RANK PROBABILITIES FOR REAL RANDOM $N \times N \times 2$ TENSORS

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Abstract
We prove that the probability $P_N$ for a real random Gaussian $N \times N \times 2$ tensor to be of real rank $N$ is

$$P_N = \frac{\Gamma((N+1)/2)}{\Gamma(N+1)} N G(N+1),$$

where $\Gamma(x)$, $G(x)$ denote the gamma and Barnes $G$-functions respectively. This is a rational number for $N$ odd and a rational number multiplied by $\pi^{N/2}$ for $N$ even. The probability to be of rank $N+1$ is $1-P_N$. The proof makes use of recent results on the probability of having $k$ real generalized eigenvalues for real random Gaussian $N \times N$ matrices. We also prove that $\log P_N = (N^2/4) \log(e/4) + (\log N - 1)/12 - \zeta'(-1) + O(1/N)$ for large $N$, where $\zeta$ is the Riemann zeta function.

1 Introduction

The (real) rank of a real $m \times n \times p$ 3-tensor or 3-way array $\mathcal{T}$ is the well defined minimal possible value of $r$ in an expansion

$$\mathcal{T} = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i \quad (u_i \in \mathbb{R}^m, v_i \in \mathbb{R}^n, w_i \in \mathbb{R}^p)$$

where $\otimes$ denotes the tensor (or outer) product [1, 3, 4, 8].

If the elements of $\mathcal{T}$ are chosen randomly according to a continuous probability distribution, there is in general (for general $m$, $n$ and $p$) no generic rank, i.e., a rank which occurs with probability 1. Ranks which occur with strictly positive probabilities are called typical ranks. We assume that all elements are independent and from a standard normal (Gaussian) distribution (mean 0, variance 1). Until now, the only analytically known probabilities for typical ranks were for $2 \times 2 \times 2$ and $3 \times 3 \times 2$ tensors [2, 7]. Thus in the $2 \times 2 \times 2$ case the probability that $r = 2$ is $\pi/4$ and the probability that $r = 3$ is $1 - \pi/4$, while in the $3 \times 3 \times 2$ case the probability of the rank equaling...
3 is the same as the probability of it equaling 4 which is 1/2. Before these analytic results the first numerical simulations were performed by Kruskal in 1989, for $2 \times 2 \times 2$ tensors [8], and the approximate values 0.79 and 0.21 obtained for the probability of ranks $r = 2$ and $r = 3$ respectively. For $N \times N \times 2$ tensors ten Berge and Kiers [10] have shown that the only typical ranks are $N$ and $N + 1$. From ten Berge [9], it follows that the probability $P_N$ for an $N \times N \times 2$ tensor to be of rank $N$ is equal to the probability that a pair of real random Gaussian $N \times N$ matrices $T_1$ and $T_2$ (the two slices of $\mathcal{T}$) has $N$ real generalized eigenvalues, i.e., the probability that $\det(T_1 - \lambda T_2) = 0$ has only real solutions $\lambda$ [2, 9]. Knowledge about the expected number of real solutions to $\det(T_1 - \lambda T_2) = 0$ obtained by Edelman et al. [5] led to the analytical results for $N = 2$ and $N = 3$ in [2]. Forrester and Mays [7] have recently determined the probabilities $p_{N,k}$ that $\det(T_1 - \lambda T_2) = 0$ has $k$ real solutions, and we here apply the results to $P_N = P_{N,N}$ to obtain explicit expressions for the probabilities for all typical ranks of $N \times N \times 2$ tensors for arbitrary $N$, hence settling this open problem for tensor decompositions. We also determine the precise asymptotic decay of $P_N$ for large $N$ and give some recursion formulas for $P_N$.

## 2 Probabilities for typical ranks of $N \times N \times 2$ tensors

As above, assume that $T_1$ and $T_2$ are real random Gaussian $N \times N$ matrices and let $p_{N,k}$ be the probability that $\det(T_1 - \lambda T_2) = 0$ has $k$ real solutions. Then Forrester and Mays [7] prove:

**Theorem 1.** Introduce the generating function

$$Z_N(\xi) = \sum_{k=0}^{N} \xi^{k} p_{N,k}$$

where the asterisk indicates that the sum is over $k$ values of the same parity as $N$. For $N$ even we have

$$Z_N(\xi) = \frac{(-1)^{N(N-2)/8} \Gamma(N+1/2)^2 \Gamma(N+2/2)^2}{2^{N(N-1)/2} \prod_{j=1}^{N} \Gamma\left(\frac{j+1}{2}\right)^2} \prod_{l=0}^{\lceil N/2 \rceil-1} (\xi^2 \alpha_l + \beta_l),$$

while for $N$ odd

$$Z_N(\xi) = \frac{(-1)^{(N-1)(N-3)/8} \Gamma(N+1/2)^2 \Gamma(N+2/2)^2 \Gamma(N-1)/2}{2^{N(N-1)/2} \prod_{j=1}^{N} \Gamma\left(\frac{j+1}{2}\right)^2} \pi \xi$$

$$\times \prod_{l=0}^{\lceil N/2 \rceil-1} (\xi^2 \alpha_l + \beta_l) \prod_{l=\lceil N/2 \rceil}^{N/2} (\xi^2 \alpha_{l+1/2} + \beta_{l+1/2}).$$

Here

$$\alpha_l = \frac{2\pi}{N-1-4l} \frac{\Gamma\left(N+1/2\right)^2}{\Gamma\left(N+2/2\right)}$$

and

$$\alpha_{l+1/2} = \frac{2\pi}{N-3-4l} \frac{\Gamma\left(N+3/2\right)^2}{\Gamma\left(N+5/2\right)}$$

The expressions for $\beta_l$ and $\beta_{l+1/2}$ are given in [7], but are not needed here, and $\lceil \cdot \rceil$ denotes the ceiling function.
The method used in [7] relies on first obtaining the explicit form of the element probability density function for

\[ G = T_1^{-1} T_2. \]  

(7)

A real Schur decomposition is used to introduce \( k \) real and \( (N-k)/2 \) complex eigenvalues, with the imaginary part of the latter required to be positive (the remaining \( (N-k)/2 \) eigenvalues are the complex conjugate of these), for \( k = 0, 2, \ldots, N \) (\( N \) even) and \( k = 1, 3, \ldots, N \) (\( N \) odd).

The variables not depending on the eigenvalues can be integrated out to give the eigenvalue probability density function, in the event that there are \( k \) real eigenvalues. And integrating this over all allowed values of the real and positive imaginary part complex eigenvalues gives \( P_{N,k} \).

From Theorem 1 we derive our main result:

**Theorem 2.** Let \( P_N \) denote the probability that a real \( N \times N \times 2 \) tensor whose elements are independent and normally distributed with mean 0 and variance 1 has rank \( N \). We have

\[ P_N = \frac{(\Gamma((N+1)/2))^N}{G(N+1)}, \]

(8)

where

\[ G(N+1) := (N-1)(N-2)! \ldots 1! \quad (N \in \mathbb{Z}^+) \]

(9)

is the Barnes G-function and \( \Gamma(x) \) denotes the gamma function. More explicitly \( P_2 = \pi/4 \), and for \( N \geq 4 \) even

\[ P_N = \frac{\pi^{N/2}(N-1)^{N-1}(N-3)^{N-3} \ldots 3^3}{2^{N^2/2}(N-2)^2(N-4)^4 \ldots 2^{N-2}}, \]

(10)

while for \( N \) odd

\[ P_N = \frac{(N-1)^{N-1}(N-3)^{N-3} \ldots 2^2}{2^{N(N-1)/2}(N-2)^2(N-4)^4 \ldots 3^{N-3}}. \]

(11)

Hence \( P_N \) for \( N \) odd is a rational number but for \( N \) even it is a rational number multiplied by \( \pi^{N/2} \).

The probability for rank \( N + 1 \) is \( 1 - P_N \).

**Proof.** From [2] we know that \( P_N = p_{N,N} \). Hence, by Theorem 1

\[ P_N = p_{N,N} = \frac{1}{N!} \frac{d^N}{d\xi^N} Z_N(\xi) \]

(12)

Since

\[ \frac{1}{N!} \frac{d^N}{d\xi^N} \prod_{l=0}^{N \choose 2} (\xi^2 \alpha_l + \beta_l) = \prod_{l=0}^{N \choose 2} \alpha_l \]

(13)

and

\[ \frac{1}{N!} \frac{d^N}{d\xi^N} \prod_{l=0}^{[N+1] \choose 2} (\xi^2 \alpha_l + \beta_l) \prod_{l=0}^{[N+1] \choose 2} (\xi^2 \alpha_{l+1/2} + \beta_{l+1/2}) = \prod_{l=0}^{[N+1] \choose 2} \alpha_l \prod_{l=0}^{[N+1] \choose 2} \alpha_{l+1/2} \]

(14)

the values of \( \beta_l \) and \( \beta_{l+1/2} \) are not needed for the determination of \( P_N \). By (3) we immediately find

\[ P_N = \frac{(-1)^{N(N-2)/2} \Gamma\left(\frac{N+1}{2}\right)^{N/2} \Gamma\left(\frac{N+2}{2}\right)^{N/2}}{2^{N(N-1)/2} \prod_{l=1}^{N} \Gamma\left(\frac{l}{2}\right)^2} \prod_{l=0}^{N \choose 2} \alpha_l \]

(15)
if \( N \) is even. For \( N \) odd we use (4) to get

\[
P_N = \frac{(-1)^{(N-1)(N-3)/8} \Gamma(N/2 + 1) \Gamma(N/2 + 2) \Gamma(N/2)}{2^{N(N-1)/2} \prod_{j=1}^{N} \Gamma(j/2)^2} \pi \prod_{l=0}^{\lfloor \frac{N-1}{2} \rfloor} \alpha_l \prod_{l=1}^{\lfloor \frac{N-3}{2} \rfloor} \alpha_{l+1/2}
\]  \hspace{1cm} (16)

Substituting the expressions for \( \alpha_l \) and \( \alpha_{l+1/2} \) into these formulas we obtain, after simplifying, for \( N \) even

\[
P_N = \frac{(-1)^{N(N-2)/8} (2\pi)^{N/2} \Gamma(N/2 + 1/2)^N}{2^{N(N-1)/2} \prod_{j=1}^{N} \Gamma(j/2)^2} \prod_{l=0}^{\lfloor \frac{N-1}{2} \rfloor} \frac{1}{N-1-4l},
\]  \hspace{1cm} (17)

and for \( N \) odd

\[
P_N = \frac{(-1)^{N(N-3)/8} (2\pi)^{N/2} \Gamma(N/2 + 1/2)^N}{2^{N(N-1)/2} + 1} \prod_{j=1}^{N} \Gamma(j/2)^2 \prod_{l=0}^{\lfloor \frac{N-1}{2} \rfloor} \frac{1}{N-1-4l} \prod_{l=0}^{\lfloor \frac{N-3}{2} \rfloor} \frac{1}{N-3-4l}.
\]  \hspace{1cm} (18)

Now

\[
\prod_{j=1}^{N} \frac{\Gamma(j/2)^2}{\Gamma((N+1)/2)} = \frac{\Gamma(1/2)}{\Gamma((N+1)/2)} \prod_{j=1}^{N} \frac{\Gamma(j/2) \Gamma((j+1)/2)}{2^{j-1} \sqrt{\pi} \Gamma(j)} = \frac{\Gamma(1/2)}{\Gamma((N+1)/2)} 2^{-N(N-1)/2} \pi^{N/2} G(N+1),
\]  \hspace{1cm} (19)

where to obtain the second equality use has been made of the duplication formula for the gamma function, and to obtain the third equality the expression (9) for the Barnes \( G \)-function has been used. Furthermore, for each \( N \) even

\[
(-1)^{N(N-2)/8} \prod_{l=0}^{\lfloor \frac{N-1}{2} \rfloor} \frac{1}{N-1-4l} = \frac{(-1)^{N(N-2)/8}}{(N-1)(N-5) \ldots (N-1-(2N-4))} = \frac{1}{(N-1)(N-3) \ldots 3 \cdot 1} \frac{\Gamma(1/2)}{2^{N/2} \Gamma((N+1)/2)},
\]  \hspace{1cm} (20)

where to obtain the final equation use is made of the fundamental gamma function recurrence

\[
\Gamma(x+1) = x \Gamma(x),
\]  \hspace{1cm} (21)
and for $N$ odd

$$
(-1)^{(N-1)(N-3)/8} \prod_{l=0}^{\lfloor N/4 \rfloor} \frac{1}{N-1-4l} \prod_{l=1}^{\lfloor (N-3)/4 \rfloor} \frac{1}{N-3-4l}
$$

$$
= (-1)^{(N-1)(N-3)/8} \left\{ \frac{1}{(N-1)(N-5)\ldots2(-4)(-8)\ldots(-N+3)} \right. \\
\left. \frac{1}{(N-1)(N-5)\ldots4(-2)(-6)\ldots(-N+3)} \right\}, \quad N = 3, 7, 11, \ldots
$$

$$
= \frac{1}{(N-1)(N-3)\ldots4\cdot2} \\
= \frac{1}{2^{(N-1)/2} \Gamma((N+1)/2)}
$$

Substituting (19) and (20) in (17) establishes (8) for $N$ even, while the $N$ odd case of (8) follows by substituting (19) and (22) in (18), and the fact that

$$
\Gamma(1/2) = \sqrt{\pi}.
$$

The forms (10) and (11) follow from (8) upon use of (9), the recurrence (21) and (for $N$ even) (23).

\[\square\]

3 Recursion formulas and asymptotic decay

By Theorem 2 it is straightforward to calculate $P_{N+1}/P_N$ from either (8) or (10) and (11), and $P_{N+2}/P_N$ from either (8) or (10) and (11).

**Corollary 3.** For general $N$

$$
P_{N+1} = P_N \cdot \frac{\Gamma((N+1)/2)^{N+1}}{\Gamma((N+1)/2)^N} \cdot \frac{1}{\Gamma((N+1)/2)}
$$

$$
P_{N+2} = P_N \cdot \frac{((N+1)/2)^{N+2} \Gamma((N+1)/2)^2}{\Gamma((N+1)/2)^3}
$$

More explicitly, making use of the double factorial

$$
N!! = \begin{cases} 
N(N-2)\ldots4\cdot2, & N \text{ even} \\
N(N-2)\ldots3\cdot1, & N \text{ odd}
\end{cases}
$$

for $N$ even we have the recursion formulas

$$
P_{N+1} = P_N \cdot \frac{(N!!)^N}{(2\pi)^{N/2}((N-1)!!)^{N+1}}
$$

$$
P_{N+2} = P_N \cdot \frac{\pi}{2} \cdot \frac{(N+1)^{N+1}}{2^{2N+1}(N!!)^2}
$$

and for $N$ odd we have

$$
P_{N+1} = P_N \cdot \frac{\pi^{(N+1)/2}(N!!)^N}{2^{(3N+1)/2}((N-1)!!)^{N+1}}
$$

$$
P_{N+2} = P_N \cdot \frac{(N+1)^{N+1}}{2^{2N+1}(N!!)^2}.
$$
We can illustrate the pattern for \( P_N \) using Theorem 2 or Corollary 3. One finds

\[
\begin{align*}
P_2 &= \frac{1}{2^2 \cdot \pi}, & P_3 &= \frac{1}{2} \\
P_4 &= \frac{3^3}{2^{10}} \cdot \pi^2, & P_5 &= \frac{1}{3^2} \\
P_6 &= \frac{5^5 \cdot 3^3}{2^{26}} \cdot \pi^3, & P_7 &= \frac{3^2}{5^2 \cdot 2^5} \\
P_8 &= \frac{7^7 \cdot 5^3 \cdot 3}{2^{48}} \cdot \pi^4, & P_9 &= \frac{2^4}{7^2 \cdot 5^4} \\
P_{10} &= \frac{7^7 \cdot 5^3 \cdot 3^{17}}{2^{80}} \cdot \pi^5, & P_{11} &= \frac{5^4}{7^4 \cdot 3^6 \cdot 2^5} \\
P_{12} &= \frac{11^{11} \cdot 7^7 \cdot 5^{315}}{2^{118}} \cdot \pi^6, & P_{13} &= \frac{5^2}{11^2 \cdot 7^6 \cdot 2^4} \cdots
\end{align*}
\]

(27)

Numerically, it is clear that \( P_N \to 0 \) as \( N \to \infty \). Some qualitative insight into the rate of decay can be obtained by recalling \( P_N = P_{N,k} \) and considering the behaviour of \( P_{N,k} \) as a function of \( k \). Thus we know from [5] that for large \( N \), the mean number of real eigenvalues \( E_N := \langle k \rangle_{P_{N,k}} \) is to leading order equal to \( \sqrt{\pi N / 2} \), and from [7] that the corresponding variance \( \sigma_N^2 := \langle k^2 \rangle_{P_{N,k}} - E_N^2 \) is to leading order equal to \((2 - \sqrt{2})E_N \). The latter reference also shows that \( \lim_{N \to \infty} \sigma_N P_{N,[\sigma_N x + E_N]} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \), and is thus \( P_{N,k} \) is a standard Gaussian distribution after centering and scaling in \( k \) by appropriate multiples of \( \sqrt{N} \). It follows that \( P_{N,N} \) is, for large \( N \), in the large deviation regime of \( P_{N,k} \).

We remark that this is similarly true of \( P_{N,N} \) in the case of eigenvalues of \( N \times N \) real random Gaussian matrices (i.e. the individual matrices \( T_1, T_2 \) of (7)), for which it is known \( P_{N,N} = 2^{-N(N-1)/4} \) [5, 6, Section 15.10].

In fact from the exact expression (8) the explicit asymptotic large \( N \) form of \( P_N \) can readily be calculated. For this, let

\[ A = e^{-\zeta(-1)+1/12} = 1.28242712... \]

(28)

denote the Glaisher-Kinkelin constant, where \( \zeta \) is the Riemann zeta function [11].

**Theorem 4.** For large \( N \),

\[
P_N = N^{1/12} \left( \frac{e}{4} \right)^{N^2/4} \cdot A e^{-1/6}(1 + O(N^{-1}))
\]

(29)

or equivalently

\[
\log P_N = (N^2/4) \log(e/4) + (\log N - 1)/12 - \zeta(-1) + O(1/N).
\]

(30)

**Proof.** We require the \( x \to \infty \) asymptotic expansions of the Barnes \( G \)-function [12] and the gamma function

\[
\log G(x + 1) = \frac{x^2}{2} \log x - \frac{3}{4} x^2 + \frac{x}{2} \log 2 \pi - \frac{1}{12} \log x + \zeta(-1) + O\left( \frac{1}{x} \right),
\]

(31)

\[
\Gamma(x + 1) = \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( 1 + \frac{1}{12x} + O\left( \frac{1}{x^2} \right) \right)
\]

(32)
For future purposes, we note that a corollary of (32), and the elementary large \( x \) expansion
\[
\left(1 + \frac{c}{x}\right)^x = e^c \left(1 - \frac{c^2}{2x} + O\left(\frac{1}{x^2}\right)\right)
\]
is the asymptotic formula
\[
\frac{\Gamma(x + 1/2)}{\Gamma(x)} = \sqrt{x} \left(1 - \frac{1}{8x} + O\left(\frac{1}{x^2}\right)\right).
\]

To make use of these expansions, we rewrite (8) as
\[
P_N = \frac{\left(\Gamma(N/2 + 1)\right)^N}{G(N + 1)} \frac{\Gamma((N + 1)/2)^N}{\Gamma(N/2 + 1)}.
\]

Now, (34) and (33) show that with
\[
y := N/2
\]
and \( y \) large we have
\[
\left(\frac{\Gamma(y + 1/2)}{\Gamma(y + 1)}\right)^N = e^{-y \log y} e^{-1/4} \left(1 + O\left(\frac{1}{y}\right)\right).
\]

Furthermore, in the notation (36) it follows from (31) and (32) and further use of (33) (only the explicit form of the leading term is now required) that
\[
\frac{\Gamma(N/2 + 1)^N}{G(N + 1)} = e^{-y^2 \log(4/e)} e^{y \log y + \frac{1}{12} \log 2} e^{1/6 - \zeta'(1)} \left(1 + O\left(\frac{1}{y}\right)\right).
\]

Multiplying together (37) and (38) as required by (35) and recalling (36) gives (29).

Recalling (28), the second stated result (30) is then immediate.

**Corollary 5.** For large \( N \),
\[
\frac{P_{N+1}}{P_N} = \left(\frac{e}{4}\right)^{(2N+1)/4} \left(1 + O(N^{-1})\right)
\]

This corollary follows trivially from Theorem 4. It can however also be derived directly from the recursion formulas in Corollary 3, without use of Theorem 4.

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