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RANDOM WALK ON PERIODIC TREES

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Abstract: Following Lyons (1990, [4]) we define a periodic tree, restate its branching number and consider a biased random walk on it. In the case of a transient walk, we describe the walk-invariant random periodic tree and calculate the asymptotic rate of escape (speed) of the walk. This is achieved by exploiting the connections between random walks and electric networks.

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RANDOM WALK ON PERIODIC TREES

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1. Introduction

This paper deals with biased random walks on infinite trees. A tree, although large, is a very simple structure to walk on. Therefore many questions concerning the walk are answered for trees. The so-called type problem, i. e., the question whether a biased walk is recurrent or transient, was solved by Lyons [4]. He proved that the critical value of the bias for the type of a biased random walk is the branching number of the underlying tree. In the case of a transient walk, the asymptotic rate of escape (speed) is the next interesting topic. For n-ary trees computation of the speed is trivial. For some special trees (e. g. the Fibonacci tree, [7]) it can also be calculated. For the so-called random environment on **Z**, which can be interpreted as a special class of random trees, the speed was calculated in 1978 by Solomon [11]. In 1995, the speed of a simple random walk on a Galton-Watson tree [6] was calculated. The last method only applies to a simple random walk, because it requires the knowledge of the walk-invariant tree, which only in the case of a simple random walk equals the augmented Galton-Watson tree. Recently, the walk-invariant random environment on **Z** was calculated (independently by Alili [1] and Roland Takacs (personal communication)), too. This in combination with the method of [6], provides another way of calculating the speed of a random walk on a random environment on **Z**.

A periodic tree $\tau(u)$ is a rooted labeled tree with N types of vertices, where the root is of type u and each vertex of type v has $g(v,w)$ successors of type w. We suppose that $((g(v,w))$ arises from a directed, weighted and di-connected graph, and denote by ρ^* its spectral radius. We consider a λ-biased random walk ($\lambda > 0$) on $\tau(u)$, which is defined by the following. Start at the root, where you choose any of the edges coming out with equal probability, and at each vertex x different from the root choose the edge pointing towards the root with probability $\frac{\lambda}{\lambda+\deg(x)-1}$ and each other edge with probability $\frac{1}{\lambda+\deg(x)-1}$. Then for large λ this walk will be recurrent and for small λ it will be transient. We choose the bias λ such that the walk is transient. On the other hand we interpret the tree $\tau(u)$ as an electric network, whose edges are weighted with conductances, where the edges incident with the root are weighted with unit conductances, and at each other vertex the conductance of the edge pointing towards the root equals λ times the conductance of another edge. We denote by $\mathcal{C}(u)$ the effective conductance of this electric network. From the connections between random walk and electric networks we know that at the exit-times, i. e., at the times, when the random walker leaves a generation of the tree forever, the transition probabilities are described by the splitting of an electric current (see [3], [4], [12]). For periodic trees the effective conductances, the current, and thus the stationary random periodic tree at the exit-times can be calculated, i. e., the tree as it is watched by the random walker at exit-times after a long time of walking. In the present paper we use the distribution of this tree to calculate the distribution of the walk-invariant random periodic tree. This connection is established by Theorem 2.2, and is the fundamental tool to achieve our main result (Corollary 3.6).

Theorem 1.1. *Let $(Y_n)_{n\in\mathbf{N}_0}$ be a λ-biased random walk on a periodic tree $\tau(u)$, where Y_0 is the root of $\tau(u)$ and let $0<\lambda<\rho^*$. Then for its speed the following equality holds almost surely*

$$\lim_{n\to\infty}\frac{1}{n}\left|Y_n-Y_0\right| = \left(\sum_{v=1}^{N}\frac{g(v)+\lambda}{\mathcal{C}(v)}\overline{q}^G(v)\right)^{-1},$$

where $|Y_n-Y_0|$ denotes the distance of the vertices Y_n and Y_0 on the tree, $g(v):=\sum_w g(v,w)$, and $\overline{q}^G(v)$ is the amount of current flowing through vertices of type v in the stationary state.

In the present paper we confine ourselves to trees without recurrent subtrees. This restriction can be overcome. So e. g. the speed of a random walk on a "Backbone with Dangling Ends" [2] can as well be calculated. A paper on this topic is in preparation.

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2. The Main Theorem

Let $\mathcal{G}$ be a finite, directed, weighted and di-connected graph with vertex-set $V:=\{1,\ldots,N\}$ and weight function $g:\{1,\ldots,N\}\times\{1,\ldots,N\}\to\mathbf{N}_0$, where $g(u,v)=0$ iff (u,v) is not an edge of $\mathcal{G}$. We call vertices with at least one directed edge between them neighbours. Let $g(u):=\sum_v g(u,v)$, $G:=(g(u,v))$ and denote by ρ^* the spectral radius (largest positive eigenvalue) of G. Let $\mathcal{C}:=(\mathcal{C}(1),\ldots,\mathcal{C}(N))^T$ and for $\lambda>0$

$$\Lambda:\mathbf{R_N}\to\mathbf{R_N} \text{ with } \Lambda\left(x_1,\ldots,x_N\right)^T=\left(\frac{x_1}{x_1+\lambda},\ldots,\frac{x_N}{x_N+\lambda}\right)^T,$$

further let $\overleftarrow{G}:=\{(\ldots,u_{-2},u_{-1},u_0):\forall i\in\mathbf{N_0}\ g(u_{-i-1},u_{-i})>0\}$ and denote by S the shift- operator, i. e., $S(\ldots,u_{-2},u_{-1},u_0)=(\ldots,u_{-2},u_{-1})$.

Proposition 2.1. *([4], Theorem 5.1) We have $\rho^*\geq 1$. For $0<\lambda<\rho^*$ equation*

$$G\Lambda(\mathcal{C})=\mathcal{C} \tag{2.1}$$

has a unique strictly positive solution. ($\mathcal{C}(v)>0$ for all $v\in V$ is a consequence of $\mathcal{G}$ being connected.)

The following theorem will be the main tool for the investigation of random walks on trees generated by $\mathcal{G}$.

Theorem 2.2. *Let $0<\lambda<\rho^*$ and $\mathcal{C}$ the unique strictly positive solution of equation 2.1.*
a) A Markov chain with state space V and transition matrix $Q:=(q^G(u,v))$, where

$$q^G(u,v):=\frac{g(u,v)}{\mathcal{C}(u)}\frac{\mathcal{C}(v)}{\mathcal{C}(v)+\lambda},$$

has a unique stationary distribution $\overline{q}^G$, which is a normalized left-eigenvector of Q belonging to the eigenvalue 1, and Q is ergodic with respect to $\overline{q}^G$.
b) A Markov chain with state space $\overleftarrow{G}$ and transition probabilities

$$q(\overleftarrow{u},\overleftarrow{v}) := \begin{cases} \frac{g(u_0,v_0)}{\mathcal{C}(u_0)}\frac{\mathcal{C}(v_0)}{\mathcal{C}(v_0)+\lambda} & \text{if } \overleftarrow{u} = S\overleftarrow{v} \\ 0 & \text{otherwise} \end{cases}$$

has a unique stationary distribution $\overline{q}$, where $\overline{q}\left((\ldots V\times V\times A_{-n}\times\ldots\times A_0)\right) = \int_{A_{-n}}\int_{A_{-n+1}}\cdots\int_{A_0} q^G(u_{-1},d\,u_0)\ldots q^G(u_{-n},d\,u_{-n+1})\overline{q}^G(d\,u_{-n})$ for all $A_{-n},\ldots,$ $A_0\subset V$. Also q is ergodic with respect to $\overline{q}$.
c) The distribution $\overline{p}$, equivalent to $\overline{q}$ with density

$$\frac{d\overline{p}}{d\overline{q}}\left(\overleftarrow{u}\right) = s\,\frac{g(u_0)+\lambda}{\mathcal{C}(u_0)},\ \text{where } s := \left(\sum_{u=1}^{N}\frac{g(u)+\lambda}{\mathcal{C}(u)}\,\overline{q}^G(u)\right)^{-1},$$

is a stationary distribution of a Markov chain with state space $\overleftarrow{G}$ and transition probabilities

$$p(\overleftarrow{u},\overleftarrow{v}) := \begin{cases} \frac{g(u_0,v_0)}{g(u_0)+\lambda} & \text{if } \overleftarrow{u} = S\overleftarrow{v} \text{ and } S\overleftarrow{u}\neq\overleftarrow{v} \\ \frac{\lambda}{g(u_0)+\lambda} & \text{if } \overleftarrow{u}\neq S\overleftarrow{v} \text{ and } S\overleftarrow{u}=\overleftarrow{v} \\ \frac{g(u_0,v_0)+\lambda}{g(u_0)+\lambda} & \text{if } \overleftarrow{u} = S\overleftarrow{v} \text{ and } S\overleftarrow{u}=\overleftarrow{v} \\ 0 & \text{otherwise} \end{cases}.$$

Also p is ergodic with respect to $\overline{p}$. Any p - invariant probability measure absolutely continuous to $\overline{q}$ equals $\overline{p}$.

Proof. a) Because of $\mathcal{G}$ being connected the Markov chain is recurrent, and the assertions hold for any recurrent chain with finite state space.
b) Each realization of a stationary Markov chain with state space $\overleftarrow{G}$ and transition probabilities q is of the form

$$(\ldots,(\ldots,u_{-2},u_{-1}),(\ldots,u_{-2},u_{-1},u_0),(\ldots,u_{-2},u_{-1},u_0,u_1),\ldots),$$

and this sequence uniquely matches the sequence $(\ldots,u_{-2},u_{-1},u_0,u_1,\ldots)$ of successively neighboring elements of V and is a realization of the (uniquely determined) stationary Markov chain with state space V, transition probabilities q^G and stationary distribution $\overline{q}^G$. This proves our claim.
c.a) We first show the p - invariance of $\overline{p}$. Let A be a measurable subset of $\overleftarrow{G}$, $\overleftarrow{u}\in\overleftarrow{G}$, $v\in V$ and denote by $\overleftarrow{u}v := (\ldots,u_{-2},u_{-1},u_0,v)$ and $Av := \{\overleftarrow{u}v : \overleftarrow{u}\in A\}$. Then q - invariance of $\overline{q}$

$$\int \mathbf{1}_A(\overleftarrow{u})\,d\overline{q}(\overleftarrow{u}) = \int \mathbf{1}_A(\overleftarrow{u})\,q(\overleftarrow{w},\overleftarrow{u})\,d\overline{q}(\overleftarrow{w})$$

together with the definition of q implies firstly

$$\begin{aligned}
&\int \mathbf{1}_A(\overleftarrow{u})\,(1+\frac{\lambda}{\mathcal{C}(u_0)})\,d\overline{q}(\overleftarrow{u})\\
=&\int \sum_{u=1}^{N} \mathbf{1}_A(\overleftarrow{w}u)\,(1+\frac{\lambda}{\mathcal{C}(u)})\,\frac{g(w_0,u)}{\mathcal{C}(w_0)}\,\frac{\mathcal{C}(u)}{\mathcal{C}(u)+\lambda}\,d\overline{q}(\overleftarrow{w})\\
=&\sum_{u=1}^{N}\int \mathbf{1}_A(\overleftarrow{w}u)\,\frac{g(w_0,u)}{\mathcal{C}(w_0)}\,d\overline{q}(\overleftarrow{w})
\end{aligned}$$

and consequently by definition of $\overline{p}$

$$\int \mathbf{1}_A(\overleftarrow{u})\,\frac{\lambda}{g(u_0)}\,d\overline{p}(\overleftarrow{u}) = s\sum_{u=1}^{N}\int \mathbf{1}_A(\overleftarrow{w}u)\,\frac{g(w_0,u)}{\mathcal{C}(w_0)}\,d\overline{q}(\overleftarrow{w}) - s\,\overline{q}(A), \tag{2.2}$$

and secondly

$$\begin{aligned}
&\int \mathbf{1}_{Av}(\overleftarrow{w})\,(1+\frac{\lambda}{\mathcal{C}(w_0)})\,d\overline{q}(\overleftarrow{w})\\
=&\int \mathbf{1}_{Av}(\overleftarrow{u}v)\,(1+\frac{\lambda}{\mathcal{C}(v)})\,\frac{g(u_0,v)}{\mathcal{C}(u_0)}\,\frac{\mathcal{C}(v)}{\mathcal{C}(v)+\lambda}\,d\overline{q}(\overleftarrow{u})\\
=&\int \mathbf{1}_A(\overleftarrow{u})\,\frac{g(u_0,v)}{\mathcal{C}(u_0)}\,d\overline{q}(\overleftarrow{u})
\end{aligned}$$

and consequently by definition of $\overline{p}$

$$\int \mathbf{1}_A(\overleftarrow{u})\,\frac{g(u_0,v)}{g(u_0)+\lambda}\,d\overline{p}(\overleftarrow{u}) = s\int \mathbf{1}_{Av}(\overleftarrow{w})\,\frac{\lambda}{\mathcal{C}(w_0)}\,d\overline{q}(\overleftarrow{w}) + s\,\overline{q}(Av). \tag{2.3}$$

Note that q - invariance of $\overline{q}$ also implies

$$\begin{aligned}
\sum_{v=1}^{N}\overline{q}(Av) &= \sum_{v=1}^{N}\int q(\overleftarrow{u},Av)\,d\overline{q}(\overleftarrow{u}) = \int q(\overleftarrow{u},\bigcup_{v=1}^{N} Av)\,d\overline{q}(\overleftarrow{u})\\
&= \int \mathbf{1}_A(\overleftarrow{u})\,d\overline{q}(\overleftarrow{u}) = \overline{q}(A)
\end{aligned}$$

Thus summation of equation (2.2) and equation (2.3) over all $v\in V$ yields the p - invariance of $\overline{p}$:

$$\begin{aligned}
\overline{p}(A) &= \sum_{u=1}^{N}\int \mathbf{1}_A(\overleftarrow{w}u)\,\frac{g(w_0,u)}{g(w_0)+\lambda}\,d\overline{p}(\overleftarrow{w}) + \int \mathbf{1}_A(S\overleftarrow{w})\,\frac{\lambda}{g(w_0)+\lambda}\,d\overline{p}(\overleftarrow{w})\\
&= \int p(\overleftarrow{w},A)\,d\overline{p}(\overleftarrow{w})
\end{aligned}$$

c.b) The concrete formula for s is a consequence of $\overline{p}$ being a (probability) distribution:

$$1=\overline{p}(\overleftarrow{G}) = s\int \frac{g(u_0)+\lambda}{\mathcal{C}(u_0)}\,d\overline{q}(\overleftarrow{u}) = s\sum_{u=1}^{N}\frac{g(u)+\lambda}{\mathcal{C}(u)}\overline{q}^{G}(u)$$

c.c) We prove ergodicity of p with respect to $\overline{p}$: By ergodicity of q with respect to $\overline{q}$ a measurable set A being q - invariant implies $\overline{q}(A)\in\{0,1\}$ and consequently $\overline{p}(A)\in\{0,1\}$ by equivalence of $\overline{p}$ and $\overline{q}$. Thus it is enough to prove that any p -

invariant set is q - invariant. Let A be p - invariant, i.e., ([10], p. 96) $\overline{p}$ almost surely

$$\begin{aligned} \overleftarrow{u} &\in A \Rightarrow \forall v \,\text{with}\, g(u_0, v) > 0, \overleftarrow{u}v \in A \wedge S\overleftarrow{u} \in A, \text{ and} \\ \overleftarrow{u} &\notin A \Rightarrow \forall v \,\text{with}\, g(u_0, v) > 0, \overleftarrow{u}v \notin A \wedge S\overleftarrow{u} \notin A \end{aligned}$$

This implies $\overline{p}$ almost surely (and thus $\overline{q}$ almost surely)

$$q(\overleftarrow{u}, A) = \begin{cases} 1 & \text{if } \overleftarrow{u} \in A \\ 0 & \text{otherwise} \end{cases},$$

i.e., A is q- invariant.
c.d) Let $\overline{p}'$ be p - invariant and absolutely continuous with respect to $\overline{q}$. Then $\overline{p}' \ll \overline{p}$ and p is ergodic with respect to $\overline{p}'$. Denote by $p_j(\overleftarrow{u}, A) := p(\overleftarrow{u}, A)$ for $j = 1$ and inductively $p_j(\overleftarrow{u}, A) := \int p_{j-1}(\overleftarrow{v}, A)\, p(\overleftarrow{u}, d\overleftarrow{v})$ for $j > 1$. Then for each measurable set A ergodicity of p with respect to $\overline{p}$ (respectively $\overline{p}'$) implies ([10], p. 94) $\overline{p}(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} p_j(\overleftarrow{u}, A) = \overline{p}'(A)$, where the left equality is true $\overline{p}$ almost surely (and thus $\overline{p}'$ almost surely) and the right $\overline{p}'$ almost surely. □

Corollary 2.3. *In the setting of Theorem 2.2 the following equation is true:*

$$\sum_{u=1}^{N} \frac{g(u) - \lambda}{\mathcal{C}(u)} \overline{q}^G(u) = 1$$

Proof. $\overline{q}^G$ being a left eigenvector of Q means that for all $v \in V$

$$\sum_{u=1}^{N} \overline{q}^G(u) \frac{g(u, v)}{\mathcal{C}(u)} \frac{\mathcal{C}(v)}{\mathcal{C}(v) + \lambda} = \overline{q}^G(v)$$

This implies

$$\sum_{u=1}^{N} \overline{q}^G(u) \frac{g(u, v)}{\mathcal{C}(u)} = \overline{q}^G(v) + \frac{\lambda}{\mathcal{C}(v)} \overline{q}^G(v)$$

Because of $\overline{q}^G$ being a (probability) distribution we conclude

$$\sum_{u=1}^{N} \frac{g(u) - \lambda}{\mathcal{C}(u)} \overline{q}^G(u) = \sum_{u=1}^{N} \overline{q}^G(u) \sum_{v=1}^{N} \frac{g(u, v)}{\mathcal{C}(u)} - \sum_{v=1}^{N} \frac{\lambda}{\mathcal{C}(v)} \overline{q}^G(v) = \sum_{v=1}^{N} \overline{q}^G(v) = 1$$

□

3. Periodic Trees, Random Walk, Speed

We use the graph $\mathcal{G}$ for the construction of trees. We refer to the vertices of $\mathcal{G}$ as types. Let u be a type. Then $\tau(u)$ is a rooted, labeled tree with the following properties:

- Each vertex has a type.
- Each vertex of type v has $g(v, w)$ neighbours of type w, which we call its successors. Additionally each vertex, except the root, has exactly one predecessor, whose successor it is.
- The root is of type u and is denoted by u.

We call $\tau(u)$ a **periodic tree** with root u (generated by $\mathcal{G}$). We call $\mathcal{G}$ the **generating graph** of $\tau(u)$. For convenience we denote by $\tau(u^*)$ a rooted, labeled tree, which arises from $\tau(u)$ by linking an additional vertex u^* to u and distinguishing this as the root of $\tau(u^*)$.
Note that our definition is slightly different from the definition in [4].

Example 3.1. *A Fibonacci tree has two types of vertices. Each vertex of type* 1 *is followed by one vertex of type* 2 *and each vertex of type* 2 *is followed by one vertex of type* 1 *and one vertex of type* 2*. Note that in this case* $\tau(1)$ *is isomorphic to* $\tau(2^*)$*.*

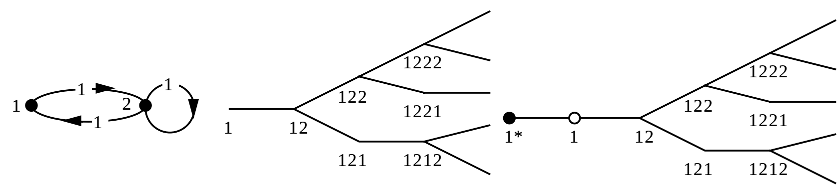

FIGURE 1. Graph and generated (Fibonacci) tree $\tau(1)$, as well as $\tau(1^*)$ with possible labelings

Example 3.2. *A tree with three types of vertices.*

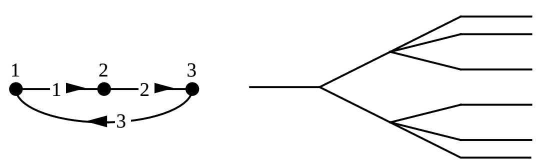

FIGURE 2. Graph and generated tree $\tau(1)$

The types of the successors of a vertex are determined by the type of the vertex itself. This is not true for the type of the predecessor. Different graphs may generate isomorphic trees.

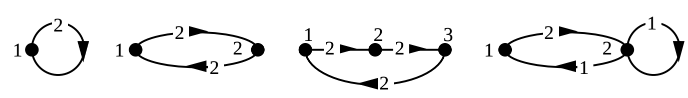

FIGURE 3. Different generating graphs of the binary tree

We consider a **biased random walk** [4] with bias λ on a periodic tree. We always assume that the random walker starts at the root of the tree. Then at each vertex the random walker moves to all successors of the vertex with equal probabilities but to the predecessor (if available) with λ times higher (lower) probability.

In case $\lambda \geq 1$ the λ-biased random walk is also called homesick, then the higher probability of moving to the predecessor balances the in general higher number of successors.
This random walk on $\tau(u)$ may now be **recurrent** or **transient**. For fixed λ this depends on the size of the tree, more precisely on its **branching number** [4] $\mathrm{br}\tau(u)$. By [4] a λ-biased random walk on $\tau(u)$ is recurrent, if $\mathrm{br}\tau(u) < \lambda$ and transient, if $\mathrm{br}\tau(u) > \lambda$. The case $\mathrm{br}\tau(u) = \lambda$ must be checked separately.
The branching number of a periodic tree (generated by $\mathcal{G}$) equals ρ^* (immediate from [4]). A λ-biased random walk with $\lambda = \rho^*$ is recurrent ([5], Theorem 4.3).

Example 3.3. *Fibonacci tree (Example 3.1)* $G = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $\rho^* = (1+\sqrt{5})/2$

Example 3.4. *Tree from Example 3.2* $G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 3 & 0 & 0 \end{pmatrix}$, $\rho^* = \sqrt[3]{6}$

In the present paper we want to calculate the **asymptotic rate of escape (speed)** of a biased random walk on a periodic tree, i. e., the asymptotic ratio of the distance covered by the random walker and the number of moves needed. Only in the case of a transient random walk, where each vertex is visited only finitely many times, we may expect a speed different from 0.
Now let a random walker start at the root u of the periodic tree $\tau(u)$, then after some time he (or she) will leave it forever moving to a successor of type u_1, later this will also be left forever towards a successor of type u_2 and so on. After some time the random walker will find him(her)self at a vertex of type u_n, which has a predecessor of type u_{n-1}, which itself has a predecessor of type u_{n-2}, ..., which finally has a predecessor of type u. Thus, if the random walk is already lasting for an infinite period of time, the tree observed by the random walker from his (her) present position, may be described (up to isomorphism) by a sequence $\overleftarrow{u} := (\dots, u_{-2}, u_{-1}, u_0) \in \overleftarrow{G}$ of types of the successive predecessors. Therefore we define a rooted labeled tree $\tau(\overleftarrow{u})$ with the following properties:

- Each vertex has a type
- Each vertex of type v has $g(v,w)$ neighbours of type w, which we call its successors. Additionally each vertex has exactly one predecessor, whose successor it is.
- The root is of type u_0 and has a predecessor of type u_{-1}, which has a predecessor of type $u_{-2}\dots$. The root indicates the position of the random walker.

We call $\tau(\overleftarrow{u})$ the **periodic tree with root** $\overleftarrow{u}$ generated by $\mathcal{G}$. We call $\mathcal{G}$ the **generating graph** of $\tau(\overleftarrow{u})$.

According to the above considerations the transition probabilities $p(\overleftarrow{u}, \overleftarrow{v})$ of Theorem 2.2 correspond to transition probabilities of an infinitely long lasting and therefore stationary λ-biased random walk. The stationary distribution $\overline{p}$ thus describes the random tree, at whose root the random walker presently finds him(her)self. In particular $\overline{p}^G(u) := (\{\overleftarrow{u} : u_0 = u\})$ is the probability of the event that the random walker is at a vertex of type u. By ergodicity of the Markov chain this equals the frequency of vertices of type u on almost every infinite path covered by the random walker. Note that the sequence of types along the path is not a Markov chain itself.

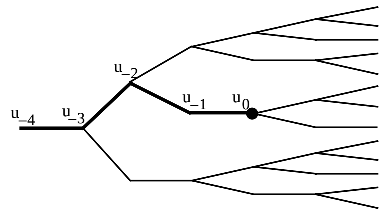

FIGURE 4. Fibonacci tree $\tau(\overleftarrow{u})$ with $\overleftarrow{u} = (\dots, 1, 2, 2, 1, 2)$

This interpretation leads to the following corollary to Theorem 2.2 concerning the speed of a biased random walk on a periodic tree.

Corollary 3.5. *Let $(X_n)_{n\in\mathbf{N}_0}$ be a λ-biased random walk on a random periodic tree $\tau(\overleftarrow{u})$, where the distribution of $\overleftarrow{u}$ is $\overline{p}$ and $X_0 = \overleftarrow{u}$ and let $0 < \lambda < \rho^*$. Then for its speed the following equality holds almost surely*

$$\lim_{n\to\infty} \frac{1}{n} |X_n - X_0| = s$$

where $|X_n - X_0|$ denotes the distance of the vertices X_n and X_0 on the tree and s is defined as in Theorem 2.2.

Proof. For all vertices x, y of a periodic tree $\tau(\overleftarrow{u})$ we define

$$[x, y] := \begin{cases} 1 & \text{if } y \text{ is a successor of } x \\ -1 & \text{if } x \text{ is a successor of } y \\ 0 & \text{otherwise} \end{cases}$$

Because of the transience of the random walk on $\tau(u_0), \tau(u_{-1}), \dots$ almost surely

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} [X_j, X_{j+1}] = \infty$$

Thus almost surely there exists an m such that

$$\sum_{j=0}^{m-1} [X_j, X_{j+1}] = min\{\sum_{j=0}^{n-1} [X_j, X_{j+1}] : n \in \mathbf{N}\}$$

With this we have

$$\sum_{j=0}^{n-1} [X_j, X_{j+1}] \le |X_n - X_0| \le 2\,|X_m - X_0| + \sum_{j=0}^{n-1} [X_j, X_{j+1}]$$

and consequently

$$\lim_{n\to\infty} \frac{1}{n} |X_n - X_0| = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} [X_j, X_{j+1}]$$

Denote by $\theta(X_n)$ the sequence of types of the successive predecessors of X_n, then $(\theta(X_n))_{n\in\mathbf{N}_0}$ equals the Markov chain of Theorem 2.2 c).
For the moment we assume that $\mathcal{G}$ has at least three edges. In this case by the zero-

one law of Hewitt-Savage $\overline{q}$ almost surely X_0 $(= \overleftarrow{u})$ does not have period two. Thus for all $n \in \mathbf{N}$ almost surely $\theta(X_{n+1}) = \theta(X_{n-1}) \Leftrightarrow X_{n+1} = X_{n-1}$ and consequently

$$[\theta(X_n), \theta(X_{n+1})] := [X_n, X_{n+1}] = \begin{cases} 1 & \text{if } S\theta(X_{n+1}) = \theta(X_n) \\ -1 & \text{if } S\theta(X_n) = \theta(X_{n+1}) \end{cases}$$

Birkhoff's ergodic theorem implies that almost surely

$$\lim_{n\to\infty} \frac{1}{n} |X_n - X_0| = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} [\theta(X_n), \theta(X_{n+1})] = \mathbf{E}[\theta(X_0), \theta(X_1)]$$

and by Theorem 2.2 and Corollary 2.3

$$\mathbf{E}[\theta(X_0), \theta(X_1)] = \sum_{u=1}^{N} \frac{g(u) - \lambda}{g(u) + \lambda} \overline{p}^G(u) = \sum_{u=1}^{N} \frac{g(u) - \lambda}{g(u) + \lambda} s \frac{g(u) + \lambda}{\mathcal{C}(u)} \overline{q}^G(u) = s$$

If $\mathcal{G}$ has at most two edges, $\overleftarrow{G}$ has at most two elements. In this case $\overline{p} = \overline{q}$ equals the uniform distribution on $\overleftarrow{G}$. The sequence $([X_n, X_{n+1}] + [X_{n+1}, X_{n+2}])_{n\in\mathbf{N}_0}$ then is a sequence of i.i.d. random variables and by the strong law of large numbers its Cesaro limit almost surely equals its expectation, which is easily calculated as $2s$. □

Corollary 3.6. *Let $(Y_n)_{n\in\mathbf{N}_0}$ be a λ-biased random walk on a periodic tree $\tau(u)$, where $u \in V$ and $Y_0 = u$ and let $0 < \lambda < \rho^*$. Then for its speed the following equality holds almost surely*

$$\lim_{n\to\infty} \frac{1}{n} |Y_n - Y_0| = s$$

where $|Y_n - Y_0|$ denotes the distance of the vertices Y_n and Y_0 on the tree and s is defined as in Theorem 2.2.

Proof. We interpret $\tau(u)$ as a subtree of $\tau(\overleftarrow{u})$ with $u_0 = u$ and $(Y_n)_{n\in\mathbf{N}_0}$ as random walk on it with $Y_0 = \overleftarrow{u}$. Let $(X_n)_{n\in\mathbf{N}_0}$ be a biased random walk on a random periodic tree as in Corollary 3.5. Let A denote the event that the predecessor of X_0 is never visited and for all $k \in \mathbf{N}$ B_k the event that X_0 is visited k-times. Then $\{(X_0)_0 = u\}$, A and B_k occur with positive probability even together and by transience of $(X_n)_{n\in\mathbf{N}_0}$ almost surely $(X_n)_{n\in\mathbf{N}_0} \in \bigcup_{k\in\mathbf{N}} B_k$. On the conditions $\{(X_0)_0 = u\}$, A and B_k the distribution of $(X_n)_{n\in\mathbf{N}_0}$ equals the distribution of $(Y_n)_{n\in\mathbf{N}_0}$ provided that Y_0 is visited k times. Thus the distribution of $(X_n)_{n\in\mathbf{N}_0}$ on conditions $\{(X_0)_0 = u\}$ and A, and the distribution of $(Y_n)_{n\in\mathbf{N}_0}$ are equivalent. This proves our claim. □

4. Electric Networks, Random Walk, Exit

To interpret a tree as an **electric network** we think of its edges as being weighted with **conductances**. We refer to a **random walk** as being **characterized by conductances**, if at each vertex the random walker traverses each adjacent edge with a probability proportional to its conductance. If we choose for all vertices x of $\tau(u)$ the conductance of the edge between x and its predecessor equal to $\lambda^{1-|x-u|}$ - we denote the arising network by $\tau(u, \lambda)$ - the random walk characterized by these conductances is a λ-biased random walk. Let $\tau(u^*, \lambda)$ arise in the same way from $\tau(u^*)$.

Example 4.1. *The electric network $\tau(1,\lambda)$ arising from the Fibonacci tree $\tau(1)$. Note that it is isomorphic to $\tau(2^*,\lambda)$.*

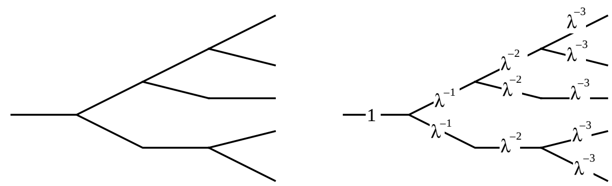

FIGURE 5. Tree and electric network

If a random walk characterized by conductances is started at the root of a tree, it is well known [3] that the **escape probability** equals the ratio of the effective conductance of the network and the sum of the conductances of the edges adjacent to the root. The **effective conductance** is the electric current flowing through the network, if a unit potential is applied between the root and infinity.
We denote the effective conductance of $\tau(u,\lambda)$ by $\mathcal{C}(u)$. Then the conductances $\mathcal{C}(1),\ldots,\mathcal{C}(N)$ are a solution of equation 2.1 by Ohm's and Kirchhoff's laws. This system may be interpreted as fixed-point-equation for the operator $G\,\Lambda$. A straight forward fixed-point-iteration [4] converges for each initial value $\mathcal{C} \geq G\,\mathbf{1}$ towards its unique strictly positive solution. Starting with $G\,\mathbf{1}$, in the n-th iteration conductances $\mathcal{C}_{(n)}(1),\ldots,\mathcal{C}_{(n)}(N)$ of the trees $\tau(1),\ldots,\tau(N)$ grounded in generation $n+1$ are computed.

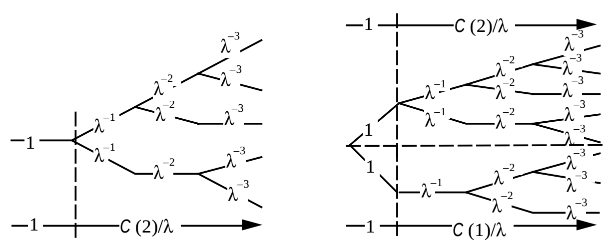

FIGURE 6. Networks $\tau(1,\lambda)$ and $\tau(2,\lambda)$ arising from Fibonacci trees

Example 4.2. *For the Fibonacci tree we demonstrate why the effective conductances are a solution of equation 2.1. Let $0<\lambda<(1+\sqrt{5})/2$. The effective conductance of $\tau(1,\lambda)$ equals a conductance 1 and a conductance $\mathcal{C}(2)/\lambda$ in series. The effective conductance of $\tau(2,\lambda)$ equals two conductances in parallel, where the first equals a conductance 1 and a conductance $\mathcal{C}(2)/\lambda$ in series and the second a conductance 1 and a conductance $\mathcal{C}(1)/\lambda$ in series.*
Thus equation 2.1 is of the form:

$$\begin{aligned}\mathcal{C}(1) &= (1+\lambda/\mathcal{C}(2))^{-1}\\ \mathcal{C}(2) &= (1+\lambda/\mathcal{C}(2))^{-1}+(1+\lambda/\mathcal{C}(1))^{-1}\end{aligned}$$

Its only strictly positive solution is $\mathcal{C}(1)=\sqrt{\lambda+1}-\lambda$ and $\mathcal{C}(2)=1/\sqrt{\lambda+1}+1-\lambda$

Example 4.3. *Continuation of Example 3.2. Let $0 < \lambda < \sqrt[3]{6}$. In this case equation 2.1 is of the form:*

$$\begin{aligned}\mathcal{C}(1) &= (1+\lambda/\mathcal{C}(2))^{-1}\\ \mathcal{C}(2) &= 2(1+\lambda/\mathcal{C}(3))^{-1}\\ \mathcal{C}(3) &= 3(1+\lambda/\mathcal{C}(1))^{-1}\end{aligned}$$

and has the only strictly positive solution

$$\begin{aligned}\mathcal{C}(1) &= (6-\lambda^3)/(6+3\lambda+\lambda^2),\\ \mathcal{C}(2) &= (6-\lambda^3)/(3+\lambda+\lambda^2),\\ \mathcal{C}(3) &= (6-\lambda^3)/(2+2\lambda+\lambda^2).\end{aligned}$$

Let $(X_n)_{n\in\mathbf{N}_0}$ be a biased random walk on a periodic tree $\tau(\overleftarrow{u})$ with $X_0 = \overleftarrow{u}$. If at a time $k \in \mathbf{N_0}$ the random walker moves towards a successor of X_k and never returns to X_k, we say, the event **exit** (E) occurs at time k, i. e., $\{(X_{n+k})_{n\in\mathbf{N}_0} \in E\}$, we call k an **exit-time** and X_k an **exit-vertex**. For every vertex x of a periodic tree we denote by $\theta_0(x)$ the type of x and note that

$$P[(X_{n+k})_{n\in\mathbf{N}_0} \in E \mid \theta_0(X_{k+1}) = v, \theta_0(X_k) = u]$$

equals the probability of the event that a biased random walker starting at the root v^* of $\tau(v^*)$ immediately leaves v^* forever, which equals

$$\frac{\mathcal{C}(v)}{\mathcal{C}(v)+\lambda}$$

by the formula for the escape probability. We then calculate

$$\begin{aligned}&P[(X_{n+k})_{n\in\mathbf{N}_0} \in E \mid \theta_0(X_k) = u]\\ =\ &\sum_{v=1}^{N} P[\theta_0(X_{k+1}) = v \mid \theta_0(X_k) = u]\, P[(X_{n+k})_{n\in\mathbf{N}_0} \in E \mid \theta_0(X_{k+1}) = v, \theta_0(X_k) = u]\\ =\ &\sum_{v=1}^{N} \frac{g(u,v)}{g(u)+\lambda}\frac{\mathcal{C}(v)}{\mathcal{C}(v)+\lambda} = \frac{\mathcal{C}(u)}{g(u)+\lambda},\end{aligned}$$

where the last equality is due to equation 2.1. Therefore conditioned on the event E the random walker being at a vertex of type u, moves to a vertex of type v with the probability

$$\begin{aligned}&P[\theta_0(X_{k+1}) = v \mid \theta_0(X_k) = u, (X_{n+k})_{n\in\mathbf{N}_0} \in E]\\ =\ &\frac{P[(X_{n+k})_{n\in\mathbf{N}_0} \in E \mid \theta_0(X_{k+1}) = v, \theta_0(X_k) = u]\, P[\theta_0(X_{k+1}) = v \mid \theta_0(X_k) = u]}{P[(X_{n+k})_{n\in\mathbf{N}_0} \in E \mid \theta_0(X_k) = u]}\\ =\ &\frac{g(u,v)}{\mathcal{C}(u)}\frac{\mathcal{C}(v)}{\mathcal{C}(v)+\lambda} = q^G(u,v)\end{aligned}$$

Note that an electric current entering a vertex of type u (from its predecessor) is reduced by even this factor towards the current leaving it via vertices of type v. Thus watching the random walker only at exit-times and noting the types of the exit-vertices yields a Markov chain with state space V and transition probabilities q^G. So $\overline{q}^G$ is the distribution of the type of an exit-vertex of an infinitely lasting and thus stationary biased random walk. Analogously $\overline{q}$ describes the random periodic tree, whose root the random walker leaves forever at an exit-time. Because the sequence of the successive exit-vertices equals the infinite ray covered by the random walker, ergodicity of q^G with respect to $\overline{q}^G$ implies that $\overline{q}^G(u)$ also equals the frequency of vertices of type u on almost every such infinite ray.

Example 4.4. *Continuation of Example 3.1:* $\overline{q}^G = (\frac{\sqrt{\lambda+1}-1}{\sqrt{\lambda+1}-1+\lambda}, \frac{\lambda}{\sqrt{\lambda+1}-1+\lambda})$

Example 4.5. *Continuation of Example 3.2: Trivially* $\overline{q}^G = (1/3, 1/3, 1/3)$

Corollary 4.6. *Let* $(X_n)_{n\in\mathbf{N}_0}$ *be a* λ*-biased random walk on a random periodic tree* $\tau(\overleftarrow{u})$*, where the distribution of* $\overleftarrow{u}$ *is* $\overline{p}$ *and* $X_0 = \overleftarrow{u}$ *and let* $0 < \lambda < \rho^*$*. Then for all* $k \in \mathbf{N_0}$

$$P[(X_{n+k})_{n\in\mathbf{N}_0} \in E] = s$$

where s *is defined as in Theorem 2.2.*

Proof. Stationarity of the chain implies

$$\begin{aligned} & P[(X_{n+k})_{n\in\mathbf{N}_0} \in E] \\ = & \sum_{u=1}^{N} P[(X_{n+k})_{n\in\mathbf{N}_0} \in E \mid \theta_0(X_k) = u]\, \overline{p}^G(u) \\ = & s \sum_{u=1}^{N} \frac{\mathcal{C}(u)}{g(u)+\lambda} \frac{g(u)+\lambda}{\mathcal{C}(u)} \overline{q}^G(u) \\ = & s \sum_{u=1}^{N} \overline{q}^G(u) = s \end{aligned}$$

□

Example 4.7. *Continuation of Example 3.1, computation of* $\overline{p}^G$ *and* s*:*

$$\begin{aligned} s^{-1} &= \overline{q}^G(1)(1+\lambda)/\mathcal{C}(1) + \overline{q}^G(2)(2+\lambda)/\mathcal{C}(2) \textit{ implies} \\ s &= \frac{(\sqrt{\lambda+1}-\lambda)(\sqrt{\lambda+1}+2)}{\lambda+1+(\lambda+2)\sqrt{\lambda+1}} \end{aligned}$$

This result was also achieved by [7] *(Ad-hoc method). The frequencies of the different types on the path are almost surely equal to*

$$\begin{aligned} \overline{p}^G(1) &= \frac{\sqrt{\lambda+1}}{\lambda+2+\sqrt{\lambda+1}} \\ \overline{p}^G(2) &= \frac{\lambda+2}{\lambda+2+\sqrt{\lambda+1}} \end{aligned}$$

Example 4.8. *Continuation of Example 3.2:*

$$\begin{aligned} s^{-1} &= \overline{q}^G(1)(1+\lambda)/\mathcal{C}(1) + \overline{q}^G(2)(2+\lambda)/\mathcal{C}(2) + \overline{q}^G(3)(3+\lambda)/\mathcal{C}(3) \textit{ implies} \\ s &= \frac{3(6-\lambda^2)}{3\lambda^3+12\lambda^2+22\lambda+18} \end{aligned}$$

We avoid the lengthy formulas for the quantities $\overline{p}^G$ *but instead only note that*

$$\overline{p}^G(2) < \overline{p}^G(1) \leq \overline{p}^G(3)$$

Corollary 4.9. *In the setting of Theorem 2.2*

$$s = \left(1 + 2\lambda \sum_{u=1}^{N} \frac{1}{\mathcal{C}(u)}\, \overline{q}^G(u)\right)^{-1} = \left(2 \sum_{u=1}^{N} \frac{g(u)}{\mathcal{C}(u)}\, \overline{q}^G(u) - 1\right)^{-1}$$

Proof. Definition of s and Corollary 2.3. □

Remark 4.1. *a) By the Kac lemma* [9] s^{-1} *equals the expected period of time between two exit-times. Thus it can be interpreted as the mean delay, necessary to overcome one generation of the tree. The expectation is taken of the same random walk as considered in Corollary 3.5.*
b) This mean delay is by 1 *lower than twice the mean effective resistance, the reciprocal of the effective conductance, of the network* $\tau(u^*, \lambda)$*, where the mean is taken over all types* u *of exit-vertices. Note that the effective resistance of* $\tau(u^*, \lambda)$ *equals the reciprocal of the escape probability from the root* u^* *as well as the expected number of visits to* u^* [3]*.*
c) The mean delay is by 1 *lower than twice the mean reciprocal of the escape probability from the root* u *of* $\tau(u)$*, where the mean is taken over all types* u *of exit-vertices.*

Remark 4.2. *The interpretation of Remark 4.1 c) is further illuminated by the following heuristic argument based on the first-hitting-time formula of* [12]*:*
Let τ *be a weighted, labeled, finite tree with root* a*. We short its leaves and denote the resulting vertex by* b*. Each edge* (x, y) *is weighted with a conductance* $c(x, y)$ *and* $c(x) := \sum_y c(x, y)$*. For each vertex* x *of* τ *let* $U_{x,\tau}$ *denote the potential between* x *and* b*, if the current from* a *to* b *is* 1*.*
A random walk on τ *characterized by the conductances and started at* a *has an expected hitting time of* b [12] *of* $\mathbf{E}_{a,\tau} T_b = \sum_x c(x)\, U_{x,\tau}$*. Note that this quantity would not be finite for an infinite tree. Let* $a_1, \ldots, a_n$ *be the neighbours of* a *in* τ *and denote by* τ_j *the subtree of* τ *with vertex-set* $\{x : |x - a| > |x - a_j|\}$ *and root* a_j*. Denote the current flowing from* a *to* a_j *by* I_j*, let* $\sum_{j=1}^n I_j = 1$ *and note that* $U_{x,\tau}/U_{x,\tau_j} = I_j$ *for all vertices* x *of* τ_j*, then*

$$\begin{aligned}
\mathbf{E}_{a,\tau} T_b &= \sum_{j=1}^{n} I_j\, \mathbf{E}_{a_j,\tau_j} T_b + \sum_{j=1}^{n} c(a, a_j)\, U_{a_j,\tau} + c(a)\, U_{a,\tau} \\
&= \sum_{j=1}^{n} I_j\, \mathbf{E}_{a_j,\tau_j} T_b + 2c(a)\, U_{a,\tau} - 1,
\end{aligned}$$

where the last equality follows from $U_{a,\tau} = U_{a_j,\tau} + I_j/c(a, a_j)$*. Using Ohm's law with the conductances corresponding to a biased random walk we arrive at* $\mathbf{E}_{a,\tau} T_b = \sum_{j=1}^n I_j\, \mathbf{E}_{a_j,\tau_j} T_b + 2(g(a)/\mathcal{C}(a)) - 1$*. Since* I_j *equals the probability of moving from* a *to* a_j *at an exit-time, the existence of* a *increases the expected hitting time of* b *by the last two terms, which thus correspond to the delay caused by the existence of* a*. In the mean over all types of exit- vertices, we conclude*

$$\text{"}s^{-1}\text{"} = 2 \sum_{u=1}^{N} \frac{g(u)}{\mathcal{C}(u)}\, \overline{q}^G(u) - 1$$

Example 4.10. *Since our previous examples could be solved by use of existing literature, we demonstrate our method for a periodic tree, say* $\tau(2)$*, which is generated*

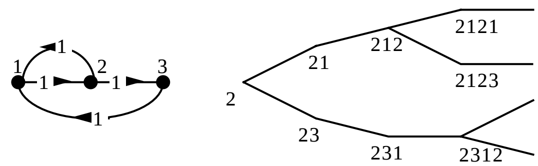

FIGURE 7. Graph $\mathcal{G}$ and the generated tree $\tau(2)$ with a possible labeling

by the graph $\mathcal{G}$ above.

$$G = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \rho^* = \mathrm{br}\tau(2) = \frac{\sqrt[3]{9-\sqrt{69}} + \sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{18}} \cong 1.3247$$

The speed is interesting for $0 < \lambda < \rho^*$. Equation 2.1 is of the form

$$\begin{aligned} \mathcal{C}(1) &= (1+\lambda/\mathcal{C}(2))^{-1} \\ \mathcal{C}(2) &= (1+\lambda/\mathcal{C}(1))^{-1} + (1+\lambda/\mathcal{C}(3))^{-1} \\ \mathcal{C}(3) &= (1+\lambda/\mathcal{C}(1))^{-1} \end{aligned}$$

The system can be solved for general λ but we avoid the lengthy formulas. Thus the following applies to the case $\lambda = 1$. The only strictly positive solution of equation 2.1 is

$$\mathcal{C}(1) = (\sqrt{6}-1)/5, \mathcal{C}(2) = 1/\sqrt{6} \text{ and } \mathcal{C}(3) = (\sqrt{6}-2)/2.$$

The frequencies of the different types along a ray are almost surely equal to

$$\overline{q}^G = (1/\sqrt{6}, 1/\sqrt{6}, 1-2/\sqrt{6}) \cong (0.408, 0.408, 0.184)$$

The frequencies of the different types along the path are almost surely equal to

$$\begin{aligned} \overline{p}^G(1) &= (24-\sqrt{6})/57 \cong 0.378 \\ \overline{p}^G(2) &= (45-9\sqrt{6})/57 \cong 0.403 \\ \overline{p}^G(3) &= (10\sqrt{6}-12)/57 \cong 0.219 \end{aligned}$$

The speed of a simple random walk on $\tau(u)$ is almost surely equal to

$$s = \frac{1}{5+\sqrt{6}}$$

The following figure shows the speed s as a function of λ.

It is now tempting to think s to be a monotone function of λ, but this is not true as an example of [7] with 32 types of vertices shows, which was pointed out to me by Y. Peres.

5. CONCLUSIONS

The method of calculating the stationary distribution $\overline{p}$ of the vertices on the path from the stationary distribution $\overline{q}$ of the exit-vertices does not only work for periodic trees. It can also be used when considering Markov chains on directed unlabeled trees, i.e., trees with a distinguished ray. The advantage of periodic trees is that $\overline{q}$ and thus $\overline{p}$ as well as s can be computed explicitly. For the random

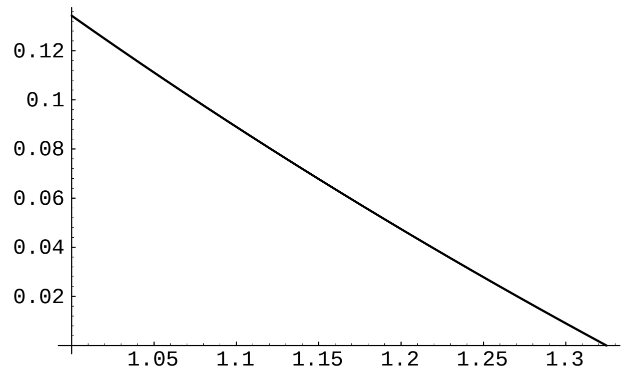

FIGURE 8. Speed of a biased random walk as a function of λ

environment the analogous connection between the two stationary distributions has already been published [1]. If $\mathcal{G}$ is not connected, new phenomena may appear, such as a random walk started at the same vertex having several possible speeds. Our method can be adapted to these more general situations.

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