+1})} \int_{\Delta^{N+1}[0,t]} (V_{i_1}\cdots V_{i_{N+1}}\pi^j)(X_{s_1}^{\epsilon}) dy_{s_1}^{i_1}\cdots dy_{s_{N+1}}^{i_{N+1}}.$$

Therefore we have

$$X_t^{\epsilon} = x_0 + \sum_{k=1}^{N} g_k(t)\epsilon^k + \epsilon^{N+1} R_N^{\epsilon}(t).$$

Computing $\frac{d^k X_t^\epsilon}{d\epsilon^k}(0)$ by using the previous expression immediately gives $h_k(t) = g_k(t)$. □

2.3 Convergence of the Taylor expansion

We now address the convergence questions related to the Taylor expansion. We use the same assumptions and the same notations as in the previous section. In particular we remind that the vector fields $V_i's$ are Lipschitz, C^∞ with Lipschitz derivatives, and analytic on the set $\{x : \|x - x_0\| \le C\}$ for some $C > 0$.

2.3.1 A general result

We first have the following general result.

Theorem 2.4. *Let $x_0 + \sum_{k=1}^{\infty} g_k(t)$ be the Taylor expansion associated with the equation (2.4). There exists $T > 0$ such that for $0 \le t < T$ the series*

$$\sum_{k=1}^{\infty} \|g_k(t)\|$$

is convergent and

$$X_t = x_0 + \sum_{k=1}^{\infty} g_k(t).$$

Proof. Let us fix $\rho > 0$. For $z \in \mathbb{C}$, $|z| < \rho$, we consider the complex differential equation

$$\begin{cases} dX_t^z = \sum_{i=0}^{d} zV_i(X_t^z)dy_t^i \\ X_0^z = x_0, \end{cases}$$

which, by analyticity of the V_i's, is well defined at least up to the strictly positive time

$$T(\rho) = \inf_{z,|z|<\rho} \inf\{t \ge 0, X_t^z \notin B(x_0, C)\}.$$

Let us recall that if we denote $z = x + iy$, then we can define the following differential operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right),$$

and the equation $\dfrac{\partial f}{\partial \bar{z}} = 0$ is equivalent to Cauchy-Riemann equations, that is equivalent to the fact that f is analytic.

As we have seen before, for a fixed $t \geq 0$, the map $z \to X_t^z$ is seen to be smooth. We claim the map $z \to X_t^z$ is even analytic. Indeed, differentiating with respect to $\bar{z}$ the integral expression

$$X_t^z = x_0 + \int_0^t \sum_{i=0}^d z V_i(X_s^z) dy_s^i,$$

we get:

$$\frac{\partial X_t^z}{\partial \bar{z}} = \int_0^t \sum_{i=0}^d z D V_i(X_s^z) \frac{\partial X_t^z}{\partial \bar{z}} dy_s^i.$$

This gives $\frac{\partial X_t^z}{\partial \bar{z}} = 0$ by uniqueness of solutions of linear equations (See Proposition 2 in [14]). As a conclusion, $z \to X_t^z$ is analytic on the disc $|z| < \rho$. On the other hand, $x_0 + \sum_{k=1}^{\infty} z^k g_k(t)$ is precisely the Taylor expansion of X_t^z. Choosing then $\rho > 1$, finishes the proof. □

2.3.2 Quantitative bounds

Theorem 2.4 is very general but gives few information concerning the radius of convergence or the speed of convergence of the Taylor expansion

$$x_0 + \sum_{k=1}^{\infty} g_k(t).$$

In this section, we shall be interested in more quantitative bounds.

Lemma 2.5. *Let* $x_0 + \sum_{k=1}^{\infty} g_k(t)$ *be the Taylor expansion associated with the equation (2.4). For* $r > 1$*, we define*

$$T_C(r) = \inf \left(t, \ \sum_{k=1}^{\infty} r^k \|g_k(t)\| \geq C \right).$$

For every $t \leq T_C(r)$ *we have*

$$X_t = x_0 + \sum_{k=1}^{\infty} g_k(t).$$

Remark 2.6. *By using a similar argument as in the proof of Theorem 2.4, it is seen that in the previous theorem, we always have* $T_C(r) > 0$.

Proof. Consider the following functions:

$$G_i(\epsilon) = V_i \left(x_0 + \sum_{k=1}^{\infty} \epsilon^k g_k(t) \right)$$

Then by the definition of $T_C(r)$, if $t < T_C(r)$, $G_i(\epsilon)$ is analytic on the disc $\{\epsilon \in \mathbb{C} : \ \|\epsilon\| < r\}$. Hence for any $r' < r$ and if $\|\epsilon\| < r'$, we have:

$$\left\| G_i(\epsilon) - (V_i(x_0) + \sum_{k=1}^{m-1} C_i^k(g_1(t), g_2(t), \cdots, g_k(t)))\epsilon^k \right\| \tag{2.7}$$

$$\leq M_{r'}^i \sum_{k=m}^{\infty} \left(\frac{\epsilon}{r'}\right)^m \tag{2.8}$$

$$\leq K^i R_m \tag{2.9}$$

where $M^i_{r'} = \sup_{\|\epsilon\|\leq r'} G_i(\epsilon) \leq \sup_{\|x\|\leq C} \|V_i(x+x_0)\| = K^i$ And R_m denotes the tail of the convergent geometric series $\sum_{k=1}^{\infty}\left(\frac{\epsilon}{r'}\right)^k$. We know that $R_m \to 0, as \;\; m \to +\infty$. Now by the definition of $g_k(t)$ we write:

$$\begin{aligned}
\sum_{k=1}^{m}\epsilon^k g_k(t) &= \sum_{i=0}^{d}\sum_{k=1}^{m}\epsilon^k \int_0^t C_i^{k-1}\left(g_1(s),\cdots,g_{k-1}(s)\right)dy_s^i \\
&= \sum_{i=0}^{d}\epsilon\sum_{k=1}^{m}\int_0^t \left(\epsilon^{k-1}C_i^{k-1}\left(g_1(s),\cdots,g_{k-1}(s)\right)\right)dy_s^i \\
&= \sum_{i=0}^{d}\epsilon\int_0^t\left(V_i(x_0)+\sum_{k=1}^{m-1}\epsilon^k C_i^k\left(g_1(s),\cdots,g_k(s)\right)\right)dy_s^i
\end{aligned}$$

Since when $t < T_C(r)$, the series $\sum_{k=1}^{\infty}\epsilon^k g_k(t) \leq C$,we replace t by $t\wedge T_C(r)$ in the previous equation:

$$\sum_{k=1}^{m}\epsilon^k g_k\left(t\wedge T_C(r)\right) = \sum_{i=0}^{d}\epsilon\int_0^{t\wedge T_C(r)}\left(V_i(x_0)+\sum_{k=1}^{m-1}\epsilon^k C_i^k\left(g_1(s),\cdots,g_k(s)\right)\right)dy_s^i \tag{2.10}$$

with the estimate (2.9), we have for $\epsilon < r'$,

$$\left\|G_i(\epsilon)-\left(V_i(x_0)+\sum_{k=1}^{m-1}C_i^k\left(g_1(t),\cdots,g_k(t)\right)\epsilon^k\right)\right\| \leq K^i R_m. \tag{2.11}$$

Therefore, since $r > 1$ we can choose $1 < r' < r$ and let $m \to \infty$, so that the right hand side of (2.11) tends to 0. Therefore, by letting $m \to \infty$ in equation (2.10) we have:

$$\begin{aligned}
x_0+\sum_{k=1}^{\infty}\epsilon^k g_k(t\wedge T_C(r)) &= x_0+\sum_{i=0}^{d}\int_0^{t\wedge T_C(r)}\epsilon G_i(t)dy_s^i \\
&= x_0+\sum_{i=0}^{d}\int_0^{t\wedge T_C(r)}\epsilon V_i\left(x_0+\sum_{k=1}^{\infty}\epsilon^k g_k(s)\right)dy_s^i
\end{aligned}$$

And the result follows when we take $\epsilon = 1$ □

The crucial estimates we need to control the speed of convergence of the Taylor expansion are provided by the following proposition.

Proposition 2.7. *Let β be the Hölder exponent of y and α be such that $1-\beta < \alpha < \frac{1}{2}$ and fix $T > 0$. For $0 < t \leq T$, we have*

$$\left|\int_{0\leq t_1\leq\ldots\leq t_n\leq t} dy_{t_1}^{i_1}\cdots dy_{t_n}^{i_n}\right| \leq \frac{\Gamma(1-2\alpha)}{\Gamma(n(1-2\alpha))}C_\alpha(T)^{n-1}\Lambda_\alpha(T,y)^{n-1}\|y\|_{\alpha,T,\infty}$$

where

$$\begin{aligned}
\Lambda_\alpha(T,y) &:= \frac{1}{\Gamma(1-\alpha)}\sup_{0<s<t<T}\|(D_{t-}^{1-\alpha}y_{t-})(s)\| \\
C_\alpha(T) &= \left(\frac{1}{\alpha(1-\alpha)}+T^\alpha\right)(1+T^\alpha)\,\Gamma(1-2\alpha)T^{1-2\alpha} \\
\|y\|_{\alpha,T,\infty} &= \sup_{0\leq t\leq T}\left(\|y(t)\|+\int_0^t\frac{\|y(t)-y(s)\|}{(t-s)^{1+\alpha}}ds\right).
\end{aligned}$$

Proof. Since $y:\mathbb{R}_{\geq 0}\to\mathbb{R}^{d+1}$ is a Hölder path with exponent $\beta>1/2$, and since $\alpha+\beta>1$, we have $y_t^i\in W_T^{1-\alpha,\infty}(0,T)$ for $i=0,1,2\cdots,d$. For $t\leq T$, we denote $I_k(t)=\int_{0\leq t_1\leq\ldots\leq t_k\leq t} dy_{t_1}^{i_1}\cdots dy_{t_k}^{i_k}$, $k=1,\cdots,n$. Clearly, $I_k(t)\in W_0^{\alpha,1}(0,T)$ for $k=1,2,\cdots,n$. Therefore, Proposition 4.1 in Nualart-Rǎşcanu ([12]) gives

$$\begin{aligned}&|I_{k+1}(t)|+\int_0^t\frac{|I_{k+1}(t)-I_{k+1}(s)|}{(t-s)^{1+\alpha}}ds\\ \leq&\Lambda_\alpha(T,y)\left(\frac{1}{\alpha(1-\alpha)}+T^\alpha\right)\int_0^t\left((t-r_k)^{-2\alpha}+r_k^{-\alpha}\right)\left(|I_k(r_k)|+\int_0^{r_k}\frac{|I_k(r_k)-I_k(s)|}{(r_k-s)^{1+\alpha}}ds\right)dr_k.\end{aligned}$$

Now notice that $0<r<t<T$ and $\alpha>0$ we have:

$$r^{2\alpha}+r^\alpha(t-r)^{2\alpha}\leq t^{2\alpha}+T^\alpha t^{2\alpha}$$

Multiplying $r^{-2\alpha}(t-r)^{-2\alpha}$ on both sides, we obtain:

$$(t-r)^{-2\alpha}+r^{-\alpha}\leq(1+T^\alpha)\,t^{2\alpha}(t-r)^{-2\alpha}r^{-2\alpha},$$

Now with the basic estimate above, we get

$$\begin{aligned}&|I_{k+1}(t)|+\int_0^t\frac{|I_{k+1}(t)-I_{k+1}(s)|}{(t-s)^{1+\alpha}}ds\\ \leq&\Lambda_\alpha(T,y)\left(\frac{1}{\alpha(1-\alpha)}+T^\alpha\right)(1+T^\alpha)\int_0^t\left(t^{2\alpha}(t-r_k)^{-2\alpha}r_k^{-2\alpha}\right)\left(|I_k(r_k)|+\int_0^{r_k}\frac{|I_k(r_k)-I_k(s)|}{(r_k-s)^{1+\alpha}}ds\right)dr_k.\end{aligned}$$

Iterating this inequality k times, we get:

$$\begin{aligned}&|I_{k+1}(t)|+\int_0^t\frac{|I_{k+1}(t)-I_{k+1}(s)|}{(t-s)^{1+\alpha}}ds\\ &\leq\Lambda_\alpha(T,y)^k\left(\frac{1}{\alpha(1-\alpha)}+T^\alpha\right)^k(1+T^\alpha)^k\|y\|_{\alpha,T,\infty}\\ &\times\int_{0\leq r_1\leq r_2\leq\cdots\leq r_k\leq t}t^{2\alpha}\left(\prod_{l=0}^{k-1}(r_{k+1-l}-r_{k-l})^{-2\alpha}\right)r_1^{-2\alpha}dr_1\cdots dr_k\end{aligned}$$

where we denote $r_{k+1}=t$. Note that by the substitution $u=\dfrac{r_1}{r_2}$, we have:

$$\int_0^{r_2}(r_2-r_1)^{-2\alpha}r_1^{-2\alpha}dr_1=r_2^{1-4\alpha}\int_0^1(1-u)^{-2\alpha}u^{-2\alpha}du=r_2^{1-4\alpha}B(1-2\alpha,1-2\alpha)$$

where

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1(1-u)^{x-1}u^{y-1}du$$

is the Beta function.
Then plug $r_2^{1-4\alpha}B(1-2\alpha,1-2\alpha)$ into the previous integrand and iterate this process, we get:

$$\begin{aligned}&|I_{k+1}(t)|+\int_0^t\frac{|I_{k+1}(t)-I_{k+1}(s)|}{(t-s)^{1+\alpha}}ds\\ &\leq\Lambda_\alpha(T,y)^k\left(\frac{1}{\alpha(1-\alpha)}+T^\alpha\right)^k(1+T^\alpha)^k\|y\|_{\alpha,T,\infty}(T^{1-2\alpha})^k\prod_{i=1}^kB(1-2\alpha,i(1-2\alpha))\\ &=\Lambda_\alpha(T,y)^k\left(\frac{1}{\alpha(1-\alpha)}+T^\alpha\right)^k(1+T^\alpha)^k\|y\|_{\alpha,T,\infty}(T^{1-2\alpha})^k\prod_{i=1}^k\frac{\Gamma(1-2\alpha)\Gamma(i(1-2\alpha))}{\Gamma((i+1)(1-2\alpha))}\\ &=\frac{\Gamma(1-2\alpha)}{\Gamma(k+1)(1-2\alpha)}\Lambda_\alpha(T,y)^k\left(\frac{1}{\alpha(1-\alpha)}+T^\alpha\right)^k(1+T^\alpha)^k\Gamma(1-2\alpha)^k(T^{1-2\alpha})^k\|y\|_{\alpha,T,\infty}\end{aligned}$$

□

We are now in position to give an estimate of the speed of convergence of the stochastic Taylor expansion. The following lemma is taylor made for that purpose:

Lemma 2.8. *Let α such that $1-\beta<\alpha<\frac{1}{2}$ and $\gamma<1-2\alpha$. There exists a constant $K_{\alpha,\gamma}>0$ such that for every $N\geq 0$ and $x\geq 0$,*

$$\sum_{k=N+1}^{+\infty}\frac{\Gamma(k\gamma)}{\Gamma(k(1-2\alpha))}x^{k-1}\leq K_{\alpha,\gamma}\begin{cases}e^{2x^{\frac{1}{1-2\alpha-\gamma}}}, & \text{if } N=0\\ \frac{x^N e^{2x^{\frac{1}{1-2\alpha-\gamma}}}}{\Gamma((1-2\alpha-\gamma)N)}, & \text{if } N\geq 1\end{cases}$$

Proof. We make the proof for $N\geq 1$ and let the reader adapt the argument when $N=0$. We have

$$\begin{aligned}\sum_{k=N+1}^{+\infty}\frac{\Gamma(k\gamma)}{\Gamma(k(1-2\alpha))}x^{k-1}&=x^N\sum_{k=0}^{+\infty}\frac{\Gamma((k+N+1)\gamma)}{\Gamma((k+N+1)(1-2\alpha))}x^k\\&=\frac{x^N}{\Gamma((1-2\alpha-\gamma)N)}\sum_{k=0}^{+\infty}\frac{\Gamma((k+N+1)\gamma)\Gamma((1-2\alpha-\gamma)N)}{\Gamma((k+N+1)(1-2\alpha))}x^k\\&\leq C_{\alpha,\gamma}\frac{x^N}{\Gamma((1-2\alpha-\gamma)N)}\sum_{k=0}^{+\infty}\frac{x^k}{\Gamma((k+1)(1-2\alpha-\gamma))}\end{aligned}$$

From Lemma 7.7 in [12], we have for every $x\geq 0$.

$$\sum_{k=0}^{\infty}\frac{x^k}{\Gamma((1+k)(1-2\alpha-\gamma))}\leq\frac{4e^2}{1-2\alpha-\gamma}e^{2x^{\frac{1}{1-2\alpha-\gamma}}}.$$

This concludes the proof. □

We can now give convenient assumptions to decide the speed of convergence of the Taylor expansion. We remind the reader that if $I=(i_1,\cdots,i_k)$, we denote

$$P_I^j=(V_{i_1}\cdots V_{i_k}\pi^j)(x_0).$$

We also use the notations introduced in Proposition 2.7.

Theorem 2.9. *Let α such that $1-\beta<\alpha<\frac{1}{2}$. Let us assume that there exist $M>0$ and $0<\gamma<1-2\alpha$ such that for every word $I\in\{0,\cdots,d\}^k$*

$$\|P_I\|\leq\Gamma(\gamma|I|)M^{|I|}. \tag{2.12}$$

Then, for every $r>1$, $T_C(r)>0$ and there exists a constant $K_{\alpha,\gamma,M,d}>0$ depending only on the subscript variables such that for every $0\leq t<T_C(r)$ and $N\geq 1$,

$$\left\|X_t-\left(x_0+\sum_{k=1}^{N}g_k(t)\right)\right\|\leq K_{\alpha,\gamma,M,d}\|y\|_{\alpha,t,\infty}\frac{(M(d+1)\Lambda_\alpha(t,y)C_\alpha(t))^N}{\Gamma((1-2\alpha-\gamma)N)}e^{2(M(d+1)\Lambda_\alpha(t,y)C_\alpha(t))^{\frac{1}{1-2\alpha-\gamma}}} \tag{2.13}$$

In particular for every $0\leq t<T_C(r)$,

$$X_t=x_0+\sum_{k=1}^{+\infty}g_k(t). \tag{2.14}$$

Moreover if $C=+\infty$, that is if the V_i's are analytic on $\mathbb{R}^n$, then (2.13) and (2.14) hold for every $t\geq 0$.

Proof. Let us fix $T > 0$ and $r > 1$. For every $t \in [0, T]$, by Proposition 2.7 we have:

$$\begin{aligned}
&\sum_{k=1}^{\infty} r^k \|g_k(t)\| \\
&\le \sum_{k=1}^{\infty} r^k \sum_{|I|=k} \|P_I\| \left| \int_{\triangle^k[0,t]} dy^I \right| \\
&\le \sum_{k=1}^{\infty} r^k (d+1)^k M^k \Gamma(k\gamma) \Lambda_\alpha(T,y)^{k-1} C_\alpha(T)^{k-1} \|y\|_{\alpha,T,\infty} \frac{\Gamma(1-2\alpha)}{\Gamma(k(1-2\alpha))} \\
&\le \|y\|_{\alpha,T,\infty} r(d+1) M \Gamma(1-2\alpha) \sum_{k=1}^{\infty} \left(r(d+1)C_\alpha(T)\Lambda_\alpha(T,y)\right)^{k-1} \frac{\Gamma(k\gamma)}{\Gamma(k(1-2\alpha))} \\
&= \|y\|_{\alpha,T,\infty} r(d+1) M \Gamma(1-2\alpha) \sum_{k=1}^{\infty} \frac{\left(r(d+1)C_\alpha(T)\Lambda_\alpha(T,y)\right)^{k-1}}{\Gamma(k(1-2\alpha-\gamma))} B(k\gamma, k(1-2\alpha-\gamma)) \\
&\le B(\gamma, 1-2\alpha-\gamma) \|y\|_{\alpha,T,\infty} r(d+1) M \Gamma(1-2\alpha) \sum_{k=1}^{\infty} \frac{\left(r(d+1)C_\alpha(T)\Lambda_\alpha(T,y)\right)^{k-1}}{\Gamma(k(1-2\alpha-\gamma))}
\end{aligned}$$

Now note that by our assumption, $1 - 2\alpha - \gamma > 0$, then the series on the right hand side of our inequality converges. This implies that

$$T_C(r) = \inf \left(t, \ \sum_{k=1}^{\infty} r^k \|g_k(t)\| \ge C \right) > 0.$$

We conclude from Lemma 2.5 that for every $0 \le t < T_C(r)$,

$$X_t = x_0 + \sum_{k=1}^{+\infty} g_k(t).$$

If moreover $C = +\infty$, that is the radius of convergence of the Taylor expansion of the V_i's is $+\infty$, the previous equality holds for every $t \ge 0$ because, $\lim_{C \to +\infty} T_C(r) = +\infty$.

We then get for $0 \le t < T_C(r)$, thanks to Lemma 2.8:

$$\begin{aligned}
&\left\| X_t - \left(x_0 + \sum_{k=1}^{N} g_k(t) \right) \right\| \\
=& \left\| \sum_{k=N+1}^{\infty} g_k(t) \right\| \\
\le& \sum_{k=N+1}^{\infty} \sum_{|I|=k} \|P_I\| \left| \int_{\triangle^k[0,t]} dy^I \right| \\
\le& \sum_{k=N+1}^{\infty} (d+1)^k M^k \Gamma(k\gamma) \Lambda_\alpha(t,y)^{k-1} C_\alpha(T)^{k-1} \|y\|_{\alpha,t,\infty} \frac{\Gamma(1-2\alpha)}{\Gamma(k(1-2\alpha))} \\
\le& K_{\alpha,\gamma,M,d} \|y\|_{\alpha,t,\infty} \frac{(M(d+1)\Lambda_\alpha(t,y)C_\alpha(t))^N}{\Gamma((1-2\alpha-\gamma)N)} e^{2(M(d+1)\Lambda_\alpha(t,y)C_\alpha(t))^{\frac{1}{1-2\alpha-\gamma}}}
\end{aligned}$$

□

2.4 The Lie group example

In this section, we will see an application of our main results in the context of Lie group. Let us denote by $\mathbb{G}$ a connected Lie group. For any $g \in \mathbb{G}$, we have the left and right translations L_g and R_g, which are the diffeomorphisms on $\mathbb{G}$, defined by $L_g(h) = gh$ and $R_g(h) = hg$ for any $h \in \mathbb{G}$. We say that a vector field V is left-invariant on $\mathbb{G}$ if for any $g \in \mathbb{G}$,

$$(L_g)_*(V) = V$$

Let us denote by $\mathfrak{g}$ the Lie algebra of $\mathbb{G}$, that is the set of left-invariant vector fields on $\mathbb{G}$ and by $T_e\mathbb{G}$ the tangent space of $\mathbb{G}$ at identity. We know that the map $V \to V(e)$ provides a linear isomorphism between $\mathfrak{g}$ and $T_e\mathbb{G}$. Now let $V_1, \cdots, V_d \in \mathfrak{g}$ be left-invariant vector fields and let us consider the following differential equation on Lie group $\mathbb{G}$,

$$\begin{cases} dX_t = \sum_{i=0}^{d} V_i(X_t) dy_t^i \\ X_0 = e \end{cases} \tag{2.15}$$

where $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^{d+1}$ is a β-Hölder path with $\beta > 1/2$. The vector fields are not necessarily globally Lipschitz, so that we are not exactly in the previous framework. However, the vector fields V_i's are smooth and analytic and the previous equation always admits a solution up to a possible positive explosion time τ.

Remark 2.10. *If $\mathbb{G} = GL(n, \mathbb{R})$, we know the associated Lie algebra is $\mathfrak{g} = M(n, \mathbb{R})$. In this case,the differential equation (2.15) is a linear equation and the Taylor expansion associated with (2.15) is convergent for every $t \geq 0$. The criterion (2.12) is satisfied with $\|P_I\| \leq M^{|I|}$ in the convergence Theorem 2.9.*

Before we state our main theorem in that context, let us introduce some notations (see [3] and [5] for further details).

- If $I = (i_1, \cdots, i_k)$ we denote by V_I the Lie commutator defined by

$$V_I = [V_{i_1}, [V_{i_2}, ..., [V_{i_{k-1}}, V_{i_k}]...].$$

- $\mathcal{S}_k$ is the set of the permutations of $\{0, ..., k\}$;
- If $\sigma \in \mathcal{S}_k$, $e(\sigma)$ is the cardinality of the set

$$\{j \in \{1, ..., k-1\}, \sigma(j) > \sigma(j+1)\},$$

-

$$\Lambda_I(y)_t = \sum_{\sigma \in \mathcal{S}_k} \frac{(-1)^{e(\sigma)}}{k^2 \begin{pmatrix} k-1 \\ e(\sigma) \end{pmatrix}} \int_{0 \leq t_1 \leq \ldots \leq t_k \leq t} dy_{t_1}^{\sigma^{-1}(i_1)} \cdots dy_{t_k}^{\sigma^{-1}(i_k)}, \quad t \geq 0.$$

In this section, our main theorem, which extends to the Hölder case results by Magnus [10] and Strichartz [16] is the following:

Theorem 2.11. *There exists $T > 0$ such that for $0 \leq t \leq T$,*

$$X_t = \exp\left(\sum_{k=1}^{\infty} \sum_{(i_1, \ldots i_k)} \Lambda_I(y)_t V_I \right),$$

is the solution of the equation 2.15.

Proof. We first remind some basic notations and results on Lie group theory. For more details, we refer the interested reader to Section 1.5 in [8]. Given a Lie group $\mathbb{G}$ and associated Lie algebra $\mathfrak{g}$, we denote by $\mathbf{ad}$ the map:

$$(\mathbf{ad}x)(z) = [x,z] = xz - zx, \ for\ \forall z \in \mathfrak{g}$$

Consider the set $\mathfrak{g}_e = \{x \in T_e\mathbb{G}\ ,\ s(x) \cap 2\pi\mathbb{Z} \subset \{0\}\}$, where $s(x)$ denotes the spectrum of $\mathbf{ad}x$. It is known that $\mathfrak{g}_e$ is an open set of $T_e\mathbb{G}$ and $\forall x \in \mathfrak{g}_e$, $\left(\frac{I-e^{-\mathbf{ad}x}}{\mathbf{ad}x}\right)$ is invertible, where $\left(\frac{I-e^{-\mathbf{ad}x}}{\mathbf{ad}x}\right)$ is defined as the power series $\sum_{k=0}^{\infty} \frac{1}{(k+1)!}(-\mathbf{ad}x)^k$. Now we define vector fields U_i on $\mathfrak{g}_e$ by $U_i(x) = \left(\frac{\mathbf{ad}x}{I-e^{-\mathbf{ad}x}}\right)(V_i(e))$ where $\left(\frac{\mathbf{ad}x}{I-e^{-\mathbf{ad}x}}\right)$ denotes the inverse of $\left(\frac{I-e^{-\mathbf{ad}x}}{\mathbf{ad}x}\right)$. We denote by $T_x \exp : \mathfrak{g} \to T_{\exp x}\mathbb{G}$ the tangent map of $\exp$. Moreover, when $x \in \mathfrak{g}_e$, we have:

$$T_x(\exp) = L(\exp x)_* \circ \left(\frac{\mathbf{ad}x}{I-e^{-\mathbf{ad}x}}\right)$$

Now we claim that for $\forall x \in \mathfrak{g}_e$ we have:

$$(T_x \exp)[U_i(x)] = V_i(\exp x).$$

In fact, since $V_i's$ are left-invariant vector field on $\mathbb{G}$, we have

$$\begin{aligned}
(T_x \exp)[U_i(x)] =& (L(\exp x))_* \circ \left(\frac{I-e^{-\mathbf{ad}x}}{\mathbf{ad}x}\right)[U_i(x)] \\
=& (L(\exp x))_* \circ \left(\frac{I-e^{-\mathbf{ad}x}}{\mathbf{ad}x}\right)\left(\frac{\mathbf{ad}x}{I-e^{-\mathbf{ad}x}}\right)(V_i(e)) \\
=& \left(L(\exp x)\right)_* (V_i(e)) \\
=& V_i(\exp x)
\end{aligned}$$

Now if we consider the differential equation on $\mathfrak{g}_e$:

$$\begin{cases} dY_t = \sum_{i=1}^{d} U_i(Y_t)dy_t^i \\ Y_0 = 0 \end{cases} \tag{2.16}$$

Without addressing the convergence questions, it is proved in [3] and [16] that the Taylor expansion $\sum_{k=1}^{\infty} g_k(t)$ of the equation defining Y_t admits the following expression:

$$g_k(t) = \sum_{(i_1,\dots i_k)} \Lambda_I(y)_t V_I.$$

Moreover, we know that $U_i(x)$ is analytic on $\mathfrak{g}_e$. Therefore from Theorem 2.4, there exists a positive time T such that for $0 \le t < T$,

$$Y_t = \sum_{k=1}^{\infty} g_k(t),$$

Now we claim that by the chain rule, we have $X_t = \exp(Y_t)$ when $t < T$. In fact, we easily compute:

$$\begin{aligned}
dX_t =&(T_{Y_t}\exp)dY_t \\
=&(T_{Y_t}\exp)[\sum_{i=1}^{d}U_i(Y_t)]dy_t^i \\
=&\sum_{i=0}^{d}T_{Y_t}\exp[U_i(Y_t)]dy_t^i \\
=&\sum_{i=0}^{d}V_i(\exp(Y_t))dy_t^i \\
=&\sum_{i=0}^{d}V_i(X_t)dy_t^i
\end{aligned}$$

where $X_0 = \exp(Y_0) = e$. □

Remark 2.12. *It seems difficult to obtain precise universal bounds on the radius of convergence of $\sum_{k=1}^{\infty}\sum_{(i_1,\dots i_k)}\Lambda_I(y)_t V_I$ as it is the case for the radius of convergence of the Magnus expansion of absolutely continuous paths. Indeed, rough estimates (see [16]) lead to*

$$\sum_{\sigma\in\mathcal{S}_k}\frac{1}{k^2\begin{pmatrix} k-1 \\ e(\sigma)\end{pmatrix}} \le \frac{C}{2^k}k!\sqrt{k},$$

for some constant $C>0$. We get therefore the bound

$$|\Lambda_I(y)_t| \le \frac{C}{2^k}k!\sqrt{k}\frac{\Gamma(1-2\alpha)}{\Gamma(k(1-2\alpha))}C_\alpha(t)^{k-1}\Lambda_\alpha(t,y)^{k-1}\|y\|_{\alpha,\infty},$$

which unfortunately gives a convergence for the series for $\alpha=0$, which corresponds to the case where y is absolutely continuous.

3 Taylor expansion for stochastic differential equations driven by fractional Brownian motions

We now turn to the second part of this work, which is of probabilistic nature, and first applies our result to stochastic differential equations driven by factional Brownian motion paths.

Let us recall that a d-dimensional fractional Brownian motion with Hurst parameter $H\in(0,1)$ is a Gaussian process

$$B_t = (B_t^1,\cdots,B_t^d),\ t\ge 0,$$

where $B^1,\cdots,B^d$ are d independent centered Gaussian processes with covariance function

$$R(t,s) = \frac{1}{2}\left(s^{2H}+t^{2H}-|t-s|^{2H}\right).$$

It can be shown that such a process admits a continuous version whose paths have finite p variation for $1/p<H$. Let us observe that for $H=\frac{1}{2}$, B is a standard Brownian motion.

In this part, we will consider the following stochastic differential equation:

$$\begin{cases} dX_t = \displaystyle\sum_{i=0}^{d}V_i(X_t)\,dB_t^i \\ X_0 = x_0 \end{cases} \tag{3.1}$$

where the V_i's are Lipschitz C^∞ vector fields on $\mathbb{R}^n$ with Lipschitz derivatives and B_t is a d-dimensional fractional Brownian motion with Hurst parameter $H > 1/2$ with the convention that $B_0 = t$.

Let us remind that when $H > 1/2$, B_t is almost surely Hölder path with Hölder exponent strictly less than H. In particular, we can pick $1/2 < \beta < H$ such that B_t is a β-Hölder path. Hence, we can define the Stochastic Taylor expansion associated with (3.1) just as in the deterministic case by

$$x_0 + \sum_{k=1}^{\infty} g_k(t),$$

where

$$g_k^j(t) = \sum_{|I|=k} P_I^j \int_{\triangle^k[0,t]} dB^I, \quad P_I^j = (V_{i_1} \cdots V_{i_k} \pi^j)(x_0)$$

and

$$\int_{\triangle^k[0,t]} dB^I = \int_{0<t_1<t_2\cdots<t_k<t} dB_{t_1}^{i_1} \cdots dB_{t_k}^{i_k}.$$

As in the previous section, we assume throughout this section that the V_i's are Lipschitz C^∞ vector fields on $\mathbb{R}^n$ with Lipschitz derivatives and analytic on the set $\{x : \|x - x_0\| \le C\}$ for some $C > 0$.

Now let us state the following convergence theorem for solution to the stochastic differential equation (3.1), which is an immediate consequence of Theorem 2.4.

Theorem 3.1. *There exists an almost surely positive stopping time T, such that for $0 \le t < T$ the series*

$$\|x_0\| + \sum_{i=1}^{\infty} \|g_k(t)\|$$

almost surely converges and almost surely

$$X_t = x_0 + \sum_{k=1}^{+\infty} g_k(t).$$

3.1 Quantitative deterministic bounds

In this part, from a direct application of the deterministic Theorem 2.9, we obtain a convergence result for the stochastic Taylor expansion associated with the SDE (3.1). Moreover, we observe that the error estimate provided by the theorem 2.9 is in L^p for every $p \ge 1$, provided that the driven signal is fractional Brownian motion with $H > 1/2$. Indeed, (see for instance Lemma 7.4 in Nualart-Răşcanu [12]), for any $\beta < H$ and $T > 0$ there exists a positive random variable $\eta_{\beta,T}$ such that $\mathbb{E}(|\eta_{\beta,T}|^p) < \infty$ for all $p \ge 1$ and for all $s, t \in [0, T]$

$$\|B(t) - B(s)\| \le \eta_{\beta,T} |t - s|^\beta \quad a.s.$$

We now have,

$$\begin{aligned}
\|B\|_{\alpha,T,\infty} &= \sup_{0\le t\le T} \left(\|B_t\| + \int_0^t \frac{\|B_t - B_s\|}{(t-s)^{1+\alpha}} ds \right) \\
&\le \sup_{0\le t\le T} \left(\eta_{\beta,T} t^\beta + \int_0^t \eta_{\beta,T} (t-s)^{\beta-1-\alpha} ds \right) \\
&\le \left(T^\beta + \frac{1}{\beta - \alpha} T^{\beta-\alpha} \right) \eta_{\beta,T}
\end{aligned}$$

So, $\mathbb{E}\|B\|^p_{\alpha,T,\infty} < \infty$ follows from the fact that $E(|\eta_{\beta,T}|^p) < \infty$ for all $p \geq 1$ and for all $s, t \in [0, T]$.
As for $\Lambda_\alpha(T, B)$, we have the following estimate

$$\begin{aligned}
\Lambda_\alpha(T,B) =&\frac{1}{\Gamma(1-\alpha)}\sup_{0\le s\le t\le T}\|D^{1-\alpha}_{t-}B_{t-}(s)\| \\
\le&\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\sup_{0\le s\le t\le T}\left(\frac{\|B_t-B_s\|}{(t-s)^{1-\alpha}}+(1-\alpha)\int_s^t\frac{\|B_u-B_s\|}{(u-s)^{2-\alpha}}du\right) \\
\le&\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\sup_{0\le s\le t\le T}\left(\eta_{\beta,T}(t-s)^{\beta+\alpha-1}+(1-\alpha)\eta_{\beta,T}\int_s^t(u-s)^{\beta+\alpha-2}du\right) \\
=&\frac{\eta_{\beta,T}}{\Gamma(1-\alpha)\Gamma(\alpha)}\sup_{0\le s\le t\le T}\left((t-s)^{\beta+\alpha-1}+\frac{1-\alpha}{\alpha+\beta-1}(t-s)^{\alpha+\beta-1}\right) \\
\le&\left(\frac{\beta T^{\alpha+\beta-1}}{(\alpha+\beta-1)\Gamma(1-\alpha)\Gamma(\alpha)}\right)\eta_{\beta,T}.
\end{aligned}$$

Note that by assumption we know $\alpha + \beta - 1 > 0$ and $\mathbb{E}|\Lambda_\alpha(B)|^p < \infty$ follows from the fact that $\eta_{\beta,T}$ has a Gaussian tail according to Fernique's theorem.

Remark 3.2. *Note that in this argument we did not include the drift term when we showed the L^p integrability. However, the argument can be modified trivially if we include drift term, since the fractional derivative of t with order $1-\alpha > 0$ is bounded in any closed interval $[0, T]$*

3.2 Quantitative probabilistic bounds, L^2 error estimate

In this part, we will develop a convergence criterion for the stochastic Taylor expansion which is based on the L_2 estimates of the iterated integrals of fractional Brownian motion. Such estimates may be found in Proposition 4.8 in [11]. They rely on the computation of $\mathbb{E}\left(\int_{\triangle^m[0,t]} dB^I\right)$ established in [4] and the isometry property of the divergence operator. Compared with the deterministic approach, the probabilistic method improves the speed of convergence in L^2 when $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$.

For convenience in this section we restrict ourselves to the case where there is no drift, that is we assume $V_0 = 0$. By using heavier notations, the arguments we use may be extended when $V_0 \neq 0$.

Let us first remind the result by Neuenkirch-Nourdin-Tindel in [11].

Proposition 3.3. *Assume that $H > 1/2$ and let $\alpha_H = H(2H-1)$ and $K = \sqrt{\frac{2}{\alpha_H}}$ we have the following inequality:*

$$\mathbb{E}\left(\left|\int_{\triangle^m[0,t]} dB^I\right|^2\right) \le \frac{K^{2m}}{m!}t^{2H|I|}.$$

It is interesting to observe that the L^2 decay of iterated integrals in m is faster than its L^∞ decay obtained by the deterministic method when $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$. Indeed, from Proposition 3.3, we know the L^2 decay of $\int_{\triangle^m[0,t]} dB^I$ is of order $\frac{1}{\sqrt{m!}}$. On the other hand, by Proposition 2.7, the L^∞ decay of $\int_{\triangle^m[0,t]} dB^I$ is of order $\frac{1}{\Gamma(m(1-2\alpha))}$. Obviously, when $H \in (1/2, 3/4)$, we can pick an appropriate α such that $\lim_{m\to\infty} \dfrac{\Gamma(m(1-2\alpha))}{\sqrt{m!}} = 0$.

Theorem 3.4. *Let us assume that there exists $M > 0$ and $0 \le \gamma < 1/2$ such that $\|P_I\| \le M^{|I|}(|I|!)^{\gamma}$. Then for every $r > 1$, $T_C(r)$ is almost surely positive and for $0 \le t < T_C(r)$, we have in L^2 and almost surely*

$$X_t = x_0 + \sum_{k=1}^{+\infty} g_k(t).$$

Moreover there exists a constant $C_\gamma > 0$ such that,

$$\mathbb{E}\left(\left\|X_t - \left(x_0 + \sum_{k=1}^{N} g_k(t)\right)\right\|^2, t < T_C(r)\right)^{1/2} \le C_\gamma \frac{(dMKt^{2H})^{N+1}}{((N+1)!)^{1/2-\gamma}} \Phi_\gamma(dKMt^{2H}),$$

where

$$\Phi_\gamma(x) = \sum_{k=0}^{+\infty} \frac{x^k}{(k!)^{1/2-\gamma}}.$$

Let us recall that the vector fields V_i's are Lipschitz together with their derivatives. Moreover if $C = +\infty$, that is if the V_i's are analytic on $\mathbb{R}^n$, then we may take $T_C(r) = +\infty$ in the above inequality.

Proof. Let $\rho > 0$. We have:

$$\begin{aligned}
\mathbb{P}\left(\sum_{|I|=m} \left\|P_I \int_{\triangle^m[0,t]} dB^I\right\| \ge \rho^m\right) \le& \frac{1}{\rho^{2m}} \mathbb{E}\left[\left(\sum_{|I|=m} \left\|P_I \int_{\triangle^m[0,t]} dB^I\right\|\right)^2\right] \\
\le& \frac{d^m}{\rho^{2m}} \sum_{|I|=m} \mathbb{E}\left(\left\|P_I \int_{\triangle^m[0,t]} dB^I\right\|^2\right) \\
\le& \left(\frac{d}{\rho^2}\right)^m \sum_{|I|=m} \|P_I\|^2 \mathbb{E}\left(\left|\int_{\triangle^m[0,t]} dB^I\right|^2\right) \\
\le& \left(\frac{d}{\rho^2}\right)^m \sum_{|I|=m} \|P_I\|^2 \frac{K^{2m}}{m!} t^{2mH} \\
\le& \left(\frac{d^2}{\rho^2}\right)^m M^{2m} \frac{K^{2m}}{(m!)^{1-2\gamma}} t^{2mH}
\end{aligned}$$

Therefore: $\sum_{m=1} \mathbb{P}\left(\sum_{|I|=m} \left\|P_I \int_{\triangle^m[0,t]} dB^I\right\| \ge \rho^m\right) < \infty$. From Borel Cantelli Lemma, we deduce that for every $r > 1$, $\sum_{m=1}^{\infty} r^m \|g_m(t)\|$ converges uniformly on any interval $[0, T]$ almost surely. From the results of the previous section, it thus implies that for every $r > 1$, $T_C(r) > 0$ and that for every $0 \le t < T_C(r)$, we have almost surely

$$X_t = x_0 + \sum_{k=1}^{+\infty} g_k(t).$$

Finally, we have

$$\begin{aligned}\mathbb{E}\left(\left\|X_t-\left(x_0+\sum_{k=1}^{N}g_k(t)\right)\right\|^2, t<T_C(r)\right)^{1/2} &\leq \sum_{k=N+1}^{+\infty}\sum_{|I|=k}\frac{K^k}{\sqrt{k!}}t^{2Hk}M^k(k!)^{\gamma}\\ &\leq \sum_{k=N+1}^{+\infty}\frac{(dKt^{2H}M)^k}{(k!)^{1/2-\gamma}}\\ &\leq C_\gamma\frac{(dMKt^{2H})^{N+1}}{((N+1)!)^{1/2-\gamma}}\Phi_\gamma(dKMt^{2H}).\end{aligned}$$

□

References

[1] R. Azencott. Formule de Taylor stochastique et développements asymptotiques d'intégrales de Feynman. *In Azéma, Yor (Eds), Séminaire de probabilités, **XVI**, LNM **921**, 237-284, Springer.* MR-0658728

[2] G. Ben Arous. Flot et séries de Taylor stochastiques. *Probab.Theory Related Fields 81, (1989), 29-77.* MR-0981567

[3] F. Baudoin. An Introduction to the geometry of stochastic flows, *Imperial College Press, 140pp, (2005).*MR-2154760

[4] F. Baudoin and L.Coutin. Operators associated with a stochastic differential equation driven by fractional Brownian motions.*Stochastic.Process.Appl. 117 (2007) 550-574.*MR-2320949

[5] F. Baudoin and L.Coutin. Self-similarity and fractional Brownian motions on Lie groups *Electron. J. Probab. 13 (2008), no. 38.*MR-2424989

[6] F. Castell. Asymptotic expansion of stochastic flows. *Probab.Theory Relat. Fields, **96**, 225-239, (1993).*MR-1227033

[7] L. Decreusefond and D. Nualart. Flow properties of differential equations driven by fractional Brownian motion. *Stochastic Differential Equations - Theory and Applications, 249–262, eds: Peter Baxendale and Sergey Lototsky. Interdiscip. Math.Sci., 2, World Sci. Publ., Hackensack, NJ, 2007.* MR-2393579

[8] J. J. Duistermaat and J. A. C. Kolk. Lie Groups,Universitext, *Springer-Verlag, Berlin, 2000.*MR-1738431

[9] Y. Hu. Multiple integrals and expansion of solutions of differential equations driven by rough paths and by fractional Brownian motions. *Journal of Theoretical Probability,* To appear.

[10] W. Magnus. On the exponential solution of differential equations for a linear operator.*Comm. Pure Appl. Math. 7, (1954). 649-673.*MR-0067873

[11] A.Neuenkirch, I.Nourdin. A. Rößler, and S. Tindel: Trees and asymptotic expansions for fractional diffusion processes. *Ann. Inst. H. Poincaré Probab. Statist.Volume 45, Number 1 (2009), 157-174.* MR-2500233

[12] D. Nualart, A.Răşcanu. Differential equations driven by fractional Brownian motion. *Collect.Math.53 (2002) 55-81.* MR-1894504

[13] Y. Hu and D. Nualart. Differential equations driven by Hölder continuous functions of order greater than 1/2. *Stochastic Analysis and Applications, 399-413, Abel Symp., 2, Springer, Berlin, 2007.*MR-2397797

[14] D. Nualart and B. Saussereau. Malliavin calculus for stochastic differential equations driven by a fractional Brownian. motion. *Stochastic Processes and their Applications 119, 391-409, 2009.*MR-2493996

[15] A.A.Ruzmaikina. Stieltjes integrals of Hölder continuous functions with application to fractional Brownian motion. *J.Statist.Phys.100, (2000), 1049-1069* MR-2062645

[16] R. Strichartz. The Campbell-Baker-Hausdorff-Dynkin formula and solutions of differential equations, *Journal of Functional Analysis Volume 72, Issue 2, June 1987, Pages 320-345*MR-0895772

[17] L.C Young. An inequality of the Hölder type connected with Stieltjes integration. *Acta Math. 67, 251-282,(1936)* MR-1555421

[18] M.Zähle. Integration with respect to fractal functions and stochastic calculus I. *Probab.Theory Related Fields 111 (1998) 333-374* MR-1640795