Electronic Journal of Probability

Electron. J. Probab. **17** (2012), no. 73, 1–20.
ISSN: 1083-6489 DOI: 10.1214/EJP.v17-1928

The compact support property for the Λ-Fleming-Viot process with underlying Brownian motion*

Huili Liu† Xiaowen Zhou†

Abstract

Using the lookdown construction of Donnelly and Kurtz we prove that, at any fixed positive time, the Λ-Fleming-Viot process with underlying Brownian motion has a compact support provided that the corresponding Λ-coalescent comes down from infinity not too slowly. We also find both upper bound and lower bound on the Hausdorff dimension for the support.

Keywords: Λ-Fleming-Viot process ; Λ-coalescent ; compact support property ; lookdown construction.
AMS MSC 2010: Primary 60G57, Secondary 60J80 ; 60G17.
Submitted to EJP on April 5, 2012, final version accepted on August 27, 2012.
Supersedes arXiv:1208.4152v1.

1 Introduction

The Fleming-Viot process is a probability-measure-valued stochastic process for population genetics. It describes the evolution of relative frequencies for different types of alleles in a large population that undergoes resampling with possible mutation. An earlier review on the classical Fleming-Viot process can be found in Ethier and Kurtz [8]. When such a process is treated as a general measure-valued stochastic process, the mutation can also be interpreted as motion. In this paper we will consider the Fleming-Viot process with underlying Brownian motion.

The support property for measure-valued stochastic processes has been an interesting topic. The earliest work on the compact support property for classical Fleming-Viot process is due to Dawson and Hochberg [4]. It was shown in [4] that at any fixed time $T > 0$ the classical Fleming-Viot process with underlying Brownian motion has a compact support and the support has a Hausdorff dimension not greater than two. Using non-standard techniques Reimers [14] refined the above result by proving that the Hausdorff dimension of support for this process is at most two simultaneously for all time t from finite interval $[0, T]$. Iscoe [11] first proved the compact support property for the related Dawson-Watanabe superprocess. The compact support property for solutions to SPDEs can be found in Mueller and Perkins [12].

*The research is supported by NSERC.
†Concordia University, Canada. E-mail: (l_huili@live,xzhou@mathstat).concordia.ca

The moments for classical Fleming-Viot process with underlying Brownian motion are determined by a dual process involving Kingman's coalescent of binary collisions and heat flow. The Λ-Fleming-Viot process generalizes the classical Fleming-Viot process by replacing in its dual process the Kingman's coalescent with the Λ-coalescent that allows multiple collisions. Blath [3] showed that the Λ-Fleming-Viot process with underlying Brownian motion does not have a compact support if the Λ-coalescent does not come down from infinity. It is then interesting to know whether such a process allows a compact support if the Λ-coalescent comes down from infinity.

In this paper we find a sufficient condition on Λ-coalescence rates for the Λ-Fleming-Viot process to have a compact support at any fixed positive time. We adapt the idea of Dawson and Hochberg [4] as follows. Given any fixed time $T > 0$, we can represent the Λ-Fleming-Viot process at time T as limit of empirical measures of the exchangeable particle systems obtained via the lookdown construction of Donnelly and Kurtz [7], see also Birkner and Blath [1]. For a sequence of random times T_n converging increasingly to T, by the lookdown construction and the property of coming down from infinity there exist a finite number of common ancestors at each time T_n for those particles at time T. Our assumption on coalescence rates allows us to estimate the number of common ancestors at time T_n. Then locations of the ancestors at time T_{n+1} are determined by a collection of possibly dependent Brownian motions starting from the locations of ancestors at time T_n and stopping after time $T_{n+1} - T_n$. By the modulus of continuity for Brownian motion we can estimate the maximal dislocation of the ancestors at time T_{n+1} from those at time T_n. Choosing (T_n) properly and applying Borel-Cantelli lemma we can show that for m large enough the maximal dislocations between T_n and T_{n+1} for all $n \geq m$ are summable. Then all the particles at time T are situated in the union of finitely many closed balls centered at the ancestors' positions at time T_m respectively. The compact support property then follows. As a byproduct of the estimates we can also find an upper bound for the Hausdorff dimension of the support at time T. The lookdown construction plays a crucial role in our arguments.

The paper is arranged as follows. In Section 2 we briefly introduce the Λ-coalescent and its coming down from infinity property. In Section 3 we present the Λ-Fleming-Viot process and its lookdown construction. In Section 4 we prove the compact support property for Λ-Fleming-Viot process with underlying Brownian motion. We also find both upper and lower bounds on the Hausdorff dimension for the compact support.

2 The Λ-coalescent

2.1 The Λ-coalescent

We first introduce some notations. Put $[n] := \{1, \ldots, n\}$ and $[\infty] := \{1, 2, \ldots\}$. An ordered *partition* of $D \subset [\infty]$ is a countable collection $\pi = \{\pi_i, i = 1, 2, \ldots\}$ of disjoint *blocks* such that $\cup_i \pi_i = D$ and $\min \pi_i < \min \pi_j$ for $i < j$. Then blocks in π are ordered by their least elements.

Denote by $\mathcal{P}_n$ the set of ordered partitions of $[n]$ and by $\mathcal{P}_\infty$ the set of ordered partitions of $[\infty]$. Write $\mathbf{0}_{[n]} := \{\{1\}, \ldots, \{n\}\}$ for the partition of $[n]$ consisting of singletons and $\mathbf{0}_{[\infty]}$ for the partition of $[\infty]$ consisting of singletons. Given $n \in [\infty]$ and $\pi \in \mathcal{P}_\infty$, let $R_n(\pi) \in \mathcal{P}_n$ be the restriction of π to $[n]$.

Kingman's coalescent is a $\mathcal{P}_\infty$-valued time homogeneous Markov process such that all different pairs of blocks independently merge at the same rate. Pitman [13] and Sagitov [15] generalized the Kingman's coalescent to the Λ-*coalescent* which allows *multiple collisions*, i.e., more than two blocks may merge at a time. The Λ-coalescent is defined as a $\mathcal{P}_\infty$-valued Markov process $\{\Pi(t) : t \geq 0\}$ such that for each $n \in [\infty]$, $\{\Pi_n(t) : t \geq 0\}$, its restriction to $[n]$, is a $\mathcal{P}_n$-valued Markov process whose transition

rates are described as follows: if there are currently b blocks in the partition, then each k-tuple of blocks ($2 \le k \le b$) independently merges to form a single block at rate

$$\lambda_{b,k} = \int_0^1 x^{k-2}(1-x)^{b-k}\Lambda(dx), \tag{2.1}$$

where Λ is a finite measure on $[0,1]$. It is easy to check that the rates $(\lambda_{b,k})$ are consistent so that for all $2 \le k \le b$,

$$\lambda_{b,k} = \lambda_{b+1,k} + \lambda_{b+1,k+1}. \tag{2.2}$$

Consequently, for all $m < n < \infty$, the coalescent process $R_m(\Pi_n(t))$ given $\Pi_n(0) = \pi_n$ has the same distribution as $\Pi_m(t)$ given $\Pi_m(0) = R_m(\pi_n)$.

With the transition rates determined by (2.1), there exists an one to one correspondence between Λ-coalescents and finite measures Λ on $[0,1]$.

For $n = 2, 3, \ldots$, denote by

$$\lambda_n = \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k}$$

the total coalescence rate starting with n blocks. In addition, denote by

$$\gamma_n = \sum_{k=2}^{n} (k-1) \binom{n}{k} \lambda_{n,k}$$

the rate at which the number of blocks decreases.

2.2 Coming down from infinity

For $n \in [\infty]$, let $\#\Pi_n(t)$ denote the number of blocks in partition $\Pi_n(t)$. The Λ-coalescent *comes down from infinity* if

$$P(\#\Pi_\infty(t) < \infty) = 1$$

for all $t > 0$. It *stays infinite* if

$$P(\#\Pi_\infty(t) = \infty) = 1$$

for all $t > 0$.

Theorem 2.1 (Schweinsberg [16])**.** *Suppose that Λ has no atom at 1. Then*

- *the Λ-coalescent comes down from infinity if and only if $\sum_{b=2}^{\infty} \gamma_b^{-1} < \infty$;*
- *the Λ-coalescent stays infinite if and only if $\sum_{b=2}^{\infty} \gamma_b^{-1} = \infty$.*

We list some examples of Λ-coalescents and identify whether they come down from infinity or stay infinite.

- If $\Lambda = \delta_1$, then $\lambda_{b,b} = 1$ and $\lambda_{b,k} = 0$ for all $2 \le k \le b-1$. The corresponding coalescent only allows all the blocks to merge into one single block after an exponential time with parameter 1. Thus it neither comes down from infinity nor stays infinite.
- If $\Lambda = \delta_0$, the corresponding coalescent degenerates to Kingman's coalescent and comes down from infinity.
- We say that a Λ-coalescent has the (c, ϵ, γ)-property, if there exist constants $c > 0$ and ϵ, $\gamma \in (0,1)$ such that the measure Λ restricted to $[0, \epsilon]$ is absolutely continuous with respect to Lebesgue measure and

  $$\Lambda(dx) \ge cx^{-\gamma}dx \text{ for all } x \in [0, \epsilon].$$

  The Λ-coalescents with the (c, ϵ, γ)-property come down from infinity.

- For $\beta \in (0,2)$, the *Beta*$(2-\beta,\beta)$*-coalescent* is the Λ-coalescent with the finite measure Λ on $[0,1]$ denoted by

$$\Lambda(dx) = \frac{\Gamma(2)}{\Gamma(2-\beta)\Gamma(\beta)} x^{1-\beta}\,(1-x)^{\beta-1}\,dx.$$

  - If $\beta \in (0,1]$, the Beta$(2-\beta,\beta)$-coalescent stays infinite.
  - If $\beta \in (1,2)$, the Beta$(2-\beta,\beta)$-coalescent has the $(c,\epsilon,\beta-1)$-property and comes down from infinity.

3 The Λ-Fleming-Viot process and its lookdown construction

In this section, we first briefly review the literatures on lookdown construction. Then we introduce the lookdown construction for Λ-Fleming-Viot process and explain how to recover the Λ-coalescent from the lookdown construction.

3.1 The lookdown construction for Λ-Fleming-Viot process

The idea of expressing the probability-measure-valued process using empirical measure of an exchangeable particle system goes back to Dawson and Hochberg [4] in which the classical Fleming-Viot process on space E can be obtained as limit of empirical measure of an E^∞-valued particle system.

Donnelly and Kurtz [5, 6, 7] made this idea explicit by introducing the celebrated lookdown construction. In [5] they showed that the classical Fleming-Viot process arises as limit of empirical measure associated with an infinite particle system obtained from the lookdown scheme. In [6] they proposed the lookdown representation for Fleming-Viot process incorporating selection and recombination. In [7] they further introduced a modified lookdown construction for a large class of measure-valued stochastic processes that include both the neutral Fleming-Viot process and the Dawson-Watanabe superprocess as examples.

Birkner and Blath [1] discussed the modified lookdown construction of [7] for the Λ-Fleming-Viot process. They also described how to recover the Λ-coalescent from their modified lookdown construction. A Poisson point process construction of the more general Ξ-lookdown model is found in Birkner et al. [2] by extending the modified lookdown construction of Donnelly and Kurtz [7]. It was shown that the empirical measure of the lookdown particle system converges almost surely on the Skorokhod space of measure-valued paths to the Ξ-Fleming-Viot process.

In the lookdown particle system each particle is attached a “level" from the set $[\infty]$. There are reproduction events in the system and the particles move independently between the reproduction times. The system is constructed in a way that the evolution of particle at level n only depends on the evolutions of particles at lower levels. This property allows us to construct approximating particle systems, and their limit as $n \to \infty$ in the same probability space. The lookdown construction is a powerful tool for studying Fleming-Viot processes.

Now we introduce the lookdown construction for Λ-Fleming-Viot process with underlying Brownian motion following Birkner and Blath [1]. Let

$$(X_1(t), X_2(t), X_3(t), \ldots)$$

be an $(\mathbb{R}^d)^\infty$-valued random variable. For any $i \in [\infty]$, $X_i(t)$ represents the spatial location of the particle at level i. We require the initial values $\{X_i(0), i \in [\infty]\}$ to be exchangeable random variables so that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(0)}$$

exists almost surely by de Finetti's theorem.

Let Λ be the finite measure associate to the Λ-coalescent. The reproduction in the particle system consists of two kinds of birth events: the events of single birth determined by measure $\Lambda(\{0\})\delta_0$ and the events of multiple births determined by measure Λ restricted to $(0,1]$ that is denoted by $\Lambda_0 \equiv \Lambda - \Lambda(\{0\})\delta_0$.

To describe the evolution of the system during events of single birth, let $\{\mathbf{N}_{ij}(t) : 1 \le i < j < \infty\}$ be independent Poisson processes with rate $\Lambda(\{0\})$. At a jump time t of $\mathbf{N}_{ij}$, the particle at level j looks down at the particle at level i and assumes its location (therefore, particle at level i gives birth to a new particle). Values of particles at levels above j are shifted accordingly, i.e., for $\Delta\mathbf{N}_{ij}(t) = 1$, we have

$$X_k(t) = \begin{cases} X_k(t-), & \text{if } k < j, \\ X_i(t-), & \text{if } k = j, \\ X_{k-1}(t-), & \text{if } k > j. \end{cases} \tag{3.1}$$

For those events of multiple births we can construct an independent Poisson point process $\mathbf{N}$ on $\mathbb{R}^+ \times (0,1]$ with intensity measure $dt \otimes x^{-2}\Lambda_0(dx)$. Let $\{U_{ij},\ i,j \in [\infty]\}$ be i.i.d. uniform $[0,1]$ random variables. Jump points $\{(t_i, x_i)\}$ for $\mathbf{N}$ correspond to the multiple birth events. For $J \subset [n]$ with $|J| \ge 2$, define

$$\mathbf{N}^n_J(t) = \sum_{i:t_i \le t} \prod_{j \in J} \mathbf{1}_{\{U_{ij} \le x_i\}} \prod_{j \in [n]\setminus J} \mathbf{1}_{\{U_{ij} > x_i\}}. \tag{3.2}$$

Then $\mathbf{N}^n_J(t)$ counts the number of birth events, among the levels $\{1, 2, \ldots, n\}$, exactly those in J were involved up to time t. Intuitively, at a jump time t_i, a uniform coin is tossed independently for each level. All the particles at levels j with $U_{ij} \le x_i$ participate in the lookdown event. More precisely, those particles involved jump to the location of the particle at the lowest level involved. The spatial locations of particles on the other levels, keeping their original order, are shifted upwards accordingly, i.e., if $t = t_i$ is the jump time and j is the lowest level involved, then

$$X_k(t) = \begin{cases} X_k(t-), \text{ for } k \le j, \\ X_j(t-), \text{ for } k > j \text{ with } U_{ik} \le x_i, \\ X_{k-J^k_t}(t-), \text{ otherwise}, \end{cases}$$

where $J^k_{t_i} \equiv \#\{m < k, U_{im} \le x_i\} - 1$.

Between jump times of the Poisson processes, particles at different levels move independently according to Brownian motions. The values of (X_i) are determined by the following system of stochastic differential equations. Let $\{\mathbf{B}_i(t) : i = 1, 2, \ldots\}$ be a sequence of independent and standard d-dimensional Brownian motions. The particle at level 1 evolves according to Brownian motion, i.e.,

$$X_1(t) = X_1(0) + \mathbf{B}_1(t).$$

For $n \geq 2$, X_n satisfies

$$\begin{aligned} X_n(t) =& X_n(0) + \mathbf{B}_n(t) + \sum_{1\leq i<j<n} \int_0^t \left(X_{n-1}(s-) - X_n(s-)\right) d\mathbf{N}_{ij}(s) \\ &+ \sum_{1\leq i<n} \int_0^t \left(X_i(s-) - X_n(s-)\right) d\mathbf{N}_{in}(s) \\ &+ \sum_{J\subset[n], n\in J} \int_0^t \left(X_{\min(J)}(s-) - X_n(s-)\right) d\mathbf{N}_J^n(s) \\ &+ \sum_{J\subset[n], n\notin J} \int_0^t \left(X_{n-J_s^n}(s-) - X_n(s-)\right) d\mathbf{N}_J^n(s), \end{aligned}$$

where the first integral concerns the lookdown event involving levels i and j both below level n; the second integral concerns the event that particle at level n looks down at particle at a lower level i; the third and fourth integrals concern the lookdown events with multiple levels involved.

For each $t > 0$, $X_1(t), X_2(t), \ldots$ are known to be exchangeable random variables so that

$$X(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(t)}$$

exists almost surely by de Finetti's theorem and follows the probability law of the Λ-Fleming-Viot process with underlying Brownian motion (cf. Donnelly and Kurtz [7], Birkner and Blath [1]).

3.2 The Λ-coalescent recovered from the lookdown construction

The birth events induce a family structure to the particle system so we can talk about its genealogy. For any $t \geq 0$, $0 \leq s \leq t$ and $n \in [\infty]$, denote by $L_n^t(s)$ the ancestor's level at time s for the particle with level n at time t. Consequently, $L_n^t(s)$ satisfies the equation:

$$\begin{aligned} L_n^t(s) = n &- \sum_{1\leq i<j<n} \int_{s-}^t \mathbf{1}_{\{L_n^t(u)>j\}} d\mathbf{N}_{ij}(u) \\ &- \sum_{1\leq i<j\leq n} \int_{s-}^t (j-i)\, \mathbf{1}_{\{L_n^t(u)=j\}} d\mathbf{N}_{ij}(u) \\ &- \sum_{J\subset[n]} \int_{s-}^t \left(L_n^t(u) - \min(J)\right) \mathbf{1}_{\{L_n^t(u)\in J\}} d\mathbf{N}_J^n(u) \\ &- \sum_{J\subset[n]} \int_{s-}^t \left(|J \cap \{1,\ldots,L_n^t(u)\}| - 1\right) \times \mathbf{1}_{\{L_n^t(u)>\min(J), L_n^t(u)\notin J\}} d\mathbf{N}_J^n(u). \end{aligned}$$

For fixed $T > 0$ and $i \in [\infty]$, $L_i^T(T-t)$ represents the ancestor's level at time $T-t$ of the particle with level i at time T and $X_{L_i^T(T-t)}((T-t)-)$ represents that ancestor's location.

Write $\left(\Pi(t)\right)_{0\leq t\leq T}$ for the $\mathcal{P}_\infty$-valued process such that i and j belong to the same block of $\Pi(t)$ if and only if $L_i^T(T-t) = L_j^T(T-t)$. Therefore, i and j belong to the same block if and only if the two particles at levels i and j, respectively, at time T share a common ancestor at time $T-t$. The process $\left(\Pi(t)\right)_{0\leq t\leq T}$ turns out to have the same law as the Λ-coalescent running up to time T (cf. Donnelly and Kurtz [7], Birkner and Blath [1]), which justifies the usage of the notion.

We end this section with an observation on $\big(\Pi(t)\big)_{0\le t\le T}$.

Lemma 3.1. *For any fixed $T>0$, let $\big(\Pi(t)\big)_{0\le t\le T}$ be the Λ-coalescent recovered from the lookdown construction. Then given $t\in[0,T]$ and the ordered random partition $\Pi(t)=\{\pi_l(t): l=1,\dots,\#\Pi(t)\}$, we have*

$$L_j^T(T-t)=l \ \text{ for any } \ j\in\pi_l(t).$$

Proof. For any $1\le l\le \#\Pi(t)$, by definition the particles with levels in block $\pi_l(t)$ at time T have the same ancestor at time $T-t$. Let $i_l=\min\pi_l(t)$.

It is trivial that $i_1=1$ and $L_{i_1}^T(T-t)=1$. Then $L_j^T(T-t)=1$ for all $j\in\pi_1(t)$.

Now we consider the case $l\ge 2$. Since $i_l=\min\pi_l(t)$, looking forwards in time the ancestor of the particle on level i_l at time T never looks down to particles of lower levels during the time interval $[T-t,T]$. As an increasing and piecewise constant function, the ancestor level $\{L_{i_l}^T(s), T-t\le s\le T\}$ only increases in s because of upward shifts. We thus have

$$L_{i_l}^T(T)-L_{i_l}^T(T-t)=\#\left\{L_j^T(T): j\in[i_l]\right\}-\#\left\{L_j^T(T-t): j\in[i_l]\right\}, \tag{3.3}$$

where we recall that $[i_l]=\{1,2,\dots,i_l\}$. By (3.3), it follows that

$$L_{i_l}^T(T-t)=\#\left\{L_j^T(T-t): j\in[i_l]\right\}.$$

Since $\{\pi_j(t), 1\le j\le\#\Pi(t)\}$ are ordered by their minimal elements, then

$$[i_l]\subset\cup_{j=1}^l\pi_j(t) \ \text{ and } \ \pi_j(t)\cap[i_l]\ne\emptyset \ \text{ for all } \ 1\le j\le l.$$

Recall that $\Pi_{i_l}(t)$ is the restriction of $\Pi(t)$ to $[i_l]$. It follows that $\#\Pi_{i_l}(t)=l$, which implies that

$$\#\left\{L_j^T(T-t): j\in[i_l]\right\}=l.$$

We then have $L_{i_l}^T(T-t)=l$. All the particles with levels from the same block have a common ancestor at time $T-t$, therefore, $L_j^T(T-t)=l$ for any $j\in\pi_l(t)$. □

4 The compact support property for the Λ-Fleming-Viot process

In this section, we proceed to prove the compact support property of Λ-Fleming-Viot process. We also find both upper bound and lower bound for the Hausdorff dimension of its support. Until the end of this section we assume that the measure Λ has no mass at 1, i.e., $\Lambda(\{1\})=0$.

Intuitively, if the corresponding Λ-coalescent comes down from infinity, then for any fixed $T>0$, the random variables $(X_1(T),X_2(T),\dots)$ in the lookdown system are highly correlated. This is because the particles at time T are offspring of the finitely many particles alive at an arbitrary time before T. Our approach is to group the countably many particles at time T into finitely many disjoint subclusters according to their respective common ancestors at an earlier time. When this earlier time is close enough to T, the distances between the particles at time T and their respective ancestors have to be small. Then each subcluster is contained in a small neighborhood of its ancestor and all the neighborhoods are contained in a compact set. The compact support property thus follows.

Throughout this paper we always write C or C with subscript for a positive constant and write $C(y)$ for a positive constant depending on y, whose values might vary from place to place.

4.1 The main result

We begin by recalling the notion of *Hausdorff dimension*. Given $A \subset \mathbb{R}^d$ and $\beta > 0$, $\eta > 0$, let

$$\mathbf{\Lambda}_\eta^\beta(A) \equiv \inf_{\{S_l\} \in \varphi_\eta} \sum_l (d(S_l))^\beta,$$

where $d(S_l)$ denotes the diameter of ball S_l in $\mathbb{R}^d$ and φ_η denotes the collection of η-*covers* of set A by balls, i.e.,

$$\varphi_\eta \equiv \big\{\{S_l\} \text{ is a cover of } A \text{ by balls with } d(S_l) < \eta \text{ for each } l\big\}.$$

The Hausdorff β-measure of A is defined by

$$\mathbf{\Lambda}^\beta(A) = \lim_{\eta \to 0} \mathbf{\Lambda}_\eta^\beta(A).$$

The Hausdorff dimension of A is defined by

$$\dim(A) \equiv \inf\big\{\beta > 0 : \mathbf{\Lambda}^\beta(A) = 0\big\} = \sup\big\{\beta > 0 : \mathbf{\Lambda}^\beta(A) = \infty\big\}.$$

For any $n > m \geq 2$, let $\Pi_n^\Lambda(t), t \geq 0$, be any Λ-coalescent with $\Pi_n^\Lambda(0) = \mathbf{0}_{[n]}$. Then the block counting process $\#\Pi_n^\Lambda(t) \vee m$ is a Markov chain with initial value n and absorbing state m. For any $n \geq b > m$, let $\{\mu_{b,k}\}_{m \leq k \leq b-1}$ be its transition rates such that

$$\begin{cases} \mu_{b,b-1} = \binom{b}{2}\lambda_{b,2}, \\ \mu_{b,b-2} = \binom{b}{3}\lambda_{b,3}, \\ \cdots\cdots \\ \mu_{b,m+1} = \binom{b}{b-m}\lambda_{b,b-m}, \\ \mu_{b,m} = \sum_{k=b-m+1}^{b} \binom{b}{k}\lambda_{b,k}. \end{cases} \tag{4.1}$$

The total transition rate is

$$\mu_b = \sum_{k=m}^{b-1} \mu_{b,k} = \sum_{k=2}^{b} \binom{b}{k} \lambda_{b,k} = \lambda_b.$$

For $b > m$, let $\gamma_{b,m}$ be the total rate at which the block counting Markov chain starting at b is decreasing. Then

$$\gamma_{b,m} = \begin{cases} \sum_{k=2}^{b-m} (k-1)\binom{b}{k}\lambda_{b,k} + \sum_{k=b-m+1}^{b} (b-m)\binom{b}{k}\lambda_{b,k}, & \text{if } b \geq m+2, \\ \sum_{k=2}^{b} \binom{b}{k}\lambda_{b,k}, & \text{if } b = m+1. \end{cases} \tag{4.2}$$

The main result of this paper is the theorem below, which gives a sufficient condition for the Λ-Fleming-Viot process to have a compact support.

Theorem 4.1. *Given any Λ-Fleming-Viot process X with underlying Brownian motion in $\mathbb{R}^d$, let $(\gamma_{b,m})$ be defined in Equation (4.2) for the corresponding Λ-coalescent. If there exist constants $C > 0$ and $\alpha > 0$ such that*

$$\sum_{b=m+1}^{\infty} {\gamma_{b,m}}^{-1} \leq C m^{-\alpha} \tag{4.3}$$

for m big enough, then for any $T > 0$, with probability one the random measure $X(T)$ has a compact support and the Hausdorff dimension for supp$X(T)$ is bounded from above by $2/\alpha$.

4.2 Some estimates on the Λ-coalescent

We first point out an immediate consequence of assumption (4.3).

Lemma 4.2. *The Λ-coalescent comes down from infinity under assumption (4.3).*

Proof. Notice that $\gamma_{b,m} \leq \gamma_b$ for any $b > m \geq 2$. Then the corresponding Λ-coalescent comes down from infinity by Theorem 2.1. □

Lemma 4.3. *For any $2 \leq m < b$, we have $\gamma_{b,m} \leq \gamma_{b+1,m}$.*

Proof. According to the different values of b and m, we consider the following three different cases separately.
Case I: $b = m + 1$. By the consistency condition (2.2) for $(\lambda_{b,k})$ and the definition of $(\gamma_{b,m})$, we have

$$\begin{aligned}\gamma_{b,m} =& \sum_{k=2}^{b} \binom{b}{k} \lambda_{b,k} = \sum_{k=2}^{b} \binom{b}{k} (\lambda_{b+1,k} + \lambda_{b+1,k+1}) \\ =& \binom{b}{2} \lambda_{b+1,2} + \sum_{k=3}^{b} \binom{b}{k} \lambda_{b+1,k} + \sum_{k=3}^{b} \binom{b}{k-1} \lambda_{b+1,k} + \lambda_{b+1,b+1} \\ =& \binom{b}{2} \lambda_{b+1,2} + \sum_{k=3}^{b} \left(\binom{b}{k} + \binom{b}{k-1} \right) \lambda_{b+1,k} + \lambda_{b+1,b+1}.\end{aligned}$$

By the identity

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}, \tag{4.4}$$

we then have

$$\gamma_{b,m} \leq \binom{b+1}{2} \lambda_{b+1,2} + \sum_{k=3}^{b} 2 \binom{b+1}{k} \lambda_{b+1,k} + 2\lambda_{b+1,b+1} = \gamma_{b+1,m}.$$

Case II: $b = m + 2$. Similarly, it follows from the consistency condition (2.2) for $(\lambda_{b,k})$ and the definition of $(\gamma_{b,m})$ that

$$\begin{aligned}\gamma_{b,m} =& \binom{b}{2} \lambda_{b,2} + \sum_{k=3}^{b} 2 \binom{b}{k} \lambda_{b,k} \\ =& \binom{b}{2} \lambda_{b+1,2} + \binom{b}{2} \lambda_{b+1,3} + \sum_{k=3}^{b} 2 \binom{b}{k} (\lambda_{b+1,k} + \lambda_{b+1,k+1}) \\ =& \binom{b}{2} \lambda_{b+1,2} + \binom{b}{2} \lambda_{b+1,3} + \sum_{k=3}^{b} 2 \binom{b}{k} \lambda_{b+1,k} + \sum_{k=4}^{b} 2 \binom{b}{k-1} \lambda_{b+1,k} + 2\lambda_{b+1,b+1} \\ \leq& \binom{b+1}{2} \lambda_{b+1,2} + 2 \left(\binom{b}{2} + \binom{b}{3} \right) \lambda_{b+1,3} + \sum_{k=4}^{b} 3 \left(\binom{b}{k} + \binom{b}{k-1} \right) \lambda_{b+1,k} + 3\lambda_{b+1,b+1}.\end{aligned}$$

Applying the identity (4.4), we have

$$\begin{aligned}\gamma_{b,m} \leq& \binom{b+1}{2} \lambda_{b+1,2} + 2 \binom{b+1}{3} \lambda_{b+1,3} + \sum_{k=4}^{b} 3 \binom{b+1}{k} \lambda_{b+1,k} + 3\lambda_{b+1,b+1} \\ =& \sum_{k=2}^{3} (k-1) \binom{b+1}{k} \lambda_{b+1,k} + \sum_{k=4}^{b+1} 3 \binom{b+1}{k} \lambda_{b+1,k} \\ =& \gamma_{b+1,m}.\end{aligned}$$

Case III: $b \geq m+3$. The proof involves similar but longer arguments as the first two cases.

$$\begin{aligned}
\gamma_{b,m} =& \sum_{k=2}^{b-m} (k-1)\binom{b}{k}\lambda_{b,k} + \sum_{k=b-m+1}^{b} (b-m)\binom{b}{k}\lambda_{b,k} \\
=& \sum_{k=2}^{b-m} (k-1)\binom{b}{k}(\lambda_{b+1,k}+\lambda_{b+1,k+1}) + \sum_{k=b-m+1}^{b} (b-m)\binom{b}{k}(\lambda_{b+1,k}+\lambda_{b+1,k+1}) \\
=& \sum_{k=2}^{b-m} (k-1)\binom{b}{k}\lambda_{b+1,k} + \sum_{k=b-m+1}^{b} (b-m)\binom{b}{k}\lambda_{b+1,k} \\
&+ \sum_{k=3}^{b-m+1} (k-2)\binom{b}{k-1}\lambda_{b+1,k} + \sum_{k=b-m+2}^{b+1} (b-m)\binom{b}{k-1}\lambda_{b+1,k} \\
=& \binom{b}{2}\lambda_{b+1,2} + \sum_{k=3}^{b-m}\left((k-1)\binom{b}{k}+(k-2)\binom{b}{k-1}\right)\lambda_{b+1,k} \\
&+ (b-m-1)\binom{b}{b-m}\lambda_{b+1,b+1-m} + (b-m)\binom{b}{b-m+1}\lambda_{b+1,b-m+1} \\
&+ \sum_{k=b-m+2}^{b} (b-m)\left(\binom{b}{k-1}+\binom{b}{k}\right)\lambda_{b+1,k} + (b-m)\lambda_{b+1,b+1}.
\end{aligned}$$

With Equation (4.4),

$$\begin{aligned}
\gamma_{b,m} \leq& \binom{b}{2}\lambda_{b+1,2} + \sum_{k=3}^{b-m} (k-1)\binom{b+1}{k}\lambda_{b+1,k} + (b-m)\binom{b+1}{b+1-m}\lambda_{b+1,b+1-m} \\
&+ \sum_{k=b-m+2}^{b} (b-m)\binom{b+1}{k}\lambda_{b+1,k} + (b-m)\lambda_{b+1,b+1} \\
\leq& \sum_{k=2}^{b+1-m} (k-1)\binom{b+1}{k}\lambda_{b+1,k} + \sum_{k=b+2-m}^{b+1} (b+1-m)\binom{b+1}{k}\lambda_{b+1,k} \\
=& \gamma_{b+1,m}.
\end{aligned}$$

□

For the Λ-coalescent $(\Pi(t))_{0\leq t\leq T}$ with $\Pi(0) = \mathbf{0}_{[\infty]}$ recovered from the lookdown construction, denote by $(\Pi_n(t))_{0\leq t\leq T}$ its restriction to $[n]$. Clearly $(\Pi_n(t))_{0\leq t\leq T}$ is exactly the Λ-coalescent recovered from the first n levels of the lookdown construction.

For any $n > m \geq 2$, put

$$T_m^n = \inf\left\{t \in [0,T] : \#\Pi_n(t) \leq m\right\}$$

and

$$T_m \equiv T_m^\infty = \inf\left\{t \in [0,T] : \#\Pi(t) \leq m\right\} \tag{4.5}$$

with the convention $\inf \emptyset = T$. From the lookdown construction, it is obvious that

$$T_m^n \leq T_m^{n+1} \leq T_m^{n+2} \leq \cdots \leq\uparrow T_m. \tag{4.6}$$

Lemma 4.4. *If there exist constants $C > 0$ and $\alpha > 0$ satisfying (4.3) for the corresponding Λ-coalescent, then we have*

$$\mathbb{E}T_m \leq Cm^{-\alpha} \tag{4.7}$$

for m big enough.

Proof. We can define a Λ-coalescent $(\Pi^\Lambda(t))_{t\geq 0}$ satisfying $\Pi^\Lambda(t)=\Pi(t)$ for all $t\leq T$. Put

$$T_m^{\Lambda,n}=\inf\left\{t\geq 0:\#\Pi_n^\Lambda(t)\leq m\right\}$$

and

$$T_m^\Lambda\equiv T_m^{\Lambda,\infty}=\inf\left\{t\geq 0:\#\Pi^\Lambda(t)\leq m\right\}$$

with the convention $\inf\emptyset=\infty$. Then $T_m\leq T_m^\Lambda$ and we only need to show that $\mathbb{E}T_m^\Lambda\leq Cm^{-\alpha}$.

We adapt the idea of Lemma 6 in Schweinsberg [16] to prove this lemma. For any $n>m$ and $1\leq k\leq n-m$, define

$$\mathcal{R}_0=0,$$
$$\mathcal{R}_k=\begin{cases}\inf\left\{t\geq 0:\#\Pi_n^\Lambda(t)<\#\Pi_n^\Lambda(\mathcal{R}_{k-1})\right\} & \text{if } \#\Pi_n^\Lambda(\mathcal{R}_{k-1})>m,\\ \mathcal{R}_{k-1} & \text{if } \#\Pi_n^\Lambda(\mathcal{R}_{k-1})=m.\end{cases}$$

Note that $T_m^{\Lambda,n}=\mathcal{R}_{n-m}$. For $i=0,1,2,\ldots,n-m$, let $\mathcal{N}_i=\#\Pi_n^\Lambda(\mathcal{R}_i)$. For $i=1,2,\ldots,n-m$, let $L_i=\mathcal{R}_i-\mathcal{R}_{i-1}$ and $\mathcal{J}_i=\mathcal{N}_{i-1}-\mathcal{N}_i$.

On the event $\{\mathcal{N}_{i-1}>m\}$, for any $n\geq b>m$, we have

$$P\left(\mathcal{J}_i=k-1|\mathcal{N}_{i-1}=b\right)=\binom{b}{k}\frac{\lambda_{b,k}}{\lambda_b}$$

for $k=2,3,\ldots,b-m$ and

$$P\left(\mathcal{J}_i=b-m|\mathcal{N}_{i-1}=b\right)=\sum_{k=b-m+1}^{b}\binom{b}{k}\frac{\lambda_{b,k}}{\lambda_b}.$$

Consequently, on the event $\{\mathcal{N}_{i-1}>m\}$, we have

$$\mathbb{E}\left(\mathcal{J}_i|\mathcal{N}_{i-1}=b\right)=\sum_{k=2}^{b-m}(k-1)\binom{b}{k}\frac{\lambda_{b,k}}{\lambda_b}+(b-m)\sum_{k=b-m+1}^{b}\binom{b}{k}\frac{\lambda_{b,k}}{\lambda_b}=\frac{\gamma_{b,m}}{\lambda_b}.$$

Therefore,

$$\begin{aligned}\mathbb{E}T_m^{\Lambda,n}=\mathbb{E}\mathcal{R}_{n-m}=\mathbb{E}\sum_{i=1}^{n-m}L_i=\sum_{i=1}^{n-m}\mathbb{E}\mathbb{E}\left(L_i|\mathcal{N}_{i-1}\right)=\sum_{i=1}^{n-m}\mathbb{E}\left(\lambda_{\mathcal{N}_{i-1}}^{-1}\mathbf{1}_{\{\mathcal{N}_{i-1}>m\}}\right)\\
=\sum_{i=1}^{n-m}\mathbb{E}\left(\gamma_{\mathcal{N}_{i-1},m}^{-1}\mathbb{E}\left(\mathcal{J}_i|\mathcal{N}_{i-1}\right)\mathbf{1}_{\{\mathcal{N}_{i-1}>m\}}\right)\\
=\sum_{i=1}^{n-m}\mathbb{E}\mathbb{E}\left(\gamma_{\mathcal{N}_{i-1},m}^{-1}\mathcal{J}_i\mathbf{1}_{\{\mathcal{N}_{i-1}>m\}}|\mathcal{N}_{i-1}\right).\end{aligned}$$

Since $\mathcal{J}_i=0$ on the event $\{\mathcal{N}_{i-1}=m\}$, we have

$$\begin{aligned}\mathbb{E}T_m^{\Lambda,n}=\sum_{i=1}^{n-m}\mathbb{E}\left(\mathbb{E}\left(\gamma_{\mathcal{N}_{i-1},m}^{-1}\mathcal{J}_i|\mathcal{N}_{i-1}\right)\right)=\sum_{i=1}^{n-m}\mathbb{E}\left(\gamma_{\mathcal{N}_{i-1},m}^{-1}\mathcal{J}_i\right)\\
=\mathbb{E}\left(\sum_{i=1}^{n-m}\gamma_{\mathcal{N}_{i-1},m}^{-1}\mathcal{J}_i\right)=\mathbb{E}\left(\sum_{i=1}^{n-m}\sum_{j=0}^{\mathcal{J}_i-1}\gamma_{\mathcal{N}_{i-1},m}^{-1}\right).\end{aligned}$$

Since $(\gamma_{b,m})_{b=m+1}^\infty$ is an increasing sequence by Lemma 4.3, it follows that

$$\mathbb{E}T_m^{\Lambda,n}\leq\mathbb{E}\left(\sum_{i=1}^{n-m}\sum_{j=0}^{\mathcal{J}_i-1}\gamma_{\mathcal{N}_{i-1}-j,m}^{-1}\right)=\mathbb{E}\left(\sum_{b=m+1}^{n}\gamma_{b,m}^{-1}\right)\leq\sum_{b=m+1}^{\infty}\gamma_{b,m}^{-1}.$$

By the Monotone Convergence Theorem, we have

$$\mathbb{E}T_m^{\Lambda} = \lim_{n\to\infty} \mathbb{E}T_m^{\Lambda,n} \leq \sum_{b=m+1}^{\infty} \gamma_{b,m}^{-1}.$$

Finally, by (4.3) we have

$$\mathbb{E}T_m^{\Lambda} \leq Cm^{-\alpha}$$

for m big enough. □

4.3 An estimate on standard Brownian motion

Write

$$(\mathbf{B}(s))_{s\geq 0} = (B_1(s), B_2(s), \ldots, B_d(s))_{s\geq 0}$$

for standard d-dimensional Brownian motion with initial value $\mathbf{0}$, where

$$(B_i(s))_{s\geq 0}, \quad i = 1, \ldots, d$$

are independent one-dimensional standard Brownian motions. For any vector $\mathbf{z} = (z_1, z_2, \ldots, z_d) \in \mathbb{R}^d$, write $\|\mathbf{z}\| = \sqrt{\sum_{i=1}^d z_i^2}$ as usual.

Lemma 4.5. *Given the above mentioned d-dimensional standard Brownian motion, for any $t > 0$ and $x > 0$, there exist positive constants C_1 and C_2 such that*

$$P\left(\sup_{0\leq s\leq t} \|\mathbf{B}(s)\| > x\right) \leq C_1\sqrt{t}x^{-1}\exp\left(-C_2x^2t^{-1}\right). \tag{4.8}$$

Proof. By the *reflection principle*, it is clear that

$$\begin{aligned} P\left(\sup_{0\leq s\leq t} \|\mathbf{B}(s)\| > x\right) &\leq 2dP\left(|B_1(t)| > x/\sqrt{d}\right) \\ &\leq \left(8d^3/\pi\right)^{1/2}\sqrt{t}x^{-1}\exp\left(-\frac{x^2}{2dt}\right) \\ &\equiv C_1\sqrt{t}x^{-1}\exp\left(-C_2x^2t^{-1}\right). \end{aligned} \tag{4.9}$$

□

4.4 The compact support property for the Λ-Fleming-Viot process

In this section, we discuss the Λ-Fleming-Viot process with the corresponding coalescent satisfying assumption (4.3). By Lemma 4.2, the corresponding coalescent comes down from infinity. Given the constant $\alpha > 0$ in (4.3), for any $k \in [\infty]$, define

$$N_k = 2^{k/\alpha}k^{2/\alpha}.$$

Because of the coming down from infinity property, if we look backwards for a small amount of positive time ΔT in the lookdown construction, there exist only finitely many ancestors at time $T - \Delta T$ whose offspring are these countably many particles existing at time T.

Recall the Λ-coalescent $(\Pi(t))_{0\leq t\leq T}$ recovered from the lookdown construction in Section 3.2 and the time T_m defined by (4.5). For all $m \in [\infty]$, the number of ancestors at time $T - T_{N_m}$ is equal to $\#\Pi(T_{N_m})$, which is almost surely finite by the coming down from infinity property.

Put

$$N_m^* = \#\Pi\left(T_{N_m}\right) \quad \text{and} \quad \Pi\left(T_{N_m}\right) = \left\{\pi_l : l = 1, \ldots, N_m^*\right\},$$

where $\{\pi_l \equiv \pi_l(m), l \in [N_m^*]\}$ are all the disjoint blocks of $\Pi(T_{N_m})$ ordered by their minimal elements. Note that $L_j^T(T - T_{N_m}) = l$ for any $j \in \pi_l$ by Lemma 3.1. The *maximal radius of subclusters* is defined as:

$$\begin{aligned} R_m &\equiv \max_{1\le l\le N_m^*} \sup_{j\in\pi_l} \left\| X_j(T) - X_{L_j^T(T-T_{N_m})}\left((T-T_{N_m})-\right)\right\| \\ &= \max_{1\le l\le N_m^*} \sup_{j\in\pi_l} \left\| X_j(T) - X_l\left((T-T_{N_m})-\right)\right\|. \end{aligned}$$

For $k \in [\infty]$, define time interval $J_k = \left[T - T_{N_k}, T - T_{N_{k+1}}\right]$. Let $|J_k|$ be the length of interval J_k. Thus

$$|J_k| = \left(T - T_{N_{k+1}}\right) - \left(T - T_{N_k}\right) = T_{N_k} - T_{N_{k+1}} \le T_{N_k}.$$

Let D_k be the maximal dislocation over time interval J_k of all the Brownian motions involved, i.e.,

$$D_k \equiv \max_{1\le l\le N_k^*} \sup_{j\in\pi_l} \left\| X_{L_j^T(T-T_{N_{k+1}})}(T - T_{N_{k+1}}) - X_l\left((T-T_{N_k})-\right)\right\|. \tag{4.10}$$

Note that for any fixed $1 \le l \le N_k^*$, the collection of ancestor levels

$$\left\{L_j^T(T - T_{N_{k+1}}) : j \in \pi_l\right\}$$

has a finite cardinality because of the coming down from infinity property. Thus the supremum in (4.10) is taken over a finite set.

Lemma 4.6. *Under the condition of Theorem 4.1, for any $\delta \in (0, 1/2)$, almost surely the maximal dislocation D_k satisfies*

$$D_k \le 2^{-k\left(\frac{1}{2}-\delta\right)}$$

for k big enough.

Proof. For the trivial case of $T_{N_{k+1}} = T$, we have $|J_k| = 0$ and the dislocation of Brownian motion over J_k is equal to 0. Consequently,

$$P\left(D_k > 2^{-k\left(\frac{1}{2}-\delta\right)}, |J_k| = 0\right) = 0.$$

In the case of $|J_k| > 0$, the total number of Brownian motions involved over J_k is no more than

$$N_{k+1} = 2^{(k+1)/\alpha}(k+1)^{2/\alpha}.$$

Thus we have

$$\begin{aligned} &P\left(D_k > 2^{-k\left(\frac{1}{2}-\delta\right)}\right) \\ =&P\left(D_k > 2^{-k\left(\frac{1}{2}-\delta\right)}, |J_k| = 0\right) + P\left(D_k > 2^{-k\left(\frac{1}{2}-\delta\right)}, |J_k| > 0\right) \\ =&P\left(D_k > 2^{-k\left(\frac{1}{2}-\delta\right)}, 0 < |J_k| \le 2^{-k}\right) + P\left(D_k > 2^{-k\left(\frac{1}{2}-\delta\right)} \middle| |J_k| > 2^{-k}\right) \times P\left(|J_k| > 2^{-k}\right) \\ \le&N_{k+1} \times P\left(\sup_{0\le s\le 2^{-k}} \|\mathbf{B}(s)\| > 2^{-k\left(\frac{1}{2}-\delta\right)}\right) + P\left(|J_k| > 2^{-k}\right) \\ \equiv&I_1(k) + I_2(k). \end{aligned}$$

By Lemma 4.5 we have

$$P\left(\sup_{0\le s\le 2^{-k}} \|\mathbf{B}(s)\| > 2^{-k\left(\frac{1}{2}-\delta\right)}\right) \le C_1 2^{-k\delta} \exp\left(-C_2 2^{2\delta k}\right).$$

Consequently,

$$\begin{aligned} I_1(k) \leq & 2^{\frac{k+1}{\alpha}} (k+1)^{\frac{2}{\alpha}} C_1 2^{-k\delta} \exp\left(-C_2 2^{2\delta k}\right) \\ \leq & C_1 2^{\frac{4k}{\alpha}} \exp\left(-C_2 2^{2\delta k}\right). \end{aligned}$$

It is clear that $\sum_k I_1(k) < \infty$.

Applying Lemma 4.4 with m replaced by N_k, we have for k large enough

$$\mathbb{E}T_{N_k} \leq C 2^{-k} k^{-2}.$$

Since $|J_k| \leq T_{N_k}$, it follows from the Markov's inequality that for k large

$$I_2(k) \leq P\left(T_{N_k} > 2^{-k}\right) \leq 2^k \mathbb{E}T_{N_k} \leq C k^{-2}.$$

Therefore,

$$\sum_k P\left(D_k > 2^{-k\left(\frac{1}{2}-\delta\right)}\right) \leq \sum_k I_1(k) + \sum_k I_2(k) < \infty.$$

Applying the Borel-Cantelli lemma, we have almost surely

$$D_k \leq 2^{-k\left(\frac{1}{2}-\delta\right)}$$

for k large enough. □

Lemma 4.7. *Under the condition of Theorem 4.1, for any $\delta \in (0, 1/2)$, there exists a positive constant $C(\delta)$ such that almost surely,*

$$R_m \leq C(\delta) 2^{-m\left(\frac{1}{2}-\delta\right)}$$

for m big enough.

Proof. Applying Lemma 4.6, we have almost surely,

$$D_k \leq 2^{-k\left(\frac{1}{2}-\delta\right)}$$

for k large enough. Then almost surely for m large enough,

$$\begin{aligned} R_m \leq & \sum_{k=m}^{\infty} D_k \leq \sum_{k=m}^{\infty} 2^{-k\left(\frac{1}{2}-\delta\right)} \\ = & \frac{2^{-m\left(\frac{1}{2}-\delta\right)}}{1-2^{-\left(\frac{1}{2}-\delta\right)}} \equiv C(\delta) 2^{-m\left(\frac{1}{2}-\delta\right)}. \end{aligned}$$

□

Proof of Theorem 4.1. We first prove the compact support property for the Λ-Fleming-Viot process with the corresponding coalescent satisfying (4.3) at any fixed time T.

For m large enough and for all $k \geq m$, by Lemma 4.7 we have

$$\begin{aligned} X_j\left((T - T_{N_k})-\right) \subseteq & \bigcup_{l=1}^{N_m^*} \mathbb{B}\left(X_l\left((T - T_{N_m})-\right), R_m\right) \\ \subseteq & \bigcup_{l=1}^{N_m^*} \mathbb{B}\left(X_l\left((T - T_{N_m})-\right), C(\delta) 2^{-m\left(\frac{1}{2}-\delta\right)}\right) \\ \equiv & \mathbf{B}, \end{aligned}$$

where $\mathbb{B}(x,r)$ denotes the closed ball centered at x with radius r. For each $n \in [\infty]$, from the lookdown construction there exists a random variable $\delta_n > 0$ such that during the time interval $[T-\delta_n, T]$, the particle at level n never looks down to those particles at lower levels $\{1,2,\ldots,n-1\}$. It then follows from Lemma 3.1 that for any $j \in [n]$, $L_j^T(s) = j$ for all $s \in [T-\delta_n, T]$. Further, the sample path continuity for Brownian motion implies that

$$X_j(T) = X_j(T-) = \lim_{k\to\infty} X_j\left((T - T_{N_k})-\right).$$

Therefore, $X_j(T)$ is a limit point for the compact set $\mathbf{B}$ and we have $X_j(T) \in \mathbf{B}$ for all j. Let

$$\hat{X}_n(T) \equiv \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i(T)}.$$

By the lookdown construction for the Λ-Fleming-Viot process we have

$$X(T) = \lim_{n\to\infty} \hat{X}_n(T).$$

Clearly,

$$\text{supp}\left(\hat{X}_n(T)\right) \subseteq \mathbf{B}$$

for all n, which implies that

$$\text{supp}\left(X\left(T\right)\right) \subseteq \mathbf{B}.$$

We now consider the Hausdorff dimension for the support at time T. The collection of closed balls

$$\left\{\mathbb{B}\left(X_l\left((T - T_{N_m})-\right), C(\delta)2^{-m\left(\frac{1}{2}-\delta\right)}\right) : l = 1,\ldots,N_m^*\right\}$$

is a cover of supp$(X(T))$ for m large enough.

For any $\epsilon > 0$, choose $\delta > 0$ small enough so that

$$\left(\frac{1}{2} - \delta\right)(2+\epsilon) > 1.$$

For all m big enough we also have $N_m^* \le N_m$. Then

$$\begin{aligned}
&\lim_{m\to\infty} N_m^* C(\delta)^{\frac{2+\epsilon}{\alpha}} 2^{-m\left(\frac{1}{2}-\delta\right)\frac{2+\epsilon}{\alpha}} \\
\le &\lim_{m\to\infty} 2^{\frac{m}{\alpha}} m^{\frac{2}{\alpha}} C(\delta)^{\frac{2+\epsilon}{\alpha}} 2^{-m\left(\frac{1}{2}-\delta\right)\frac{2+\epsilon}{\alpha}} \\
= &C(\delta)^{\frac{2+\epsilon}{\alpha}} \lim_{m\to\infty} m^{\frac{2}{\alpha}} 2^{-\frac{m}{\alpha}\left[\left(\frac{1}{2}-\delta\right)(2+\epsilon)-1\right]} \\
= &0
\end{aligned}$$

and we have

$$\dim\left(\text{supp}(X(T))\right) \le (2+\epsilon)/\alpha.$$

ϵ is arbitrary, so the Hausdorff dimension for the support is bounded from above by $2/\alpha$. □

Corollary 4.8. *If there exist constants $C > 0$ and $\alpha > 0$ such that the total coalescence rates $(\lambda_b)_{b\ge 2}$ of the corresponding Λ-coalescent $(\Pi(t))_{0\le t\le T}$ satisfy*

$$\sum_{b=m+1}^{\infty} \lambda_b^{-1} \le C m^{-\alpha}$$

for m big enough, then for any $T > 0$, with probability one the Λ-Fleming-Viot process has a compact support at time T and the Hausdorff dimension for supp$(X(T))$ is bounded from above by $2/\alpha$.

Proof. From the definitions of λ_b and $\gamma_{b,m}$, we have $\lambda_b \leq \gamma_{b,m}$ for any $b > m$. Thus $\sum_{b=m+1}^{\infty} \lambda_b^{-1} \leq Cm^{-\alpha}$ implies $\sum_{b=m+1}^{\infty} \gamma_{b,m}^{-1} \leq Cm^{-\alpha}$. Then the conclusion is directly obtained from Theorem 4.1. □

The lemma below on a lower bound for the Hausdorff dimension can be found in Falconer [10].

Lemma 4.9. *Let A be any Borel subset of $\mathbb{R}^n$. If there is a mass distribution μ, supported by A such that*

$$I_a(A) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{\|x-y\|^a} \mu(dx)\, \mu(dy) < \infty,$$

then $\dim(A) \geq a$.

By adapting the approach of Proposition 6.14 in Etheridge [9], we could also find a lower bound on the Hausdorff dimension for the support of Λ-Fleming-Viot process at a fixed time.

Proposition 4.10. *Let X be the Λ-Fleming-Viot process with underlying Brownian motion in $\mathbb{R}^d$ for $d \geq 2$. Then for any $T > 0$, with probability one the Hausdorff dimension of supp$(X(T))$ is at least 2.*

Proof. By Lemma 4.9, we need to show that at fixed $T > 0$,

$$\mathbb{E}\left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{\|x-y\|^a} X(T)(dx)\, X(T)(dy)\right] < \infty,$$

where $1 < a < 2$. Write $\langle \mu, f \rangle$ for the integral of function f with respect to measure μ. It is well-known that moments of the Λ-Fleming-Viot process can be expressed in terms of a dual process involving Λ-coalescent and heat flow, see Section 5.2 of [2] for such a dual process. For lack of multiple collisions, expression for the second moment of the Λ-Fleming-Viot process is the same as that for classical Fleming-Viot process given in Proposition 2.27 of [9]. Then for any $\phi_1, \phi_2 \in C_b(\mathbb{R}^d)$, we have

$$\begin{aligned}\mathbb{E}\left[\langle X(T), \phi_1\rangle \langle X(T), \phi_2\rangle\right] =& e^{-rT} \langle X(0), P_T\phi_1\rangle \langle X(0), P_T\phi_2\rangle \\ &+ \left\langle X(0), \int_0^T re^{-rs} P_{T-s}\left(P_s\phi_1 P_s\phi_2\right) ds \right\rangle,\end{aligned}$$

where P_s is the heat flow and r is the total coalescence rate when the number of existing blocks is 2, i.e., $r = \lambda_2$.

Following arguments similar to Proposition 6.14 of [9], we can show that for any nonnegative function of the form $\psi(x, y)$,

$$\begin{aligned}&\mathbb{E}\left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(x,y)\, X(T)(dx) X(T)(dy)\right] \\ =& e^{-rT} \int \cdots \int p(T,z,w)\, p\left(T, z', w'\right) \psi\left(w, w'\right) dw dw' X(0)(dz) X(0)(dz') \\ &+ r \int_0^T \int \cdots \int e^{-rs} p(T-s, z, w)\, p(s, w, y)\, p\left(s, w, y'\right) \psi\left(y, y'\right) dy dy' dw X(0)(dz) ds,\end{aligned}$$

where $p(\cdot,\cdot,\cdot)$ denotes the heat kernel.

Choose $\psi(x,y) = 1/\|x-y\|^a$ for $1 < a < 2$. Following the hint in proof of Proposition 6.14 in [9] we can show that both integrals on the right hand side of the above equation are finite. Therefore, the Hausdorff dimension for the support is at least 2. □

Corollary 4.11. *Suppose that $d \geq 2$ and $\Lambda(\{0\}) > 0$, i.e., the Λ-coalescent has a nontrivial Kingman component. Then at any fixed time $T > 0$, with probability one the Λ-Fleming-Viot process has a compact support of Hausdorff dimension 2.*

Proof. Since $\Lambda(\{0\}) > 0$, the Λ-coalescent has a nontrivial Kingman component. Then

$$\lambda_b \geq \frac{\Lambda(\{0\})b(b-1)}{2}$$

and

$$\sum_{b=m+1}^{\infty} \frac{1}{\lambda_b} \leq \sum_{b=m+1}^{\infty} \frac{2}{\Lambda(\{0\})b(b-1)} = \frac{2}{\Lambda(\{0\})m}.$$

Applying Corollary 4.8 with $\alpha = 1$, the Λ-Fleming-Viot process has a compact support and the Hausdorff dimension for the support is bounded from above by 2 at any fixed time T. This together with Proposition 4.10 implies the desired result. □

Remark 4.12. *Corollary 4.11 complements the result on Hausdorff dimension for the classical Fleming-Viot process in Dawson and Hochberg [4].*

4.5 Examples

4.5.1 The Λ-Fleming-Viot process with its coalescent having the (c, ϵ, γ)-property

Lemma 4.13. *For $n \geq 2$, there exists a positive constant $C(c, \gamma, \epsilon)$ such that the total coalescence rate of the Λ-coalescent with the (c, ϵ, γ)-property satisfies*

$$\lambda_n \geq C(c, \gamma, \epsilon) n^{1+\gamma},$$

where

$$C(c, \gamma, \epsilon) = \frac{c\epsilon^{1-\gamma}}{2\,(1-\gamma)} \left(\frac{1}{3\,(2-\gamma)}\right)^{\gamma} e^{-\frac{\gamma^2}{2(1-\gamma)}}.$$

Proof. By the definition of λ_n, we have

$$\begin{aligned}
\lambda_n &= \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k} \geq \binom{n}{2} \lambda_{n,2} \\
&\geq c \binom{n}{2} \int_0^{\epsilon} x^{-\gamma} (1-x)^{n-2} dx \\
&= c \binom{n}{2} \int_0^{1} (y\epsilon)^{-\gamma} (1-y\epsilon)^{n-2} \epsilon dy \\
&\geq c \binom{n}{2} \epsilon^{1-\gamma} \int_0^{1} y^{-\gamma} (1-y)^{n-2} dy \\
&= \frac{c\epsilon^{1-\gamma} n}{2} \frac{(n-1)!\Gamma(1-\gamma)}{\Gamma(n-\gamma)} \\
&\equiv \frac{c\epsilon^{1-\gamma} n}{2(1-\gamma)} \times B,
\end{aligned}$$

where

$$B = \frac{n-1}{n-1-\gamma} \times \frac{n-2}{n-2-\gamma} \times \cdots \times \frac{3}{3-\gamma} \times \frac{2}{2-\gamma}.$$

It follows from the inequality $\ln(1+x) \geq x - x^2/2$ for $0 < x < 1$ that

$$\begin{aligned}
\ln B &= \sum_{l=2}^{n-1} \ln\left(\frac{l}{l-\gamma}\right) = \sum_{l=2}^{n-1} \ln\left(1+\frac{\gamma}{l-\gamma}\right) \\
&\geq \sum_{l=2}^{n-1} \frac{\gamma}{l-\gamma} - \frac{\gamma^2}{2}\sum_{l=2}^{n-1} \frac{1}{(l-\gamma)^2} \\
&\geq \int_2^n \frac{\gamma}{x-\gamma}dx - \frac{\gamma^2}{2}\int_1^{n-1} \frac{1}{(x-\gamma)^2}dx \\
&= \gamma \ln \frac{n-\gamma}{2-\gamma} - \frac{\gamma^2}{2}\left(\frac{1}{1-\gamma} - \frac{1}{n-1-\gamma}\right) \\
&\geq \gamma \ln \frac{n-\gamma}{2-\gamma} - \frac{\gamma^2}{2(1-\gamma)}.
\end{aligned}$$

Consequently,

$$\lambda_n \geq \frac{c\epsilon^{1-\gamma}n}{2(1-\gamma)}\left(\frac{n-\gamma}{2-\gamma}\right)^{\gamma} e^{-\frac{\gamma^2}{2(1-\gamma)}}.$$

Since $\gamma \in (0,1)$, then $n-\gamma \geq n/3$ for any $n \geq 2$. Therefore,

$$\lambda_n \geq \frac{c\epsilon^{1-\gamma}n}{2(1-\gamma)}\left(\frac{n}{3(2-\gamma)}\right)^{\gamma} e^{-\frac{\gamma^2}{2(1-\gamma)}} \equiv C(c,\gamma,\epsilon)\, n^{1+\gamma}.$$

□

Proposition 4.14. *Let X be any Λ-Fleming-Viot process with underlying Brownian motion in $\mathbb{R}^d$ for $d \geq 2$. If the corresponding Λ-coalescent has the (c,ϵ,γ)-property, then for any $T > 0$, with probability one the random measure X has a compact support at time T. Further,*

$$2 \leq \mathit{dim}\left(\mathit{supp}\left(X(T)\right)\right) \leq 2/\gamma.$$

Proof. It follows from Lemma 4.13 that

$$\begin{aligned}
\sum_{k=m+1}^{\infty} \lambda_k^{-1} &\leq \sum_{k=m+1}^{\infty} \frac{1}{C(c,\gamma,\epsilon)\,k^{1+\gamma}} \\
&\leq \int_m^{\infty} \frac{1}{C(c,\gamma,\epsilon)\,x^{1+\gamma}}dx \\
&= \frac{1}{\gamma C(c,\gamma,\epsilon)\,m^{\gamma}}.
\end{aligned}$$

Applying Corollary 4.8 and Proposition 4.10, the conclusion is immediately available. □

4.5.2 The Beta$(2-\beta,\beta)$-Fleming-Viot process with underlying Brownian motion

Proposition 4.15. *Suppose that $d \geq 2$. For any $T > 0$, with probability one the Beta$(2-\beta,\beta)$-Fleming-Viot process X with underlying Brownian motion in $\mathbb{R}^d$ has a compact support at time T if and only if $\beta \in (1,2)$. Further, for $\beta \in (1,2)$,*

$$2 \leq \mathit{dim}\left(\mathit{supp}\left(X(T)\right)\right) \leq 2/(\beta-1).$$

Proof. For $\beta \in (0,1]$, the corresponding Beta$(2-\beta,\beta)$-coalescent does not come down from infinity.

For $\beta \in (1,2)$, then $\beta - 1 \in (0,1)$ and given $\epsilon \in (0,1)$, for all $x \in [0,\epsilon]$, we have

$$\begin{aligned}\Lambda(dx) &= \frac{\Gamma(2)}{\Gamma(2-\beta)\Gamma(\beta)} x^{1-\beta} (1-x)^{\beta-1} dx \\ &\geq \frac{\Gamma(2)(1-\epsilon)^{\beta-1}}{\Gamma(2-\beta)\Gamma(\beta)} x^{1-\beta} dx,\end{aligned}$$

which implies the Beta$(2-\beta,\beta)$-coalescent has the $(c,\epsilon,\beta-1)$-property.

By Proposition 7.2 of [3] and Proposition 4.14, the Beta$(2-\beta,\beta)$-Fleming-Viot process has a compact support if and only if $\beta \in (1,2)$ and the Hausdorff dimension for its support is between 2 and $2/(\beta-1)$. □

Remark 4.16. *Intuitively, since the Beta-coalescent comes down from infinity at a speed slower than Kingman's coalescent, the particles in the lookdown representation are less correlated. So we expect a higher Hausdorff dimension for the support of Beta-Fleming-Viot process with underlying Brownian motion.*

Remark 4.17. *By Proposition 4.15 the coming down from infinity property is equivalent to the compact support property for Beta$(2-\beta,\beta)$-Fleming-Viot processes, which suggests that the assumption (4.3) is rather mild.*

References

[1] Birkner, M. and Blath, J.: Measure-valued diffusions, general coalescents and population genetic inference. *Trends in Stochastic Analysis*, LMS, Cambridge University Press **353**, (2009), 329-363. MR-2562160

[2] Birkner, M.; Blath, J.; Möhle, M.; Steinrücken, M. and Tams, J.: A modified lookdown construction for the Xi-Fleming-Viot process with mutation and populations with recurrent bottlenecks. *Alea* **6**, (2009), 25-61. MR-2485878

[3] Blath, J.: Measure-valued Processes, Self-similarity and Flickering Random Measures. *Progress in Probability* **61**, (2009), 175-196. MR-2762677

[4] Dawson, D. A. and Hochberg, K. J.: Wandering random measures in the Fleming-Viot model. *The Annals of Probability* **10**, No. 3, (1982), 554-580. MR-0659528

[5] Donnelly, P. and Kurtz, T. G.: A countable representation of the Fleming-Viot measure-valued diffusion. *The Annals of Probability* **24**, No. 2, (1996), 698-742. MR-1404525

[6] Donnelly, P. and Kurtz, T. G.: Genealogical processes for Fleming-Viot models with selection and recombination. *The Annals of Applied Probability* **9**, No. 4, (1999), 1091-1148. MR-1728556

[7] Donnelly, P. and Kurtz, T. G.: Particle representations for measure-valued population models. *The Annals of Probability* **27**, No. 1, (1999), 166-205. MR-1681126

[8] Ethier, S. N. and Kurtz, T. G.: Fleming-Viot processes in population genetics. *SIAM Journal on Control and Optimization* **31**, No. 2, (1993), 345-386. MR-1205982

[9] Etheridge, A. M.: An introduction to superprocesses. *University Lecture Series*, **20**, 2000. MR-1779100

[10] Falconer, K. J.: The geometry of fractal sets. *Cambridge University Press*, 1985. MR-0867284

[11] Iscoe, I.: On the supports of measure-valued critical branching Brownian motion. *The Annals of Probability* **16**, No. 1, (1988), 200-221. MR-0920265

[12] Mueller, C. and Perkins, E.: The compact support property for solutions to the heat equation with noise. *Probability Theory and Related Fields* **93**, (1992), 325-358. MR-1180704

[13] Pitman, J.: Coalescents with multiple collisions. *The Annals of Probability* **27**, No. 4, (1999), 1870-1902. MR-1742892

[14] Reimers, M.: A new result on the support of the Fleming-Viot process, proved by nonstandard construction. *Stochastics and Stochastics Reports* **44**, (1993), 213-223. MR-1277170

[15] Sagitov, S.: The general coalescent with asynchronous mergers of ancestral lines. *Journal of Applied Probability* **36**, No. 4, (1999), 1116-1125. MR-1742154

[16] Schweinsberg, J.: A necessary and sufficient condition for the Λ-coalescent to come down from infinity. *Electronic Communications in Probability* **5**, (2000), 1-11. MR-1736720

Acknowledgments. The authors are thankful to anonymous referees for very helpful suggestions and detailed comments.