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Random walks with unbounded jumps among random conductances I: Uniform quenched CLT

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Abstract

We study a one-dimensional random walk among random conductances, with unbounded jumps. Assuming the ergodicity of the collection of conductances and a few other technical conditions (uniform ellipticity and polynomial bounds on the tails of the jumps) we prove a quenched *uniform* invariance principle for the random walk. This means that the rescaled trajectory of length n is (in a certain sense) close enough to the Brownian motion, uniformly with respect to the choice of the starting location in an interval of length $O(\sqrt{n})$ around the origin.

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1 Introduction and results

Suppose that for each pair of integers we are given a nonnegative number. One may think that sites of $\mathbb{Z}$ are nodes of an electrical network where any site can be connected to any other site, and those numbers are thought of as the *conductances* of the corresponding links. The conductances are initially chosen at random, and we call the set of the conductances *random environment*. For the random environment, we assume that it is stationary and ergodic. Given the conductances, one then defines a (reversible) discrete-time random walk in the usual way: the transition probability from x to y is proportional to the conductance between x and y.

Here and in the companion paper [15] we study one-dimensional random walks among random conductances informally described above, with unbounded jumps (we impose a condition that implies that the conductances can decay polynomially in the distance between the sites, but with sufficiently large power). The main result of [15] concerns the (quenched) limiting law of the trajectory of the random walk $(X_n, n = 0, 1, 2, \ldots)$ starting from the origin up to time n, under condition that it remains positive at the moments $1, \ldots, n$. In [15] we prove that, after suitable rescaling, for a.e. environment it converges to the *Brownian meander* process, which is, roughly speaking, a Brownian motion conditioned on staying positive up to some finite time. It turns out

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that one of the main ingredients for the proof of the conditional CLT is the *uniform* quenched CLT, which is the main result (Theorem 1.2) of the present paper.

Our main motivation for considering one-dimensional random walks with unbounded jumps among random conductances and with minimal assumptions on the environment comes from Knudsen billiards in random tubes, see [9, 10, 11, 12]. This model can be regarded as a discrete-time Markov chain with continuous space, the positions of the walker correspond to places where a billiard ball with a *random* law of reflection of certain form hits the boundary. This model has some nice reversibility properties that makes it in some sense a continuous space analogue of the random walk among random conductances. In (rather long and technical) Section 3 of [12] the following problem was treated: given that the particle, injected at the left boundary of the tube, crosses the tube of length H without returning to the starting boundary, then the crossing time exceeds εH^2 with high probability, as $\varepsilon \to 0$, $H \to \infty$. Of course, such a fact would be an easy consequence of a *conditional* limit theorem similar to the one described above (since the probability that the Brownian meander behaves that way is high). We decided to study the discrete-space model because it presents less technical difficulties than random billiards in random tubes, and therefore allows to obtain finer results (such as the conditional CLT).

(Unconditional) quenched Central Limit Theorems for this and related models (even in the many-dimensional case) received much attention in the recent years, see e.g. [2, 1, 5, 3, 6, 20, 19]. Mainly, the modern approach consists in constructing a so-called corrector function which turns the random walk to a martingale, and then using the CLT for martingales. To construct the corrector, one can use the method of orthogonal projections [6, 20, 19]. While the corrector method is powerful enough to yield quenched CLTs, its construction by itself is not very explicit, and, in particular, it does not say a lot about the speed of convergence. Besides that, it is, in principle, not very clear how the speed of convergence depends on the starting point. For instance, one may imagine that there are "distant" regions where the environment is "atypical", and so is the behavior of the random walk starting at a point from such a region until the time when it comes to "normal" parts of the environment. In Theorem 1.2 we prove that, for rather general ergodic environments that admit unbounded jumps, if the rescaled trajectory by time n is "close" enough to the Brownian motion, then so are the trajectories starting from points of an interval of length $O(\sqrt{n})$ centered in the origin. In our opinion, this result is interesting by its own, but for us the main motivation for investigating this question was that it provides an important tool for proving the conditional CLT. Indeed, the strategy of the proof of the conditional CLT is to force it a bit away (around $\varepsilon\sqrt{n}$) from the origin in a "controlled" way and then use the usual (unconditional) CLT; but then it is clear that it is quite convenient to have the CLT for *all* starting positions in an interval of length $O(\sqrt{n})$ at once.

It is important to note that in the most papers about random walks with random conductances one assumes that the jumps are uniformly bounded, usually nearest-neighbor (one can mention e.g. [1, 7] that consider the case of unbounded jumps). When there is no uniform bound on the size of the jumps, this of course brings some complications to the proofs, as one can see in the proof of Theorem 1.2 below. Still, in our opinion, it is important to be able to obtain "usual" results for the case of long-range jumps as well; for example, in some related models, such as the above-mentioned reversible random billiards in random tubes [11, 12] the jumps are naturally unbounded.

In the case when (in dimension 1) the jumps are uniformly bounded, the proofs become much simpler, mainly because one does not need to bother about the exit distributions, as in Section 2.3. The case of nearest-neighbor jumps is, of course, even simpler, since many quantities of interest have explicit expressions. We will not discuss this case separately, since it is (in some sense) “too easy” and does not provide a lot of clues about how the walk with unbounded jumps should be treated. Let us make an observation that a random walk with nearest-neighbor jumps becomes a very interesting and complex object to study if one samples at random not the conductances, but the transition probabilities themselves (i.e., the transition probabilities from n to $n+1$ are chosen independently before the process starts). The resulting random walk, while still reversible, behaves quite differently (in particular, diffusive limits are unusual for that model). We only mention that *conditional* (on being at the origin at time $2t$) behavior of this random walk in the transient case was studied in [16], and a similar result for the recurrent case can be obtained from Corollary 2.1 of [8].

Of course, a natural question is whether the result analogous to Theorem 1.2 also holds for the many-dimensional nearest-neighbor random walk among random conductances. We postpone the discussion about that to the end of this section.

Now, we define the model formally. For $x, y \in \mathbb{Z}$, let us denote by $\omega_{x,y} = \omega_{y,x}$ the conductance between x and y. Define $\theta_z \omega_{x,y} := \omega_{x+z,y+z}$, for all $z \in \mathbb{Z}$. Note that, by Condition K below, the vectors $\omega_{x,\cdot}$ are elements of the Polish space $\ell^2(\mathbb{Z})$. We assume that $(\omega_{x,\cdot})_{x\in\mathbb{Z}}$ is a stationary ergodic (with respect to the family of shifts θ) sequence of random vectors; $\mathbb{P}$ stands for the law of this sequence and $\langle \cdot \rangle_{\mathbb{P}}$ is the expectation with respect to $\mathbb{P}$. The collection of all conductances $\omega = (\omega_{x,y}, x, y \in \mathbb{Z})$ is called the *environment*. For all $x \in \mathbb{Z}$, define $C_x := \sum_y \omega_{x,y}$. Given that $C_x < \infty$ for all $x \in \mathbb{Z}$ (which is always so by Condition K below), the random walk X in random environment ω is defined through its transition probabilities

$$p_\omega(x, y) = \frac{\omega_{x,y}}{C_x};$$

that is, if $\mathtt{P}^x_\omega$ is the quenched law of the random walk starting from x, we have

$$\mathtt{P}^x_\omega[X_0 = x] = 1, \quad \mathtt{P}^x_\omega[X_{k+1} = z \mid X_k = y] = p_\omega(y, z).$$

Clearly, this random walk is reversible with the reversible measure $(C_x, x \in \mathbb{Z})$. Also, we denote by $\mathtt{E}^x_\omega$ the quenched expectation for the process starting from x. When the random walk starts from 0, we use shortened notations $\mathtt{P}_\omega, \mathtt{E}_\omega$.

In order to prove our results, we need to make two technical assumptions on the environment:

Condition E. There exists $\kappa > 0$ such that, $\mathbb{P}$-a.s., $\omega_{0,1} \geq \kappa$.

Condition K. There exist constants $K, \beta > 0$ such that, $\mathbb{P}$-a.s., $\omega_{0,y} \leq \frac{K}{1+y^{3+\beta}}$, for all $y \geq 0$. Note, for future reference, that the stationarity of $\mathbb{P}$ and Conditions E and K together imply that there exists $\hat{\kappa} > 0$ such that, $\mathbb{P}$-a.s.,

$$\hat{\kappa} \leq \sum_{y\in\mathbb{Z}} \omega_{0,y} \leq \hat{\kappa}^{-1}. \tag{1.1}$$

We decided to formulate Condition E this way because, due to the fact that this work was motivated by random billiards, the main challenge was to deal with the long-range jumps. It is plausible that Condition E could be relaxed to some extent; however, for the sake of cleaner presentation of the argument, we prefer not trying to deal with *both* long-range jumps and the lack of nearest-neighbor ellipticity.

Next, for all $n \geq 1$, we define the continuous map $Z^n = (Z^n(t), t \in \mathbb{R}_+)$ as the natural polygonal interpolation of the map $k/n \mapsto \sigma^{-1} n^{-1/2} X_k$ (with σ from Theorem 1.1 below). In other words,

$$\sigma\sqrt{n} Z^n_t = X_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1}$$

with $\lfloor \cdot \rfloor$ the integer part. Also, we denote by W the standard Brownian motion.

First, we state the following result, which is the usual quenched invariance principle:

Theorem 1.1. *Assume Conditions E and K. Then, there exists a finite (nonrandom) constant $\sigma > 0$ such that for $\mathbb{P}$-almost all ω, Z^n converges in law, under $\mathtt{P}_\omega$, to Brownian motion W as $n \to \infty$.*

Of course, with the current state of the art in this field, obtaining the proof of Theorem 1.1 is a mere exercise (one can follow e.g. the argument of [6]); for this reason, we do not write the proof of Theorem 1.1 in this paper. The key observation, though, is that Condition K implies that

$$\Big\langle \sum_{y\in\mathbb{Z}} y^2 \omega_{0,y} \Big\rangle_{\mathbb{P}} < \infty.$$

Let $C(\mathbb{R}_+)$ be the space of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$. Let us denote by $\mathfrak{C}_b(C(\mathbb{R}_+),\mathbb{R})$ (respectively, $\mathfrak{C}_b^u(C(\mathbb{R}_+),\mathbb{R})$) the space of bounded continuous (respectively, bounded uniformly continuous) functionals from $C(\mathbb{R}_+)$ into $\mathbb{R}$ and by $\mathcal{B}$ the Borel σ-field on $C(\mathbb{R}_+)$. We have the following result, which is referred to as quenched Uniform Central Limit Theorem (UCLT):

Theorem 1.2. *Under Conditions E and K, the following statements hold:*

(i) we have $\mathbb{P}$-a.s., for all $H > 0$ and any $F \in \mathfrak{C}_b(C(\mathbb{R}_+),\mathbb{R})$,

$$\lim_{n\to\infty} \sup_{x\in[-H\sqrt{n},H\sqrt{n}]} \Big| \mathtt{E}_{\theta_x\omega}[F(Z^n)] - E[F(W)] \Big| = 0;$$

(ii) we have $\mathbb{P}$-a.s., for all $H > 0$ and any $F \in \mathfrak{C}_b^u(C(\mathbb{R}_+),\mathbb{R})$,

$$\lim_{n\to\infty} \sup_{x\in[-H\sqrt{n},H\sqrt{n}]} \Big| \mathtt{E}_{\theta_x\omega}[F(Z^n)] - E[F(W)] \Big| = 0;$$

(iii) we have $\mathbb{P}$-a.s., for all $H > 0$ and any closed set $B \in \mathcal{B}$,

$$\limsup_{n\to\infty} \sup_{x\in[-H\sqrt{n},H\sqrt{n}]} \mathtt{P}_{\theta_x\omega}[Z^n \in B] \le P[W \in B];$$

(iv) we have $\mathbb{P}$-a.s., for all $H > 0$ and any open set $G \in \mathcal{B}$,

$$\liminf_{n\to\infty} \inf_{x\in[-H\sqrt{n},H\sqrt{n}]} \mathtt{P}_{\theta_x\omega}[Z^n \in G] \ge P[W \in G];$$

(v) we have $\mathbb{P}$-a.s., for all $H > 0$ and any $A \in \mathcal{B}$ such that $P[W \in \partial A] = 0$,

$$\lim_{n\to\infty} \sup_{x\in[-H\sqrt{n},H\sqrt{n}]} \Big| \mathtt{P}_{\theta_x\omega}[Z^n \in A] - P[W \in A] \Big| = 0.$$

Even though it may be possible to find a concise formulation of our main result with only one "final" statement and not a list of equivalent ones (the authors did not succeed in finding it), we content ourselves in writing it in this form because, in our opinion, possible situations where it can be useful are covered by the list. Of course, item (ii) is redundant (it follows trivially from (i)), and (iii) and (iv) are equivalent by complementation.

For the model of this paper, it is not possible to generalize Theorem 1.2 by considering a wider interval $[-Hn^\alpha, Hn^\alpha]$ for some $\alpha > 1/2$, where the starting point is taken; this is because we only assume the ergodicity of the environment of conductances. Indeed, consider e.g. a nearest-neighbor random walk, and suppose that the conductances can assume only two possible values, say, 1 and 2. To construct the stationary ergodic random environment, we first construct its cycle-stationary version in the following way. Fix $\varepsilon > 0$ such that $\frac{\alpha}{1+\varepsilon} > \frac{1}{2}$, and let us divide the edges of $\mathbb{Z}$ into blocks of random i.i.d. sizes $(V_i, i \in \mathbb{Z})$, with $P[V_1 > s] = O(s^{-(1+\varepsilon)})$. Inside each block, we toss a fair coin and, depending on the result, either place all 1s, or an alternating sequence of 2s and 1s. Since the expected size of the block is finite, it is clear that this environment can be made stationary (and, of course, ergodic) by a standard random shift procedure, see e.g. Chapter 8 of [22]. Then, one readily obtains that in the interval $[-Hn^\alpha, Hn^\alpha]$, with large probability, one finds both $1\dots1$-blocks and $212\dots12$-blocks of length at least $\sqrt{n}$. So, the UCLT cannot be valid: just consider starting points in the middle of two blocks of different type. It is, in our opinion, an interesting problem to obtain a stronger form of Theorem 1.2 in the case when the environment has mixing or independence properties. It seems plausible that one can make the above interval at least polynomially (with any power) wide, but we prefer not to discuss further questions of this type in this paper: in any case, for the results of [15], Theorem 1.2 is already enough.

Let us also comment on possible many-dimensional variants of Theorem 1.2. For the case of nearest-neighbor random walks in $\mathbb{Z}^d$ with random conductances bounded from both sides by two positive constants, an analogous result was obtained in [14] (Theorem 1.1). The proof of Theorem 1.1 of [14] relies on the uniform heat-kernel bounds of [13]; one uses these bounds to obtain that, regardless of the starting point, with probability close to 1 the walk will enter to the set of "good" sites (i.e., the sites from where the convergence is good enough). Naturally, this poses the question of what to do with unbounded conductances (and/or unbounded jumps), to which we have no answer for now (although one can expect, as usual, that the case $d = 2$ should be more accessible, since in this case each site is "surrounded" by "good" sites, cf. e.g. the proof of Theorem 4.7 in [5]).

The paper is organized in the following way: in the next section, we obtain some auxiliary facts which are necessary for the proof of Theorem 1.2 (recurrence, estimates on the probability of confinement in an interval, estimates on the exit measure from an interval). Then, in Section 3, we give the proof of Theorem 1.2.

We will denote by K_1, K_2, ... the "global" constants, that is, those that are used all along the paper, and by γ_1, γ_2, ... the "local" constants, that is, those that are used only in the subsection in which they appear for the first time. For the local constants, we restart the numeration in the beginning of each subsection. Depending on the context, expressions like $x \in [-H\sqrt{n}, H\sqrt{n}]$ should be understood as $x \in [-H\sqrt{n}, H\sqrt{n}] \cap \mathbb{Z}$.

2 Auxiliary results

In this section, we will prove some technical results that will be needed later to prove Theorem 1.2. Let us introduce the following notations. If $A \subset \mathbb{Z}$,

$$\tau_A := \inf\{n \geq 0 : X_n \in A\} \quad \text{and} \quad \tau_A^+ := \inf\{n \geq 1 : X_n \in A\}.$$

2.1 Recurrence of the random walk

Lemma 2.1. *Under Conditions E and K the random walk X is $\mathbb{P}$-a.s. recurrent.*

Proof. To show the recurrence of the random walk, we will show that the probability of escape to infinity is zero. First, let us consider the finite interval $I_L = [-L, L]$ for some $L > 0$. Consider the time $\tau_{I_L^c}$ of exit from the interval I_L. By the Dirichlet variational principle for reversible Markov chains (see, for example, Theorem 6.1 Chap. II of [18]) we have that

$$2C_0 \mathrm{P}_\omega[\tau_{I_L^c} < \tau_0^+] = \min_{f \in \mathcal{H}} \Phi(f) \tag{2.1}$$

where Φ is the Dirichlet form defined by

$$\Phi(f) := \sum_{x,y\in\mathbb{Z}} \omega_{x,y}[f(x) - f(y)]^2$$

and $\mathcal{H}$ is the following set of functions:

$$\mathcal{H} := \{f : \mathbb{Z} \to [0,1] : f(0) = 0 \text{ and } f(x) = 1 \text{ for } x \notin I_L\}.$$

In order to estimate $\mathtt{P}_\omega[\tau_{I_L^c} < \tau_0^+]$ from above let us consider the function h in $\mathcal{H}$ defined by

$$h(x) = \begin{cases} L^{-1}|x|, & \text{if } |x| \le L, \\ 1, & \text{if } |x| > L. \end{cases}$$

Now, let us estimate $\Phi(h)$. We start by writing

$$\begin{aligned}\Phi(h) &= \sum_{x,y\in\mathbb{Z}} \omega_{x,y}[h(x) - h(y)]^2 \\ &= \sum_{-L<x,y<L} \omega_{x,y}[h(x) - h(y)]^2 + 2 \sum_{x\in(-\infty,-L]\cup[L,\infty)} \sum_{y\in(-L,L)} \omega_{x,y}[h(x) - h(y)]^2. \quad (2.2)\end{aligned}$$

We are going to show that both terms in the decomposition (2.2) of $\Phi(h)$ are of order smaller or equal to L^{-1}. Indeed, it is not difficult to see that each of both terms in the decomposition (2.2) is smaller than

$$\frac{2}{L^2} \sum_{-L<x<L} \sum_{y\in\mathbb{Z}} \omega_{x,y}(y - x)^2.$$

By Condition K, we obtain that there exists a constant γ_1 such that $\mathbb{P}$-a.s., $\sum_{y\in\mathbb{Z}} \omega_{x,y}(y - x)^2 \le \gamma_1$ for all x. Then, we deduce

$$\Phi(h) \le \frac{8\gamma_1}{L}.$$

Using (2.1), we obtain that,

$$2C_0\mathtt{P}_\omega[\tau_{I_L^c} < \tau_0^+] \le \frac{8\gamma_1}{L}.$$

By (1.1), we have $C_0 \ge \hat{\kappa}$ so that

$$\mathtt{P}_\omega[\tau_{I_L^c} < \tau_0^+] \le \frac{4\gamma_1}{\hat{\kappa}L}. \quad (2.3)$$

Now, let p_{esc} be the probability that the walk started at 0 escapes to infinity. We have $p_{esc} = \lim_{L\to\infty} \mathtt{P}_\omega[\tau_{I_L^c} < \tau_0^+]$, and so, by (2.3), $p_{esc} = 0$. Hence, the random walk X is $\mathbb{P}$-a.s. recurrent. □

2.2 Probability of confinement

Let $I = [a, b] \subset \mathbb{Z}$ be a finite interval containing at least 3 points and let $B = (-\infty, a]$ and $E = [b, \infty)$. In this subsection we shall prove the following

Proposition 2.2. *There exist constants $K_1 > 0$ and $K_2 > 0$ such that we have $\mathbb{P}$-a.s.,*

$$\max_{x\in(a,b)} \mathtt{P}^x_\omega[\tau_{B\cup E} > n] \le \exp\Big\{ - \frac{n}{K_1(b - a)^2}\Big\}$$

for all $n > K_2(b - a)^2$.

Proof. Let ω be a realization of the random environment. Consider the new environment obtained from ω by deleting all the conductances $\omega_{x,y}$ if x and y belong to $B \cup E$. The reversible measure C' on this new environment is given by

$$C'_x = \begin{cases} C_x, & \text{if } x \in (a,b), \\ \sum_{y \notin B \cup E} \omega_{x,y}, & \text{otherwise.} \end{cases}$$

Next, we define $C'_B := \sum_{y \in B} C'_y$ and $\pi_B(x) := C'_x / C'_B$ for all $x \in B$. Observe that, by Conditions E and K, C'_B is positive and finite $\mathbb{P}$-a.s. Hence, it holds that π_B is $\mathbb{P}$-a.s. a probability measure on B. In the same way we define the probability measure π_E on E. Now, we introduce a new Markov chain X' on a finite state space $\mathcal{S}' := (a,b) \cup \{\Delta_B, \Delta_E\}$ (Δ_B and Δ_E are the states corresponding to B and E). On $\mathcal{S}'$, we define the following transition probabilities: if $x \notin \{\Delta_B, \Delta_E\}$

$$P_{x,\Delta_E} = \sum_{y \in E} \frac{\omega_{x,y}}{C'_x}, \quad P_{x,\Delta_B} = \sum_{y \in B} \frac{\omega_{x,y}}{C'_x}$$

and

$$P_{\Delta_E,x} = \sum_{y \in E} \pi_E(y) \frac{\omega_{x,y}}{C'_y}, \quad P_{\Delta_B,x} = \sum_{y \in B} \pi_B(y) \frac{\omega_{y,x}}{C'_y}.$$

Then, set $P_{\Delta_E,\Delta_B} = P_{\Delta_B,\Delta_E} = P_{\Delta_B,\Delta_B} = P_{\Delta_E,\Delta_E} = 0$. For $x \notin \{\Delta_B, \Delta_E\}$ and $y \notin \{\Delta_B, \Delta_E\}$ we just set $P_{x,y} = \omega_{x,y}/C'_x$. Defining $C'_{\Delta_B} := C'_B$ and $C'_{\Delta_E} := C'_E$, we can easily check that the detailed balance equations are satisfied, that is, on $\mathcal{S}'$ we have a new set of conductances ω' specified by $\omega'_{x,y} := C'_x P_{x,y} = C'_y P_{y,x}$. Observe also that by Condition K, there exists a constant $\gamma_1 > 0$ such that $\mathbb{P}$-a.s., $C'_x \le \gamma_1$ for all $x \in \mathcal{S}'$. By the commute time identity (see for example Proposition 10.6 of [17]) we have that

$$\mathtt{E}^x_\omega[\tau_{B\cup E}] \le \mathtt{E}^x_{\omega'}[\tau_{\Delta_B}] + \mathtt{E}^{\Delta_B}_{\omega'}[\tau_x] = \Big(\sum_{y \in \mathcal{S}'} C'_y\Big) R_{\text{eff}}(\Delta_B, x)$$

where $R_{\text{eff}}(\Delta_B, x)$ is the effective resistance between Δ_B and x. We have

$$\Big(\sum_{y \in \mathcal{S}'} C'_y\Big) \le \gamma_1(b-a+1)$$

and

$$R_{\text{eff}}(\Delta_B, x) \le \sum_{y=\Delta_B}^{x-1} \omega^{-1}_{y,y+1} \le \kappa^{-1}(b-a+1).$$

Thus,

$$\mathtt{E}^x_\omega[\tau_{B\cup E}] \le \gamma_1 \kappa^{-1}(b-a+1)^2 \le \gamma_2(b-a)^2$$

for some positive constant γ_2. By the Chebyshev inequality, we can choose a large enough constant $\gamma_3 > 0$ in such a way that

$$\mathtt{P}^x_\omega\big[\tau_{B\cup E} > \lfloor \gamma_3(b-a)^2 \rfloor\big] \le \frac{\mathtt{E}^x_\omega[\tau_{B\cup E}]}{\lfloor \gamma_3(b-a)^2 \rfloor} \le \frac{\gamma_2(b-a)^2}{\lfloor \gamma_3(b-a)^2 \rfloor} < 1. \tag{2.4}$$

Let us denote $s := \lfloor \gamma_3(b-a)^2 \rfloor$ and $p := \gamma_2(b-a)^2 \lfloor \gamma_3(b-a)^2 \rfloor^{-1}$. For $n \ge s$ divide the time interval $[0,n]$ into $N := \lfloor \frac{n}{s} \rfloor$ subintervals of length s. Using (2.4) and the Markov property we obtain

$$\begin{aligned} \mathtt{P}^x_\omega[\tau_{B\cup E} > n] &\le \mathtt{P}^x_\omega\big[X'(sj) \notin \{\Delta_B, \Delta_E\}, j = 1, \dots, N\big] \\ &\le (1-p)^N \\ &\le \exp\Big(-\frac{n}{\gamma_4(b-a)^2}\Big) \end{aligned}$$

for some positive constant γ_4. This concludes the proof of Proposition 2.2. □

2.3 Estimates on the exit distribution

Let $I = [a,b] \subset \mathbb{Z}$ be a finite interval and $E = (-\infty, a] \cup [b, +\infty)$. We prove the following

Proposition 2.3. *For all $\eta > 0$ there exists $M > 0$ such that $\mathbb{P}$-a.s., for each interval $[a,b] \subset \mathbb{Z}$ containing at least three points we have*

$$\min_{x\in(a,b)} \mathrm{P}_\omega^x[X_{\tau_E} \in I_M] \geq 1-\eta$$

with $I_M := [a-M, a] \cup [b, b+M]$.

Proof. Fix an arbitrary $\eta \in (0,1)$. For intervals $[a,b]$ of length 2, there exists only one point x in (a,b). By the Markov property we have that

$$\mathrm{P}_\omega^x[X_{\tau_E} \in I_M] = \mathrm{P}_\omega^x[X_1 \in I_M \mid X_1 \neq x].$$

This implies that

$$\mathrm{P}_\omega^x[X_{\tau_E} \in I_M] = 1 - \frac{\mathrm{P}_\omega^x[X_1 \in (-\infty, a-M) \cup (b+M, \infty)]}{\mathrm{P}_\omega^x[X_1 \neq x]}.$$

Then, Condition K and (1.1) guarantee the existence of a constant $M > 0$ such that $\mathbb{P}$-a.s.,

$$\mathrm{P}_\omega^x[X_{\tau_E} \in I_M] \geq 1-\eta.$$

For intervals $[a,b]$ of length greater or equal to 3, let us do the following. Fix some $x \in (a,b)$. Let $\zeta_0 = 0$ and for $i \geq 1$, $\zeta_i := \inf\{n > \zeta_{i-1} : X_n = x\}$ with the convention $\inf\{\emptyset\} = +\infty$. Since by Lemma 2.1 our random walk is $\mathbb{P}$-a.s. recurrent, the sequence $(\zeta_i)_{i\geq 1}$ is $\mathbb{P}$-a.s. strictly increasing and we have by the Markov property

$$\begin{aligned} \mathrm{P}_\omega^x[X_{\tau_E} \in I_M] &= \sum_{i=0}^{\infty} \mathrm{P}_\omega^x[X_{\tau_E} \in I_M \mid \tau_E \in [\zeta_i, \zeta_{i+1})]\mathrm{P}_\omega^x[\tau_E \in [\zeta_i, \zeta_{i+1})] \\ &= \mathrm{P}_\omega^x[X_{\tau_E} \in I_M \mid \tau_E < \tau_x^+]. \end{aligned} \tag{2.5}$$

Let us define $A_E := \{\tau_E < \tau_x^+\}$. First, we write

$$\begin{aligned} \mathrm{P}_\omega^x[X_{\tau_E} \in I_M \mid A_E] &= 1 - \mathrm{P}_\omega^x[X_{\tau_E} \notin I_M \mid A_E] \\ &= 1 - \sum_{y\in(-\infty,a-M)\cup(b+M,\infty)} \mathrm{P}_\omega^x[X_{\tau_E} = y \mid A_E]. \end{aligned} \tag{2.6}$$

Then, consider the new environment ω' obtained from ω by deleting all the conductances $\omega_{y,z}$ when both y and z belong to E. The reversible measure on this new environment ω' is given by

$$C'_y := \begin{cases} C_y, & \text{if } y \in (a,b), \\ \sum_{z\notin E} \omega_{y,z}, & \text{otherwise.} \end{cases}$$

We define $C'_E := \sum_{y\in E} C'_y$ and for all $y \in E$, $\pi_E(y) := C'_y / C'_E$. Observe that by Condition K, $C'_E \in (0,\infty)$, $\mathbb{P}$-a.s. Hence, π_E is a probability measure on E. For the sake of simplicity we write $\mathrm{P}_{\omega'}^E$ instead of $\mathrm{P}_{\omega'}^{\pi_E}$ for the random walk on ω' starting with probability π_E. We can couple the random walks in the environments ω and ω' so that $\mathrm{P}_{\omega'}^x[X_{\tau_E} = y \mid A_E] = \mathrm{P}_\omega^x[X_{\tau_E} = y \mid A_E]$.

Now, let us find an upper bound for the term $\mathrm{P}_{\omega'}^x[X_{\tau_E} = y \mid A_E]$ with $y \in (-\infty, a-M) \cup (b+M, \infty)$. By definition of A_E we have

$$\mathrm{P}_{\omega'}^x[X_{\tau_E} = y \mid A_E] = \frac{\mathrm{P}_{\omega'}^x[X_{\tau_E} = y, \tau_E < \tau_x^+]}{\mathrm{P}_{\omega'}^x[\tau_E < \tau_x^+]}. \tag{2.7}$$

Let us denote by $\Gamma_{z',z''}$ the set of finite paths $(z',z_1,\dots,z_k,z'')$ such that $z_i\notin E\cup\{z',z''\}$ for all $i=1,\dots,k$. Let $\boldsymbol{\gamma}=(z',z_1,\dots,z_k,z'')\in\Gamma_{z',z''}$ and define

$$\mathtt{P}^{z'}_{\omega'}[\boldsymbol{\gamma}]:=\mathtt{P}^{z'}_{\omega'}[X_1=z_1,\dots,X_k=z_k,X_{k+1}=z''].$$

By reversibility we obtain

$$\mathtt{P}^x_{\omega'}[X_{\tau_E}=y,\tau_E<\tau_x^+]=\sum_{\boldsymbol{\gamma}\in\Gamma_{x,y}}\mathtt{P}^x_{\omega'}[\boldsymbol{\gamma}]=\frac{1}{C'_x}\sum_{\boldsymbol{\gamma}\in\Gamma_{y,x}}C'_y\mathtt{P}^y_{\omega'}[\boldsymbol{\gamma}]=\frac{C'_y}{C'_x}\mathtt{P}^y_{\omega'}[\tau_x<\tau_E^+]$$

and

$$\begin{aligned}\mathtt{P}^x_{\omega'}[\tau_E<\tau_x^+]&=\sum_{z\in E}\sum_{\boldsymbol{\gamma}\in\Gamma_{x,z}}\mathtt{P}^x_{\omega'}[\boldsymbol{\gamma}]=\sum_{z\in E}\sum_{\boldsymbol{\gamma}\in\Gamma_{z,x}}\frac{C'_z}{C'_x}\mathtt{P}^z_{\omega'}[\boldsymbol{\gamma}]=\frac{C'_E}{C'_x}\sum_{z\in E}\pi_E(z)\sum_{\boldsymbol{\gamma}\in\Gamma_{z,x}}\mathtt{P}^z_{\omega'}[\boldsymbol{\gamma}]\\&=\frac{C'_E}{C'_x}\mathtt{P}^E_{\omega'}[\tau_x<\tau_E^+].\end{aligned}\tag{2.8}$$

Thus, by (2.7) we have

$$\mathtt{P}^x_{\omega'}[X_{\tau_E}=y\mid A_E]=\frac{C'_y\mathtt{P}^y_{\omega'}[\tau_x<\tau_E^+]}{C'_E\mathtt{P}^E_{\omega'}[\tau_x<\tau_E^+]}.\tag{2.9}$$

To bound from below the term $\mathtt{P}^E_{\omega'}[\tau_x<\tau_E^+]$ we use an electric networks argument. To this end, we will define a Markov chain on a new state space for which it will be easy to compute the effective conductance. First, we introduce a point Δ_E and the state space $\mathcal{S}:=(\mathbb{Z}\setminus E)\cup\{\Delta_E\}$. For $z\notin E$, we define the transition probabilities

$$P_{z,\Delta_E}=\sum_{u\in E}\frac{\omega'_{z,u}}{C'_z},\qquad P_{\Delta_E,z}=\sum_{u\in E}\pi_E(u)\frac{\omega'_{z,u}}{C'_u}.$$

For $z\notin E$ and $u\notin E$ we set $P_{z,u}=\omega'_{z,u}/C'_z$, and, we put $P_{\Delta_E,\Delta_E}=0$. By defining $C'_{\Delta_E}:=C'_E$, we can easily check that the detailed balance equations are satisfied, i.e., for all $z\in\mathcal{S}$ we have $C'_zP_{z,u}=C'_uP_{u,z}$. We have that

$$\mathtt{P}^E_{\omega'}[\tau_x<\tau_E^+]=\mathtt{P}^{\Delta_E}_{\omega'}[\tau_x<\tau_{\Delta_E}^+]=\frac{C_{\text{eff}}(\Delta_E,x)}{C'_{\Delta_E}}=\frac{C_{\text{eff}}(\Delta_E,x)}{C'_E}\tag{2.10}$$

where $C_{\text{eff}}(\Delta_E,x)$ is the effective conductance between Δ_E and x. Observe that

$$C_{\text{eff}}(\Delta_E,x)\geq\Big(\sum_{i=a}^{x-1}\omega_{i,i+1}^{-1}\Big)^{-1}+\Big(\sum_{i=x}^{b-1}\omega_{i,i+1}^{-1}\Big)^{-1}.$$

Using Condition E, we obtain

$$C'_E\mathtt{P}^E_{\omega'}[\tau_x<\tau_E^+]\geq\kappa\Big(\frac{1}{x-a}+\frac{1}{b-x}\Big).\tag{2.11}$$

Then, we have to treat the term $C'_y\mathtt{P}^y_{\omega'}[\tau_x<\tau_E^+]$. By construction of ω'

$$\begin{aligned}C'_y\mathtt{P}^y_{\omega'}[\tau_x<\tau_E^+]&=C'_y\sum_{z\in(a,b)}p_{\omega'}(y,z)\mathtt{P}^z_{\omega'}[\tau_x<\tau_E]\\&=\sum_{z\in(a,b)}\omega'_{y,z}\mathtt{P}^z_{\omega'}[\tau_x<\tau_E]\\&=\sum_{z\in(a,b)}\omega_{y,z}\mathtt{P}^z_{\omega}[\tau_x<\tau_E].\end{aligned}$$

Finally, we have to estimate $\mathtt{P}^z_\omega[\tau_x < \tau_E]$ for $z \in (a,b) \setminus \{x\}$. To this end, we define the following sequence of stopping times. Let $\Upsilon_0 = 0$ and for $i \geq 1$, $\Upsilon_i := \inf\{n > \Upsilon_{i-1} : X_n = z\}$ with the convention $\inf\{\emptyset\} = +\infty$. The sequence $(\Upsilon_i)_{i\geq 1}$ is a.s. strictly increasing and we have

$$\mathtt{P}^z_\omega[\tau_x < \tau_E] = \mathtt{P}^z_\omega[\tau_x < \tau_E \mid \tau_{E\cup\{x\}} \in [0, \Upsilon_1)]. \tag{2.12}$$

Then, we have

$$\mathtt{P}^z_\omega[\tau_x < \tau_E \mid \tau_{E\cup\{x\}} \in [0,\Upsilon_1)] = \frac{\mathtt{P}^z_\omega[\tau_x < \tau_E, \tau_{E\cup\{x\}} \in [0,\Upsilon_1)]}{\mathtt{P}^z_\omega[\tau_{E\cup\{x\}} \in [0,\Upsilon_1)]} \leq \frac{\mathtt{P}^z_\omega[\tau_x < \tau^+_z]}{\mathtt{P}^z_\omega[\tau_E < \tau^+_z]} \wedge 1. \tag{2.13}$$

We estimate $\mathtt{P}^z_\omega[\tau_E < \tau^+_z]$ in the following way,

$$\begin{aligned}
\mathtt{P}^z_\omega[\tau_E < \tau^+_z] &= \frac{C_{\text{eff}}(z,E)}{C_z} \\
&\geq \frac{1}{C_z}\Big(\Big(\sum_{i=z}^{b-1}\omega^{-1}_{i,i+1}\Big)^{-1} + \Big(\sum_{i=a}^{z-1}\omega^{-1}_{i,i+1}\Big)^{-1}\Big) \\
&\geq \hat\kappa\kappa\Big(\frac{1}{b-z} + \frac{1}{z-a}\Big).
\end{aligned} \tag{2.14}$$

Now, using the Dirichlet variational principle, we obtain an upper bound for $\mathtt{P}^z_\omega[\tau_x < \tau^+_z]$. Suppose that the interval $(x,b) \neq \emptyset$ and that $z \in (x,b)$, consider the function h given by

$$h(u) = \begin{cases} 1, & \text{if } u < x \text{ or } u > 2z - x, \\ \frac{|z-u|}{z-x}, & \text{if } x \leq u \leq z. \end{cases}$$

Hence, we have $2C_z\mathtt{P}^z_\omega[\tau_x < \tau^+_z] \leq \Phi(h)$. By the same reasoning as we used in order to obtain (2.3) in the proof of Lemma 2.1, we deduce that there exists a constant $\gamma_2 > 0$ such that

$$\mathtt{P}^z_\omega[\tau_x < \tau^+_z] \leq \frac{\gamma_2}{z-x} \tag{2.15}$$

for $z \in (x,b)$. Similarly, if we suppose that $(a,x) \neq \emptyset$ and $z \in (a,x)$, we obtain a bound similar to (2.15) for $\mathtt{P}^z_\omega[\tau_x < \tau^+_z]$. Then, we obtain that for $z \in (a,b) \setminus \{x\}$,

$$\mathtt{P}^z_\omega[\tau_x < \tau^+_z] \leq \frac{\gamma_2}{|z-x|}. \tag{2.16}$$

Note that we can choose γ_2 in such a way that it does not depend on the size of the interval $[a,b]$. By (2.12), (2.13), (2.14) and (2.16) we obtain

$$C'_y\mathtt{P}^y_{\omega'}[\tau_x < \tau^+_E] \leq \sum_{z\in(a,b)} \omega_{y,z}\Big(\frac{\gamma_2(z-a)(b-z)}{\hat\kappa\kappa(b-a)|z-x|} \wedge 1\Big).$$

Thus, by (2.9) and (2.11) we obtain

$$\begin{aligned}
\mathtt{P}^x_{\omega'}[X_{\tau_E} = y \mid A_E] &\leq \frac{1}{\kappa}\frac{(x-a)(b-x)}{b-a}\sum_{z\in(a,b)}\omega_{y,z}\Big(\frac{\gamma_2(z-a)(b-z)}{\hat\kappa\kappa(b-a)|z-x|} \wedge 1\Big), \\
\mathtt{P}^x_\omega[X_{\tau_E} \in I_M \mid A_E] &\geq 1 - \frac{1}{\kappa}\sum_{y\in I'_M}\frac{(x-a)(b-x)}{b-a}\sum_{z\in(a,b)}\omega_{y,z}\Big(\frac{\gamma_2(z-a)(b-z)}{\hat\kappa\kappa(b-a)|z-x|} \wedge 1\Big)
\end{aligned} \tag{2.17}$$

with $I'_M = (-\infty, a-M) \cup (b+M, \infty)$. Let us divide the set I'_M into the subintervals $J_1(M) = (b+M,\infty)$ and $J_2(M) = (-\infty, a-M)$. Denote

$$\begin{aligned}
H_1(M) &= \sum_{y\in J_1}\frac{(x-a)(b-x)}{b-a}\sum_{z\in(a,b)}\omega_{y,z}\Big(\frac{\gamma_2(z-a)(b-z)}{\hat\kappa\kappa(b-a)|z-x|} \wedge 1\Big), \\
H_2(M) &= \sum_{y\in J_2}\frac{(x-a)(b-x)}{b-a}\sum_{z\in(a,b)}\omega_{y,z}\Big(\frac{\gamma_2(z-a)(b-z)}{\hat\kappa\kappa(b-a)|z-x|} \wedge 1\Big).
\end{aligned}$$

We have

$$H_1(M) \le \sum_{y\in J_1} \frac{(x-a)(b-x)}{b-a}\Big\{ \sum_{z\in(a,\frac{x+a}{2}]} \frac{\gamma_2(z-a)(b-z)}{\hat\kappa\kappa(b-a)|z-x|}\omega_{y,z} + \sum_{z\in(\frac{x+a}{2},\frac{b+x}{2})} \omega_{y,z} + \sum_{z\in[\frac{b+x}{2},b)} \frac{\gamma_2(z-a)(b-z)}{\hat\kappa\kappa(b-a)|z-x|}\omega_{y,z}\Big\}.$$

Now, observe that

$$\begin{aligned}\frac{(x-a)(b-x)}{b-a} \sum_{z\in(a,\frac{x+a}{2}]} \frac{(z-a)(b-z)}{(b-a)|z-x|}\omega_{y,z} &\le \sum_{z\in(a,\frac{x+a}{2}]} (b-z)\omega_{y,z} \\ &\le \sum_{z<b}(b-z)\omega_{y,z}, \end{aligned} \tag{2.18}$$

$$\begin{aligned}\frac{(x-a)(b-x)}{b-a} \sum_{z\in(\frac{x+a}{2},\frac{b+x}{2})} \omega_{y,z} &\le (b-x) \sum_{z\in(\frac{x+a}{2},\frac{b+x}{2})} \omega_{y,z} \\ &\le 2 \sum_{z\in(\frac{x+a}{2},\frac{b+x}{2})} (b-z)\omega_{y,z} \\ &\le 2\sum_{z<b}(b-z)\omega_{y,z} \end{aligned} \tag{2.19}$$

and

$$\begin{aligned}\frac{(x-a)(b-x)}{b-a} \sum_{z\in[\frac{x+b}{2},b)} \frac{(z-a)(b-z)}{(b-a)|z-x|}\omega_{y,z} &\le 2 \sum_{z\in[\frac{x+b}{2},b)} (b-z)\omega_{y,z} \\ &\le 2\sum_{z<b}(b-z)\omega_{y,z}. \end{aligned} \tag{2.20}$$

Putting (2.18), (2.19) and (2.20) together leads to

$$H_1(M) \le 2\Big(\frac{2\gamma_2}{\hat\kappa\kappa}+1\Big)\sum_{y\in J_1}\sum_{z<b}(b-z)\omega_{y,z}. \tag{2.21}$$

Observe that this last upper bound on $H_1(M)$ does not depend on x anymore. Now, by Condition K, for any $\eta>0$, we can take $M_1>0$ sufficiently large such that $\mathbb{P}$-a.s. for all $u\in\mathbb{Z}$ we have

$$\sum_{v>u+M_1}\sum_{w<u}\omega_{v,w}(u-w) < \frac{\kappa\eta}{4}\Big(\frac{2\gamma_2}{\hat\kappa\kappa}+1\Big)^{-1}.$$

For this M_1, we have $H_1(M_1)<\kappa\eta/2$. By symmetry, we also have that $H_2(M_1)<\kappa\eta/2$. Combining these two last results with (2.5) and (2.17) and the case of intervals of length 2 treated at the beginning of the proof, we obtain that for every $\eta>0$ there exists M such that $\mathbb{P}$-a.s., for any interval $[a,b]$,

$$\min_{x\in(a,b)} \mathrm{P}^x_\omega[X_{\tau_E}\in I_M] \ge 1-\eta.$$

This concludes the proof of Proposition 2.3. □

3 Proof of Theorem 1.2

In this section we prove the UCLT. Let $\mathfrak{C}_b^u(C(\mathbb{R}_+),\mathbb{R})$ be the space of bounded uniformly continuous functionals from $C(\mathbb{R}_+)$ into $\mathbb{R}$. First, let us prove the apparently weaker statement:

Proposition 3.1. *For all $F \in \mathfrak{C}_b^u(C(\mathbb{R}_+),\mathbb{R})$, we have $\mathbb{P}$-a.s., for every $H > 0$,*

$$\lim_{n\to\infty} \sup_{x\in[-H\sqrt{n},H\sqrt{n}]} \Big| \mathrm{E}_{\theta_x\omega}[F(Z^n)] - E[F(W)] \Big| = 0.$$

The difficult part of the proof of Theorem 1.2 is to show Proposition 3.1. To prove this proposition, we will introduce the notion of “good site" in $\mathbb{Z}$. The set of good sites is by definition the set of sites in $\mathbb{Z}$ from which we can guarantee that the random walk converges uniformly to Brownian motion. Due to the ergodicity of the random environment, we will then prove that starting a random walk from any site in $[-H\sqrt{n}, H\sqrt{n}]$, with high probability, it will meet a close good site quickly enough to derive a uniform CLT. This part will be done in two steps, introducing the intermediate concept of “nice site". More precisely, the sequence of steps we will follow in this section to prove Proposition 3.1 is the following:

- In Definition 3.2, we formally define the notion of “good sites”.
- In Definition 3.3, we introduce the notion of “nice sites". Heuristically, x is a nice site if for some $\delta > 0$ and $h > 0$, the range of the random walk starting from x until time hn is greater than $\delta h^{1/2}\sqrt{n}$ with high probability, so that the random walk cannot stay “too close” to its starting location (it holds that good sites are nice).
- Right after Definition 3.3, we show that any interval $I \in [-2H\sqrt{n}, 2H\sqrt{n}]$ of length n^ν with $\nu \in (1/(2+\beta), \frac{1}{2})$ (here β is from Condition K) must contain at least one “nice site".
- In Lemma 3.4 we show that, starting from a site $x \in [-H\sqrt{n}, H\sqrt{n}]$, with high probability the random walk meets a nice site at a distance at most n^μ in time at most $n^{2\mu}$ with $\mu \in (\nu, \frac{1}{2})$.
- In Lemma 3.5 we show that, starting from a nice site $x \in [-(3/2)H\sqrt{n}, (3/2)H\sqrt{n}]$, the random walk meets with high probability a good site at a distance less than $h\sqrt{n}$ before time hn.
- We combine Lemmas 3.4 and 3.5 to obtain that, starting from any $x \in [-H\sqrt{n}, H\sqrt{n}]$ the random walk meets a good site at a distance less than $h\sqrt{n}$ before time hn. This is the statement of Lemma 3.6.
- Proposition 3.1 then follows from Lemma 3.6, since we know essentially that the random walk will quickly reach a nearby good site, and from this good site the convergence properties are “good” by definition.

From Proposition 3.1, we obtain Theorem 1.2 in the following way. In Proposition 3.7, we first show that Proposition 3.1 implies a corresponding statement in which we substitute uniformly continuous functionals F by open sets of $C(\mathbb{R}_+)$. Then, in Proposition 3.8, we use the separability of the space $C(\mathbb{R}_+)$ to show that we can interchange the terms “for any open set G" and “$\mathbb{P}$-a.s." in (ii) of Proposition 3.7. Then, we use standard arguments as in the proof of the Portmanteau theorem of [4] to conclude the proof of Theorem 1.2.

Now, fix $F \in \mathfrak{C}_b^u(C(\mathbb{R}_+),\mathbb{R})$. Our first goal is to prove Proposition 3.1, that is, $\mathbb{P}$-a.s., for every $\tilde{\varepsilon}, H > 0$,

$$\sup_{x\in[-H\sqrt{n},H\sqrt{n}]} \Big| \mathrm{E}_{\theta_x\omega}[F(Z^n)] - E[F(W)] \Big| \le \tilde{\varepsilon} \tag{3.1}$$

for all large enough n. To start, we need to write some definitions and prove some intermediate results. From now on, we suppose that $\sigma = 1$ (otherwise replace X by $\sigma^{-1}X$).

Denote

$$\begin{aligned} R_n^+(m) &= \max_{s\le m}(X_{n+s} - X_n),\\ R_n^-(m) &= \min_{s\le m}(X_{n+s} - X_n),\\ R_n(m) &= R_n^+(m) - R_n^-(m), \end{aligned}$$

and

$$\begin{aligned} \mathfrak{R}_t^+(u) &= \max_{s\le u}(W_{t+s} - W_t),\\ \mathfrak{R}_t^-(u) &= \min_{s\le u}(W_{t+s} - W_t),\\ \mathfrak{R}_t(u) &= \mathfrak{R}_t^+(u) - \mathfrak{R}_t^-(u). \end{aligned}$$

Let $\mathtt{d}$ be the distance on the space $C_{\mathbb{R}_+}$ defined by

$$\mathtt{d}(x,y) = \sum_{n=1}^{\infty} 2^{-n+1}\min\Big\{1, \sup_{s\in[0,n]}|x(s)-y(s)|\Big\}.$$

Now, for any given $\varepsilon > 0$, we define

$$\begin{aligned} \delta_\varepsilon := \max\Big\{\delta_1 \in (0,1] : P[\mathfrak{R}_0(1/2) < \delta_1] + P[\mathfrak{R}_{1/2}(1/2) < \delta_1] \\ + P[\mathfrak{R}_1(1/2) < \delta_1] + P[\mathfrak{R}_0^+(1) < \delta_1] + P[\mathfrak{R}_0^-(1) < \delta_1] \le \frac{\varepsilon}{2}\Big\} \end{aligned} \tag{3.2}$$

and

$$h_\varepsilon := \max\Big\{h_1 \in (0,1] : P\Big[\sup_{s\le h_1}|W(s)| > \varepsilon\Big] + P\Big[\sup_{s\le h_1}\mathtt{d}(\theta_s W, W) > \varepsilon\Big] \le \frac{\varepsilon}{2}\Big\}. \tag{3.3}$$

Observe that δ_ε and h_ε are positive for all $\varepsilon > 0$ and decrease to 0 as $\varepsilon \to 0$. For (3.3), the positivity of h_ε for $\varepsilon > 0$ follows from the properties of the modulus of continuity of Brownian motion (see e.g. Theorem 1.12 of [21]).

Definition 3.2. *For a given realization ω of the environment and $N \in \mathbb{N}$, we say that $x \in \mathbb{Z}$ is (ε, N)-good, if*

(i) $\min\Big\{n \ge 1 : \big|\mathtt{E}_\omega[F(Z^m)] - E[F(W)]\big| \le \varepsilon,$ *for all* $m \ge n\Big\} \le N$*;*

(ii) $\mathtt{P}_\omega^x\Big[R_k(h_\varepsilon m) \ge \delta_\varepsilon h_\varepsilon^{1/2}\sqrt{m}$ *for all* $k \le h_\varepsilon m, R_0^{\pm}(h_\varepsilon m) \ge \delta_\varepsilon h_\varepsilon^{1/2}\sqrt{m}\Big] \ge 1-\varepsilon$*, for all* $m \ge N$*;*

(iii) $\mathtt{P}_{\theta_x\omega}\Big[\sup_{s\le h_\varepsilon}|Z^m(s)| \le \varepsilon, \sup_{s\le h_\varepsilon}\mathtt{d}(\theta_s Z^m, Z^m) \le \varepsilon\Big] \ge 1-\varepsilon$*, for all* $m \ge N$*.*

For any given $\varepsilon > 0$, it follows from Theorem 1.1, (3.2) and (3.3) that for any $\varepsilon' > 0$ there exists N such that

$$\mathbb{P}[0 \text{ is } (\varepsilon, N)\text{-good}] > 1 - \varepsilon'.$$

Then, by the Ergodic Theorem, $\mathbb{P}$-a.s., for all n large enough, it holds that

$$\big|\{x \in [-2H\sqrt{n}, 2H\sqrt{n}] : x \text{ is not } (\varepsilon, N)\text{-good}\}\big| < 5\varepsilon' H\sqrt{n}. \tag{3.4}$$

Next, we need the following

Definition 3.3. *We say that a site x is (ε, n)-nice, if*

$$\mathrm{P}_\omega^x\Big[R_0(h_\varepsilon n) \geq \delta_\varepsilon h_\varepsilon^{1/2}\sqrt{n}\Big] \geq 1 - 3\varepsilon.$$

In particular, note that, if for some $N \leq n$ a site x is (ε, N)-good, then it is (ε, n)-nice.

Now, fix some $\nu \in (\frac{1}{2+\beta}, \frac{1}{2})$ (so that $\nu(2+\beta) - 1 > 0$), where β is from Condition K. Observe that by Condition K there exists $\gamma_1 > 0$ such that for any starting point $x \in \mathbb{Z}$

$$\mathrm{P}_\omega^x[|X_{k+1} - X_k| < n^\nu \text{ for all } k \leq h_\varepsilon n] \geq 1 - \gamma_1 n^{-(\nu(2+\beta)-1)}. \tag{3.5}$$

We argue by contradiction that, $\mathbb{P}$-a.s., there exists $n_1 = n_1(\omega, \varepsilon)$ such that any interval of length at least n^ν contains a (ε, n)-nice site for $n > n_1$. For this, choose $\varepsilon' > 0$ such that $5\varepsilon' H < \delta_\varepsilon h_\varepsilon^{1/2}$ and let n large enough such that $n^\nu < 5\varepsilon' H\sqrt{n}$ and (3.4) hold. Let $I \subset [-2H\sqrt{n}, 2H\sqrt{n}]$ be an interval of length n^ν such that it does not contain any (ε, n)-nice site. Observe that, by (3.4), there exists a (ε, N)-good site x_0 such that $|x_0 - y| < \delta_\varepsilon h_\varepsilon^{1/2}\sqrt{n}$ for all $y \in I$. Note that, by (3.5) and (ii) of Definition 3.2,

$$\mathrm{P}_\omega^{x_0}[\text{there exists } k \leq h_\varepsilon n \text{ such that } X_k \in I] \geq 1 - \varepsilon - \gamma_1 n^{-(\nu(2+\beta)-1)}$$

(the particle crosses I without jumping over it entirely), hence

$$\begin{aligned}\mathrm{P}_\omega^{x_0}[\text{there exists } k \leq h_\varepsilon n \text{ such that } R_k(h_\varepsilon n) < \delta_\varepsilon h_\varepsilon^{1/2}\sqrt{n}] &\geq 3\varepsilon(1 - \varepsilon - \gamma_1 n^{-(\nu(2+\beta)-1)}) \\ &\geq 2\varepsilon\end{aligned}$$

if n is large enough. But this contradicts the fact that x_0 is (ε, N)-good. So, we see that, $\mathbb{P}$-a.s., any interval $I \subset [-2H\sqrt{n}, 2H\sqrt{n}]$ of length n^ν should contain at least one (ε, n)-nice site for n large enough.

Let $\mu \in (\nu, \frac{1}{2})$. In the next Lemma, we show that starting from a site $x \in [-H\sqrt{n}, H\sqrt{n}]$, with high probability the random walk will meet a (ε, n)-nice site at a distance at most n^μ in time at most $n^{2\mu}$.

For $x \in \mathbb{Z}$ and $n \in \mathbb{N}$, let us denote by $\mathcal{P}_n^l(x)$ the largest $y \leq x$ such that y is a (ε, n)-nice site and by $\mathcal{P}_n^r(x)$ the smallest $y \geq x$ such that y is a (ε, n)-nice site. Furthermore, we denote by $\mathcal{N}_{\varepsilon,n}$ the set of (ε, n)-nice sites in $\mathbb{Z}$.

Lemma 3.4. *For any $\varepsilon_1 > 0$ and $\varepsilon > 0$, we have $\mathbb{P}$-a.s., for all sufficiently large n, for all $x \in [-H\sqrt{n}, H\sqrt{n}]$,*

$$\mathrm{P}_\omega^x\Big[\tau_{\mathcal{N}_{\varepsilon,n}} \leq n^{2\mu}, \max_{j \leq \tau_{\mathcal{N}_{\varepsilon,n}}} |X_j - X_0| \leq n^\mu\Big] \geq 1 - \varepsilon_1. \tag{3.6}$$

Proof. First, suppose that x is not (ε, n)-nice, otherwise the proof of (3.6) is trivial. For some integer $M_1 > 0$, define the intervals

$$I_{M_1}(x) := [\mathcal{P}_n^l(x) - M_1, \mathcal{P}_n^r(x) + M_1].$$

Let us also define the following increasing sequence of stopping times: $\xi_0 := 0$ and for $i \geq 1$,

$$\xi_i := \inf\big\{k > \xi_{i-1} : X_k \notin (\mathcal{P}_n^l(X_{\xi_{i-1}}), \mathcal{P}_n^r(X_{\xi_{i-1}}))\big\} + M_1.$$

Then, we define the events

$$\begin{aligned}A_i &:= \big\{\text{there exists } k \in (\xi_i, \xi_{i+1}] \text{ such that } X_k \in \{\mathcal{P}_n^l(X_{\xi_i}), \mathcal{P}_n^r(X_{\xi_i})\}\big\}, \\ B_i &:= \Big\{X_{\xi_{i+1} - M_1} \in I_{M_1}(X_{\xi_i}), \max_{(k,l) \in [\xi_{i+1} - M_1, \xi_{i+1}]^2} |X_k - X_l| \leq M_1 n^\nu\Big\}\end{aligned}$$

for $i \geq 0$. Now, fix temporarily $\tilde{\varepsilon} \in (0, \frac{1}{2})$. By Proposition 2.3, we can choose M_1 large enough in such a way that

$$\min_{y \in (\mathcal{P}_n^l(x), \mathcal{P}_n^r(x))} \mathtt{P}_\omega^y[X_{\xi_1 - M_1} \in I_{M_1}(x)] \geq 1 - \tilde{\varepsilon}. \tag{3.7}$$

Note that by Condition E and Proposition 2.3 and (3.7) we have $\mathtt{P}_\omega[A_0] \geq \frac{1}{2}\kappa^{M_1}$. By the Markov property, we obtain for some integer $L > 0$,

$$\mathtt{P}_\omega^x\Big[\bigcap_{i=0}^{L-1} A_i^c\Big] = \mathtt{P}_\omega^x[A_0^c] \dots \mathtt{P}_\omega^x[A_{L-1}^c \mid A_0^c \dots A_{L-2}^c] \leq \Big(1 - \frac{1}{2}\kappa^{M_1}\Big)^L. \tag{3.8}$$

Since the event $\{\max_{(k,l)\in[\xi_1 - M_1, \xi_1]^2} |X_k - X_l| > M_1 n^\nu\}$ implies that there is at least one jump of size n^ν during the time interval $[\xi_1 - M_1, \xi_1]$, using Condition K, (1.1), (3.7) and the Markov property, we have that $\mathtt{P}_\omega^x[B_0^c] \leq \tilde{\varepsilon} + \gamma_1 n^{-\nu(2+\beta)}$ for some positive constant γ_1. We obtain by the Markov property,

$$\mathtt{P}_\omega^x\Big[\bigcup_{i=0}^{L-1} B_i^c\Big] \leq L(\tilde{\varepsilon} + \gamma_1 n^{-\nu(2+\beta)}). \tag{3.9}$$

Observe that each event $\{\max_{j\in[\xi_i - M_1, \xi_i]} |X_j - X_{\xi_i - M_1 - 1}| > (M_1+1)n^\nu + M_1\}$, $i \geq 1$, implies either $\{|X_{\xi_i - M_1} - X_{\xi_i - M_1 - 1}| > n^\nu + M_1\}$ (which implies that the first jump after time $\xi_i - M_1 - 1$ is out of the interval $I_{M_1}(\xi_i - M_1 - 1)$) or $\{\max_{(k,l)\in[\xi_i - M_1, \xi_i]^2} |X_k - X_l| > M_1 n^\nu\}$. Then, combining (3.8) and (3.9), we have

$$\begin{aligned}\mathtt{P}_\omega^x\Big[\tau_{\mathcal{N}_{\varepsilon,n}} \in [0, \xi_L], \max_{j \leq \tau_{\mathcal{N}_{\varepsilon,n}}} |X_j - X_0| \leq L((M_1+1)n^\nu + M_1)\Big] \\ \geq 1 - \Big(1 - \frac{1}{2}\kappa^{M_1}\Big)^L - L\tilde{\varepsilon} - L\gamma_1 n^{-\nu(2+\beta)}.\end{aligned} \tag{3.10}$$

Now, let $\mu' > 0$ and denote $G_i := \{\xi_i - \xi_{i-1} \leq n^{2\nu+\mu'} + M_1\}$ for $1 \leq i \leq L$. We have

$$\begin{aligned}\mathtt{P}_\omega^x[\xi_L \leq L(n^{2\nu+\mu'} + M_1)] &\geq \mathtt{P}_\omega^x[G_1, G_2, \dots, G_L] \\ &= \mathtt{P}_\omega^x[G_1]\mathtt{P}_\omega^x[G_2 \mid G_1] \dots \mathtt{P}_\omega^x[G_L \mid G_1 \dots G_{L-1}].\end{aligned} \tag{3.11}$$

By Proposition 2.2 and the fact that any interval of length n^ν in $[-2H\sqrt{n}, 2H\sqrt{n}]$ should contain at least one (ε, n)-nice site, we obtain

$$\mathtt{P}_\omega^x[G_1] \geq 1 - \exp\Big(-\frac{n^{\mu'}}{K_1}\Big)$$

for sufficiently large n. By (3.11) and the Markov property we have

$$\mathtt{P}_\omega^x[\xi_L \leq L(n^{2\nu+\mu'} + M_1)] \geq \Big[1 - \exp\Big(-\frac{n^{\mu'}}{K_1}\Big)\Big]^L \geq 1 - L\exp\Big(-\frac{n^{\mu'}}{K_1}\Big). \tag{3.12}$$

Now, choose L sufficiently large so that

$$\Big(1 - \frac{1}{2}\kappa^{M_1}\Big)^L \leq \frac{\varepsilon_1}{3} \tag{3.13}$$

and $\tilde{\varepsilon}$ sufficiently small such that

$$L\tilde{\varepsilon} \leq \frac{\varepsilon_1}{3}. \tag{3.14}$$

Then, combining (3.10), (3.12), (3.13) and (3.14), we obtain

$$\begin{aligned}\mathtt{P}_\omega^x\Big[\tau_{\mathcal{N}_{\varepsilon,n}} \leq L(n^{2\nu+\mu'} + M_1), \max_{j \leq \tau_{\mathcal{N}_{\varepsilon,n}}} |X_j - X_0| \leq L((M_1+1)n^\nu + M_1)\Big] \\ \geq 1 - \frac{2\varepsilon_1}{3} - L\gamma_1 n^{-\nu(2+\beta)} - L\exp\Big(-\frac{n^{\mu'}}{K_1}\Big).\end{aligned}$$

Taking $\mu' < 2(\mu - \nu)$ we obtain for all sufficiently large n

$$\mathrm{P}_\omega^x\Big[\tau_{\mathcal{N}_{\varepsilon,n}} \le n^{2\mu}, \max_{j\le\tau_{\mathcal{N}_{\varepsilon,n}}} |X_j - X_0| \le n^\mu\Big] \ge 1-\varepsilon_1.$$

This concludes the proof of Lemma 3.4. □

We now show that we can find $\varepsilon > 0$ small enough such that starting from a (ε, n)-nice site $x \in [-\frac{3}{2}H\sqrt{n}, \frac{3}{2}H\sqrt{n}]$, with high probability, the random walk will meet a (ε, n)-good site at a distance at most $h_\varepsilon^{1/2}\sqrt{n}$ before time $h_\varepsilon n$. We denote by $\mathcal{G}_{\varepsilon,N}$ the set of (ε, N)-good sites in $\mathbb{Z}$.

Lemma 3.5. *For any $\varepsilon_1 > 0$ and $\varepsilon \in (0, \frac{\varepsilon_1}{6}]$, we have that $\mathbb{P}$-a.s., for all sufficiently large n, for all $x \in [-\frac{3}{2}H\sqrt{n}, \frac{3}{2}H\sqrt{n}] \cap \mathcal{N}_{\varepsilon,n}$,*

$$\mathrm{P}_\omega^x\Big[\tau_{\mathcal{G}_{\varepsilon,n}} \le h_\varepsilon n, \max_{j\le\tau_{\mathcal{G}_{\varepsilon,n}}} |X_j - X_0| \le h_\varepsilon^{1/2}\sqrt{n}\Big] \ge 1-\varepsilon_1.$$

Proof. Fix some integer $M > 1$ and consider the following partition of $\mathbb{Z}$ into intervals of size M:

$$J_j = [jM, (j+1)M), \quad j \in \mathbb{Z}.$$

We say that an interval J_j is (ε, N)-good if all the points inside J_j are (ε, N)-good (otherwise we call the interval "bad"). Fix $\varepsilon \le \frac{\varepsilon_1}{6}$. Then,we have that, for any $\varepsilon' > 0$, there exists N such that

$$\mathbb{P}[J_0 \text{ is } (\varepsilon, N)\text{-good}] > 1-\varepsilon'.$$

Then, by the Ergodic Theorem, $\mathbb{P}$-a.s., for all n large enough it holds that

$$\big|\{J_j \text{ such that } j \in [-H\sqrt{n}, H\sqrt{n}] \text{ and } J_j \text{ is not } (\varepsilon, N)\text{-good}\}\big| < 3\varepsilon' H\sqrt{n}. \tag{3.15}$$

In particular, from this last inequality, we deduce that the length of the largest subinterval of $[-2H\sqrt{n}, 2H\sqrt{n}]$ that is a union of bad intervals is smaller than $3\varepsilon' HM\sqrt{n}$. Let $x \in [-\frac{3}{2}H\sqrt{n}, \frac{3}{2}H\sqrt{n}]$ be a (ε, n)-nice site that belongs to an interval $I = [a, b] \subset [-2H\sqrt{n}, 2H\sqrt{n}]$ that is a maximal union of bad J_j's (so that the adjacent J_j's to I are necessarily good). Then, choose ε' such that $3\varepsilon' HM < h^{1/2}\delta_\varepsilon$. Thus, in a time of order $h_\varepsilon n$ a random walk starting at x will leave the interval I with high probability. When this happens, to guarantee that the random walk will hit a (ε, N)-good site with high probability, we can choose a large enough M in such a way that

$$\mathrm{P}_\omega^x[X_{\tau_{I^c}} \in I_M] \ge 1 - \frac{\varepsilon_1}{2}, \tag{3.16}$$

with $I_M = [a-M, a] \cup [b, b+M]$. By definition of a (ε, n)-nice site, since $\varepsilon \le \frac{\varepsilon_1}{6}$ we have

$$\mathrm{P}_\omega^x[\tau_{I^c} \le h_\varepsilon n] \ge \mathrm{P}_\omega^x[R_0(h_\varepsilon n) \ge \delta_\varepsilon h_\varepsilon^{1/2}\sqrt{n}] \ge 1 - 3\varepsilon \ge 1 - \frac{\varepsilon_1}{2}. \tag{3.17}$$

Thus, combining (3.16) and (3.17), and using the fact that $\delta_\varepsilon \in (0, 1]$, we obtain $\mathbb{P}$-a.s.,

$$\mathrm{P}_\omega^x\Big[\tau_{\mathcal{G}_{\varepsilon,n}} \le h_\varepsilon n, \max_{j\le\tau_{\mathcal{G}_{\varepsilon,n}}} |X_j - X_0| \le h_\varepsilon^{1/2}\sqrt{n}\Big] \ge 1-\varepsilon_1.$$

for all n large enough and $x \in [-\frac{3}{2}H\sqrt{n}, \frac{3}{2}H\sqrt{n}] \cap \mathcal{N}_{\varepsilon,n}$. This concludes the proof of Lemma 3.5. □

Then, combining Lemmas 3.4 and 3.5, we can deduce (considering for example rational values for ε_1 and ε):

Lemma 3.6. *The following statement holds $\mathbb{P}$-a.s.: for any $\varepsilon_1 > 0$, we can choose $\varepsilon > 0$ arbitrary small in such a way that for all sufficiently large n and for all $x \in [-H\sqrt{n}, H\sqrt{n}]$,*

$$\mathtt{P}^x_\omega\Big[\tau_{\mathcal{G}_{\varepsilon,n}} \le h_\varepsilon n, \max_{j\le \tau_{\mathcal{G}_{\varepsilon,n}}} |X_j - X_0| \le h_\varepsilon^{1/2}\sqrt{n}\Big] \ge 1-\varepsilon_1.$$

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. Let us prove (3.1). Let $x \in [-H\sqrt{n}, H\sqrt{n}]$. In this last part, for the sake of brevity, we denote by $\mathcal{G}$ the set of (ε, n)-good sites. Let us denote by

$$R := \Big|\mathtt{E}_{\theta_x\omega}[F(Z^n)] - E[F(W)]\Big|$$

the quantity we have to bound. Let $\varepsilon \le \frac{\tilde{\varepsilon}}{2}$, we have by definition of a (ε, n)-good site,

$$\begin{aligned} R &\le \Big|\mathtt{E}_{\theta_x\omega}\Big(F(Z^n) - \mathtt{E}_{\theta_{X_{\tau_\mathcal{G}}}\omega}[F(Z^n)]\Big)\Big| + \Big|\mathtt{E}_{\theta_x\omega}\Big(\mathtt{E}_{\theta_{X_{\tau_\mathcal{G}}}\omega}[F(Z^n)] - E[F(W)]\Big)\Big| \\ &\le \Big|\mathtt{E}_{\theta_x\omega}\Big(F(Z^n) - \mathtt{E}_{\theta_{X_{\tau_\mathcal{G}}}\omega}[F(Z^n)]\Big)\Big| + \frac{\tilde{\varepsilon}}{2}. \end{aligned} \tag{3.18}$$

Denote $X' := X - x$ and observe that, by the Markov property

$$\begin{aligned} \Big|\mathtt{E}_{\theta_x\omega}\Big(F(Z^n) - \mathtt{E}_{\theta_{X_{\tau_\mathcal{G}}}\omega}[F(Z^n)]\Big)\Big| &= \Big|\mathtt{E}_{\theta_x\omega}\Big(F(Z^n) - \mathtt{E}_{\theta_{X'_{\tau_\mathcal{G}}}(\theta_x\omega)}[F(Z^n)]\Big)\Big| \\ &= \Big|\mathtt{E}_{\theta_x\omega}[F\circ Z^n - F\circ\theta_{n^{-1}\tau_\mathcal{G}}(Z^n - n^{-\frac12}X'_{\tau_\mathcal{G}})]\Big| \\ &\le \mathtt{E}_{\theta_x\omega}\Big|F\circ Z^n - F\circ\theta_{n^{-1}\tau_\mathcal{G}}(Z^n - n^{-\frac12}X'_{\tau_\mathcal{G}})\Big|. \end{aligned} \tag{3.19}$$

We are going to show that for all sufficiently large n we have uniformly in $x \in [-H\sqrt{n}, H\sqrt{n}]$

$$\mathtt{E}_{\theta_x\omega}\Big|F\circ Z^n - F\circ\theta_{n^{-1}\tau_\mathcal{G}}(Z^n - n^{-\frac12}X'_{\tau_\mathcal{G}})\Big| \le \frac{\tilde{\varepsilon}}{2}$$

for $\varepsilon > 0$ small enough. Let $M^n := Z^n - n^{-\frac12}X'_{\tau_\mathcal{G}}$. Since F is uniformly continuous, we can choose $\eta > 0$ in such a way that if $\mathtt{d}(x,y) \le \eta$ then $|F(x) - F(y)| \le \frac{\tilde{\varepsilon}}{4}$. Then, we have

$$\begin{aligned} \mathtt{E}_{\theta_x\omega}\Big|F\circ Z^n - F\circ\theta_{n^{-1}\tau_\mathcal{G}}M^n\Big| &= \mathtt{E}_{\theta_x\omega}\Big[\Big|F\circ Z^n - F\circ\theta_{n^{-1}\tau_\mathcal{G}}M^n\Big|\mathbf{1}\{\mathtt{d}(Z^n, \theta_{n^{-1}\tau_\mathcal{G}}M^n) \le \eta\}\Big] \\ &\quad + \mathtt{E}_{\theta_x\omega}\Big[\Big|F\circ Z^n - F\circ\theta_{n^{-1}\tau_\mathcal{G}}M^n\Big|\mathbf{1}\{\mathtt{d}(Z^n, \theta_{n^{-1}\tau_\mathcal{G}}M^n) > \eta\}\Big] \\ &\le \frac{\tilde{\varepsilon}}{4} + 2\|F\|_\infty \mathtt{P}_{\theta_x\omega}\Big[\mathtt{d}(Z^n, \theta_{n^{-1}\tau_\mathcal{G}}M^n) > \eta\Big]. \end{aligned} \tag{3.20}$$

Since $h_\varepsilon \le 1$, we have

$$\begin{aligned} \mathtt{P}_{\theta_x\omega}\Big[\mathtt{d}(Z^n, \theta_{n^{-1}\tau_\mathcal{G}}M^n) > \eta\Big] &\le \mathtt{P}_{\theta_x\omega}\Big[\mathtt{d}(Z^n, \theta_{n^{-1}\tau_\mathcal{G}}M^n) > \eta, \tau_\mathcal{G} \le h_\varepsilon n\Big] + \mathtt{P}_{\theta_x\omega}[\tau_\mathcal{G} > h_\varepsilon n] \\ &\le \mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_\mathcal{G}]} |Z^n - \theta_{n^{-1}\tau_\mathcal{G}}M^n| > \frac{\eta}{2}, \tau_\mathcal{G} \le h_\varepsilon n\Big] \\ &\quad + \mathtt{P}_{\theta_x\omega}\Big[\mathtt{d}(\theta_{n^{-1}\tau_\mathcal{G}}Z^n, \theta^2_{n^{-1}\tau_\mathcal{G}}M^n) > \frac{\eta}{2}, \tau_\mathcal{G} \le h_\varepsilon n\Big] \\ &\quad + \mathtt{P}_{\theta_x\omega}[\tau_\mathcal{G} > h_\varepsilon n]. \end{aligned} \tag{3.21}$$

Let $\mathcal{F}_{\tau_\mathcal{G}}$ be the σ-field generated by X until time $\tau_\mathcal{G}$. We decompose the first term in the right-hand side of (3.21) in the following way

$$\begin{aligned}
&\mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_{\mathcal G}]}|Z^n-\theta_{n^{-1}\tau_{\mathcal G}}M^n|>\frac{\eta}{2},\tau_{\mathcal G}\le h_\varepsilon n\Big]\\
&\quad\le \mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_{\mathcal G}]}|Z^n|>\frac{\eta}{4}\Big]+\mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,h_\varepsilon]}|\theta_{n^{-1}\tau_{\mathcal G}}M^n|>\frac{\eta}{4}\Big]\\
&\quad= \mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_{\mathcal G}]}|Z^n|>\frac{\eta}{4}\Big]+\mathtt{E}_{\theta_x\omega}\Big(\mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,h_\varepsilon]}|\theta_{n^{-1}\tau_{\mathcal G}}M^n|>\frac{\eta}{4}\;\Big|\;\mathcal F_{\tau_{\mathcal G}}\Big]\Big)\\
&\quad= \mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_{\mathcal G}]}|Z^n|>\frac{\eta}{4}\Big]+\mathtt{E}_{\theta_x\omega}\Big(\mathtt{P}_{\theta_{X_{\tau_{\mathcal G}}}\omega}\Big[\sup_{t\in[0,h_\varepsilon]}|Z^n|>\frac{\eta}{4}\Big]\Big). \qquad (3.22)
\end{aligned}$$

We now deal with the second term of the right-hand side of (3.21)

$$\begin{aligned}
&\mathtt{P}_{\theta_x\omega}\Big[\mathtt{d}(\theta_{n^{-1}\tau_{\mathcal G}}Z^n,\theta^2_{n^{-1}\tau_{\mathcal G}}M^n)>\frac{\eta}{2},\tau_{\mathcal G}\le h_\varepsilon n\Big]\\
&\quad\le \mathtt{P}_{\theta_x\omega}\Big[|X'_{\tau_{\mathcal G}}|>\frac{\eta}{4}\sqrt{n}\Big]+\mathtt{P}_{\theta_x\omega}\Big[\mathtt{d}(\theta_{n^{-1}\tau_{\mathcal G}}M^n,\theta^2_{n^{-1}\tau_{\mathcal G}}M^n)>\frac{\eta}{4},\tau_{\mathcal G}\le h_\varepsilon n\Big]\\
&\quad\le \mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_{\mathcal G}]}|Z^n|>\frac{\eta}{4}\Big]\\
&\qquad+\mathtt{E}_{\theta_x\omega}\Big(\mathbf 1\{\tau_{\mathcal G}\le h_\varepsilon n\}\mathtt{P}_{\theta_x\omega}\Big[\mathtt{d}(\theta_{n^{-1}\tau_{\mathcal G}}M^n,\theta^2_{n^{-1}\tau_{\mathcal G}}M^n)>\frac{\eta}{4}\;\Big|\;\mathcal F_{\tau_{\mathcal G}}\Big]\Big)\\
&\quad= \mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_{\mathcal G}]}|Z^n|>\frac{\eta}{4}\Big]+\mathtt{E}_{\theta_x\omega}\Big(\mathbf 1\{\tau_{\mathcal G}\le h_\varepsilon n\}\mathtt{P}_{\theta_{X_{\tau_{\mathcal G}}}\omega}\Big[\mathtt{d}(Z^n,\theta_{n^{-1}\tau_{\mathcal G}}Z^n)>\frac{\eta}{4}\Big]\Big).
\end{aligned}$$

(3.23)

Combining (3.21), (3.22) and (3.23), we obtain

$$\begin{aligned}
\mathtt{P}_{\theta_x\omega}\Big[\mathtt{d}(Z^n,\theta_{n^{-1}\tau_{\mathcal G}}M^n)>\eta\Big]&\le \mathtt{P}_{\theta_x\omega}[\tau_{\mathcal G}>h_\varepsilon n]+2\mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_{\mathcal G}]}|Z^n|>\frac{\eta}{4}\Big]\\
&\quad+\mathtt{E}_{\theta_x\omega}\Big(\mathtt{P}_{\theta_{X_{\tau_{\mathcal G}}}\omega}\Big[\sup_{t\in[0,h_\varepsilon]}|Z^n|>\frac{\eta}{4}\Big]\\
&\qquad+\mathbf 1\{\tau_{\mathcal G}\le h_\varepsilon n\}\mathtt{P}_{\theta_{X_{\tau_{\mathcal G}}}\omega}\Big[\mathtt{d}(Z^n,\theta_{n^{-1}\tau_{\mathcal G}}Z^n)>\frac{\eta}{4}\Big]\Big).
\end{aligned}$$

(3.24)

By definition of a (ε,n)-good site, we can choose $\varepsilon>0$ small enough in such a way that $\varepsilon\le\min\{\eta/4,\tilde\varepsilon(32\|F\|_\infty)^{-1}\}$ and $h_\varepsilon\le\eta^2/16$. Therefore, we have uniformly in $x\in[-H\sqrt{n},H\sqrt{n}]$,

$$\mathtt{E}_{\theta_x\omega}\Big(\mathtt{P}_{\theta_{X_{\tau_{\mathcal G}}}\omega}\Big[\sup_{t\in[0,h_\varepsilon]}|Z^n|>\frac{\eta}{4}\Big]+\mathbf 1\{\tau_{\mathcal G}\le h_\varepsilon n\}\mathtt{P}_{\theta_{X_{\tau_{\mathcal G}}}\omega}\Big[\mathtt{d}(Z^n,\theta_{n^{-1}\tau_{\mathcal G}}Z^n)>\frac{\eta}{4}\Big]\Big)\le\frac{\tilde\varepsilon}{32\|F\|_\infty}$$

(3.25)

for all sufficiently large n. On the other hand, by Lemma 3.6, we have uniformly in $x\in[-H\sqrt{n},H\sqrt{n}]$,

$$\mathtt{P}_{\theta_x\omega}[\tau_{\mathcal G}>h_\varepsilon n]\le\frac{\tilde\varepsilon}{32\|F\|_\infty} \qquad (3.26)$$

and

$$\mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_{\mathcal G}]}|Z^n|>\frac{\eta}{4}\Big]\le\frac{\tilde\varepsilon}{32\|F\|_\infty} \qquad (3.27)$$

for sufficiently large n. Combining (3.25), (3.26), (3.27) with (3.24), (3.21), (3.20) and (3.19), we have

$$\Big|\mathtt{E}_{\theta_x\omega}\Big(F(Z^n)-\mathtt{E}_{\theta_{X_{\tau_{\mathcal G}}}\omega}[F(Z^n)]\Big)\Big|\le\tilde\varepsilon/2.$$

Together with (3.18), we obtain that $R\le\tilde\varepsilon$ which concludes the proof of Proposition 3.1. □

Next, we prove

Proposition 3.7. *The first statement implies the second one:*

(i) for any $F \in \mathfrak{C}_b^u(C(\mathbb{R}_+), \mathbb{R})$, we have $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty} \sup_{x\in[-H\sqrt{n},H\sqrt{n}]} \Big| \mathtt{E}_{\theta_x\omega}[F(Z^n)] - E[F(W)] \Big| = 0;$$

(ii) for any open set $G \subset C(\mathbb{R}_+)$, we have $\mathbb{P}$-a.s.,

$$\liminf_{n\to\infty} \inf_{x\in[-H\sqrt{n},H\sqrt{n}]} \mathtt{P}_{\theta_x\omega}[Z^n \in G] \geq P[W \in G].$$

Proof. Let G be an open set. Then, there exists a sequence $(F_k, k \geq 1) \subset \mathfrak{C}_b^u(C(\mathbb{R}_+), \mathbb{R})$ such that $F_k \uparrow \mathbf{1}_G$ pointwise as $k \to \infty$. Thus, we have for all ω, n, k and $x \in [-H\sqrt{n}, H\sqrt{n}]$

$$\mathtt{P}_{\theta_x\omega}[Z^n \in G] \geq \mathtt{E}_{\theta_x\omega}[F_k(Z^n)]. \tag{3.28}$$

Then, fix $\varepsilon > 0$. By the monotone convergence theorem, there exists k_0 such that for all $k \geq k_0$,

$$E[F_k(W)] \geq P[W \in G] - \frac{\varepsilon}{2}. \tag{3.29}$$

Now, by (i), $\mathbb{P}$-a.s., for all $k \geq k_0$, we have that for $n \geq n_0(k,\omega)$ and all $x \in [-H\sqrt{n}, H\sqrt{n}]$,

$$\mathtt{E}_{\theta_x\omega}[F_k(Z^n)] \geq E[F_k(W)] - \frac{\varepsilon}{2}. \tag{3.30}$$

Combining (3.28) and (3.30), we have, $\mathbb{P}$-a.s., for all $k \geq k_0$, for all $n \geq n_0(k,\omega)$ and all $x \in [-H\sqrt{n}, H\sqrt{n}]$

$$\mathtt{P}_{\theta_x\omega}[Z^n \in G] \geq E[F_k(W)] - \frac{\varepsilon}{2}. \tag{3.31}$$

Then, combining (3.29) and (3.31) we obtain $\mathbb{P}$-a.s., for all sufficiently large n,

$$\inf_{x\in[-H\sqrt{n},H\sqrt{n}]} \mathtt{P}_{\theta_x\omega}[Z^n \in G] \geq P[W \in G] - \varepsilon. \tag{3.32}$$

As ε is arbitrary, take the $\liminf_{n\to\infty}$ in the last inequality to show that (i) ⇒ (ii). □

Then, in the next proposition, we show that "for every" and "$\mathbb{P}$-a.s." can be interchanged:

Proposition 3.8. *The following statements are equivalent:*

(i) we have $\mathbb{P}$-a.s., for every open set G,

$$\liminf_{n\to\infty} \inf_{x\in[-H\sqrt{n},H\sqrt{n}]} \mathtt{P}_{\theta_x\omega}[Z^n \in G] \geq P[W \in G];$$

(ii) for every open set G, we have $\mathbb{P}$-a.s.,

$$\liminf_{n\to\infty} \inf_{x\in[-H\sqrt{n},H\sqrt{n}]} \mathtt{P}_{\theta_x\omega}[Z^n \in G] \geq P[W \in G].$$

Proof. We only have to show that (ii) ⇒ (i). Since a basis of open sets for the topology of $C(\mathbb{R}_+)$ is formed by open balls of rational radii about piecewise linear functions connecting rational points, there exists a countable family $\mathcal{G}$ of open sets such that for every open set G there exists a sequence $(O_n, n \geq 1) \subset \mathcal{G}$ such that $\mathbf{1}_{O_n} \uparrow \mathbf{1}_G$ pointwise as $n \to \infty$. By (ii), since the family $\mathcal{G}$ is countable we have, $\mathbb{P}$-a.s., for all $O \in \mathcal{G}$,

$$\liminf_{n\to\infty} \inf_{x\in[-H\sqrt{n},H\sqrt{n}]} \mathtt{P}_{\theta_x\omega}[Z^n \in O] \geq P[W \in O]. \tag{3.33}$$

Then, the same kind of reasoning as that used in the proof of Proposition 3.7 would provide the desired result. □

Now we are ready to finish the proof of Theorem 1.2.

Proof of Theorem 1.2. Recall that we have to prove that the following statements hold:

(i) we have $\mathbb{P}$-a.s., for any $F \in \mathfrak{C}_b(C(\mathbb{R}_+), \mathbb{R})$,

$$\lim_{n\to\infty} \sup_{x\in[-H\sqrt{n}, H\sqrt{n}]} \Big| \mathtt{E}_{\theta_x\omega}[F(Z^n)] - E[F(W)] \Big| = 0;$$

(ii) we have $\mathbb{P}$-a.s., for any $F \in \mathfrak{C}_b^u(C(\mathbb{R}_+), \mathbb{R})$,

$$\lim_{n\to\infty} \sup_{x\in[-H\sqrt{n}, H\sqrt{n}]} \Big| \mathtt{E}_{\theta_x\omega}[F(Z^n)] - E[F(W)] \Big| = 0;$$

(iii) we have $\mathbb{P}$-a.s., for any closed set B,

$$\limsup_{n\to\infty} \sup_{x\in[-H\sqrt{n}, H\sqrt{n}]} \mathtt{P}_{\theta_x\omega}[Z^n \in B] \le P[W \in B];$$

(iv) we have $\mathbb{P}$-a.s., for any open set G,

$$\liminf_{n\to\infty} \inf_{x\in[-H\sqrt{n}, H\sqrt{n}]} \mathtt{P}_{\theta_x\omega}[Z^n \in G] \ge P[W \in G];$$

(v) we have $\mathbb{P}$-a.s., for any $A \in \mathcal{B}$ such that $P[W \in \partial A] = 0$,

$$\lim_{n\to\infty} \sup_{x\in[-H\sqrt{n}, H\sqrt{n}]} \Big| \mathtt{P}_{\theta_x\omega}[Z^n \in A] - P[W \in A] \Big| = 0.$$

Essentially, we follow the proof of Theorem 2.1 of [4]. Of course, (i) $\Rightarrow$ (ii) is trivial. The proof of the fact that (ii) $\Rightarrow$ (iii) (and, by complementation, that (ii) $\Rightarrow$ (iv)) is a consequence of Propositions 3.1, 3.7 and 3.8. Let us show that (iii) $\Leftrightarrow$ (v). We start by showing (iii) $\Rightarrow$ (v). Let $\mathring{A}$ denote the interior of A and $\bar{A}$ denote its closure. If (iii) holds, then so does (iv), and hence $\mathbb{P}$-a.s.,

$$\begin{aligned} P[W \in \bar{A}] &\ge \limsup_{n\to\infty} \sup_{x\in[-H\sqrt{n}, H\sqrt{n}]} \mathtt{P}_{\theta_x\omega}[Z^n \in \bar{A}] \ge \limsup_{n\to\infty} \sup_{x\in[-H\sqrt{n}, H\sqrt{n}]} \mathtt{P}_{\theta_x\omega}[Z^n \in A] \\ &\ge \liminf_{n\to\infty} \sup_{x\in[-H\sqrt{n}, H\sqrt{n}]} \mathtt{P}_{\theta_x\omega}[Z^n \in A] \ge \liminf_{n\to\infty} \sup_{x\in[-H\sqrt{n}, H\sqrt{n}]} \mathtt{P}_{\theta_x\omega}[Z^n \in \mathring{A}] \\ &\ge P[W \in \mathring{A}]. \end{aligned} \tag{3.34}$$

Since $P[W \in \partial A] = 0$, the first and the last terms in (3.34) are both equal to $P[W \in A]$, and (v) follows. We continue by showing that (v) $\Rightarrow$ (iii). Since $\partial\{w \in C(\mathbb{R}_+) : \mathtt{d}(w, B) \le \delta\} = \{w \in C(\mathbb{R}_+) : \mathtt{d}(w, B) = \delta\}$, the sets $\partial\{w \in C(\mathbb{R}_+) : \mathtt{d}(w, B) \le \delta\}$ are disjoint for distinct δ, hence at most countably many of them can have positive $P[W \in \cdot\,]$-measure. Thus, for some sequence of positive δ_k such that $\delta_k \to 0$ as $k \to \infty$, the sets $B_k = \{w : \mathtt{d}(w, B) \le \delta_k\}$ are such that $P[W \in \partial B_k] = 0$. If (v) holds, then we can apply the same sequence of arguments used to show Proposition 3.7 with the sequence $(\mathbf{1}_{B_k}, k \ge 1)$ instead of $(F_k, k \ge 1)$.

Finally, we show that (iii) $\Rightarrow$ (i). Suppose that (iii) holds and that $F \in \mathfrak{C}_b(C(\mathbb{R}_+), \mathbb{R})$. By transforming F linearly (with positive coefficient for the first-degree term) we can reduce the problem to the case in which $0 \le F < 1$. For a fixed integer k, let B_i be the closed set $B_i = \{w : i/k \le F(w)\}$, $i = 0, \dots, k$. Since $0 \le F < 1$, we have for all ω, n and all $x \in [-H\sqrt{n}, H\sqrt{n}]$

$$\mathtt{E}_{\theta_x\omega}[F(Z^n)] < \sum_{i=1}^{k} \frac{i}{k} \mathtt{P}_{\theta_x\omega}\Big[\frac{i-1}{k} \le F(Z^n) < \frac{i}{k}\Big]$$

which implies

$$\mathrm{E}_{\theta_x\omega}[F(Z^n)] < \frac{1}{k} + \frac{1}{k}\sum_{i=1}^{k} \mathrm{P}_{\theta_x\omega}[Z^n \in B_i].$$

If (iii) holds then we have $\mathbb{P}$-a.s., $\limsup_{n\to\infty} \sup_{x\in[-H\sqrt{n},H\sqrt{n}]} \mathrm{P}_{\theta_x\omega}[Z^n \in B_i] \le P[W \in B_i]$ for all i, hence we can deduce that we have $\mathbb{P}$-a.s.,

$$\limsup_{n\to\infty} \sup_{x\in[-H\sqrt{n},H\sqrt{n}]} \mathrm{E}_{\theta_x\omega}[F(Z^n)] \le \frac{1}{k} + E[F(W)].$$

Since k is arbitrary, we obtain for all $F \in \mathfrak{C}_b(C(\mathbb{R}_+),\mathbb{R})$,

$$\limsup_{n\to\infty} \sup_{x\in[-H\sqrt{n},H\sqrt{n}]} \mathrm{E}_{\theta_x\omega}[F(Z^n)] \le E[F(W)]. \tag{3.35}$$

Note that (3.35) implies

$$\limsup_{n\to\infty} \inf_{x\in[-H\sqrt{n},H\sqrt{n}]} \mathrm{E}_{\theta_x\omega}[F(Z^n)] \le E[F(W)]. \tag{3.36}$$

Applying (3.36) to $(-F)$ yields $\liminf_{n\to\infty} \sup_{x\in[-H\sqrt{n},H\sqrt{n}]} \mathrm{E}_{\theta_x\omega}[F(Z^n)] \ge E[F(W)]$ which together with (3.35) implies (i), and thus the proof of Theorem 1.2 is concluded. □

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