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Uniqueness of the representation for G-martingales with finite variation*

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Abstract

Letting $\{\delta_n\}$ be a refining sequence of Rademacher functions on the interval $[0,T]$, we introduce a functional on processes in the G-expectation space by

$$d(K)=\limsup_n \hat{E}[\int_0^T \delta_n(s)dK_s].$$

We prove that $d(K)>0$ if $K_t=\int_0^t \eta_s d\langle B\rangle_s$ with nontrivial $\eta\in M_G^1(0,T)$ and that $d(K)=0$ if $K_t=\int_0^t \eta_s ds$ with $\eta\in M_G^1(0,T)$. This implies the uniqueness of the representation for G-martingales with finite variation, which is the main purpose of this article.

Keywords: uniqueness; representation theorem; G-martingale; finite variation; G-expectation.
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1 Introduction

Recently, [4], [5], [6] introduced the notion of G-expectation space, which is a generalization of probability space. As the counterpart of Wiener space in the linear case, the notions of G-Brownian motion, G-martingale, and Itô integral w.r.t G-Brownian motion were also introduced.

In this article, we consider only the G-expectation space $(\Omega_T, L_G^1(\Omega_T), \hat{E})$ with $\Omega_T=C_0([0,T],R)$ and $\overline{\sigma}^2=\hat{E}(B_1^2)>-\hat{E}(-B_1^2)=\underline{\sigma}^2\geq 0$. Here, the canonical process $\{B_t\}_{t\in[0,T]}$ is called a G-Brownian motion. In this 1-dimensional case, the function $G: R\to R$ is defined by $G(a)=1/2(\overline{\sigma}^2a^+-\underline{\sigma}^2a^-)$.

[5] proposed one fundamental and challenging question: to show the G-martingale representation theorem. More precisely, for any $\xi\in L_G^2(\Omega_T)$, can we have the following representation: for any $t\in[0,T]$,

$$X_t:=\hat{E}_t(\xi)=\hat{E}(\xi)+\int_0^t Z_s dB_s+\int_0^t \eta_s d\langle B\rangle_s-\int_0^t 2G(\eta_s)ds. \tag{1.1}$$

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Remark 1.1. *Clearly,* $K_t := \int_0^t \eta_s d\langle B\rangle_s - \int_0^t 2G(\eta_s)ds$ *is a nonincreasing process and* $K_t \equiv 0$ *if the* G*-expectation reduces to the classical linear case* ($\underline{\sigma} = \overline{\sigma}$)*. In the case* $\underline{\sigma} > \overline{\sigma}$*,* G*-martingales with finite variation have very rich and interesting new structures which nontrivially generalize the classical ones.*

[5] proved the representation (1.1) for cylindrical functions $\xi \in L_{ip}(\Omega_T)$ by Itô's formula in the setting of G-expectation space. [10] and [11] generalized this result and proved a decomposition theorem for G-martingales. The following theorem is from [11]:

For $\xi \in L_G^\beta(\Omega_T)$ with some $\beta > 1$, $X_t = \hat{E}_t(\xi)$, $t \in [0,T]$ has the following decomposition:

$$X_t = X_0 + \int_0^t Z_s dB_s + K_t,\ q.s.,$$

where $\{K_t\}$ is a continuous nonincreasing process with $K_0 = 0$ and $\{K_t\}_{t\in[0,T]}$ a G-martingale. Furthermore, the above decomposition is unique and $\{Z_t\} \in H_G^\alpha(0,T)$, $K_T \in L_G^\alpha(\Omega_T)$ for any $1 \le \alpha < \beta$.

So in order to prove the representation theorem for general G-martiangales, it suffices to prove the representation for G-martingales with finite variation. The main purpose of this article is to prove the uniqueness of the representation.

In [5], processes in form of $K_t(\eta) := \int_0^t \eta_s d\langle B\rangle_s - \int_0^t 2G(\eta_s)ds$, $\eta \in M_G^1(0,T)$ were proved to be G-martingales. However, the uniqueness of the representation remains unresolved. Since $\underline{\sigma}^2(t-s) \le \langle B\rangle_t - \langle B\rangle_s \le \overline{\sigma}^2(t-s)$ for any $0 \le s < t \le T$, we set $\theta_s = \frac{d\langle B\rangle_s}{ds}$. Setting $\eta_s = \theta_s - \overline{\sigma}^2, \eta_s^* = \theta_s - \underline{\sigma}^2$, it's easy to check that $K_t(\eta) = K_t(\eta^*)$. This leads to a popular misunderstanding that the representation is not unique, and that the two parts of the representation are essentially the same things. Note that the counterexample is based on the assumption that θ belongs to $M_G^1(0,T)$, which seems "natural" since in the linear case $M_G^1(0,T)$ consists of all adapted measurable processes with the norm finite. However, in [13] it was proved that θ does not belong to $M_G^1(0,T)$.

In order to prove the uniqueness, we must find ways to distinguish the two classes of processes in forms of $\int_0^t \eta_s d\langle B\rangle_s$ and $\int_0^t \zeta_s ds$, $\eta, \zeta \in M_G^1(0,T)$, which are both processes with absolutely continuous paths.

For a process $\{K_t\}$ with finite variation, motivated by [13], we define

$$d(K) := \limsup_{n\to\infty} \hat{E}[\int_0^T \delta_n(s)dK_s], \tag{1.2}$$

where, for $n \in N$, $\delta_n(s)$ is defined in the following way:

$$\delta_n(s) = \sum_{i=0}^{n-1} (-1)^i 1_{]\frac{iT}{n},\frac{(i+1)T}{n}]}(s),\ \text{for all}\ s \in [0,T].$$

We prove that $d(K) = 0$ if $K_t = \int_0^t \zeta_s ds$ for some $\zeta \in M_G^1(0,T)$ and that $d(K) > 0$ if $K_t = \int_0^t \eta_s d\langle B\rangle_s$ for some $\eta \in M_G^1(0,T)$ such that $\hat{E}[\int_0^T |\eta_s|ds] > 0$.

Remark 1.2. *Assume* $\overline{\sigma} > \underline{\sigma}$*. If* $K_t = \langle B\rangle_t$*, it's easy to prove that*

$$d(K) = \frac{(\overline{\sigma}^2 - \underline{\sigma}^2)T}{2} > 0$$

since $\{\langle B\rangle_t\}$ *is a process with stationary and independent increments. However, for general nontrivial* $\eta \in M_G^1(0,T)$*, it turns out quite difficult to prove*

$$\limsup_{n\to\infty} \hat{E}[\int_0^T \delta_n(s)\eta_s d\langle B\rangle_s] > 0.$$

Due to the results stated above, we can distinguish these two classes of processes completely:

If $\int_0^t \eta_s d\langle B\rangle_s = \int_0^t \zeta_s ds$, for some $\eta, \zeta \in M_G^1(0,T)$, then we have

$$\hat{E}[\int_0^T |\eta_s|ds] = \hat{E}[\int_0^T |\zeta_s|ds] = 0.$$

As an application, we obtain the uniqueness of the representation for G-martingales with finite variation(see Corollary 3.6).

We refer the reader to [3] for a result on the uniqueness of the representation in a different situation. More recently, some developments have been made in this field. [9] presented a result on the existence of the representation. However, in that setting, the representation is not unique. In [8] more properties were developed about the functional d defined in (1.2) and gave a complete representation for G-martingales. In particular, the multidimensional case was considered there.

This article is organized as follows: In section 2, we recall some basic notions and results of G-expectation and the related space of random variables. In section 3, we present the main results and some corollaries. In section 4, we give the proofs to the main results.

2 Preliminaries

We review some basic notions and results of G-expectation and the related space of random variables. More details of this section can be found in [4, 5, 6, 7].

Definition 2.1. *Let Ω be a given set and let $\mathcal{H}$ be a vector lattice of real valued functions defined on Ω with $c \in \mathcal{H}$ for all constants c. $\mathcal{H}$ is considered as the space of "random variables". A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \to R$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have*

(a) Monotonicity: If $X \geq Y$ then $\hat{E}(X) \geq \hat{E}(Y)$.

(b) Constant preservation: $\hat{E}(c) = c$.

(c) Sub-additivity: $\hat{E}(X) - \hat{E}(Y) \leq \hat{E}(X - Y)$.

(d) Positive homogeneity: $\hat{E}(\lambda X) = \lambda \hat{E}(X)$, $\lambda \geq 0$.

$(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

Definition 2.2. *Let X_1 and X_2 be two n-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed, denoted by $X_1 \sim X_2$, if $\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)]$, for all $\varphi \in C_{l,Lip}(R^n)$, where $C_{l,Lip}(R^n)$ is the space of real continuous functions defined on R^n such that*

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|, \text{ for all } x, y \in R^n,$$

where k and C depend only on φ.

Definition 2.3. *In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ a random vector $Y = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$, is said to be independent of another random vector $X = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{E}(\cdot)$, denoted by $Y \perp X$, if for every test function $\varphi \in C_{l,Lip}(R^m \times R^n)$ we have $\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, Y)]_{x=X}]$.*

Definition 2.4. *(G-normal distribution) A d-dimensional random vector $X = (X_1, \cdots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called G-normally distributed if for each $a, b \in R_+$ we have*

$$aX + b\hat{X} \sim \sqrt{a^2 + b^2}X,$$

where $\hat{X}$ is an independent copy of X. Here the letter G denotes the function

$$G(A) := \frac{1}{2}\hat{E}[(AX, X)] : S_d \to R,$$

where S_d denotes the collection of $d \times d$ symmetric matrices.

The function $G(\cdot) : S_d \to R$ is a monotonic, sublinear mapping on S_d and $G(A) = \frac{1}{2}\hat{E}[(AX, X)] \leq \frac{1}{2}|A|\hat{E}[|X|^2] =: \frac{1}{2}|A|\bar{\sigma}^2$ implies that there exists a bounded, convex and closed subset $\Gamma \subset S_d^+$ such that

$$G(A) = \frac{1}{2}\sup_{\gamma \in \Gamma} Tr(\gamma A), \tag{2.1}$$

where S_d^+ denotes the collection of nonnegative elements in S_d.

If there exists some $\beta > 0$ such that $G(A) - G(B) \geq \beta Tr(A - B)$ for any $A \geq B$, we call the G-normal distribution non-degenerate.

Definition 2.5. *i) Let $\Omega_T = C_0([0, T]; R^d)$, the space of real valued continuous functions on $[0, T]$ with $\omega_0 = 0$, be endowed with the supremum norm and let $B_t(\omega) = \omega_t$ be the canonical process. Set $\mathcal{H}_T^0 := \{\varphi(B_{t_1}, ..., B_{t_n}) | n \geq 1, t_1, ..., t_n \in [0, T], \varphi \in C_{l,Lip}(R^{d\times n})\}$. G-expectation is a sublinear expectation defined by*

$$\hat{E}[X] = \tilde{E}[\varphi(\sqrt{t_1 - t_0}\xi_1, \cdots, \sqrt{t_m - t_{m-1}}\xi_m)],$$

for all $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})$, where $\xi_1, \cdots, \xi_n$ are identically distributed d-dimensional G-normally distributed random vectors in a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{E})$ such that ξ_{i+1} is independent of $(\xi_1, \cdots, \xi_i)$ for every $i = 1, \cdots, m$. $(\Omega_T, \mathcal{H}_T^0, \hat{E})$ is called a G-expectation space.

ii) Let us define the conditional G-expectation $\hat{E}_t$ of $\xi \in \mathcal{H}_T^0$ knowing $\mathcal{H}_t^0$, for $t \in [0, T]$. Without loss of generality we can assume that ξ has the representation $\xi = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})$ with $t = t_i$, for some $1 \leq i \leq n$, and we put

$$\hat{E}_{t_i}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})]$$
$$= \tilde{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_i} - B_{t_{i-1}}),$$

where

$$\tilde{\varphi}(x_1, \cdots, x_i) = \hat{E}[\varphi(x_1, \cdots, x_i, B_{t_{i+1}} - B_{t_i}, \cdots, B_{t_m} - B_{t_{m-1}})].$$

Define $\|\xi\|_{p,G} = [\hat{E}(|\xi|^p)]^{1/p}$ for $\xi \in \mathcal{H}_T^0$ and $p \geq 1$. Then for all $t \in [0, T]$, $\hat{E}_t(\cdot)$ is a continuous mapping on $\mathcal{H}_T^0$ w.r.t. the norm $\|\cdot\|_{1,G}$. Therefore it can be extended continuously to the completion $L_G^1(\Omega_T)$ of $\mathcal{H}_T^0$ under the norm $\|\cdot\|_{1,G}$.

Let $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, ..., B_{t_n}) | n \geq 1, t_1, ..., t_n \in [0, T], \varphi \in C_{b,Lip}(R^{d\times n})\}$, where $C_{b,Lip}(R^{d\times n})$ denotes the set of bounded Lipschitz functions on $R^{d\times n}$. [1] proved that the completions of $C_b(\Omega_T)$(the totality of bounded continuous function on Ω_T), $\mathcal{H}_T^0$ and $L_{ip}(\Omega_T)$ under $\|\cdot\|_{p,G}$ are the same and we denote them by $L_G^p(\Omega_T)$.

Definition 2.6. *Let $M_G^0(0, T)$ be the collection of processes in the following form: for a given partition $\{t_0, \cdots, t_N\} = \pi_T$ of $[0, T]$,*

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{]t_j, t_{j+1}]}(t),$$

where $\xi_i \in L_{ip}(\Omega_{t_i})$, $i = 0, 1, 2, \cdots, N-1$. For $p \geq 1$ and $\eta \in M_G^0(0, T)$, let $\|\eta\|_{H_G^p} = \{\hat{E}(\int_0^T |\eta_s|^2 ds)^{p/2}\}^{1/p}$, $\|\eta\|_{M_G^p} = \{\hat{E}(\int_0^T |\eta_s|^p ds)\}^{1/p}$ and denote $H_G^p(0, T)$, $M_G^p(0, T)$ the completions of $M_G^0(0, T)$ under the norms $\|\cdot\|_{H_G^p}$, $\|\cdot\|_{M_G^p}$ respectively.

Theorem 2.7. *([1]) There exists a tight subset $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$, the totality of probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$, such that*

$$\hat{E}(\xi) = \sup_{P \in \mathcal{P}} E_P(\xi) \ \text{ for } \ \text{all } \xi \in \mathcal{H}_T^0.$$

$\mathcal{P}$ is called a set that represents $\hat{E}$.

Remark 2.8. *(i) Let $(\Omega^0, \mathcal{F}^0, P^0)$ be a probability space and $\{W_t\}$ be a d-dimensional Brownian motion under P^0. Let $F^0 = \{\mathcal{F}_t^0\}$ be the augmented filtration generated by W. [1] proved that*

$$\mathcal{P}_M := \{P_h | P_h = P^0 \circ X^{-1}, X_t = \int_0^t h_s dW_s, h \in L^2_{F^0}([0,T]; \Gamma^{1/2})\}$$

is a set that represents $\hat{E}$, where $\Gamma^{1/2} := \{\gamma^{1/2} | \gamma \in \Gamma\}$, Γ is the set in the representation of $G(\cdot)$ in the formula (2.1) and $L^2_{F^0}([0,T]; \Gamma^{1/2})$ is the set of F^0-adapted measurable processes with values in $\Gamma^{1/2}$.

(ii) For the 1-dimensional case, $L^2_{F^0}([0,T]; \Gamma^{1/2})$ reduces to the form below:

$$\{h | \ h \text{ is an adapted measurable process w.r.t. } F^0 \text{ and } \underline{\sigma} \leq |h_s| \leq \overline{\sigma}\}.$$

3 Main results

In the following two sections, the function sgn : $R \to \{1, -1\}$ is defined by $\text{sgn}(x) = 1$ if $x \geq 0$ and $\text{sgn}(x) = -1$ if $x < 0$. The main result in this section is Theorem 3.3, relying on which we prove the uniqueness of the representation for G-martingales(Corollary 3.6).

Proposition 3.1. *For each $\eta \in M_G^1(0,T)$, by abuse of notation, let*

$$d(\eta) = \limsup_{n \to \infty} \hat{E}[\int_0^T \delta_n(s) \eta_s d\langle B \rangle_s].$$

Then

$$-\frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2} \hat{E}[-\int_0^T |\eta_s| ds] \leq d(\eta) \leq \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2} \hat{E}[\int_0^T |\eta_s| ds]. \tag{3.1}$$

Proof. For $\eta, \zeta \in M_G^1(0,T)$, we have

$$|\hat{E}[-\int_0^T |\eta_s| ds] - \hat{E}[-\int_0^T |\zeta_s| ds]| \leq \hat{E}[\int_0^T |\eta_s - \zeta_s| ds]$$

and

$$|d(\eta) - d(\zeta)| \leq \hat{E}[\int_0^T |\eta_s - \zeta_s| d\langle B \rangle_s].$$

Hence both $\hat{E}[-\int_0^T |\cdot| ds]$ and $d(\cdot)$ are continuous functionals on $M_G^1(0,T)$ w.r.t. the norm $\|\cdot\|_{M_G^1}$, and consequently it suffices to prove the assertion for $\eta \in M_G^0(0,T)$. Let $\eta_s = \sum_{i=0}^{m-1} \xi_{t_i} 1_{]t_i, t_{i+1}]}(s)$, $\xi_{t_i} \in L_G^1(\Omega_{t_i})$, $i = 0, \cdots, m-1$. We first prove the inequality on the right-hand side. Note that

$$\begin{aligned}
&\hat{E}[\int_0^T \delta_n(s)\eta_s d\langle B\rangle_s] - \frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}\hat{E}[\int_0^T |\eta_s|ds] \\
= \;& \hat{E}[\sum_{i=0}^{m-1}|\xi_{t_i}|\int_{t_i}^{t_{i+1}}\delta_n(s)\mathrm{sgn}(\xi_{t_i})d\langle B\rangle_s] - \hat{E}[\sum_{i=0}^{m-1}|\xi_{t_i}|\int_{t_i}^{t_{i+1}}\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}ds] \\
\leq \;& \sum_{i=0}^{m-1}\hat{E}[|\xi_{t_i}|(\int_{t_i}^{t_{i+1}}\delta_n(s)\mathrm{sgn}(\xi_{t_i})d\langle B\rangle_s - \int_{t_i}^{t_{i+1}}\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}ds)] \\
\leq \;& \sum_{i=0}^{m-1}[\hat{E}(|\xi_{t_i}|)a_i(n)],
\end{aligned}$$

where $a_i(n)=\max\{b_i(n),c_i(n)\}\to 0$ as n goes to infinity. Here

$$\begin{aligned}
b_i(n) &= |\hat{E}(\int_{t_i}^{t_{i+1}}\delta_n(s)d\langle B\rangle_s - \int_{t_i}^{t_{i+1}}\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}ds)|, \\
c_i(n) &= |\hat{E}(-\int_{t_i}^{t_{i+1}}\delta_n(s)d\langle B\rangle_s - \int_{t_i}^{t_{i+1}}\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}ds)|.
\end{aligned}$$

So we have

$$d(\eta)\leq\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}\hat{E}[\int_0^T|\eta_s|ds].$$

On the other hand,

$$\begin{aligned}
&\hat{E}[\int_0^T \delta_n(s)\eta_s d\langle B\rangle_s] + \frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}\hat{E}[-\int_0^T |\eta_s|ds] \\
= \;& \hat{E}[\sum_{i=0}^{m-1}|\xi_{t_i}|\int_{t_i}^{t_{i+1}}\delta_n(s)\mathrm{sgn}(\xi_{t_i})d\langle B\rangle_s] + \hat{E}[\sum_{i=0}^{m-1}(-|\xi_{t_i}|)\int_{t_i}^{t_{i+1}}\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}ds] \\
\geq \;& \hat{E}[\sum_{i=0}^{m-1}|\xi_{t_i}|(\int_{t_i}^{t_{i+1}}\delta_n(s)\mathrm{sgn}(\xi_{t_i})d\langle B\rangle_s - \int_{t_i}^{t_{i+1}}\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}ds)] \\
\geq \;& \hat{E}[\sum_{i=0}^{m-2}|\xi_{t_i}|(\int_{t_i}^{t_{i+1}}\delta_n(s)\mathrm{sgn}(\xi_{t_i})d\langle B\rangle_s - \int_{t_i}^{t_{i+1}}\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}ds) - |\xi_{t_{m-1}}|a_{m-1}(n)] \\
\geq \;& \hat{E}[\sum_{i=0}^{m-2}|\xi_{t_i}|(\int_{t_i}^{t_{i+1}}\delta_n(s)\mathrm{sgn}(\xi_{t_i})d\langle B\rangle_s - \int_{t_i}^{t_{i+1}}\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}ds)] - \hat{E}[|\xi_{t_{m-1}}|a_{m-1}(n)] \\
\geq \;& \sum_{i=0}^{m-1}[-\hat{E}(|\xi_{t_i}|)a_i(n)].
\end{aligned}$$

So by the same arguments as above we have

$$-\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}\hat{E}[-\int_0^T|\eta_s|ds]\leq d(\eta).$$

□

Remark 3.2. *(i) A straightforward corollary of Proposition 3.1 is that if $\int_0^T|\eta_s|ds$ is symmetric (i.e., $\hat{E}[\int_0^T|\eta_s|ds]=-\hat{E}[-\int_0^T|\eta_s|ds]$), the equality below holds:*

$$d(\eta)=\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}\hat{E}[\int_0^T|\eta_s|ds].$$

(ii) The inequalities in (3.1) *may be strict:*
Let $\eta_s = \langle B\rangle_{T/2}1_{]T/2,T]}(s) + a1_{[0,T/2]}(s)$, $a = T(\overline{\sigma}^2 - \underline{\sigma}^2)/4$.
Then

$$d(\eta) = \lim_{n\to\infty} \hat{E}[\int_0^T \delta_{2n}(s)\eta_s d\langle B\rangle_s] = a\overline{\sigma}^2T/2,$$

$$\frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}\hat{E}[\int_0^T |\eta_s|ds] = a^2 + a\overline{\sigma}^2T/2,$$

$$-\frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}\hat{E}[-\int_0^T |\eta_s|ds] = -a^2 + a\overline{\sigma}^2T/2.$$

Now, we shall state the main result of this article, whose proof is postponed to Section 4.

Theorem 3.3. *(i) For* $\eta \in M_G^1(0,T)$*, we have*

$$\lim_{n\to\infty} \hat{E}[\int_0^T \delta_n(s)\eta_s ds] = 0;$$

ii) Assume $\overline{\sigma} > \underline{\sigma}$*. For* $\eta \in M_G^1(0,T)$ *with* $\hat{E}[\int_0^T |\eta_s|ds] > 0$*, we have*

$$d(\eta) = \limsup_{n\to\infty} \hat{E}[\int_0^T \delta_n(s)\eta_s d\langle B\rangle_s] > 0.$$

Remark 3.4. *(i) Let* $(\Omega, F, \mathcal{F}, P)$ *be a filtered probability space. We recall that for any progressively measurable process* η *such that* $E[\int_0^T |\eta_s|ds] < \infty$*, we have*

$$\lim_{n\to\infty} E[\int_0^T \delta_n(s)\eta_s ds] = 0.$$

Therefore, ii) of Theorem 3.3 presents a particular property of G*-expectation space relative to probability space.*

(ii) The second assertion of Theorem 3.3 is motivated by the following simple case: for any $n \in N$*, we have*

$$\hat{E}[\int_0^T \delta_{2n}(s)d\langle B\rangle_s] = \hat{E}[\sum_{i=0}^{n-1}(-(\langle B\rangle_{\frac{(2i+2)T}{2n}} - \langle B\rangle_{\frac{(2i+1)T}{2n}}) + (\langle B\rangle_{\frac{(2i+1)T}{2n}} - \langle B\rangle_{\frac{2iT}{2n}}))].$$

Since $\langle B\rangle$ *is a process with stationary and independent increments, we have*

$$\begin{aligned} &\hat{E}[\sum_{i=0}^{n-1}(-(\langle B\rangle_{\frac{(2i+2)T}{2n}} - \langle B\rangle_{\frac{(2i+1)T}{2n}}) + (\langle B\rangle_{\frac{(2i+1)T}{2n}} - \langle B\rangle_{\frac{2iT}{2n}}))] \\ = &\sum_{i=0}^{n-1}\{\hat{E}[-(\langle B\rangle_{\frac{(2i+2)T}{2n}} - \langle B\rangle_{\frac{(2i+1)T}{2n}})] + \hat{E}[\langle B\rangle_{\frac{(2i+1)T}{2n}} - B_{\frac{2iT}{2n}}]\}. \end{aligned}$$

Noting that

$$\hat{E}[-(\langle B\rangle_{\frac{(2i+2)T}{2n}} - \langle B\rangle_{\frac{(2i+1)T}{2n}})] = -\frac{\underline{\sigma}^2T}{2n},\ \hat{E}[\langle B\rangle_{\frac{(2i+1)T}{2n}} - B_{\frac{2iT}{2n}}] = \frac{\overline{\sigma}^2T}{2n},$$

we have

$$\hat{E}[\int_0^T \delta_{2n}(s)d\langle B\rangle_s] = \frac{(\overline{\sigma}^2 - \underline{\sigma}^2)T}{2}.$$

Corollary 3.5. *Let $\zeta, \eta \in M_G^1(0,T)$. If $\int_0^t \eta_s d\langle B\rangle_s = \int_0^t \zeta_s ds$ for all $t \in [0,T]$, we have $E[\int_0^T |\eta_s| ds] = \hat{E}[\int_0^T |\zeta_s| ds] = 0$.*

Proof. By i) of Theorem 3.3, we have

$$\limsup_{n\to\infty} \hat{E}[\int_0^T \delta_n(s)\eta_s d\langle B\rangle_s] = \lim_{n\to\infty} \hat{E}[\int_0^T \delta_n(s)\zeta_s ds] = 0.$$

By ii) of Theorem 3.3, we have $\hat{E}[\int_0^T |\eta_s| ds] = 0$, which leads to $\hat{E}[\int_0^T |\zeta_s| ds] = 0$. □

The following corollary is about the uniqueness of representation for G-martingales with finite variation.

Corollary 3.6. *Let $\zeta, \eta \in M_G^1(0,T)$. If for all $t \in [0,T]$,*

$$\int_0^t \eta_s d\langle B\rangle_s - \int_0^t 2G(\eta_s) ds = \int_0^t \zeta_s d\langle B\rangle_s - \int_0^t 2G(\zeta_s) ds, \tag{3.2}$$

we have $\hat{E}[\int_0^T |\eta_s - \zeta_s| ds] = 0$.

Proof. By the assumption, we have

$$\int_0^t (\eta_s - \zeta_s) d\langle B\rangle_s = \int_0^t 2[G(\eta_s) - G(\zeta_s)] ds, \text{ for all } t \in [0,T].$$

Since $\eta - \zeta, 2[G(\eta) - G(\zeta)] \in M_G^1(0,T)$, we have $\hat{E}[\int_0^T |\eta_s - \zeta_s| ds] = 0$ by Corollary 3.5. □

Remark 3.7. *(i) Recall that $G(a) = \frac{1}{2}(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$. For $\varepsilon \in (0, \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2})$, [3] defined G_ε in the following way:*

$$G_\varepsilon(a) = G(a) - \frac{\varepsilon}{2}|a|, \text{ for all } a \in R.$$

Note that in the proof to the second assertion of Theorem 3.3 in the next section we have actually proved that

$$d(\eta) \geq \varepsilon \hat{E}_{G_\varepsilon}[\int_0^T |\eta_s| ds]. \tag{3.3}$$

(ii) For $\eta \in M_G^1(0,T)$, let $K_t = \int_0^t \eta_s d\langle B\rangle_s - \int_0^t 2G(\eta_s) ds$. Then, by i) of Theorem 3.3, we have

$$\hat{E}(-K_T) \geq \limsup_{n\to\infty} \hat{E}(\int_0^T \delta_n(s) dK_s) = d(\eta). \tag{3.4}$$

This, combined with (3.3), leads to the following estimate:

$$\hat{E}[-K_T] \geq \varepsilon \hat{E}_{G_\varepsilon}[\int_0^T |\eta_s| ds], \tag{3.5}$$

which was already proved in [3]. So for $\eta \in M_G^1(0,T)$ such that $K_T := \int_0^T \eta_s d\langle B\rangle_s - \int_0^T 2G(\eta_s) ds = 0$ we have, by (3.5), $\eta \equiv 0$. However, for $\eta, \zeta \in M_G^1(0,T)$ such that (3.2) holds, (3.5) does not lead to $\eta \equiv \zeta$ since the nonlinearity of G, which is the main difficulty in dealing with the uniqueness of the representation.

4 Proof to Theorem 3.3

In order to prove Theorem 3.3, we first introduce two lemmas.

Let $\Omega_T = C_b([0,T];R)$ be endowed with the supremum norm and let $\sigma : [0,T] \times \Omega_T \to R$ be a measurable mapping satisfying

i) σ is bounded;

ii) There exists $L > 0$ such that $|\sigma(s,\omega) - \sigma(s,\omega')| \le L\|\omega - \omega'\|$ for any $s \in [0,T]$ and $\omega, \omega' \in C_b([0,T];R)$;

iii) For $t \in [0,T]$, $\sigma(t,\cdot)$ is $\mathcal{B}_t(\Omega_T)$ measurable.

Then the following lemma is easy.

Lemma 4.1. *Let $(\Omega, F, \mathcal{F}, P)$ be a filtered probability space and let M be a continuous F-martingale with $\langle M\rangle_t - \langle M\rangle_s \le C(t-s)$ for some $C > 0$ and any $0 \le s < t \le T$. Let $F^M = \{\mathcal{F}_t^M\}$ be the augmented filtration generated by M. Then for any $Y_0 \in L^2(\mathcal{F}_0^M)$, there exists a unique F-adapted continuous process Y with $E[\sup_{t\in[0,T]} |Y_t|^2] < \infty$ such that $Y_t = Y_0 + \int_0^t \sigma(s,Y)dM_s$. Moreover, Y is F^M-adapted.*

We believe that Lemma 4.1 must be covered by some more general result. For readers' convenience, we give a brief proof here.

Proof. Uniqueness. Assume that X^i, $i = 1,2$ are F-adapted continuous processes such that $E[\sup_{t\in[0,T]} |X_t^i|^2] < \infty$ and $X_t^i = Y_0 + \int_0^t \sigma(s,X^i)dM_s$. Set $\hat{X} = X^1 - X^2$ and $\hat{\sigma}_s = \sigma(s,X^1) - \sigma(s,X^2)$. Then we have

$$\sup_{0\le s\le t} \hat{X}_s^2 \le \sup_{0\le s\le t} |\int_0^s \hat{\sigma}_r dM_r|^2.$$

By Doob's inequality, we have

$$A_t := E[\sup_{0\le s\le t} \hat{X}_s^2] \le 4CL^2 \int_0^t E[\hat{X}_s^2]ds \le 4CL^2 \int_0^t A_s ds.$$

By Gronwall's inequality, we have $A_T = 0$.

Existence. Let $Y^0 \equiv Y_0$. For $m \ge 0$, set

$$Y_t^{m+1} = Y_0 + \int_0^t \sigma(s,Y^m)dM_s. \tag{4.1}$$

Clearly, for any m, Y^m is a continuous F^M-adapted process.

For any $m,n \in N$, set $\hat{X} = Y^{m+1} - Y^{n+1}$, $\hat{x} = Y^m - Y^n$ and $\hat{\sigma}_s = \sigma(s,Y^m) - \sigma(s,Y^n)$. Choose $\beta > CL^2/2$ and apply Itô's formula to $e^{-2\beta t}|\hat{X}_t|^2$:

$$e^{-2\beta t}|\hat{X}_t|^2 = -2\beta \int_0^t e^{-2\beta s}|\hat{X}_s|^2 ds + \int_0^t 2e^{-2\beta s}\hat{X}_s d\hat{X}_s + \int_0^t e^{-2\beta s}\hat{\sigma}_s^2 d\langle M\rangle_s.$$

Since $\hat{X}$ is a square integrable martingale and $\hat{\sigma}$ is bounded, we know that

$$\langle \int_0^\cdot 2e^{-2\beta s}\hat{X}_s d\hat{X}_s\rangle_t = \int_0^t 4e^{-4\beta s}|\hat{X}_s|^2 \hat{\sigma}_s^2 d\langle M\rangle_s$$

is L^1-integrable, which implies that $\int_0^t 2e^{-2\beta s}\hat{X}_s d\hat{X}_s$ is a square integrable martingale. So we have

$$\frac{1}{2\beta}e^{-2\beta T}E[|\hat{X}_T|^2] + E[\int_0^T e^{-2\beta s}|\hat{X}_s|^2 ds] \le \frac{CL^2}{2\beta}E[\int_0^T e^{-2\beta s}|\hat{x}_s|^2 ds]. \tag{4.2}$$

Noting that $\alpha = \sqrt{\frac{CL^2}{2\beta}} < 1$, by (4.2), we conclude that

$$\|Y^{k+l} - Y^l\|_{2,\beta} \leq \sum_{i=l}^{\infty} \|Y^{i+1} - Y^i\|_{2,\beta} \leq \sum_{i=l}^{\infty} \alpha^i \|Y^1 - Y^0\|_{2,\beta} = \frac{\alpha^l}{1-\alpha}\|Y^1 - Y^0\|_{2,\beta},$$

where $\|X\|^2_{2,\beta} = E[\int_0^T e^{-2\beta s}|X_s|^2 ds]$. So $\{Y^m\}$ is a Cauchy sequence under the norm $\|\cdot\|_{2,\beta}$. By (4.2) and Doob's inequality, we conclude that

$$\sup_{m>0} E[\sup_{t\in[0,T]} |Y_t^{m+n} - Y_t^n|^2] \to 0$$

as n goes to infinity. So there exists a continuous F^M-adapted process Y such that $E[\sup_{t\in[0,T]}|Y_t|^2] < \infty$ and $Y_t = Y_0 + \int_0^t \sigma(s,Y)dM_s$. □

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{W_t\}$ be a standard 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$. Let F^W be the augmented filtration generated by W.

Denote by $\mathcal{A}^0([c,C])$, for some $0 \leq c \leq C < \infty$, the collection of F^W adapted measurable processes in the following form

$$h_s = \sum_{i=0}^{m-1} \xi_i 1_{]\frac{iT}{m}, \frac{(i+1)T}{m}]}(s), \tag{4.3}$$

where $\xi_i = \psi_i(\int_{\frac{(i-1)T}{m}}^{\frac{iT}{m}} h_s dW_s, \cdots, \int_0^{\frac{T}{m}} h_s dW_s)$, $\psi_i \in C_{b,lip}(R^i)$, $c \leq |\psi_i| \leq C$. Denote by $\mathcal{A}([c,C])$ the collection of F^W adapted measurable processes such that $c \leq |h_s| \leq C$.

Lemma 4.2. *$\mathcal{A}^0([c,C])$ is dense in $\mathcal{A}([c,C])$ under the norm*

$$\|h\|_2 = [E(\int_0^T |h_s|^2 ds)]^{1/2}.$$

Proof. Since $\cup_{C>\varepsilon>0}\mathcal{A}([\varepsilon,C])$ is dense in $\mathcal{A}([0,C])$ under the norm $\|\cdot\|_2$, it suffices to show the $c > 0$ case.

Let $h_s = H_s(W) = \sum_{i=0}^{m-1} \xi_i 1_{]\frac{iT}{m}, \frac{(i+1)T}{m}]}(s)$, where

$$\xi_i = \varphi_i(W_{\frac{iT}{m}} - W_{\frac{(i-1)T}{m}}, \cdots, W_{\frac{T}{m}} - W_0),$$

$$\varphi_i \in C_{b,lip}(R^i), c \leq |\varphi_i| \leq C.$$

Then $\sigma(s,\omega) = H_s^{-1}(\omega)$ is a bounded Lipschitz function. Let $X_t := \int_0^t h_s dW_s$. Since $W_t = \int_0^t \sigma(s,W)dX_s$, we conclude, by Lemma 4.1, that W is F^X-adapted.

For a process $\{X_t\}$, we denote the vector $(X_T - X_{\frac{(m-1)T}{m}}, \cdots, X_{\frac{T}{m}} - X_0)$ by $X^m_{[0,T]}$.

For arbitrary $\varepsilon_i > 0$, $i = 0, \cdots, m-1$, since X is a process with continuous paths, there exists $n_i \in N$ and $\psi_i \in C_{b,lip}(R^{in_i})$ with the Lipschitz constant $L_i = L_i(\varepsilon_i)$ such that $E[|\xi_i - \widetilde{\xi}_i|^2] < \varepsilon_i^2$. Here $\widetilde{\xi}_i = \psi_i(X^{in_i}_{[0,\frac{iT}{m}]})$, $c \leq |\psi_i| \leq C$.

Define $\widehat{\xi}_i$ in the following way:

Set $\widehat{\xi}_0 = \widetilde{\xi}_0$;

For $s \in]0, \frac{T}{m}]$, set $\widehat{h}_s = \widehat{\xi}_0$;

Assuming that we have defined $\widehat{h}_s$ for all $s \in [0, \frac{iT}{m}]$, $0 \leq i \leq m-1$,

set $\widehat{X}_t := \int_0^t \widehat{h}_s dW_s$, for $t \in [0, \frac{iT}{m}]$, and $\widehat{\xi}_i = \psi_i(\widehat{X}^{in_i}_{[0,\frac{iT}{m}]})$;

For $s \in]\frac{iT}{m}, \frac{(i+1)T}{m}]$, set $\widehat{h}_s = \widehat{\xi}_i$.

It is obvious that $\hat{h}$ belongs to $\mathcal{A}^0([c,C])$.

We claim that for any $m-1 \geq i \geq 1$,

$$E[|\widehat{\xi}_i - \widetilde{\xi}_i|^2] \leq \sum_{j=0}^{i-1} A_j^i \varepsilon_j^2, \tag{4.4}$$

where $A_j^i = 2TL_i^2(\sum_{k=j+1}^{i-1} A_j^k + 1)$, for $i \geq j+2$, $A_{i-1}^i = 2TL_i^2$, which shows that A_j^i depends only on $L_{j+1}, \cdots, L_i$ and T.

Indeed, $E[|\widehat{\xi}_1 - \widetilde{\xi}_1|^2] \leq L_1^2 E[|\widehat{\xi}_0 - \xi_0|^2|]E[|W^{n_1}_{[0,\frac{T}{m}]}|^2] \leq \frac{T}{m} L_1^2 \varepsilon_0^2 \leq A_0^1 \varepsilon_0^2$. Assume (4.4) holds for $1 \leq i \leq l$; we will prove it for $i = l+1$.

$$\begin{aligned} E[|\widehat{\xi}_{l+1} - \widetilde{\xi}_{l+1}|^2] &\leq L_{l+1}^2 \sum_{i=0}^{l} E[|\widehat{\xi}_i - \xi_i|^2] E[|W^{n_{l+1}}_{[\frac{iT}{m}, \frac{(i+1)T}{m}]}|^2] \\ &\leq TL_{l+1}^2 \sum_{i=0}^{l} E[|\widehat{\xi}_i - \xi_i|^2] \\ &\leq 2TL_{l+1}^2 \sum_{i=0}^{l} E[(|\widehat{\xi}_i - \widetilde{\xi}_i|^2 + |\widetilde{\xi}_i - \xi_i|^2)]. \end{aligned}$$

By the assumption, we have

$$\begin{aligned} & 2TL_{l+1}^2 \sum_{i=0}^{l} E[(|\widehat{\xi}_i - \widetilde{\xi}_i|^2 + |\widetilde{\xi}_i - \xi_i|^2)] \\ \leq\ & 2TL_{l+1}^2 (\sum_{i=0}^{l} \varepsilon_i^2 + \sum_{i=1}^{l} \sum_{j=0}^{i-1} A_j^i \varepsilon_j^2) \\ =\ & 2TL_{l+1}^2 [\sum_{j=0}^{l-1} (\sum_{i=j+1}^{l} A_j^i + 1) \varepsilon_j^2 + \varepsilon_l^2] \\ =\ & \sum_{j=0}^{l} A_j^{l+1} \varepsilon_j^2. \end{aligned}$$

Then

$$\begin{aligned} E[|\widehat{\xi}_i - \xi_i|^2] &\leq 2(E[|\widehat{\xi}_i - \widetilde{\xi}_i|^2] + E[|\widetilde{\xi}_i - \xi_i|^2]) \\ &\leq 2\varepsilon_i^2 + 2\sum_{j=0}^{i-1} A_j^i \varepsilon_j^2 =: \sum_{j=0}^{i} B_j^i \varepsilon_j^2, \end{aligned}$$

which shows that B_j^i depends only on $L_{j+1}, \cdots, L_i$, T for $j < i$ and $B_i^i = 2$. So for any $\varepsilon > 0$, we can choose $\widehat{\xi}_i$, $i = 0, \cdots, m-1$ defined above such that $E[|\widehat{\xi}_i - \xi_i|^2] < \varepsilon$ for all $i = 0, \cdots, m-1$. Then

$$E[\int_0^T |h_s - \widehat{h}_s|^2] < T\varepsilon.$$

$\square$

Remark 4.3. *According to Lemma 4.2, the collection of F^W adapted measurable processes in the form of* (4.3) *with $\xi_i = \psi_i(\int_{\frac{(i-1)T}{m}}^{\frac{iT}{m}} h_s dW_s, \cdots, \int_0^{\frac{T}{m}} h_s dW_s)$, $\psi_i \in C_{b,lip}(R^i)$ is dense in $L^2_{F^W}([0,T] \times \Omega)$, the space of F^W adapted measurable processes endowed with the L^2-norm.*

Proof to Theorem 3.3. i) The proof to the first assertion is easy. For $\eta \in M_G^0(0,T)$, the claim is obvious. For $\eta \in M_G^1(0,T)$, there exists a sequence of $\{\eta^m\} \subset M_G^0(0,T)$ such that $\hat{E}[\int_0^T |\eta_s^m - \eta_s|ds] \to 0$ as $m \to \infty$. Then $|\hat{E}[\int_0^T \delta_n(s)\eta_s ds]| \leq |\hat{E}[\int_0^T \delta_n(s)\eta_s^m ds]| + \hat{E}[\int_0^T |\eta_s^m - \eta_s|ds]$. First let $n \to \infty$, then let $m \to \infty$, and we get the desired result.

ii) Now we prove the second assertion. Let $(\Omega^0, \mathcal{F}^0, P^0)$ be a probability space and $\{W_t\}$ be a 1-dimensional standard Brownian motion under P^0. Let $F^0 = \{\mathcal{F}_t^0\}$ be the augmented filtration generated by W.

For $\eta \in M_G^1(0,T)$ with $\hat{E}[\int_0^T |\eta_s|ds] > 0$, by Theorem 2.7 and Remark 2.8, there exists an F^0 adapted measurable process g with $\underline{\sigma} \leq |g_s| \leq \overline{\sigma}$ such that $E_{P_g}[\int_0^T |\eta_s|ds] > 0$, where $P_g = P^0 \circ [\int_0^\cdot g_s dW_s]^{-1}$. Set $g_s^n = \mathrm{sgn}(g_s)(|g_s| \vee \sqrt{\underline{\sigma}^2 + \frac{1}{n}}) \wedge \sqrt{\overline{\sigma}^2 - \frac{1}{n}}$. Since P_{g^n} converges weekly to P_g and $\int_0^T |\eta_s|ds$ belongs to $L_G^1(\Omega_T)$, we have

$$\lim_n E_{P_{g^n}}[\int_0^T |\eta_s|ds] = E_{P_g}[\int_0^T |\eta_s|ds] > 0.$$

So, by Theorem 2.7 and Remark 2.8 again, there exists $\varepsilon > 0$ such that $\hat{E}_{G_\varepsilon}[\int_0^T |\eta_s|ds] > 0$, and consequently, for any ϵ with $\hat{E}_{G_\varepsilon}[\int_0^T |\eta_s|ds] > \epsilon > 0$, there exists an F^0 adapted measuable process h with $\underline{\sigma}^2 + \varepsilon \leq h_s^2 \leq \overline{\sigma}^2 - \varepsilon$ such that $E_{P_h}[\int_0^T |\eta_s|ds] \geq \hat{E}_{G_\varepsilon}[\int_0^T |\eta_s|ds] - \epsilon =: A > 0$. By the definition of $M_G^1(0,T)$, for any $\frac{A\varepsilon}{(\overline{\sigma}^2+\varepsilon)} > \delta > 0$, there exists $\zeta \in M_G^0(0,T)$ such that

$$\hat{E}[\int_0^T |\eta_s - \zeta_s|ds] < \delta.$$

Without loss of generality, by Lemma 4.2, we assume that there exists $m \in N$ such that

$$\zeta_s = \sum_{i=0}^{m-1} \xi_{\frac{iT}{m}} 1_{]\frac{iT}{m}, \frac{(i+1)T}{m}]}(s)$$

where $\xi_{\frac{iT}{m}} = \varphi_i(B_{\frac{iT}{m}} - B_{\frac{(i-1)T}{m}}, \cdots, B_{\frac{T}{m}} - B_0)$, $\varphi_i \in C_{b,lip}(R^i)$, for all $0 \leq i \leq m-1$; and that

$$h_s = \sum_{i=0}^{m-1} a_{\frac{iT}{m}} 1_{]\frac{iT}{m}, \frac{(i+1)T}{m}]}(s)$$

where $a_{\frac{iT}{m}} = \psi_i(\int_{\frac{(i-1)T}{m}}^{\frac{iT}{m}} h_s dW_s, \cdots, \int_0^{\frac{T}{m}} h_s dW_s)$, $\underline{\sigma}^2 + \varepsilon \leq |\psi_i|^2 \leq \overline{\sigma}^2 - \varepsilon$, $\psi_i \in C_{b,lip}(R^i)$, for all $0 \leq i \leq m-1$.

We have divided the following proof into four steps:

1. Here we give some notations those will be used in the sequel steps.

Define $H^i : [\underline{\sigma}^2 + \varepsilon, \overline{\sigma}^2 - \varepsilon] \to [\underline{\sigma}, \overline{\sigma}]$, i=1, -1 in the following way:

$$H^1(x)^2 = \overline{\sigma}^2 1_{[x \geq \frac{\overline{\sigma}^2 + \underline{\sigma}^2}{2}]} + (2x - \underline{\sigma}^2) 1_{[x < \frac{\overline{\sigma}^2 + \underline{\sigma}^2}{2}]};$$

$$H^{-1}(x)^2 = (2x - \overline{\sigma}^2) 1_{[x \geq \frac{\overline{\sigma}^2 + \underline{\sigma}^2}{2}]} + \underline{\sigma}^2 1_{[x < \frac{\overline{\sigma}^2 + \underline{\sigma}^2}{2}]}.$$

It's easily seen that $H^1(x)^2 + H^{-1}(x)^2 = 2x$ and $H^1(x)^2 - H^{-1}(x)^2 \geq 2\varepsilon$.

For $n \in N$, define $H_n^i :]0, T/m] \times [\underline{\sigma}^2 + \varepsilon, \overline{\sigma}^2 - \varepsilon] \to [\underline{\sigma}, \overline{\sigma}]$, $i = 1, -1$ by

$$H_n^i(s,x) = \sum_{j=0}^{2n-1} 1_{]\frac{jT}{2mn}, \frac{(j+1)T}{2mn}]}(s) H^{(-1)^j i}(x).$$

2. Fix $n \in N$. We construct an adapted measurable process h^n based on h, ζ.

Set $a_0^n = a_0$, $\xi_0^n = \xi_0$;

For $s \in]0, \frac{T}{m}]$, set $h^n_s = H_n^{\mathrm{sgn}(\xi^n_0)}(s, (a^n_0)^2)$;

Assume that we have defined h^n_s for all $s \in [0, \frac{iT}{m}]$, $0 \le i \le m-1$. Set

$a^n_{\frac{iT}{m}} = \psi_i(\int_{\frac{(i-1)T}{m}}^{\frac{iT}{m}} h^n_s dW_s, \cdots, \int_0^{\frac{T}{m}} h^n_s dW_s)$,

$\xi^n_{\frac{iT}{m}} = \varphi_i(\int_{\frac{(i-1)T}{m}}^{\frac{iT}{m}} h^n_s dW_s, \cdots, \int_0^{\frac{T}{m}} h^n_s dW_s)$;

For $s \in]\frac{iT}{m}, \frac{(i+1)T}{m}]$, set $h^n_s = H_n^{\mathrm{sgn}(\xi^n_{\frac{iT}{m}})}(s - \frac{iT}{m}, (a^n_{\frac{iT}{m}})^2)$.

3. We claim that $E_{P_h}[\int_0^T |\zeta_s| ds] = E_{P_{h^n}}[\int_0^T |\zeta_s| ds]$.

Actually, we have

$$\begin{aligned} E_{P_h}[\int_0^T |\zeta_s| ds] &= \frac{T}{m} E_{P^0}[\sum_{i=0}^{m-1} |\varphi_i(\int_{\frac{(i-1)T}{m}}^{\frac{iT}{m}} h_s dW_s, \cdots, \int_0^{\frac{T}{m}} h_s dW_s)|] \\ &=: E_{P^0}[\Phi(\int_{\frac{(m-1)T}{m}}^{T} h_s dW_s, \cdots, \int_0^{\frac{T}{m}} h_s dW_s)] \end{aligned}$$

and

$$E_{P_{h^n}}[\int_0^T |\zeta_s| ds] = E_{P^0}[\Phi(\int_{\frac{(m-1)T}{m}}^{T} h^n_s dW_s, \cdots, \int_0^{\frac{T}{m}} h^n_s dW_s)],$$

where $\Phi(x_m, \cdots, x_1) = \sum_{i=0}^{m-1} |\varphi_i(x_i, \cdots, x_1)|$.

Now it suffices to show that for any $0 \le i \le m$ and $\Phi_i \in C_b(R^i)$, we have

$$E_{P^0}[\Phi_i(\int_{\frac{(i-1)T}{m}}^{\frac{iT}{m}} h^n_s dW_s, \cdots, \int_0^{\frac{T}{m}} h^n_s dW_s)] = E_{P^0}[\Phi_i(\int_{\frac{(i-1)T}{m}}^{\frac{iT}{m}} h_s dW_s, \cdots, \int_0^{\frac{T}{m}} h_s dW_s)].$$

We prove the assertion by induction on i. Clearly, the assertion holds for $i = 0$. Assuming the assertion to hold for $i \ge 0$, we shall prove it for $i+1$.

Let $x = (x_i, \cdots, x_1)$. Noting that

$$\begin{aligned} \Phi_i(x) &:= E_{P^0}[\Phi_{i+1}(\int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} H_n^{\mathrm{sgn}(\varphi_i(x))}(s - \frac{iT}{m}, \psi_i(x)^2) dW_s, x)] \\ &= E_{P^0}[\Phi_{i+1}(\int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} \psi_i(x) dW_s, x)], \end{aligned}$$

we have

$$E_{P^0}[\Phi_{i+1}(\int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} h_s dW_s, \cdots, \int_0^{\frac{T}{m}} h_s dW_s)] = E_{P^0}[\Phi_i(\int_{\frac{(i-1)T}{m}}^{\frac{iT}{m}} h_s dW_s, \cdots, \int_0^{\frac{T}{m}} h_s dW_s)]$$

$$E_{P^0}[\Phi_{i+1}(\int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} h^n_s dW_s, \cdots, \int_0^{\frac{T}{m}} h^n_s dW_s)] = E_{P^0}[\Phi_i(\int_{\frac{(i-1)T}{m}}^{\frac{iT}{m}} h^n_s dW_s, \cdots, \int_0^{\frac{T}{m}} h^n_s dW_s)].$$

By the assumption, we get the desired result.

4. Based on the above arguments, we can prove the desired conclusion by simple computations.

By the sub-additivity of $\hat{E}$, we have

$$\begin{aligned}
& \hat{E}[\int_0^T \delta_{2mn}(s)\eta_s d\langle B\rangle_s] \\
\geq\ & \hat{E}[\int_0^T \delta_{2mn}(s)\zeta_s d\langle B\rangle_s] - \hat{E}[\int_0^T |\eta_s - \zeta_s| d\langle B\rangle_s] \\
\geq\ & E_{P_{h^n}}[\int_0^T \delta_{2mn}(s)\zeta_s d\langle B\rangle_s] - \overline{\sigma}^2\delta \\
=\ & E_{P_{h^n}}[\sum_{i=0}^{m-1} \xi_{\frac{iT}{m}} \int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} \delta_{2mn}(s) d\langle B\rangle_s] - \overline{\sigma}^2\delta.
\end{aligned}$$

By the definition of h^n, we have

$$\begin{aligned}
& E_{P_{h^n}}[\sum_{i=0}^{m-1} \xi_{\frac{iT}{m}} \int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} \delta_{2mn}(s) d\langle B\rangle_s] - \overline{\sigma}^2\delta \\
\geq\ & \frac{T}{m}\varepsilon E_{P_{h^n}}[\sum_{i=0}^{m-1} |\xi_{\frac{iT}{m}}|] - \overline{\sigma}^2\delta \\
=\ & \varepsilon E_{P_{h^n}}[\int_0^T |\zeta_s| ds] - \overline{\sigma}^2\delta.
\end{aligned}$$

Due to the assertion of Step 3, we have

$$\begin{aligned}
\varepsilon E_{P_{h^n}}[\int_0^T |\zeta_s| ds] - \overline{\sigma}^2\delta &= \varepsilon E_{P_h}[\int_0^T |\zeta_s| ds] - \overline{\sigma}^2\delta \\
&\geq \varepsilon E_{P_h}[\int_0^T |\eta_s| ds] - \varepsilon\delta - \overline{\sigma}^2\delta \\
&\geq A\varepsilon - \varepsilon\delta - \overline{\sigma}^2\delta > 0.
\end{aligned}$$

Since A, ε, δ do not depend on n, we have $d(\eta) \geq A\varepsilon - \varepsilon\delta - \overline{\sigma}^2\delta > 0$. Noting that ϵ, δ are arbitrary, we have $d(\eta) \geq \varepsilon \hat{E}_{G_\varepsilon}[\int_0^T |\eta_s| ds]$. The proof is completed. □

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