Electronic Journal of Probability

Electron. J. Probab. **17** (2012), no. 105, 1–25.
ISSN: 1083-6489 DOI: 10.1214/EJP.v17-1866

Bridges of quadratic harnesses

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Abstract

Quadratic harnesses are typically non-homogeneous Markov processes with time-dependent state space. Motivated by a question raised in [14, (4.4)] we give explicit formulas for bridges of such processes. Using an appropriately defined f-transformation we show that all bridges of a given quadratic harness can be transformed into other standard quadratic harnesses. Conversely, each such bridge is an f-transformation of a standard quadratic harness. We describe quadratic harnesses that correspond to bridges of some Lévy processes. We determine all quadratic harnesses that may arise from stitching together a pair of q-Meixner processes.

Keywords: bridges; harnesses; Lévy-Meixner processes; quadratic conditional variances .
AMS MSC 2010: 60J25.
Submitted to EJP on March 10, 2012, final version accepted on December 17, 2012.
Supersedes arXiv:0903.0150.

1 Introduction

The celebrated Paul Lévy's Brownian bridge of the Wiener process (W_t) is $X_t = W_t - tW_1$, $t \in [0,1]$. It is clear that $X_0 = X_1 = 0$, that the trajectories are continuous, and that (X_t) is a Gaussian process with mean zero and covariance $E(X_sX_t) = s(1-t)$ for $0 \le s \le t \le 1$. A well-known transformation (see, e.g., [3, pg 68])

$$Y_t = (1+t)X_{t/(1+t)} \tag{1.1}$$

converts the Brownian bridge into another Wiener process.

In this paper we extend this representation to bridges of a class of processes with linear regressions and quadratic conditional variances which we call quadratic harnesses, using an *f-transformation* of a stochastic process introduced in Definition 2.1. Our main result, Theorem 2.2, shows how to transform a bridge of a quadratic harness into another quadratic harness on $(0,\infty)$. The inverse of this f-transformation gives an explicit formula for the bridge in terms of a quadratic harness; the formula is more involved but similar in spirit to [14, (4.4)]. It is conjectured, and confirmed in many specific instances, that such quadratic harnesses are determined uniquely. We give two

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applications of Theorem 2.2: in Section 2.1 we describe quadratic harnesses that arise as f-transformations of bridges of Meixner processes. In Section 2.2 we determine parameters of all quadratic harnesses that may arise from a construction that stitches together pairs of Meixner processes with random parameters. Theorem 3.1 shows how the parameters of a quadratic harness change under group action of the affine transformations. Section 4 contains more technical proofs.

1.1 Bridges

A heuristic description of a bridge of a process (Z_t) is that it behaves like (Z_t) conditioned to start at time r at a prescribed point z_r and to end at time v at a prescribed point z_v.

For $t_1 < t_2 < \cdots < t_m$ in an open interval $\mathcal{T} \subset (-\infty, \infty)$, denote $\underline{\mathbf{Z}}_{\mathbf{t}} = (Z_{t_1}, \ldots, Z_{t_m})$.

The bridges are often constructed as families $(X_t^{(z_r,z_v)})_{t\in[r,v]}$ of processes indexed by pairs (z_r, z_v) and the above intuition can then be formalized as the requirement that

$$\mathbb{E}\left(f(Z_r)g(\underline{\mathbf{Z}}_{\mathbf{t}})h(Z_v)\right) = \int \mathbb{E}\left(g(\underline{\mathbf{X}}_{\mathbf{t}}^{(z_r,z_v)})\right) f(z_r)h(z_v)\Pr(Z_r \in dz_r, Z_v \in dz_v) \tag{1.2}$$

for $r < t_1 < t_2 < \cdots < t_m < v$ and positive measurable functions f, g, h, together with the requirement that $X_r^{(z_r,z_s)} = z_r$ and $X_v^{(z_r,z_s)} = z_v$.

Unfortunately, adopting (1.2) as a definition of a bridge would make it difficult to make judgements for specific pairs (z_r, z_v); in particular, we would not be able to use this definition to construct quadratic harnesses with prescribed parameters, as in Proposition 2.7. This issue is less important in specific constructions of bridges, which often produce a bridge for all admissible pairs (z_r, z_v). However, our goal is to study representations of the bridges for a class of processes that we expect (but do not know) to be Markov. In our main results we assume that a bridge exists rather than construct it. So we choose the weakest useful definition of a bridge.

Definition 1.1. *A stochastic process $(X_t)_{t\in(r,v)}$ is a bridge between points (r, z_r) and (v, z_v) of a process $(Z_t)_{t\in\mathcal{T}}$ such that $(r, v) \subset \mathcal{T}$, if:*

(i) $\lim_{t\to r+} X_t = z_r$ *and* $\lim_{t\to v-} X_t = z_v$ *in probability.*

(ii) For $r < t_1 < t_2 < \cdots < t_m < v$ the distribution of the vector $\underline{\mathbf{X}}_{\mathbf{t}}$ is absolutely continuous with respect to the distribution of the vector $\underline{\mathbf{Z}}_{\mathbf{t}}$.

(iii) For $r < s_1 < \cdots < s_k < t_1 < t_2 < \cdots < t_m < u_1 < u_2 < \cdots < u_n < v$ and integrable function g, if

$$\mathbb{E}(g(\underline{\mathbf{Z}}_{\mathbf{t}})|\underline{\mathbf{Z}}_{\mathbf{s}}, \underline{\mathbf{Z}}_{\mathbf{u}}) = h(\underline{\mathbf{Z}}_{\mathbf{s}}, \underline{\mathbf{Z}}_{\mathbf{u}}), \tag{1.3}$$

for some measurable function h (that depends on $\mathbf{s}, \mathbf{t}, \mathbf{u}$) then

$$\mathbb{E}(g(\underline{\mathbf{X}}_{\mathbf{t}})|\underline{\mathbf{X}}_{\mathbf{s}}, \underline{\mathbf{X}}_{\mathbf{u}}) = h(\underline{\mathbf{X}}_{\mathbf{s}}, \underline{\mathbf{X}}_{\mathbf{u}}). \tag{1.4}$$

(The equalities hold almost surely on the respective probability spaces.)

The "standard" construction of bridges of Markov processes based on Doob h-transform and duality is presented in [17, Proposition 1]. A Feller property framework for existence of bridges appears in [12], see also [1]. There is also a related construction of more general "reciprocal processes" in [21], where one may prescribe the initial and final laws instead of the point masses. These constructions usually assume time-homogeneous Markov property, and/or existence of a σ-finite reference measure. However, the processes we are interested in are often not time-homogeneous, and the densities with respect to a fixed σ-finite measure may fail to exist. We therefore state the following proposition which applies e.g. to many examples of Markov quadratic harnesses, see Remark 1.3.

Proposition 1.2. *Suppose $(Z_t)_{t\in\mathcal{T}}$ is a (non-homogeneous) Markov process with univariate distributions π_t. We assume that there is a family of Borel sets $\{M_t : t \in \mathcal{T}\}$ such that $\pi_t(M_t) = 1$ and that Z_t has transition probabilities $P_{s,t}(x, dy)$ defined for $x \in M_s$. We assume that*

(i) transition probabilities $P_{s,t}(x, dy)$ are absolutely continuous with respect to the univariate laws π_t: for $x \in M_s$ and $s < t$, the transition probabilities are

$$P_{s,t}(x, dy) = p(s, x; t, y)\pi_t(dy), \tag{1.5}$$

where $(x, y) \mapsto p(s, x; t, y)$ is a measurable function $M_s \times M_t \to [0,\infty)$.

(ii) there are $r < v$ in $\mathcal{T}$ and a pair $(z_r, z_v) \in M_r \times M_v$ such that $0 < p(r, z_r; v, z_v) < \infty$, and for any $\varepsilon > 0$, we have

$$\lim_{t\to r^+}\int_{\{y:|y-z_r|>\varepsilon\}} p(r, z_r; t, y)p(t, y; v, z_v)\pi_t(dy) = 0, \tag{1.6}$$

$$\lim_{t\to v^-}\int_{\{y:|y-z_v|>\varepsilon\}} p(r, z_r; t, y)p(t, y; v, z_v)\pi_t(dy) = 0. \tag{1.7}$$

Then there is a Markov process $(X_t)_{t\in(r,v)}$ which is a bridge between points (r, z_r) and (v, z_v) of the process (Z_t). If in addition $0 < p(r, z_r; v, z_v) < \infty$ for all $(z_r, z_v) \in M_r \times M_v$, then (1.2) *holds.*

In the proof, which appears in Section 4.1, we give explicit formulas for the transition probabilities of such a bridge.

Remark 1.3. *Condition (ii) of Proposition 1.2 holds under the following assumptions. Suppose that the following integrals exist:*

$$a(t) = \int yp(r, z_r; t, y)p(t, y; v, z_v)\pi_t(dy)$$

$$b(t) = \int (y - m_t)^2 p(r, z_r; t, y)p(t, y; v, z_v)\pi_t(dy)$$

and that with $m(t) = a(t)/p(r, z_r; v, z_v)$, $\sigma^2(t) = b(t)/p(r, z_r; v, z_v)$ we have

$$\lim_{t\to r^+} m(t) = z_r,\ \lim_{t\to v^-} m(t) = z_v,\ \lim_{t\to r^+}\sigma^2(t) = \lim_{t\to v^-}\sigma^2(t) = 0. \tag{1.8}$$

Then (1.6) *and* (1.7) *hold.*

In particular, conditions (1.8) *are easy to verify for quadratic harnesses, since* (1.11) *below implies*

$$m(t) = \frac{t-r}{v-r}z_v + \frac{v-t}{v-r}z_r, \tag{1.9}$$

and (1.14) *below implies that there is a constant $C = C(r, v, z_r, z_v)$ such that*

$$\sigma^2(t) = C(v - t)(t - r). \tag{1.10}$$

Remark 1.4. *We will also want to consider one-sided "bridges" corresponding to $v = \infty$. For a process (Z_t) such that $\lim_{t\to\infty} Z_t/t = 0$ in probability, we define a one sided bridge from (r, z_r) as the time-inversion of the bridge between $(0, 0)$ and $(1/r, z_r/r)$ of the time-inverted process $(tZ_{1/t})_{t>0}$.*

1.2 Quadratic harnesses

Throughout the paper the past-future filtration $(\mathcal{F}_{s,t})$ is a family of sigma fields with $s < t$ from a nonempty open interval $\mathcal{T} = (T_0, T_1) \subset (-\infty, \infty)$ such that $\mathcal{F}_{r,u} \subset \mathcal{F}_{s,t}$ for $r, s, t, u \in \mathcal{T}$ with $r \leq s \leq t \leq u$. We allow $T_0 = -\infty$ or $T_1 = \infty$.

An integrable stochastic process $\mathbf{X} = \{X_t : t \in \mathcal{T}\}$ such that X_s, X_t are $\mathcal{F}_{s,t}$-measurable, is called a harness [18, 22, 28] on $\mathcal{T}$ with respect to $(\mathcal{F}_{s,t})$ if for any $s, t, u \in \mathcal{T}$ with $s < t < u$,

$$\mathbb{E}(X_t|\mathcal{F}_{s,u}) = \frac{u-t}{u-s}X_s + \frac{t-s}{u-s}X_u. \tag{1.11}$$

All integrable Lévy processes are harnesses with respect to their natural past-future filtration ([19, (2.8)]); additional examples are mentioned after Definition 1.5.

For a square-integrable process, a natural second-order analog of (1.11) is the additional requirement that $\mathbb{V}\mathrm{ar}(X_t|\mathcal{F}_{s,u})$ is a quadratic function of X_s, X_u. It turns out that under additional assumptions, conditional variance of such a process is given by expressions (1.14) and (1.15) below, see [7, Theorem 2.2]. This motivates the following definition, in which we specify the form of the conditional variance in terms of six parameters.

Definition 1.5. *We will say that a square-integrable stochastic process* $\mathbf{X} = (X_t)_{t\in\mathcal{T}}$ *is a* quadratic harness *on* $\mathcal{T}$ *with respect to* $(\mathcal{F}_{s,t})$*, if* (X_t) *is a harness, and there are six constants* $\chi, \eta, \theta, \sigma, \tau, \rho$ *and a non-random function* $F_{t,s,u}$ *of* $s < t < u$ *such that*

$$\mathbb{V}\mathrm{ar}[X_t|\mathcal{F}_{s,u}] = F_{t,s,u}\left(\chi + \eta\frac{uX_s - sX_u}{u-s} + \theta\frac{X_u - X_s}{u-s} + \sigma\frac{(uX_s - sX_u)^2}{(u-s)^2} + \tau\frac{(X_u - X_s)^2}{(u-s)^2} + \rho\frac{(X_u - X_s)(uX_s - sX_u)}{(u-s)^2}\right). \tag{1.12}$$

We will say that (X_t) *is a* standard *quadratic harness, if* $\chi \neq 0$, $\mathcal{T} \subset (0, \infty)$*, and*

$$\mathbb{E}(X_t) = 0,\ \mathbb{E}(X_sX_t) = \min\{s, t\}. \tag{1.13}$$

Examples of quadratic harnesses on $(0, \infty)$ are five Lévy processes with quadratic conditional variances from [27]. Other examples include the classical versions of some free Lévy processes ([10, Theorem 4.3]), classical versions of q-Brownian motion ([10, Theorem 4.1]), bi-Poisson process [8, 11], and Markov processes with Askey-Wilson laws ([5, Theorem 1.1]).

For standard quadratic harnesses it will be convenient to re-write (1.12) with $\rho = \gamma - 1$, so that

$$\mathbb{V}\mathrm{ar}[X_t|\mathcal{F}_{s,u}] = F_{t,s,u}\left(1 + \eta\frac{uX_s - sX_u}{u-s} + \theta\frac{X_u - X_s}{u-s} + \sigma\frac{(uX_s - sX_u)^2}{(u-s)^2} + \tau\frac{(X_u - X_s)^2}{(u-s)^2} - (1-\gamma)\frac{(X_u - X_s)(uX_s - sX_u)}{(u-s)^2}\right). \tag{1.14}$$

(This change of notation matches better matrix representation in Section 3 and is consistent with [7, Theorem 2.2].) By taking the expected value of (1.14), see Proposition 3.6, we have

$$F_{t,s,u} = \frac{(u-t)(t-s)}{u(1+s\sigma) + \tau - s\gamma}. \tag{1.15}$$

When we want to indicate the parameters of a standard quadratic harness, referring to representation (1.14) we shall write $\mathbf{X} \in QH(\eta, \theta; \sigma, \tau; \gamma)$. Unless specified otherwise, we will use the natural past-future filtration $(\mathcal{F}_{s,t})$, that is $\mathcal{F}_{s,t} = \sigma(X_r : r \in ((0, s] \cup [t, \infty)) \cap \mathcal{T})$.

The importance of the standard form of a quadratic harness lies in the following.

Conjecture 1.6. *The parameters* $\eta, \theta, \sigma, \tau, \gamma$ *of a standard quadratic harness on* $(0, \infty)$ *determine uniquely its finite dimensional distributions.*

This conjecture is known to be true under additional, seemingly technical, conditions that include restricting the range of values of the parameters, see [7, 10, 9, 27]. In particular, for $\tau \geq 0$ and $-1 \leq q \leq 1$, a quadratic harness in $QH(0, \theta; 0, \tau; q)$ is a Markov process, called q-Meixner process in [10]; for the Meixner processes with $\gamma = 1$, see [27].

We now introduce additional notation for the right hand side of (1.14). For a (random or deterministic) real function $\mathbf{X}$ of real parameter t, denote

$$\underline{\mathbf{\Delta}}_{s,u}(\mathbf{X}) = \begin{bmatrix} \Delta_{s,u}(\mathbf{X}) \\ \widetilde{\Delta}_{s,u}(\mathbf{X}) \end{bmatrix}$$

with

$$\Delta_{s,u}(\mathbf{X}) = \frac{X_u - X_s}{u - s} \text{ and } \widetilde{\Delta}_{s,u}(\mathbf{X}) = \frac{uX_s - sX_u}{u - s}. \tag{1.16}$$

In the sequel, if $\mathbf{X}$ is clear from the context, we will write $\underline{\mathbf{\Delta}}_{s,u}$ instead of $\underline{\mathbf{\Delta}}_{s,u}(\mathbf{X})$.

The right hand side of (1.14) is a multiple of a quadratic polynomial in two real variables which we will write as

$$K(\underline{\mathbf{\Delta}}_{s,u}) = 1 + \eta\widetilde{\Delta}_{s,u} + \theta\Delta_{s,u} + \sigma\widetilde{\Delta}^2_{s,u} + \tau\Delta^2_{s,u} - (1-\gamma)\Delta_{s,u}\widetilde{\Delta}_{s,u}. \tag{1.17}$$

We will write $K(a, b)$ instead of $K\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$.

2 f-transformation of a bridge

Formula (1.1) describes a bridge $\mathbf{X}$ between points $(0, 0)$ and $(1, 0)$ of a quadratic harness $(W_t) \in QH(0, 0; 0, 0; 1)$ by indicating how to transform it into another quadratic harness $\mathbf{V} \in QH(0, 0; 0, 0; 1)$ on $(0, \infty)$. This relation can be generalized as follows.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-degenerate affine function $f : \mathbb{R}^2 \to \mathbb{R}^2$, written in matrix notation as

$$f(x, y) = [x, y]A + \underline{\mathbf{m}}^T , \tag{2.1}$$

where $\underline{\mathbf{m}}^T = [m_1, m_2]$ and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R}). \tag{2.2}$$

Let $\varphi(t) = (at + b)/(ct + d)$ be the associated Möbius transform, $t \in \mathbb{R} \setminus \{-d/c\}$. We note that φ is increasing on subintervals of $\mathbb{R} \setminus \{-d/c\}$ if $\det(A) > 0$, and φ is decreasing otherwise.

Definition 2.1. *If* $\mathbf{X} = (X_t)_{t \in \mathcal{T}}$ *is a stochastic process on an open interval* $\mathcal{T} \subset \varphi(\mathbb{R} \setminus \{-d/c\})$*, and* f *is the function in* (2.1)*, we define the* f-transformation $\mathbf{X}^f$ *of the stochastic process* $\mathbf{X}$ *as the process* $\mathbf{Y} = \mathbf{X}^f$ *on the open interval* $\mathcal{S} = \varphi^{-1}(\mathcal{T})$ *such that*

$$Y_t = (ct + d)X_{\varphi(t)} + \langle \underline{\mathbf{t}}, \underline{\mathbf{m}} \rangle, \ t \in \mathcal{S}, \tag{2.3}$$

where $\langle \underline{\mathbf{a}}, \underline{\mathbf{b}} \rangle = \underline{\mathbf{a}}^T \underline{\mathbf{b}}$ *and* $\underline{\mathbf{t}} = [t, 1]^T$*.*

For example, formula (1.1) describes f-transformation $\mathbf{V} = \mathbf{X}^f$ with $a = c = d = 1$, $b = m_1 = m_2 = 0$. Another well known f-transformation is time inversion $Y_t = tX_{1/t}$ which can be written as $\mathbf{Y} = \mathbf{X}^f$ with $b = c = 1$, $a = d = m_1 = m_2 = 0$. Similarly, re-scaling transformation $Y_t = \alpha X_{\beta t}$ with $\alpha, \beta \neq 0$ can be written as $\mathbf{Y} = \mathbf{X}^f$ with non-zero parameters given by $a = \alpha\beta$, $d = \alpha$.

A calculation verifies that as long as the time domains of the processes match,

$$(\mathbf{X}^f)^g = \mathbf{X}^{g\circ f}. \tag{2.4}$$

This allows us to build more complicated transformations from simpler components, and gives us the flexibility to consider either $\mathbf{Y} = \mathbf{X}^f$ or $\mathbf{X} = \mathbf{Y}^{f^{-1}}$ as needed.

By [27, Theorem 1(i)], see also [10, Theorem 4.2] and [24], if $\mathbf{V}$ is a quadratic harness in $QH(0,0;0,0;1)$ on $(0,\infty)$ then $\mathbf{V}$ is the Wiener process. So formula (1.1) says that if $\mathbf{X}$ is a Brownian bridge then an appropriate f-transformation $\mathbf{V} = \mathbf{X}^f$ gives $\mathbf{V} \in QH(0,0;0,0;1)$. Our next result is an analogous property of bridges of more general quadratic harnesses. For every bridge of a quadratic harnesses we give an f-transformation turning this bridge into a standard quadratic harness on $(0,\infty)$.

Theorem 2.2. *Let $\mathbf{Z} = (Z_t)_{t\in\mathcal{T}}$ be a standard quadratic harness with respect to its natural past-future filtration, with parameters specified by $\mathbf{Z} \in QH(\eta,\theta;\sigma,\tau;\gamma)$. Fix $r < v$ in $\mathcal{T}$ and z_r, z_v such that there exists a bridge $\mathbf{X} = (X_t)_{t\in(r,v)}$ of process $\mathbf{Z}$ between points (r, z_r) and (v, z_v). Denote*

$$\Delta_{r,v} = \frac{z_v - z_r}{v - r},\ \widetilde{\Delta}_{r,v} = \frac{vz_r - rz_v}{v - r}, \tag{2.5}$$

and assume that $K(\Delta_{r,v}, \widetilde{\Delta}_{r,v}) > 0$ (recall (1.17)).

Then $v(1 + r\sigma) + \tau - r\gamma > 0$, so $\mathbf{Y} = \mathbf{X}^f$ with f given by

$$A = \frac{\sqrt{v}}{(v-r)M}\begin{bmatrix} 1 & r \\ 1/v & 1\end{bmatrix},\ \underline{\mathbf{m}}^T = -\frac{\sqrt{v}}{(v-r)M}\left[\frac{z_v}{v}, z_r\right],\ M^2 = \frac{K(\Delta_{r,v}, \widetilde{\Delta}_{r,v})}{v(1+r\sigma)+\tau-r\gamma} \tag{2.6}$$

is well defined. Moreover, $\mathbf{Y} \in QH(\widetilde{\eta},\widetilde{\theta};\widetilde{\sigma},\widetilde{\tau};\widetilde{\gamma})$ on $(0,\infty)$ with

$$\widetilde{\eta} = \frac{v\eta - \theta - 2\tau\Delta_{r,v} + 2\sigma v\widetilde{\Delta}_{r,v} - (1-\gamma)(v\Delta_{r,v} - \widetilde{\Delta}_{r,v})}{\sqrt{v}M(v(r\sigma+1)+\tau-r\gamma)}, \tag{2.7}$$

$$\widetilde{\theta} = \frac{\sqrt{v}\left(\theta - r\eta + 2\tau\Delta_{r,v} - 2r\sigma\widetilde{\Delta}_{r,v} - (1-\gamma)(\widetilde{\Delta}_{r,v} - r\Delta_{r,v})\right)}{M(v(r\sigma+1)+\tau-r\gamma)}, \tag{2.8}$$

$$\widetilde{\sigma} = \frac{\sigma v^2 + (1-\gamma)v + \tau}{v(v(r\sigma+1)+\tau-r\gamma)}, \tag{2.9}$$

$$\widetilde{\tau} = v\frac{\sigma r^2 + (1-\gamma)r + \tau}{v(r\sigma+1)+\tau-r\gamma}, \tag{2.10}$$

$$\widetilde{\gamma} = \frac{v\gamma - r(v\sigma+1) - \tau}{v(r\sigma+1)+\tau-r\gamma}. \tag{2.11}$$

The simplest way to handle the calculations needed for the proof of Theorem 2.2 is to use matrix notation from Section 3.1; the proof appears in Section 4.4.

The explicit formula for $\mathbf{Y}$ is

$$Y_t = \frac{\sqrt{v}}{(v-r)M}\left(\left(1+\tfrac{t}{v}\right)X_{(t+r)/(1+\frac{t}{v})} - \left(t\tfrac{z_v}{v} + z_r\right)\right).$$

Conversely, we can express the bridge $\mathbf{X}$ in terms of a standard quadratic harness $\mathbf{Y} \in QH(\widetilde{\eta},\widetilde{\theta};\widetilde{\sigma},\widetilde{\tau};\widetilde{\gamma})$ by an explicit formula

$$X_t = a(t,v,r,z_r,z_v)Y_{v(t-r)/(v-t)} + b(t,v,r,z_r,z_v), \tag{2.12}$$

compare [14, Section (4.4)]. In Proposition 2.10 we describe bridges that can be obtained in this way from Poisson, negative binomial, gamma and hyperbolic secant processes.

Next we give an example of how one can use Theorem 2.2 to construct quadratic harnesses. (This subject is continued in Section 2.1.)

Example 2.3. *Markov process $\mathbf{Z} \in QH(0,0;0,0;0)$ is a classical version of the free Brownian motion, see [4, Example 4.9] and [2, 5.3]. One can check that the assumptions of Proposition 1.2 are satisfied, so the bridges of this process between (r, z_r) and (v, z_v) exist for all $r < v$, with $z_r \in (-2\sqrt{r}, 2\sqrt{r})$ and $z_v \in (-2\sqrt{v}, 2\sqrt{v})$. If $\mathbf{X}$ is such a bridge, then with f from (2.6), the transformation $\mathbf{X}^f$ is a quadratic harness in $QH(\eta, \theta; \sigma, \tau; \gamma)$ with*

$$\eta = \frac{\widetilde{\Delta}_{r,v} - v\Delta_{r,v}}{v\sqrt{1 - \Delta_{r,v}\widetilde{\Delta}_{r,v}}},\ \theta = \frac{r\Delta_{r,v} - \widetilde{\Delta}_{r,v}}{\sqrt{1 - \Delta_{r,v}\widetilde{\Delta}_{r,v}}},\ \sigma = \frac{1}{v},\ \tau = r,\ \gamma = -\frac{r}{v}.$$

These are free quadratic harnesses, $\gamma = -\sigma\tau$, without atoms from [9, 26].

For bridges between points $(0,0)$ and (v, z_v) and for one-sided bridges from (r, z_r), formulas for parameters correspond to Theorem 2.2 after taking $(r, z_r) = (0,0)$ or by taking the limit as $v \to \infty$, while keeping z_r, z_v fixed. These two situations are described in the following remark.

Remark 2.4. *Suppose $\mathbf{Z}$ is a quadratic harness on $(0,\infty)$ in $QH(\eta, \theta; \sigma, \tau; \gamma)$.*

(i) Let $\mathbf{X} = (X_t)_{t\in(0,v)}$ be the bridge between points $(0,0)$ and (v, z_v) of the process $\mathbf{Z}$. Assume that

$$\kappa^2 = \left(1 + \frac{\tau}{v}\right)\left(1 + \theta\frac{z_v}{v} + \tau\frac{z_v^2}{v^2}\right) > 0.$$

Then process $\mathbf{Y} = \mathbf{X}^f$ with

$$A = \frac{v+\tau}{v\kappa}\begin{bmatrix} 1 & 0 \\ 1/v & 1 \end{bmatrix},\ \underline{\mathbf{m}} = -\frac{v+\tau}{v\kappa}\begin{bmatrix} z_v/v \\ 0 \end{bmatrix},$$

written explicitly as

$$Y_t = \frac{v+\tau}{v\kappa}\left((1 + \tfrac{t}{v})X_{vt/(t+v)} - \tfrac{t}{v}z_v\right), \tag{2.13}$$

is a quadratic harness with parameters

$$\theta_Y = \frac{\theta + 2\tau z_v/v}{\kappa}, \tag{2.14}$$

$$\eta_Y = \frac{v\eta - \theta - 2\tau z_v/v - (1-\gamma)z_v}{v\kappa}, \tag{2.15}$$

$$\tau_Y = \frac{v\tau}{v+\tau}, \tag{2.16}$$

$$\sigma_Y = \frac{\sigma v^2 + (1-\gamma)v + \tau}{v(v+\tau)}, \tag{2.17}$$

$$\gamma_Y = \frac{v\gamma - \tau}{v+\tau}. \tag{2.18}$$

(ii) Let $\mathbf{X} = (X_t)_{t\in(r,\infty)}$ be the one-sided bridge from (r, z_r) of process $\mathbf{Z}$, see Remark 1.4. Assume that $\kappa^2 = (1 + r\sigma)(1 + \eta z_r + \sigma z_r^2) > 0$. For $0 < t < \infty$, let $\mathbf{Y} = \mathbf{X}^f$ with

$$A = \tfrac{1+r\sigma}{\kappa}\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix},\ \underline{\mathbf{m}} = -\tfrac{1+r\sigma}{\kappa}\begin{bmatrix} 0 \\ z_r \end{bmatrix},$$

which can be explicitly written as

$$Y_t = \tfrac{1+r\sigma}{\kappa}(X_{t+r} - z_r). \tag{2.19}$$

Then $\mathbf{Y}$ *is a quadratic harness with parameters*

$$\theta_Y = \frac{\theta - r\eta - 2r\sigma z_r - (1-\gamma)z_r}{\kappa}, \tag{2.20}$$

$$\eta_Y = \frac{\eta + 2\sigma z_r}{\kappa}, \tag{2.21}$$

$$\tau_Y = \frac{\sigma r^2 + (1-\gamma)r + \tau}{1 + r\sigma}, \tag{2.22}$$

$$\sigma_Y = \frac{\sigma}{1 + r\sigma}, \tag{2.23}$$

$$\gamma_Y = \frac{\gamma - r\sigma}{1 + r\sigma}. \tag{2.24}$$

The f-transformation into the "standard form" is unique up to some special f-transformation.

Proposition 2.5. *Suppose f is given by* (2.1) *with* $\underline{\mathbf{m}} = 0$*. If* $\mathbf{Y}$ *is a standard quadratic harness then* $\mathbf{Y}^f$ *is also a standard quadratic harness iff either* $b = c = 0$ *and* $ad = 1$ *or* $a = d = 0$ *and* $bc = 1$.

In addition, if $\mathbf{Y} \in QH(\eta, \theta; \sigma, \tau; \rho)$*, then*

$$\mathbf{Y}^f \in QH(\eta/d, \theta d; \sigma/d^2, d^2\tau; \rho). \tag{2.25}$$

in the first case, and

$$\mathbf{Y}^f \in QH(\theta/c, \eta c; \tau/c^2, c^2\sigma; \rho)$$

with swapped parameters θ, η *and* τ, σ.

Proof. This is of course an elementary calculation, but the simplest way is to apply (3.17) to check that

$$\mathbb{E}(Y_s^f Y_t^f) = \begin{cases} (as+b)(ct+d) & \text{if } ad - bc > 0 \\ (at+b)(cs+d) & \text{if } ad - bc < 0 \end{cases}$$

This proves the first part. The second part is again an elementary calculation, which is also covered by the more general formula (3.18). □

Next we present relations between the parameters of the original quadratic harness that are preserved by the f-transformation (2.6) of its bridge.

Proposition 2.6. *Let* $\mathbf{X}$ *be a bridge between points* (r, z_r) *and* (v, z_v) *of a process* $\mathbf{Z} \in QH(\eta_Z, \theta_Z; \sigma_Z, \tau_Z; \gamma_Z)$*. Let* $\mathbf{Y} = \mathbf{X}^f$ *with f defined in* (2.6)*, so that* $\mathbf{Y} \in QH(\eta_Y, \theta_Y; \sigma_Y, \tau_Y; \gamma_Y)$.

(i) Then $\gamma_Z = -1$ *if and only if* $\gamma_Y = -1$.

(ii) If $\gamma_Z > -1$*, then*

$$\frac{(1-\gamma_Y)^2 - 4\sigma_Y\tau_Y}{(1+\gamma_Y)^2} = \frac{(1-\gamma_Z)^2 - 4\sigma_Z\tau_Z}{(1+\gamma_Z)^2}. \tag{2.26}$$

(iii) If $(r, z_r) = (0, 0)$*, then*

$$\operatorname{sign}(\theta_{\mathrm{Y}}^2 - 4\tau_{\mathrm{Y}}) = \operatorname{sign}(\theta_{\mathrm{Z}}^2 - 4\tau_{\mathrm{Z}}). \tag{2.27}$$

Proof. The first statement follows from (2.11). Formula (2.26) follows by a direct computation from (2.9–2.11). Formula (2.27) follows from formulas (2.14) and (2.16), which give $\theta_Y^2 - 4\tau_Y = (\theta_Z^2 - 4\tau_Z)/\kappa^2$. □

2.1 Application: bridges of Meixner processes

By the Meixner processes we mean Lévy processes for which the univariate laws belong to the natural exponential families with quadratic variance functions, [23]. This class consists of the Wiener, Poisson, negative binomial, gamma, and hyperbolic secant processes. Some authors use the name Meixner process for the hyperbolic secant process only; [25, Section 4.3] calls the whole class the Lévy-Meixner systems. According to [27], up to affine transformations this class coincides with the standard quadratic harnesses such that $\eta = \sigma = 0$ and $\gamma = 1$.

From Proposition 1.2 together with Remark 1.3, see also [17, Proposition 1], it follows that bridges with endpoints in the support of the univariate laws exist for all five quadratic harnesses in $QH(0, \theta; 0, \tau; 1)$. Such bridges are quadratic harnesses, and from Theorem 2.2 we read out their standard form. It turns out that their parameters satisfy relations $0 \leq \sigma\tau < 1$, $\gamma = 1 - 2\sqrt{\sigma\tau}$ and $\eta\sqrt{\tau} + \theta\sqrt{\sigma} = 0$.

Proposition 2.7. *Fix $\tau \geq 0$ and $\theta \in \mathbb{R}$. Suppose $\mathbf{Z} \in QH(0, \theta; 0, \tau; 1)$ i.e., $\mathbf{Z}$ is a Meixner process. Let $\mathbf{X}$ be a bridge between points (r, z_r) and (v, z_v) of $\mathbf{Z}$, and $\mathbf{Y} = \mathbf{X}^f$ with*

$$A = \frac{\sqrt{v - r + \tau}}{(v - r)\sqrt{1 + \theta\Delta_{r,v} + \tau\Delta_{r,v}^2}} \begin{bmatrix} v & r \\ 1 & 1 \end{bmatrix},$$

$$\underline{\mathbf{m}} = -\frac{\sqrt{v - r + \tau}}{(v - r)\sqrt{1 + \theta\Delta_{r,v} + \tau\Delta_{r,v}^2}} \begin{bmatrix} z_v \\ z_r \end{bmatrix},$$

(recall (2.5)), i.e., explicitly

$$Y_t = \frac{\sqrt{v - r + \tau}}{(v - r)\sqrt{1 + \theta\Delta_{r,v} + \tau\Delta_{r,v}^2}} \left((1 + t)X_{(vt+r)/(1+t)} - tz_v - z_r\right). \tag{2.28}$$

Then $\mathbf{Y} \in QH(\eta_Y, \theta_Y; \sigma_Y, \tau_Y; \gamma_Y)$ with parameters

$$\gamma_Y = \frac{v - r - \tau}{v - r + \tau},\ \tau_Y = \sigma_Y = \frac{\tau}{v - r + \tau}, \tag{2.29}$$

$$\theta_Y = -\eta_Y = \frac{\theta + 2\tau\Delta_{r,v}}{\sqrt{v - r + \tau}\sqrt{1 + \theta\Delta_{r,v} + \tau\Delta_{r,v}^2}}. \tag{2.30}$$

Proof. From Theorem 2.2 we read out the transformation that leads to parameters

$$\gamma' = \frac{v - r - \tau}{v - r + \tau},\ \tau' = \frac{\tau v}{v - r + \tau},\ \sigma' = \frac{\tau}{v(v - r + \tau)},$$

$$\theta' = \sqrt{v}\frac{\theta + 2\tau\Delta_{r,v}}{\sqrt{v - r + \tau}\sqrt{1 + \theta\Delta_{r,v} + \tau\Delta_{r,v}^2}},$$

$$\eta' = -\frac{\theta + 2\tau\Delta_{r,v}}{\sqrt{v}\sqrt{v - r + \tau}\sqrt{1 + \theta\Delta_{r,v} + \tau\Delta_{r,v}^2}}.$$

Composing the f-transformation from (2.25) with the f-transformation from Theorem 2.2, see (2.4), we get (2.28) and parameters as claimed. ∎

Example 2.8 (Dirichlet process). *For any $\sigma_0, \tau_0 > 0$ with $\sigma_0\tau_0 < 1$, there exists a standard quadratic harness $\mathbf{Y}$ (namely, a Dirichlet process) on $(0, \infty)$ which has parameters $\sigma_Y = \sigma_0$, $\tau_Y = \tau_0$, $\theta_Y = 2\sqrt{\tau_0}$, $\eta_Y = -2\sqrt{\sigma_0}$, and $\gamma_Y = 1 - 2\sqrt{\sigma_0\tau_0}$.*

Indeed, consider a bridge of a gamma process $(G_t)_{t>0}$*. The gamma process* (G_t) *is a non-negative Lévy processes with two parameters* $\alpha, \beta > 0$*. The density of* G_t *is given by*

$$\tfrac{\beta^{\alpha t}}{\Gamma(\alpha)} x^{\alpha t-1} e^{-\beta x} \mathbf{1}_{(0,\infty)}(x). \tag{2.31}$$

As a Lévy process, (G_t) *is a harness with mean* $\mathbb{E}(G_t) = t\alpha/\beta$ *and variance* $\mathbb{V}\mathrm{ar}(G_t) = t\alpha/\beta^2$*. It is also known (see* (2.34) *below) that*

$$\mathbb{V}\mathrm{ar}(G_t|\mathcal{F}_{s,u}) = \frac{(t-s)(u-t)}{(u-s)^2((u-s)/\alpha+1)}(G_u - G_s)^2 \,. \tag{2.32}$$

Then

$$Z_t = \frac{\beta}{\alpha} G_{\alpha t} - \alpha t$$

is a quadratic harness in $QH(0, 2/\alpha; 0, 1/\alpha^2; 1)$ *which by further scaling as in* (2.25) *can be transformed into a quadratic harness in* $QH(0, 2; 0, 1; 1)$*. So instead of considering bridges of* (G_t)*, we consider bridges of* $\mathbf{Z} \in QH(0, 2; 0, 1; 1)$*. To apply Proposition 2.7 we choose* $v - r = 1/\sqrt{\sigma_0\tau_0} - 1$ *so that* $1/(1 + v - r) = \sqrt{\sigma_0\tau_0}$*. From* (2.29) *we have*

$$\gamma_Y = 1 - \frac{2}{v-r+1} = 1 - 2\sqrt{\sigma_0\tau_0}\,.$$

Transformation (2.25) *with* $d^2 = \sqrt{\sigma_0/\tau_0}$ *gives*

$$\tau_Y = \sqrt{\sigma_0\tau_0}/d^2 = \tau_0,\ \sigma_Y = d^2\sqrt{\sigma_0\tau_0} = \sigma_0.$$

By (2.30) *with any* $\Delta = \Delta_{r,v} \geq 0$*,*

$$\theta_Y d = \frac{2(1+\Delta)}{\sqrt{v-r+1}\sqrt{1+2\Delta+\Delta^2}} = \frac{2}{\sqrt{v-r+1}} = 2\sqrt[4]{\sigma_0\tau_0},$$

so $\theta_Y = 2\sqrt{\tau_0}$*. Similarly,* $\eta_Y/d = -2\sqrt[4]{\sigma_0\tau_0}$*, so* $\eta_Y = -2\sqrt{\sigma_0}$*.*

Since bridges of the gamma process are Dirichlet processes, the same conclusion can be obtained directly by a fairly natural reparameterization (2.35) *without invoking explicitly any of the transformations. Let* $a_1, \ldots, a_n, a_{n+1}$ *be positive numbers. A Dirichlet distribution is defined through its density*

$$f(x_1, \ldots, x_n) = \frac{\Gamma(a_1 + \ldots + a_{n+1})}{\prod_{i=1}^{n+1}\Gamma(a_i)} \prod_{i=1}^{n} x_i^{a_i-1}\left(1 - \sum_{i=1}^{n} x_i\right)^{a_{n+1}} \mathbf{1}_{U_n}(x_1, \ldots, x_n)\,,$$

where $U_n = \{(x_1, \ldots, x_n) \in (0,\infty)^n :\ \sum_{i=1}^n x_i < 1\}$*. A stochastic process* $\mathbf{X} = (X_t)_{t\in(0,v)}$ *is called a Dirichlet process if there exists a finite nonzero measure* μ *on* $(0, v)$ *such that for any* n *and any* $0 = t_0 < t_1 < \ldots < t_n < v$ *the distribution of the vector of increments* $(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}})$ *is Dirichlet with* $a_j = \mu((t_{j-1}, t_j])$*,* $1 \leq j \leq n$ *and* $a_{n+1} = \mu((t_n, v))$*. This is one of the basic objects of non-parametric Bayesian statistics - see [13, 15]. Let* $\mu = c\lambda$*, where* λ *is a Lebesgue measure on* $(0, v)$ *and* $c = 1/\alpha > 0$ *is a number. Recall that the beta distribution,* $B_I(a, b)$*, is defined by the density*

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}\mathbf{1}_{(0,1)}(x)\,,$$

and if $X \sim B_I(a, b)$ *then*

$$\mathbb{E}(X) = \frac{a}{a+b}, \quad \textit{and} \quad \mathbb{V}\mathrm{ar}(X) = \frac{ab}{(a+b)^2(a+b+1)}\,. \tag{2.33}$$

Since X_t has the beta distribution $B_I(ct, c(v-t))$ the formulas (2.33) give

$$\mathbb{E}(X_t) = \tfrac{t}{v} \quad \text{and} \quad \mathbb{C}\mathrm{ov}(X_s, X_t) = \tfrac{s(v-t)}{v^2(cv+1)} \ .$$

Note that to compute $\mathbb{E}(X_s X_t)$ it is convenient to use the classical fact, that X_s/X_t and X_t are independent and X_s/X_t is a beta $B_I(cs, c(t-s))$ random variable. Note also that $\mathbf{X}$ is a Markov process with transition distribution defined by the fact that $(X_t - X_s)/(1 - X_s)$ and X_s are independent, and $(X_t - X_s)/(1 - X_s)$ is beta $B_I(c(t-s), c(v-t))$. It is also known that

$$\frac{X_t - X_s}{X_u - X_s} \sim B_I(c(t-s), c(u-t)).$$

Moreover, $(X_t - X_s)/(X_u - X_s)$ and (X_s, X_u) are independent. Therefore, from (2.33) we get

$$\mathbb{E}(X_t | \mathcal{F}_{s,u}) = X_s + (X_u - X_s)\frac{t-s}{u-s}$$

and thus $\mathbf{X}$ is a harness. The second formula in (2.33) gives

$$\mathbb{V}\mathrm{ar}(X_t | \mathcal{F}_{s,u}) = (X_u - X_s)^2 \frac{(t-s)(u-t)}{(u-s)^2(c(u-s)+1)} \ . \tag{2.34}$$

This is an example of a quadratic harness with $\chi = 0$.

Define now

$$Y_t = \sqrt{\tfrac{1+cv}{v}} \left((v+t) X_{\frac{tv}{v+t}} - t \right), \quad t \in (0, \infty) \ . \tag{2.35}$$

Note that $\mathbf{Y}$ is an f-transformation of $\mathbf{X}$ with f defined by

$$A = \sqrt{\tfrac{1+cv}{v}} \begin{bmatrix} v & 0 \\ 1 & v \end{bmatrix}, \ \underline{\mathbf{m}} = -\sqrt{\tfrac{1+cv}{v}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

It is elementary to check that $(Y_t)_{t \in (0,\infty)}$ is a quadratic harness and the parameters are as follows

$$\theta_Y = \frac{2\sqrt{v}}{\sqrt{1+cv}} \ , \quad \eta_Y = \frac{-2}{\sqrt{v}\sqrt{1+cv}} \ ,$$

$$\tau_Y = \frac{v}{1+cv} \ , \quad \sigma_Y = \frac{1}{v(1+cv)} \ , \quad \gamma_Y = 1 - \frac{2}{(1+cv)^2} \ .$$

(Note that this agrees with the answers deduced from Proposition 2.7 which implies that $\theta_Y^2 = 4\tau_Y$, $\eta_Y^2 = 4\sigma_Y$ and $\gamma_Y = 1 - 2\sqrt{\sigma_Y \tau_Y}$.)

On the other hand, it can be easily seen that the process $\mathbf{X}$ is a bridge of the gamma process $(G_t)_{t\in(0,\infty)}$ governed by the gamma distribution with the shape parameter $1/c$ and the scale equal 1. More precisely, process $\mathbf{X}$ is equal in distribution to the gamma bridge $(G_t/G_v)_{t\in(0,v)} | G_v \sim (G_t/G_v)_{t\in(0,v)}$ between points $(0,0)$ and $(v,1)$, see [16, Definition 2], see also [14].

Example 2.9 (Binomial process)**.** *Fix $n \in \mathbb{N}$ and real η_0, θ_0 such that $\eta_0\theta_0 = -1/n$. Then there exist a standard quadratic harness $\mathbf{Y}$ (namely, the Binomial process described here) on $(0,\infty)$ which has parameters $\sigma_Y = \tau_Y = 0$, $\theta_Y = \theta_0$, $\eta_Y = \eta_0$, and $\gamma_Y = 1$.*

Indeed, consider standard quadratic harnesses arising from bridges of a Poisson process. Poisson process N_t with parameter $\lambda > 0$ is a harness with mean $\mathbb{E}(N_t) = \lambda t$ variance $\mathbb{V}\mathrm{ar}(N_t) = \lambda t$, and with conditional variance with respect to the natural past-future filtration given by

$$\mathbb{V}\mathrm{ar}(N_t | \mathcal{F}_{s,u}) = \frac{(u-t)(t-s)}{(u-s)^2}(N_u - N_s).$$

Then

$$Z_t = N_{t/\lambda} - t$$

is in $QH(0,1;0,0;1)$*. So instead of considering bridges of* $(N_t)_{t>0}$ *we consider a bridge of* $\mathbf{Z}$ *between points* $(0,0)$ *and* (v,n)*. From* (2.29) *we see that* $\gamma_Y = 1$*,* $\sigma_Y = \tau_Y = 0$*. Since* $\Delta_{0,v} = (n-v)/v$*, from* (2.30) *we see that* $\theta_Y = -\eta_Y = 1/\sqrt{n}$*. So* $\eta_Y\theta_Y = -1/n$*.*

The same conclusion can be obtained more directly without invoking explicitly any of the transformations. Let $b(m,p)$ *denote the binomial distribution with sample size* m *and probability of success* p*. For a fixed* $n \in \mathbb{N}$*, define a Markov process* $\mathbf{X} = (X_t)_{t\in(0,v)}$ *by the following (consistent) family of marginal and conditional distributions:*

$$X_t \sim b\left(n, \tfrac{t}{v}\right) \quad \text{and} \quad X_t - X_s | X_s \sim b\left(n - X_s, \tfrac{t-s}{v-s}\right),\ 0 < s \le t < v\ .$$

Then process $\mathbf{X}$ *is called a binomial process with parameter* n*. (Compare [10, Proposition 4.4].) It is elementary to see that the conditional distribution* $X_t|\mathcal{F}_{s,u} \sim b\left(X_u - X_s, \tfrac{t-s}{u-s}\right)$*. Therefore* $\mathbf{X}$ *is a harness i.e.* (1.11) *holds, and for any* $s,t,u \in (0,v)$*,* $s<t<u$

$$\mathbb{V}\mathrm{ar}(X_t|\mathcal{F}_{s,u}) = \frac{(u-t)(t-s)}{(u-s)^2}(X_u - X_s)\ .$$

This is again an example of quadratic harness with $\chi = 0$*.*

Let

$$Y_t = \frac{(v+t)X_{\frac{tv}{v+t}} - nt}{\sqrt{nv}}\ , \quad t \in (0,\infty),$$

i.e., $\mathbf{Y}$ *is an* f*-transformation of* $\mathbf{X}$ *with* f *defined by*

$$A = \tfrac{1}{\sqrt{nv}}\begin{bmatrix} v & 0 \\ 1 & v\end{bmatrix},\ \underline{\mathbf{m}} = \sqrt{\tfrac{n}{v}}\ \begin{bmatrix}1\\0\end{bmatrix}.$$

Then an easy computation shows that process $(Y_t)_{t>0}$ *is a quadratic harness and the parameters are* $\theta_Y = \sqrt{v/n}$*,* $\eta_Y = -1/\sqrt{nv}$*,* $\tau_Y = \sigma_Y = 0$ *and* $\gamma_Y = 1$*.*

On the other hand, it is immediate that $\mathbf{X}$ *is a bridge obtained by conditioning a Poisson process* $(N_t)_{t>0}$ *at* $N_v = n$*.*

2.2 Application: stitching construction

This section is motivated by the construction of a classical bi-Poisson process from a pair of two conditionally independent Poisson processes in [11, Proposition 4.1], and by the construction of a quadratic harness from two conditionally independent negative binomial processes in [20, Proposition 5.1]. These constructions essentially consist of choosing an appropriate time T and an appropriate law for random variable Z_T. For z_T in the support of Z_T, the bridge $\mathbf{X}_+$ between $(0,0)$ and (T,z_T) of process $\mathbf{Z}$ transforms into a Poisson process (a negative binomial process) by f-transformation (2.13). Similarly, the one-sided bridge $\mathbf{X}_-$ from (T,z_T) transforms into another Poisson process (another negative binomial process). The two Poisson processes, or the two negative binomial processes, used in the stitching construction are Z_T-conditionally independent.

We use Theorem 2.2 and Remark 2.4(i) to determine parameters of all quadratic harnesses that might arise from such a stitching construction from more general q-Meixner processes, which are quadratic harnesses that generalize the Meixner processes by allowing arbitrary $\gamma \in [-1,1]$, see [10]. Namely, if a quadratic harness $\mathbf{Z}$ comes from stitching together two q-Meixner processes, then there is at least one pair (T,z_T) such that the bridge $\mathbf{X}_-$ between $(0,0)$ and (T,z_T) exists and can be transformed back into a q-Meixner process $\mathbf{Y}$ with parameters given in Remark 2.4(i). The following result describes the parameters of the standard quadratic harness $\mathbf{Z}$ in such situation.

Proposition 2.10. *Let* $\mathbf{Z} \in QH(\eta, \theta; \sigma, \tau; \gamma)$ *be defined on* $(0, \infty)$*. Suppose that there are real numbers* $T > 0$ *and* z_T *such that the bridge* $\mathbf{X}_-$ *between points* $(0, 0)$ *and* (T, z_T) *of* $\mathbf{Z}$*, transforms by formula* (2.13) *into a* q*-Meixner process* $\mathbf{Y}$*. Then* $\mathbf{Y}$ *is a Meixner process and one of the following cases must happen:*

(i) $\gamma = 1$, $\sigma = \tau = 0$ *and* $\eta = \theta = 0$;

(ii) $\gamma = 1$, $\sigma = \tau = 0$ *and* $\eta\theta > 0$;

(iii) $\sigma, \tau > 0$, $\gamma = 1 + 2\sqrt{\sigma\tau}$ *and* $\eta\sqrt{\tau} = \theta\sqrt{\sigma}$.

Proof. The only possibility for (2.17) to correspond to a q-Meixner process is when the parameters of $\mathbf{Z}$ satisfy

$$\sigma T^2 + (1 - \gamma)T + \tau = 0. \tag{2.36}$$

Since $\sigma, \tau \geq 0$ (see [7, Theorem 2.2]), the only solution with $\gamma \leq 1$ is $\gamma = 1$, $\sigma = \tau = 0$. Then from (2.15) we see that $\mathbf{Y}$ is indeed a Meixner process when we set $T = \theta/\eta$ or when $T > 0$ is arbitrary but $\eta = \theta = 0$.

Other solutions of (2.36) interpreted as a quadratic equation in T are possible only when $(1 - \gamma)^2 \geq 4\sigma\tau$. However, since $\gamma \leq 1 + 2\sqrt{\sigma\tau}$ by [7, Theorem 2.2], this gives $\gamma = 1 + 2\sqrt{\sigma\tau}$ and $T = \sqrt{\tau/\sigma}$. Then from (2.15), the coefficient at z_T vanishes when $\eta\sqrt{\tau} = \theta\sqrt{\sigma}$, so $\mathbf{Y}$ is indeed a Meixner process.

□

Remark 2.11. *As expected, stitching constructions work in all of the cases described in Proposition 2.10. Case (i) is trivial, with* $\mathbf{Y}$ *being the Wiener process. In case (ii)* $T = \theta/\eta$, $\mathbf{Y}$ *is the Poisson processes with parameter* λ *which depends on* z_T *and the stitching construction is described in [11, Proposition 4.1]. In case (iii) with* $T = \sqrt{\tau/\sigma}$, $\gamma = 1 + 2\sqrt{\sigma\tau}$, *by* (2.27) *the sign of* $\theta^2 - 4\tau$ *is preserved. From [27, Theorem 1] we see that* $\mathbf{Y}$ *is either a negative binomial (*$\theta^2 > 4\tau$*), or a gamma (*$\theta^2 = 4\tau$*), or a hyperbolic secant (*$\theta^2 < 4\tau$*) process. The stitching construction for the negative binomial process appears in [20, Proposition 5.1]. The general case appears in [6].*

3 Transforming quadratic harnesses into standard form

Bridges of standard quadratic harnesses are quadratic harness but they are not in the standard form because their means are affine functions of time and they have product covariances. In our next theorem we present f-transformations (2.3) that convert such quadratic harnesses into standard form.

Theorem 3.1. *Let* $\mathbf{X}$ *be a harness* (1.11) *with respect to a past-future filtration* $(\mathcal{F}_{s,t})$ *on an interval* $(T_0, T_1) \subset \mathbb{R}$ *with mean*

$$\mathbb{E}(X_t) = \alpha + \beta t$$

and with covariance

$$\mathbb{C}\mathrm{ov}(X_s, X_t) = (as + b)(ct + d), \; s < t, \tag{3.1}$$

such that $ad - bc > 0$ *and* $(at + b)(ct + d) > 0$ *for* $t \in (T_0, T_1)$*. Suppose that* (1.12) *holds, and that*

$$\widetilde{\chi} := \chi + \alpha\eta + \theta\beta + \sigma\alpha^2 + \tau\beta^2 + \rho\alpha\beta > 0. \tag{3.2}$$

Let $\psi(t) = (dt - b)/(a - ct)$*. Then stochastic process*

$$Y_t = \frac{a - ct}{ad - bc}\left(X_{\psi(t)} - \alpha - \beta\psi(t)\right) \tag{3.3}$$

is a quadratic harness in $QH(\eta',\theta';\sigma',\tau';\rho')$ *on the interval* $\left(\frac{aT_0+b}{cT_0+d},\frac{aT_1+b}{cT_1+d}\right)\subset(0,\infty)$*, and has parameters*

$$\eta' = (d(\eta+\beta\rho+2\alpha\sigma)+c(\theta+\alpha\rho+2\beta\tau))/\widetilde{\chi}\,, \tag{3.4}$$

$$\theta' = (b(\eta+\beta\rho+2\alpha\sigma)+a(\theta+\alpha\rho+2\beta\tau))/\widetilde{\chi}\,, \tag{3.5}$$

$$\sigma' = (\tau c^2+d\rho c+d^2\sigma)/\widetilde{\chi}\,, \tag{3.6}$$

$$\tau' = (\tau a^2+b\rho a+b^2\sigma)/\widetilde{\chi}\,, \tag{3.7}$$

$$\rho' = (bc\rho+ad\rho+2bd\sigma+2ac\tau)/\widetilde{\chi}\,. \tag{3.8}$$

We remark that the f-transformation used in Theorem 3.1 is reversible, so $\mathbf{X}=\mathbf{Y}^f$ with

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix},\ \underline{\mathbf{m}}=\begin{bmatrix}\beta\\ \alpha\end{bmatrix},$$

that is

$$X_t=(ct+d)Y_{(at+b)/(ct+d)}+\alpha+\beta t.$$

The proof of Theorem 3.1 uses matrix notation and is postponed until Section 4.2. Here we use Theorem 3.1 to give examples which show that Conjecture 1.6 does not hold on finite intervals.

Example 3.2. *Let* $(W_t)_{t>0}$ *be the Wiener process and* ξ *be a centered random variable independent of* W *with* $\mathbb{E}\xi^2=v^2$*. Let*

$$X_t=W_t+\xi t,\ t>0.$$

Then $E(X_t)=0$ *and* $\mathbb{C}\mathrm{ov}(X_s,X_t)=s(1+v^2t)$*. Furthermore,* X_t *is a harness with respect to its natural past-future filtration, and*

$$\mathbb{V}\mathrm{ar}(X_t|\mathcal{F}_{s,u})=\mathbb{V}\mathrm{ar}(W_t|W_s,W_u)=F_{t,s,u}.$$

So from Theorem 3.1 (or by direct calculation) we see that

$$Y_t=(1-tv^2)X_{t/(1-tv^2)}$$

is a standard quadratic harness on $(0,1/v^2)$ *and has parameters* $\eta=\theta=\sigma=\tau=0$*,* $\gamma=1$*.*

Next, we give a simple example of a quadratic harnesses with $\gamma>1$ and $\sigma\tau>1$; such examples are interesting because most of the general theory developed in [7] does not apply.

Example 3.3. *Suppose* $(G_t)_{t>0}$ *is a gamma process with both parameters in* (2.31) *equal* 1*. Let* ξ *be an independent random variable with mean* $\mathbb{E}(\xi)=\beta>0$ *and* $\mathbb{E}\xi^2=v^2$*. Let*

$$X_t=\xi G_t\,.$$

Then $\mathbb{E}(X_t)=\beta t$*. From*

$$\mathbb{C}\mathrm{ov}(X_s,X_t)=\mathbb{E}(\mathbb{C}\mathrm{ov}(X_s,X_t|\xi))+\mathbb{C}\mathrm{ov}(\mathbb{E}(X_s|\xi),\mathbb{E}(X_t|\xi))$$

we see that for $s\le t$*,* $\mathbb{C}\mathrm{ov}(X_s,X_t)=s(v^2t+\beta)$*. Let* $(\mathcal{F}_{s,u})$ *be the natural past-future filtration associated with* (X_t)*. Consider auxiliary* σ*-fields* $\widetilde{\mathcal{F}}_{s,u}$ *generated by* ξ *and*

$\{G_t : t \in (0,s] \cup [u,\infty)\}$. Then $\mathbb{E}(X_t|\widetilde{\mathcal{F}}_{s,u}) = \xi\left(\frac{u-t}{u-s}G_s + \frac{t-s}{u-s}G_u\right) = \frac{u-t}{u-s}X_s + \frac{t-s}{u-s}X_u$ so $\mathbb{E}(X_t|\mathcal{F}_{s,u}) = \frac{u-t}{u-s}X_s + \frac{t-s}{u-s}X_u$. Similarly, using (2.32) with $\alpha = 1$ we get

$$\begin{aligned}\mathbb{V}\mathrm{ar}(X_t|\widetilde{\mathcal{F}}_{s,u}) = \xi^2\mathbb{V}\mathrm{ar}(G_t|G_s,G_u) &= \xi^2\frac{(u-t)(t-s)}{(u-s+1)(u-s)^2}(G_u-G_s)^2 \\ &= \frac{(u-t)(t-s)}{(u-s+1)(u-s)^2}(X_u-X_s)^2,\end{aligned}$$

so

$$\mathbb{V}\mathrm{ar}(X_t|\mathcal{F}_{s,u}) = \frac{(u-t)(t-s)}{(u-s+1)(u-s)^2}(X_u-X_s)^2.$$

From Theorem 3.1 applied with

$$a = v,\ b = 0,\ c = v,\ d = \beta/v\,,$$

we see that

$$Z_t = v(1-t)X_{\frac{\beta t}{v^2(1-t)}} - \frac{\beta^2}{v}t$$

is a standard quadratic harness on $(0,1)$ with parameters

$$\eta = \theta = 2v/\beta,\ \sigma = \tau = v^2/\beta^2,\ \gamma = 1 + 2\sqrt{\sigma\tau}.$$

In particular, $\gamma = 1 + 2\sqrt{\sigma\tau}$ and $\sigma\tau = v^4/\beta^4 = (\mathbb{E}(\xi^2))^2/(\mathbb{E}(\xi))^4 \geq 1$ can be arbitrarily large.

Of course, the distribution of ξ is arbitrary so the higher moments of Z_t are not determined uniquely and may fail to exist. In particular, Conjecture 1.6 does not hold for quadratic harnesses on finite intervals.

3.1 Matrix notation

For calculations, it will be convenient to parameterize time as a subset of the projective plane, i.e. using $\underline{\mathbf{t}} = \left[t,1\right]^T$. Throughout this section, letters $s,t,u \in \mathcal{T}$ are reserved to denote time, and $\underline{\mathbf{s}}$, $\underline{\mathbf{t}}$, and also $\underline{\mathbf{u}} = [u,1]^T$ have this special meaning also when used with subscripts or primed. We also use the convention that $s \leq t \leq u$.

We rewrite (1.11) in vector form as

$$\mathbb{E}(X_t|\mathcal{F}_{s,u}) = \langle \underline{\mathbf{t}}, \underline{\mathbf{\Delta}}_{s,u}(\mathbf{X})\rangle\,, \tag{3.9}$$

where the components of $\underline{\mathbf{\Delta}}_{s,u}(\mathbf{X})$ are defined by (1.16).

It follows from (1.11) that admissible expectations of a harness $\mathbf{X}$ are affine in t, i.e.,

$$\mathbb{E}(X_t) = \langle \underline{\mathbf{t}}, \underline{\mu}\rangle\,, \quad t \in \mathcal{T}, \tag{3.10}$$

where $\underline{\mu} = [\mu_1,\mu_2]^T$. Moreover, if $\mathbf{X}$ is a square integrable harness then by [7, Proposition 2.1] the admissible covariances are of the form

$$\mathbb{C}\mathrm{ov}(X_s,X_t) = \langle \underline{\mathbf{s}},\ \Sigma\underline{\mathbf{t}}\rangle\,, \quad s,t \in \mathcal{T}, \ \ s \leq t\,, \tag{3.11}$$

where

$$\Sigma = \begin{bmatrix} c_0 & c_1 \\ c_2 & c_3\end{bmatrix}.$$

Note that under our convention $s \leq t$ so Σ is not a symmetric matrix; for example, covariance $\min\{s,t\}$ is represented by matrix $\Sigma = \begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}$. We also remark that if $c_3 \geq 0$,

$c_1 > c_2$, and $c_0c_3 > c_2^2$ then the right hand side of (3.11) indeed defines a positive definite function on $\mathcal{T} = (0, \infty)$, and that processes with $c_1 = c_2$ are degenerate in the sense that X_t is a linear combination of X_s, X_u.

Formula (1.14) can be written in matrix form as

$$\mathbb{V}\mathrm{ar}(X_t|\mathcal{F}_{s,u}) = F_{t,s,u}\left(1 + \langle \underline{\theta}, \underline{\boldsymbol{\Delta}}_{s,u}\rangle + \langle \underline{\boldsymbol{\Delta}}_{s,u}, \Gamma \underline{\boldsymbol{\Delta}}_{s,u}\rangle\right) , \tag{3.12}$$

where

$$\underline{\theta} = \begin{bmatrix} \theta \\ \eta \end{bmatrix}, \ \Gamma = \begin{bmatrix} \tau & -1 \\ \gamma & \sigma \end{bmatrix} . \tag{3.13}$$

Here $\eta, \theta, \sigma, \tau, \gamma$ are constants independent of s, t, u.

Remark 3.4. *Of course, any matrix*

$$\Gamma = \begin{bmatrix} \tau & \Gamma_{12} \\ \Gamma_{21} & \sigma \end{bmatrix}$$

with $\Gamma_{12} + \Gamma_{21} = \gamma - 1$ *gives the same right hand side of* (3.12)*. The standard choice of symmetric* Γ *is in fact inconvenient, see Proposition 3.6. The choice made in* (3.13) *matches the notation we used in previous papers: after substituting* q *for* γ*, the resulting parametrization of the conditional variance is identical to [7, (2.14)].*

The non-random constant $F_{t,s,u}$ is determined uniquely by taking the average of both sides of (3.12). According to [7, (2.15)], with the choice of Γ as in (3.13), formula (1.15) holds.

3.2 Transformations of quadratic harnesses

For a non-degenerate affine function f, as defined in (2.1) and (2.2) with Möbius transform $\varphi : \mathcal{S} \to \mathcal{T}$, and for $s < u$ in $\mathcal{S}$, the transformed σ-fields are

$$\mathcal{F}^f_{s,u} = \begin{cases} \mathcal{F}_{\varphi(s),\varphi(u)} & \text{if } \det(A) > 0\,, \\ \mathcal{F}_{\varphi(u),\varphi(s)} & \text{if } \det(A) < 0\,. \end{cases} \tag{3.14}$$

It is clear that if $\mathbf{X}$ has linear regressions and quadratic conditional variances with respect to past-future filtration $(\mathcal{F}_{s,u})$, then $\mathbf{X}^f$ has linear regressions and quadratic conditional variances with respect to the past-future filtration $(\mathcal{F}^f_{s,u})$.

The following technical result describes how the parameters of a quadratic harness change under the f-transformation.

Proposition 3.5. *Let* $\mathbf{X}$ *be a harness* (1.11) *with respect to a past-future filtration* $(\mathcal{F}_{s,u})$ *on an open interval* $\mathcal{T}$ *with the first two moments given by* (3.10) *and* (3.11)*. Suppose that*

$$\mathbb{V}\mathrm{ar}(X_t|\mathcal{F}_{s,u}) = F_{t,s,u}\left(\chi + \langle \underline{\theta}, \underline{\boldsymbol{\Delta}}_{s,u}\rangle + \langle \underline{\boldsymbol{\Delta}}_{s,u}, \Gamma \underline{\boldsymbol{\Delta}}_{s,u}\rangle\right) , \tag{3.15}$$

with non-random $F_{t,s,u}$*,* $\chi \in \mathbb{R}$*,* $\underline{\theta} \in \mathbb{R}^2$*, and arbitrary* 2×2 *matrix* Γ*. Let* f *be a non-degenerate affine function* (2.1) *such that* φ *is onto* $\mathcal{T}$*.*

Then the process $\widetilde{\mathbf{X}} := \mathbf{X}^f$ *on* $\mathcal{S} = \varphi^{-1}(\mathcal{T})$*, see* (2.3)*, satisfies* (3.10)*,* (3.11) *with* $\underline{\mu}$ *and* Σ *replaced by*

$$\widetilde{\underline{\mu}} = A^T \underline{\mu} + \underline{\mathbf{m}}. \tag{3.16}$$

$$\widetilde{\Sigma} = \begin{cases} A^T \Sigma A & \textit{if } \det(A) > 0\,, \\ A^T \Sigma^T A & \textit{if } \det(A) < 0\,. \end{cases} \tag{3.17}$$

With respect to past-future filtration $(\mathcal{F}^f_{s,u})$*, formulas* (1.11) *and* (3.15) *hold for* $\widetilde{\mathbf{X}}$ *with*

$$\widetilde{\Gamma} = \begin{cases} A^{-1}\Gamma(A^{-1})^T & \text{if } \det(A) > 0\,, \\ (A^{-1})^T\Gamma A^{-1} & \text{if } \det(A) < 0\,, \end{cases} \tag{3.18}$$

$$\underline{\widetilde{\theta}} = A^{-1}\underline{\theta} - (\widetilde{\Gamma} + \widetilde{\Gamma}^T)\underline{\mathbf{m}}\,, \tag{3.19}$$

and

$$\widetilde{\chi} = \chi - \langle \underline{\widetilde{\theta}}, \underline{\mathbf{m}} \rangle - \langle \underline{\mathbf{m}}, \widetilde{\Gamma}\underline{\mathbf{m}} \rangle\,. \tag{3.20}$$

We remark that transformation (3.17) preserves a product form of the covariance. That is, suppose that

$$\mathbb{C}\mathrm{ov}(X_s, X_t) = (\varepsilon s + \delta)(\phi t + \psi),\ \ s < t, \tag{3.21}$$

so that

$$\Sigma = \begin{bmatrix} \varepsilon\phi & \varepsilon\psi \\ \delta\phi & \delta\psi \end{bmatrix}.$$

If for book-keeping we write the coefficients of (3.21) as

$$\Theta = \begin{bmatrix} \varepsilon & \delta \\ \phi & \psi \end{bmatrix}, \tag{3.22}$$

then a calculation based on (3.17) shows that for $\det A > 0$ the covariance of $\widetilde{\mathbf{X}}$ corresponds to $\widetilde{\Theta} = \Theta A$.

We postpone the proof of Proposition 3.5 until Section 4.2, so that we can first clarify the role of non-random constant $F_{t,s,u}$. The main point is that in the non-degenerate case with $c_1 > c_2$, this constant is determined uniquely by taking the average of both sides of (3.15). Furthermore, we explain when $F_{t,s,u}$ is given by formula (1.15).

Proposition 3.6. *Suppose a harness* $\mathbf{X}$ *has mean* (3.10) *and non-degenerate covariance* (3.11) *with* $c_1 > c_2$*. If* $\mathbf{X}$ *has quadratic conditional variance* (3.15) *and the off-diagonal entries of matrix* Γ*, see Remark 3.4, are chosen so that*

$$\chi + \langle \underline{\theta}, \underline{\mu} \rangle + \langle \underline{\mu}, \Gamma\underline{\mu} \rangle + \mathrm{tr}(\Gamma\Sigma^T) = 0, \tag{3.23}$$

then $u(1+s\sigma)+\tau-s\gamma \neq 0$ *and* $F_{t,s,u}$ *is given by formula* (1.15)*. Moreover, transformation formulas in Proposition 3.5 preserve* (1.15).

Formulas (1.14) and (3.13) illustrate the choice of such Γ.

4 Proofs

4.1 Proof of Proposition 1.2

Let $N_t = \{y : p(t, y; v, z_v) > 0\} \subset M_t$ and denote by A a generic Borel set. Let

$$f_{s,t}(x, y) = \frac{p(s, x; t, y)p(t, y; v, z_v)}{p(s, x; v, z_v)} 1_{N_s}(x). \tag{4.1}$$

For any $s < t$ in $[r, v)$ and $x \in N_s$, define probability measure $\nu_{s,x,t}$ by

$$\nu_{s,x,t}(A) = \int_A f_{s,t}(x, y)\pi_t(dy). \tag{4.2}$$

For any $t \in (r, v)$, we let $\nu_t = \nu_{r,z_r,t}$.

To prove that the above probabilities define a Markov process we verify Chapman-Kolmogorov equations. We need to show that for $x \in N_s$,

$$\int_A f_{s,u}(x,z)\pi_u(dz) = \int_{N_t} f_{s,t}(x,y)\left(\int_A f_{t,u}(y,z)\pi_u(dz)\right)\pi_t(dy). \tag{4.3}$$

To this end, we use algebraic identity that holds for all $(x,y) \in N_s \times N_t$,

$$f_{s,t}(x,y)f_{t,u}(y,z) = \frac{p(s,x;t,y)p(t,y;u,z)p(u,z;v,z_v)}{p(s,x;v,z_v)}. \tag{4.4}$$

By (4.4) and Chapman-Kolmogorov equations for process (Z_t), the right hand side of (4.3) is

$$\begin{aligned}\int_{N_t}\left(\int_A \frac{p(s,x;t,y)p(t,y;u,z)p(u,z;v,z_v)}{p(s,x;v,z_v)}\pi_u(dz)\right)\pi_t(dy) = \int_A f_{s,u}(x,z)\pi_u(dz)\\ - \int_{N_t^c}\left(\int_A \frac{p(s,x;t,y)p(t,y;u,z)p(u,z;v,z_v)}{p(s,x;v,z_v)}\pi_u(dz)\right)\pi_t(dy). \end{aligned}\tag{4.5}$$

So to end the proof of (4.3), it is enough to show that for $y \in N_t^c$, $A \subset N_u \subset M_u$,

$$\int_A p(t,y;u,z)p(u,z;v,z_v)\pi_u(dz) = 0. \tag{4.6}$$

To see this, note that for $y \in N_t^c$ we have $p(t,y;v,z_v) = 0$. Therefore,

$$\begin{aligned}0 \le \int_A p(t,y;u,z)p(u,z;v,z_v)\pi_u(dz)\\ \le \int_{M_u} p(t,y;u,z)p(u,z;v,z_v)\pi_u(dz) = p(t,y;v,z_v) = 0.\end{aligned} \tag{4.7}$$

The same argument with $(s,x) = (r,z_r)$ shows that Chapman-Kolmogorov equations hold for ν_t.

Let $(X_t)_{t\in(r,v)}$ be a Markov process with univariate laws (ν_t) and transition probabilities $(\nu_{s,x,t})$. We now verify that (X_t) is a bridge. Since $\nu_t = \nu_{r,z_r,t}$, assumption (1.6) implies that $X_t \xrightarrow{P} z_r$ as $t \to r^+$, and similarly (1.7) implies that $X_t \xrightarrow{P} z_v$ as $t \to v^-$.

Due to Markov property, it suffices to verify the implication (1.3)⇒(1.4) for $s < t < u$. Fix $g \ge 0$. Assumption (1.3) implies that for any measurable function $\psi : M_s \times M_u \to [0,\infty)$,

$$\begin{aligned}\int_{M_s\times M_t\times M_u} g(y)\psi(x,z)p(s,x;t,y)p(t,y;u,z)\pi_s(dx)\pi_t(dy)\pi_u(dz)\\ = \int_{M_s\times M_u} h(x,z)\psi(x,z)p(s,x;u,z)\pi_s(dx)\pi_u(dz).\end{aligned} \tag{4.8}$$

To prove (1.4), it is enough to show that for any measurable $\varphi : N_s \times N_u \to [0,1]$, we have

$$\mathbb{E}(g(X_t)\varphi(X_s,X_u)) = \mathbb{E}(h(X_s,X_u)\varphi(X_s,X_u)),$$

which is the same as

$$\begin{aligned}\int_{N_s\times N_t\times N_u} g(y)\varphi(x,z)f_{s,t}(x,y)f_{t,u}(y,z)f_{r,s}(z_r,x)\pi_s(dx)\pi_t(dy)\pi_u(dz)\\ = \int_{N_s\times N_u} h(x,z)\varphi(x,z)f_{s,u}(x,z)f_{r,s}(z_r,x)\pi_s(dx)\pi_u(dz).\end{aligned} \tag{4.9}$$

Since $p(r,z_r;v,z_v)>0$, from (4.1) and (4.4) we see that (4.9) is equivalent to

$$\int_{N_s\times N_t\times N_u} g(y)\varphi(x,z)p(r,z_r;s,x)p(u,z;v,z_v) \\ \times p(s,x;t,y)p(t,y;u,z)\pi_s(dx)\pi_t(dy)\pi_u(dz) \\ = \int_{N_s\times N_u} h(x,z)\varphi(x,z)p(r,z_r;s,x)p(u,z;v,z_v)p(s,x;u,z)\pi_s(dx)\pi_u(dz).$$

Using (4.6), we can enlarge the region of integration on the left hand side to $N_s\times M_t\times N_u$. Thus, the identity follows from (4.8) applied to

$$\psi(x,z)=\varphi(x,z)p(r,z_r;s,x)p(u,z;v,z_v)1_{N_s}(x)1_{N_u}(z).$$

Now we prove (1.2). By Markov property we can consider a single moment of time. That is, writing $\mu_t^{(x,z)}$ for the univariate law of the bridge when $z_r=x$, $z_v=z$ (note that $\mu_t^{(x,z)}=\nu_{r,x,t}(dy)$), we want to prove that

$$\int_{M_r\times\mathbb{R}\times M_v} f(x)g(y)h(z)\mu_t^{(x,z)}(dy)\pi_r(dx)p(r,x;v,z)\pi_v(dz) \\ = \int_{M_r\times M_t\times M_v} f(x)g(y)h(z)\pi_r(dx)p(r,x;t,y)\pi_t(dy)p(t,y;v,z)\pi_v(dz).$$

Since we assume $p(r,z_r;v,z_v)>0$, this holds true by (4.1) and (4.2).

□

4.2 Proof of Propositions 3.5 and 3.6

We re-write formula (1.15) in matrix notation using a special matrix

$$J=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}. \tag{4.10}$$

It is easy to see that $J^2=-I$, $J^T=-J$. For ease of reference we state also two less obvious properties: for $A\in GL_2(\mathbb{R})$,

$$A^TJA=\det(A)J \ \text{ and } J^TAJ=\det(A)(A^{-1})^T. \tag{4.11}$$

Formula (1.15) can now be written as

$$F_{t,s,u}=\frac{\langle\underline{\mathbf{t}},J\underline{\mathbf{u}}\rangle\langle\underline{\mathbf{s}},J\underline{\mathbf{t}}\rangle}{\langle\underline{\mathbf{s}},J^T\Gamma J\underline{\mathbf{u}}\rangle}\,. \tag{4.12}$$

This formula makes sense for any 2×2 matrix Γ as long as the denominator is non-zero.

Lemma 4.1. *Let f be a non-degenerate affine function* (2.1) *with Möbius transform φ. If $s'=\varphi(s)$, $u'=\varphi(u)$, then*

$$\underline{\mathbf{\Delta}}_{s,u}(\mathbf{X}^f)=A^T\underline{\mathbf{\Delta}}_{s',u'}(\mathbf{X})+\underline{\mathbf{m}}\,. \tag{4.13}$$

Proof. Let $g(x,y)=[x,y]A$ denote the linear part of f. Since $\underline{\mathbf{\Delta}}_{s,u}(a)=\underline{\mathbf{m}}$ on a linear function $a(t)=\langle\underline{\mathbf{t}},\underline{\mathbf{m}}\rangle$, and $\mathbf{X}^f(t)=\mathbf{X}^g(t)+\langle\underline{\mathbf{t}},\underline{\mathbf{m}}\rangle$, we have $\underline{\mathbf{\Delta}}_{s,u}(\mathbf{X}^f)=\underline{\mathbf{\Delta}}_{s,u}(\mathbf{X}^g)+\underline{\mathbf{m}}$. Since $X_{s'}=\mathbf{X}^g(s)/(cs+d)$, and from the matrix form of (1.16) we have

$$\underline{\mathbf{\Delta}}_{s',u'}(\mathbf{X})=\frac{J(X_{u'}\underline{\mathbf{s}}'-X_{s'}\underline{\mathbf{u}}')}{\langle\underline{\mathbf{u}}',J\underline{\mathbf{s}}'\rangle}, \tag{4.14}$$

we get

$$\underline{\mathbf{\Delta}}_{s',u'}(\mathbf{X}) = J\frac{\frac{\mathbf{X}^g(u)}{cu+d}\underline{\mathbf{s}}' - \frac{\mathbf{X}^g(s)}{cs+d}\underline{\mathbf{u}}'}{\langle \underline{\mathbf{u}}', J\underline{\mathbf{s}}'\rangle} = J\frac{\mathbf{X}^g(u)(cs+d)\underline{\mathbf{s}}' - \mathbf{X}^g(s)(cu+d)\underline{\mathbf{u}}'}{\langle (cu+d)\underline{\mathbf{u}}', J(cs+d)\underline{\mathbf{s}}'\rangle}\,.$$

Noting that

$$(cs+d)\underline{\mathbf{s}}' = A\underline{\mathbf{s}}, \tag{4.15}$$

and using (4.11) we get

$$\underline{\mathbf{\Delta}}_{s',u'}(\mathbf{X}) = JA\frac{\mathbf{X}^g(u)\underline{\mathbf{s}} - \mathbf{X}^g(s)\underline{\mathbf{u}}}{\langle \underline{\mathbf{u}}, A^TJA\underline{\mathbf{s}}\rangle} = (A^{-1})^TA^TJA\frac{\mathbf{X}^g(u)\underline{\mathbf{s}} - \mathbf{X}^g(s)\underline{\mathbf{u}}}{\langle \underline{\mathbf{u}}, A^TJA\underline{\mathbf{s}}\rangle}$$
$$= (A^{-1})^T\underline{\mathbf{\Delta}}_{s,u}(\mathbf{X}^g).$$

Thus $\underline{\mathbf{\Delta}}_{s,u}(\mathbf{X}^f) = \underline{\mathbf{\Delta}}_{s,u}(\mathbf{X}^g) + \underline{\mathbf{m}} = A^T\underline{\mathbf{\Delta}}_{s',u'}(\mathbf{X}) + \underline{\mathbf{m}}$. □

Proof of Proposition 3.5. Throughout the proof we write $t' = \varphi(t)$ as in Lemma 4.1. If φ is increasing, by (3.9) and the definition of $\widetilde{\mathbf{X}} = \mathbf{X}^f$ we have

$$\mathbb{E}(\widetilde{X}_t|\mathcal{F}^f_{s,u}) = (ct+d)\mathbb{E}(X_{t'}|\mathcal{F}_{s',u'}) + \langle \underline{\mathbf{t}}, \underline{\mathbf{m}}\rangle \tag{4.16}$$
$$= (ct+d)\langle \underline{\mathbf{t}}', \underline{\mathbf{\Delta}}_{s',u'}(\mathbf{X})\rangle + \langle \underline{\mathbf{t}}, \underline{\mathbf{m}}\rangle\,.$$

By (4.15) and (4.13) we get

$$\mathbb{E}(\widetilde{X}_t|\mathcal{F}^f_{s,u}) = \langle A\underline{\mathbf{t}}, (A^{-1})^T\left(\underline{\mathbf{\Delta}}_{s,u}(\widetilde{\mathbf{X}}) - \underline{\mathbf{m}}\right)\rangle + \langle \underline{\mathbf{t}}, \underline{\mathbf{m}}\rangle = \langle \underline{\mathbf{t}}, \underline{\mathbf{\Delta}}_{s,u}(\widetilde{\mathbf{X}})\rangle\,.$$

Thus the condition (1.11) holds true and $\widetilde{\mathbf{X}}$ is a harness. Similarly, one can verify that (1.11) holds when φ is a decreasing function.

We use (4.15) to compute the mean of $\widetilde{\mathbf{X}}$

$$\mathbb{E}(\widetilde{X}_t) = (ct+d)\langle \underline{\mathbf{t}}', \underline{\mu}\rangle + \langle \underline{\mathbf{t}}, \underline{\mathbf{m}}\rangle = \langle A\underline{\mathbf{t}}, \underline{\mu}\rangle + \langle \underline{\mathbf{t}}, \underline{\mathbf{m}}\rangle = \langle \underline{\mathbf{t}}, \underline{\mathbf{m}} + A^T\underline{\mu}\rangle\,,$$

and (3.16) follows.

To find the covariance we again use (4.15) and the fact that $\mathbb{C}\mathrm{ov}\,(X_{s'}, X_{t'})$ is either $\langle \underline{\mathbf{s}}', \Sigma\underline{\mathbf{t}}'\rangle$ or $\langle \underline{\mathbf{t}}', \Sigma\underline{\mathbf{s}}'\rangle = \langle \underline{\mathbf{s}}', \Sigma^T\underline{\mathbf{t}}'\rangle$ depending whether $s' < t'$ (case $\det(A) > 0$) or $s' > t'$ (case $\det(A) < 0$). For example, if $\det(A) > 0$ then

$$\mathbb{C}\mathrm{ov}(\widetilde{X}_s, \widetilde{X}_t) = (cs+d)(ct+d)\mathbb{C}\mathrm{ov}\,(X_{s'}, X_{t'})$$
$$= (cs+d)(ct+d)\langle \underline{\mathbf{s}}', \Sigma\underline{\mathbf{t}}'\rangle = \langle A\underline{\mathbf{s}}, \Sigma A\underline{\mathbf{t}}\rangle = \langle \underline{\mathbf{s}}, A^T\Sigma A\underline{\mathbf{t}}\rangle\,,$$

and thus (3.17) follows. (We omit the proof when $\det A < 0$.)

Next we tackle the conditional variance. Since ϕ is monotone on $\mathcal{S}$,

$$\mathbb{V}\mathrm{ar}(\widetilde{X}_t|\mathcal{F}^f_{s,u}) = \begin{cases} (ct+d)^2\mathbb{V}\mathrm{ar}(X_{t'}|\mathcal{F}_{s',u'}) & \text{if } \det(A) > 0\,, \\ \\ (ct+d)^2\mathbb{V}\mathrm{ar}(X_{t'}|\mathcal{F}_{u',s'}) & \text{if } \det(A) < 0\,. \end{cases} \tag{4.17}$$

Consider the case $\det(A) < 0$ so that $u' < t' < s'$. Since $\underline{\mathbf{\Delta}}_{a,b} = \underline{\mathbf{\Delta}}_{b,a}$, by (4.17) and Lemma 4.1, the conditional variance is

$$\mathbb{V}\mathrm{ar}(\widetilde{X}_t|\mathcal{F}^f_{s,u}) = (ct+d)^2F_{t',u',s'}\Big(\chi + \Big\langle A^{-1}\underline{\theta}, \underline{\mathbf{\Delta}}_{s,u}(\widetilde{\mathbf{X}}) - \underline{\mathbf{m}}\Big\rangle$$
$$+ \Big\langle \underline{\mathbf{\Delta}}_{s,u}(\widetilde{\mathbf{X}}) - \underline{\mathbf{m}}, A^{-1}\Gamma(A^{-1})^T(\underline{\mathbf{\Delta}}_{s,u}(\widetilde{\mathbf{X}}) - \underline{\mathbf{m}})\Big\rangle\Big). \tag{4.18}$$

Using (4.12), (4.15) and (4.11), we get

$$(ct+d)^2 F_{t',u',s'} = (ct+d)^2 \frac{\langle \underline{\mathbf{u}'}, J\underline{\mathbf{t}'}\rangle\langle \underline{\mathbf{t}'}, J\underline{\mathbf{s}'}\rangle}{\langle \underline{\mathbf{u}'}, J^T\Gamma J\underline{\mathbf{s}'}\rangle}$$
$$= \frac{\langle (cu+d)\underline{\mathbf{u}'}, J(ct+d)\underline{\mathbf{t}'}\rangle\langle (ct+d)\underline{\mathbf{t}'}, J(cs+d)\underline{\mathbf{s}'}\rangle}{\langle (cu+d)\underline{\mathbf{u}'}, J^T\Gamma J(cs+d)\underline{\mathbf{s}'}\rangle}$$
$$= \frac{\langle A\underline{\mathbf{u}}, JA\underline{\mathbf{t}}\rangle\langle A\underline{\mathbf{t}}, JA\underline{\mathbf{s}}\rangle}{\langle A\underline{\mathbf{u}}, J^T\Gamma JA\underline{\mathbf{s}}\rangle} = \frac{\langle \underline{\mathbf{s}}, J\underline{\mathbf{t}}\rangle\langle \underline{\mathbf{t}}, J\underline{\mathbf{u}}\rangle}{\langle \underline{\mathbf{s}}, J^T\widetilde{\Gamma} J\underline{\mathbf{u}}\rangle}.$$

So formula (4.18) rewrites as

$$\mathbb{V}\mathrm{ar}(\widetilde{X}_t|\mathcal{F}^f_{s,u}) = (ct+d)^2 F_{t',u',s'}\Big(\chi - \langle A^{-1}\underline{\theta}, \underline{\mathbf{m}}\rangle + \langle \underline{\mathbf{m}}, \widetilde{\Gamma}^T\underline{\mathbf{m}}\rangle$$
$$+ \Big\langle \underline{\mathbf{\Delta}}_{s,u}(\widetilde{\mathbf{X}}), A^{-1}\underline{\theta} - (\widetilde{\Gamma}+\widetilde{\Gamma}^T)\underline{\mathbf{m}}\Big\rangle + \langle \underline{\mathbf{\Delta}}_{s,u}(\widetilde{\mathbf{X}}), \widetilde{\Gamma}\underline{\mathbf{\Delta}}_{s,u}(\widetilde{\mathbf{X}})\rangle\Big)$$
$$= \frac{\langle \underline{\mathbf{s}}, J\underline{\mathbf{t}}\rangle\langle \underline{\mathbf{t}}, J\underline{\mathbf{u}}\rangle}{\langle \underline{\mathbf{s}}, J^T\widetilde{\Gamma} J\underline{\mathbf{u}}\rangle}\Big(\widetilde{\chi} + \Big\langle \underline{\mathbf{\Delta}}_{s,u}(\widetilde{\mathbf{X}}), A^{-1}\underline{\theta} - (\widetilde{\Gamma}+\widetilde{\Gamma}^T)\underline{\mathbf{m}}\Big\rangle$$
$$+ \langle \underline{\mathbf{\Delta}}_{s,u}(\widetilde{\mathbf{X}}), \widetilde{\Gamma}^T\underline{\mathbf{\Delta}}_{s,u}(\widetilde{\mathbf{X}})\rangle\Big).$$

Since the last term is invariant under transposition, we get (3.19) and (3.18). The case $\det(A) > 0$ is handled similarly and the proof is omitted. □

The proof of Proposition 3.6 is based on the formula for the covariance matrix of vector $\underline{\mathbf{\Delta}}_{s,u}$.

Lemma 4.2.

$$\mathbb{C}\mathrm{ov}\,\underline{\mathbf{\Delta}}_{s,u} = \frac{c_1-c_2}{u-s}\,J\underline{\mathbf{u}}\,\underline{\mathbf{s}}^T J^T + \Sigma^T. \tag{4.19}$$

Proof. From (4.14) we get

$$\mathbb{C}\mathrm{ov}\,\underline{\mathbf{\Delta}}_{s,u} = \mathbb{E}\left(\underline{\mathbf{\Delta}}_{s,u}\underline{\mathbf{\Delta}}_{s,u}^T\right) - \mathbb{E}\left(\underline{\mathbf{\Delta}}_{s,u}\right)\mathbb{E}\left(\underline{\mathbf{\Delta}}_{s,u}^T\right)$$
$$= \frac{J\underline{\mathbf{s}}\underline{\mathbf{u}}^T\Sigma\underline{\mathbf{u}}\underline{\mathbf{s}}^T J^T - J\underline{\mathbf{u}}\underline{\mathbf{s}}^T\Sigma\underline{\mathbf{u}}\underline{\mathbf{s}}^T J^T + J\underline{\mathbf{u}}\underline{\mathbf{s}}^T\Sigma\underline{\mathbf{s}}\underline{\mathbf{u}}^t J^T - J\underline{\mathbf{s}}\underline{\mathbf{s}}^T\Sigma\underline{\mathbf{u}}\underline{\mathbf{u}}^T J^T}{(u-s)^2}.$$

Note that since $\underline{\mathbf{s}}^T\Sigma\underline{\mathbf{u}} = \underline{\mathbf{u}}^T\Sigma^T\underline{\mathbf{s}}$ and $\underline{\mathbf{s}}\,\underline{\mathbf{u}}^T - \underline{\mathbf{u}}\,\underline{\mathbf{s}}^T = (u-s)J^T$, the numerator can be written as

$$J(\underline{\mathbf{s}}\,\underline{\mathbf{u}}^T - \underline{\mathbf{u}}\,\underline{\mathbf{s}}^T)\Sigma\underline{\mathbf{u}}\,\underline{\mathbf{s}}^T J^T - J(\underline{\mathbf{s}}\,\underline{\mathbf{u}}^T - \underline{\mathbf{u}}\,\underline{\mathbf{s}}^T)\Sigma^T\underline{\mathbf{s}}\,\underline{\mathbf{u}}^T J^T = (u-s)(\Sigma\underline{\mathbf{u}}\,\underline{\mathbf{s}}^T - \Sigma^T\underline{\mathbf{s}}\,\underline{\mathbf{u}}^T)J^T$$

(recall that $J^T = -J$ and $JJ^T = I$). Further we write the above expression as

$$(u-s)[(\Sigma-\Sigma^T)\underline{\mathbf{u}}\,\underline{\mathbf{s}}^T + \Sigma^T(\underline{\mathbf{u}}\,\underline{\mathbf{s}}^T - \underline{\mathbf{s}}\,\underline{\mathbf{u}}^T)]J^T = (u-s)(c_1-c_2)J\underline{\mathbf{u}}\,\underline{\mathbf{s}}^T J^T + (u-s)^2\Sigma^T JJ^T$$

and thus (4.19) follows. □

Proof of Proposition 3.6. We first remark that formulas (3.17) and (3.18) imply that

$$\mathrm{tr}(\Gamma\Sigma^T) = \mathrm{tr}(\widetilde{\Gamma}\widetilde{\Sigma}).$$

Next, we note that

$$\chi + \langle \underline{\theta}, \underline{\mu}\rangle + \langle \underline{\mu}, \Gamma\underline{\mu}\rangle = \widetilde{\chi} + \langle \widetilde{\underline{\theta}}, \widetilde{\underline{\mu}}\rangle + \langle \widetilde{\underline{\mu}}, \widetilde{\Gamma}\widetilde{\underline{\mu}}\rangle.$$

(This follows from a longer calculation based on the formulas from Proposition 3.5.) Therefore, transformation formulas preserve (3.23).

Next, we show that (3.23) implies (4.12). This will be accomplished by computing the averages of both sides of (3.15).

We first note that for any harness with covariance (3.11), the expected value of the left hand side of (3.15) is

$$\mathbb{E}\left(\mathbb{V}\mathrm{ar}(X_t|\mathcal{F}_{s,u})\right) = \frac{(t-s)(u-t)}{u-s}(c_1-c_2). \tag{4.20}$$

To prove (4.20), we use (4.19). From (3.9) we get

$$\begin{aligned}\mathbb{E}\,\mathbb{V}\mathrm{ar}(X_t|\mathcal{F}_{s,u}) &= \mathbb{V}\mathrm{ar}\,X_t - \mathbb{V}\mathrm{ar}(\mathbb{E}(X_t|\mathcal{F}_{s,u})) = \mathbb{V}\mathrm{ar}\,X_t - \underline{\mathbf{t}}^T\mathbb{C}\mathrm{ov}(\underline{\mathbf{\Delta}}_{s,u})\underline{\mathbf{t}}\\ &= \underline{\mathbf{t}}^T\Sigma\underline{\mathbf{t}} - \frac{c_1-c_2}{u-s}\underline{\mathbf{t}}^TJ\underline{\mathbf{u}}\,\underline{\mathbf{s}}^TJ^T\underline{\mathbf{t}} - \underline{\mathbf{t}}^T\Sigma^T\underline{\mathbf{t}} = \frac{c_1-c_2}{u-s}(\underline{\mathbf{s}}^TJ\underline{\mathbf{t}})(\underline{\mathbf{t}}^TJ\underline{\mathbf{u}})\\ &= \frac{c_1-c_2}{u-s}(t-s)(u-s).\end{aligned}$$

Next, we compute the right hand side of (3.15). With $K(\underline{\mathbf{\Delta}}_{s,u}) = \chi + \langle\underline{\theta}, \underline{\mathbf{\Delta}}_{s,u}\rangle + \langle\underline{\mathbf{\Delta}}_{s,u}, \Gamma\underline{\mathbf{\Delta}}_{s,u}\rangle$, we have

$$\mathbb{E}\left(K(\underline{\mathbf{\Delta}}_{s,u})\right) = \mathrm{tr}(\Gamma\Sigma^T) + K(\underline{\mu}) + (c_1-c_2)\frac{\langle\underline{\mathbf{s}}, J^T\Gamma J\underline{\mathbf{u}}\rangle}{u-s}. \tag{4.21}$$

To prove (4.21) we note that $\mathbb{E}\,\underline{\mathbf{\Delta}}_{s,u} = \underline{\mu}$, so

$$\begin{aligned}\mathbb{E}\,K(\underline{\mathbf{\Delta}}_{s,u}) &= \chi + \mathbb{E}\,\theta^T\underline{\mathbf{\Delta}}_{s,u} + \mathbb{E}\left(\underline{\mathbf{\Delta}}_{s,u}^T\Gamma\underline{\mathbf{\Delta}}_{s,u}\right)\\ &= \chi + \theta^T\underline{\mu} + \underline{\mu}^T\Gamma\underline{\mu} + \mathrm{tr}\,\left(\Gamma\mathbb{C}\mathrm{ov}\,\underline{\mathbf{\Delta}}_{s,u}\right) = K(\underline{\mu}) + \mathrm{tr}\left(\Gamma\mathbb{C}\mathrm{ov}\,\underline{\mathbf{\Delta}}_{s,u}\right).\end{aligned} \tag{4.22}$$

From (4.19) we get

$$\mathrm{tr}\left(\Gamma\mathbb{C}\mathrm{ov}\,\underline{\mathbf{\Delta}}_{s,u}\right) = \frac{c_1-c_2}{u-s}\mathrm{tr}\left(\Gamma\,J\underline{\mathbf{u}}\,\underline{\mathbf{s}}^TJ^T\right) + \mathrm{tr}\left(\Gamma\Sigma^T\right). \tag{4.23}$$

Since

$$\mathrm{tr}\left(\Gamma\,J\underline{\mathbf{u}}\,\underline{\mathbf{s}}^TJ^T\right) = \mathrm{tr}\left(\underline{\mathbf{s}}^TJ^T\Gamma J\underline{\mathbf{u}}\right) = \langle\underline{\mathbf{s}}, J^T\Gamma J\underline{\mathbf{u}}\rangle,$$

(4.21) follows from (4.22) and (4.23).

Since $\mathrm{tr}(\Gamma\Sigma^T) + K(\underline{\mu}) = 0$ by (3.23), and $\mathbb{E}\left(\mathbb{V}\mathrm{ar}(X_t|\mathcal{F}_{s,u})\right) = F_{t,s,u}\mathbb{E}\left(K(\underline{\mathbf{\Delta}}_{s,u})\right)$, in the non-degenerate case $c_1 > c_2$, formula (4.20) implies that $\mathbb{E}\left(K(\underline{\mathbf{\Delta}}_{s,u})\right) \neq 0$ so from (4.21) we see that $\langle\underline{\mathbf{s}}, J^T\Gamma J\underline{\mathbf{u}}\rangle = u(1+s\sigma) + \tau - s\gamma \neq 0$. We also see that $F_{t,s,u}$ is given by formula (4.12), which is just a matrix form of (1.15).

□

4.3 Proof of Theorem 3.1

Let $A = \frac{1}{ad-bc}\begin{bmatrix} d & -b\\ -c & a\end{bmatrix}$ so that its inverse is $B = \begin{bmatrix} a & b\\ c & d\end{bmatrix}$. We apply Proposition 3.5 with $f(x,y) = ([x,y]-[\alpha,\beta])B$ to

$$\underline{\mu} = \begin{bmatrix}\beta\\ \alpha\end{bmatrix},\ \Sigma = \begin{bmatrix} ac & ad\\ bc & bd\end{bmatrix},\ \Gamma = \begin{bmatrix}\tau & \rho/2\\ \rho/2 & \sigma\end{bmatrix},\ \underline{\theta} = \begin{bmatrix}\theta\\ \eta\end{bmatrix}.$$

From the transformation formulas we get $\widetilde{\mu} = 0$, $\widetilde{\Sigma} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, and $\widetilde{\chi} = \chi + \alpha\eta + \theta\beta + \sigma\alpha^2 + \tau\beta^2 + 2\rho\alpha\beta > 0$ by (3.2). We also get

$$\widetilde{\underline{\theta}} = \begin{bmatrix} b(\eta+\beta\rho+2\alpha\sigma) + a(\theta+\alpha\rho+2\beta\tau)\\ d(\eta+\beta\rho+2\alpha\sigma) + c(\theta+\alpha\rho+2\beta\tau)\end{bmatrix}$$

and
$$\widetilde{\Gamma} = \begin{bmatrix} \tau a^2 + b\rho a + b^2\sigma & \frac{1}{2}(bc\rho + ad\rho + 2bd\sigma + 2ac\tau) \\ \frac{1}{2}(bc\rho + ad\rho + 2bd\sigma + 2ac\tau) & \tau c^2 + d\rho c + d^2\sigma \end{bmatrix}.$$

The quadratic polynomial K remains unchanged if we replace $\widetilde{\Gamma}$ by

$$\Gamma' = \begin{bmatrix} \tau a^2 + b\rho a + b^2\sigma & -\widetilde{\chi} \\ \widetilde{\chi} + bc\rho + ad\rho + 2bd\sigma + 2ac\tau & \tau c^2 + d\rho c + d^2\sigma \end{bmatrix},$$

see Remark 3.4. Rewriting K as

$$K(\underline{\mathbf{x}}) = \widetilde{\chi} + \langle \underline{\widetilde{\theta}}, \underline{\mathbf{x}} \rangle + \langle \underline{\mathbf{x}}, \Gamma' \underline{\mathbf{x}} \rangle = \widetilde{\chi} \left(1 + \langle \frac{1}{\widetilde{\chi}} \underline{\widetilde{\theta}}, \underline{\mathbf{x}} \rangle + \langle \underline{\mathbf{x}}, \frac{1}{\widetilde{\chi}} \Gamma' \underline{\mathbf{x}} \rangle \right),$$

we get the parameters as claimed.

□

4.4 Proof of Theorem 2.2

With $s < t_1 < t_2 < u$, the conditional covariance of a standard quadratic harness is

$$\mathbb{C}\mathrm{ov}\big(X_{t_1}, X_{t_2} \big| \mathcal{F}_{s,u}\big) = \frac{\langle \underline{\mathbf{t}}_2, J\underline{\mathbf{u}} \rangle \langle \underline{\mathbf{s}}, J\underline{\mathbf{t}}_1 \rangle}{\langle \underline{\mathbf{s}}, J^T \Gamma J \underline{\mathbf{u}} \rangle} K(\underline{\boldsymbol{\Delta}}_{s,u}), \tag{4.24}$$

where $K(\underline{\mathbf{a}}) = 1 + \langle \underline{\theta}, \underline{\mathbf{a}} \rangle + \langle \underline{\mathbf{a}}, \Gamma \underline{\mathbf{a}} \rangle$ is the quadratic polynomial from (1.17) and (3.12). A quick way to see this is to notice that (1.11) implies

$$\mathbb{C}\mathrm{ov}\big(X_{t_1}, X_{t_2} \big| \mathcal{F}_{s,u}\big) = \frac{u - t_2}{u - t_1} \mathbb{V}\mathrm{ar}(X_{t_1} | \mathcal{F}_{s,u}) = \frac{\langle \underline{\mathbf{t}}_2, J\underline{\mathbf{u}} \rangle}{\langle \underline{\mathbf{t}}_1, J\underline{\mathbf{u}} \rangle} \mathbb{V}\mathrm{ar}(X_{t_1} | \mathcal{F}_{s,u}).$$

Proof of Theorem 2.2. To prove that $v(1 + r\sigma) + \tau - r\gamma > 0$ we note that $v \mapsto v(1 + r\sigma) + \tau - r\gamma$ is a continuous function on $\mathbb{R}$ which by Proposition 3.6 applied to $\mathbf{Z}$ cannot cross zero on (r, ∞).

To determine the mean and the variance of $\mathbf{X}$, we use Definition 1.1. Denote by $\mathcal{F}^X_{s,u}$ the natural past-future filtration of $\mathbf{X}$. By taking $g(x) = x$, from Definition 1.1(iii) we see that for $t \in (r, v)$, we have $\mathbb{E}(X_t | \mathcal{F}^X_{s,u}) = h_{s,t,u}(X_s, X_u)$, where

$$h_{s,t,u}(x, y) = \frac{u - t}{u - s} x + \frac{t - s}{u - s} y.$$

Thus by Definition 1.1(i), $\mathbb{E}(X_t | \mathcal{F}^X_{s,u}) \xrightarrow{P} h_{r,t,v}(z_r, z_v)$ as $(s, u) \to (r, v)$.

We note that as $s \searrow r$ and $u \nearrow v$, say over rational numbers, the filtration $\mathcal{F}^X_{s,u}$ decrease. So by the martingale convergence theorem,

$$\mathbb{E}(X_t) = \mathbb{E}\left(\lim_{(s,u) \to (r^+, v^-)} \mathbb{E}\left(X_t | \mathcal{F}^X_{s,u}\right) \right) = \frac{v - t}{v - r} z_r + \frac{t - r}{v - r} z_v = \langle \underline{\boldsymbol{\Delta}}_{r,v}, \underline{\mathbf{t}} \rangle.$$

Similarly, by martingale convergence theorem, for fixed $t_1 < t_2$ in (r, v),

$$\mathbb{E}(X_{t_1} X_{t_2}) = \mathbb{E}\left(\lim_{(s,u) \to (r^+, v^-)} \mathbb{E}\left(X_{t_1} X_{t_2} | \mathcal{F}^X_{s,u}\right) \right)$$

Using again Definition 1.1 with $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x_1, x_2) = x_1 x_2$, we see that

$$\mathbb{E}\left(X_{t_1} X_{t_2} | \mathcal{F}^X_{s,u}\right) = \tilde{h}_{s,u}(X_s, X_u),$$

where $\tilde{h}$, defined through

$$\mathbb{E}\left(Z_{t_1} Z_{t_2} | \mathcal{F}_{s,u}\right) = \tilde{h}_{s,u}(Z_s, Z_u),$$

is computed from (4.24) as

$$\tilde{h}_{s,u}(x,y)=\tfrac{(u-t_2)(t_1-s)}{u(1+s\sigma)+\tau-s\gamma}K\left(\tfrac{y-x}{u-s},\tfrac{ux-sy}{u-s}\right)+h_{s,t_1,u}(x,y)h_{s,t_2,u}(x,y).$$

Since $\tilde{h}_{s,u}(X_s,X_u)\xrightarrow{P}\tilde{h}_{r,v}(z_r,z_v)$ as $(s,u)\to(r^+,v^-)$, we get $\mathbb{C}\mathrm{ov}(X_{t_1},X_{t_2})=M^2(v-t_2)(t_1-r)$ with

$$M=\frac{\sqrt{K(\Delta_{r,v},\widetilde{\Delta}_{r,v})}}{\sqrt{v(1+r\sigma)+\tau-r\gamma}}>0\,. \tag{4.25}$$

The above reasoning also shows that for $r<s<t<u<v$ the conditional moments $\mathbb{E}(X_t|\mathcal{F}^X_{s,u})$ and $\mathbb{V}\mathrm{ar}(X_t|\mathcal{F}^X_{s,u})$ are given by the same polynomials as the corresponding conditional moments of **Z**. Thus the assumptions of Theorem 3.1 hold, and we apply it with

$$\alpha=\widetilde{\Delta}_{r,v},\ \beta=\Delta_{r,v},\ \chi=1,\ \rho=\gamma-1,$$
$$a=M\sqrt{v},\ b=-rM\sqrt{v},\ c=-M/\sqrt{v},\ d=M\sqrt{v}.$$

(There are other possible choices that lead to "equivalent" quadratic harnesses as in (2.25).) With the above choice of a,b,c,d, formula (3.3) gives (2.6). Since $\widetilde{\chi}=K(\Delta_{r,v},\widetilde{\Delta}_{r,v})>0$, assumption (3.2) holds, and the parameters of the resulting quadratic harness are as claimed.

□

Acknowledgments. Maja Jamiołkowska worked out the construction [20] which inspired our Proposition 2.10. We also benefited from discussions with Ryszard Szwarc and Wojciech Matysiak. This research was partially supported by NSF grant #DMS-0904720, and by Taft Research Seminar 2008/09.

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