POPULATION MODELS WITH NONLINEAR BOUNDARY CONDITIONS

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ABSTRACT. We study a two point boundary-value problem describing the steady states of a Logistic growth population model with diffusion and constant yield harvesting. In particular, we focus on a model when a certain nonlinear boundary condition is satisfied.

1. INTRODUCTION

Consider the Logistic growth population dynamics model with nonlinear boundary conditions:

\begin{align*}
    u_t &= d \Delta u + au - bu^2 - ch(x) \quad \text{in } \Omega, \\
    d\alpha(x, u) \frac{\partial u}{\partial \eta} + [1 - \alpha(x, u)]u &= 0 \quad \text{on } \partial \Omega,
\end{align*}

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with \( n \geq 1 \), \( \Delta \) is the Laplace operator, \( d \) is the diffusion coefficient, \( a, b \) are positive parameters, \( c \geq 0 \) is the harvesting parameter, \( h(x) : \overline{\Omega} \to \mathbb{R} \) is a \( C^1 \) function, \( \frac{\partial u}{\partial \eta} \) is the outward normal derivative, and \( \alpha(x, u) : \Omega \times \mathbb{R} \to [0, 1] \) is a nondecreasing \( C^1 \) function.

The parameter \( c \geq 0 \) represents the level of harvesting, \( h(x) \geq 0 \) for \( x \in \Omega \), \( h(x) = 0 \) for \( x \in \partial \Omega \), and \( \|h\|_\infty = 1 \). Here \( ch(x) \) can be understood as the rate of the harvesting distribution. The nonlinear boundary condition (1.2) has only been recently studied by such authors as [1, 2, 3], among others. Here

\[ \alpha(x, u) = \alpha(u) = \frac{u}{u - d \frac{\partial u}{\partial \eta}} \]

represents the fraction of the population that remains on the boundary when reached. For the case when \( \alpha(x, u) \equiv 0 \), (1.2) becomes the well known Dirichlet boundary condition. If \( \alpha(x, u) \equiv 1 \) then (1.2) becomes the Neumann boundary condition. Here we will be interested in the study of positive steady state solutions

\[ \begin{array}{c}
    u_t = d \Delta u + au - bu^2 - ch(x) \\
    d\alpha(x, u) \frac{\partial u}{\partial \eta} + [1 - \alpha(x, u)]u = 0
\end{array} \]

in \( \Omega \), on \( \partial \Omega \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with \( n \geq 1 \), \( \Delta \) is the Laplace operator, \( d \) is the diffusion coefficient, \( a, b \) are positive parameters, \( c \geq 0 \) is the harvesting parameter, \( h(x) : \overline{\Omega} \to \mathbb{R} \) is a \( C^1 \) function, \( \frac{\partial u}{\partial \eta} \) is the outward normal derivative, and \( \alpha(x, u) : \Omega \times \mathbb{R} \to [0, 1] \) is a nondecreasing \( C^1 \) function.

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of $\text{(1.1)}$–$\text{(1.2)}$ when $d = 1$ and
$$
\alpha(x, u) = \frac{u}{u + 1} \quad \text{on } \partial \Omega.
$$

Hence, we consider the model
\begin{align*}
-\Delta u &= au - bu^2 - ch(x) =: f(x, u) \quad \text{in } \Omega, \quad (1.3) \\
\frac{\partial u}{\partial \eta} + 1 &= 0 \quad \text{on } \partial \Omega. \quad (1.4)
\end{align*}

We will present the results of the case when $n = 1$, $\Omega = (0, 1)$, and $h(x) \equiv 1$.

Thus, we study the nonlinear boundary-value problem
\begin{align*}
-\Delta u &= au - bu^2 - c, \quad x \in (0, 1), \quad (1.5) \\
-u'(0) + 1 &= u(0) = 0, \quad (1.6) \\
[u'(1) + 1]u(1) &= 0. \quad (1.7)
\end{align*}

It is easy to see that analyzing the positive solutions of $\text{(1.5)}$–$\text{(1.7)}$ is equivalent to studying the four boundary-value problems
\begin{align*}
-u'' &= au - bu^2 - c, \quad x \in (0, 1), \quad (1.8) \\
u(0) &= 0, \quad u(1) = 0; \quad (1.9) \\
-u'' &= au - bu^2 - c, \quad x \in (0, 1), \quad (1.10) \\
u(0) &= 0, \quad u'(1) = -1; \quad (1.11) \\
-u'' &= au - bu^2 - c, \quad x \in (0, 1), \quad (1.12) \\
u'(0) &= 1, \quad u(1) = 0; \quad (1.13) \\
-u'' &= au - bu^2 - c, \quad x \in (0, 1), \quad (1.14) \\
u'(0) &= 1, \quad u'(1) = -1. \quad (1.15)
\end{align*}

Hence, the positive solutions of these four BVPs are the positive solutions of $\text{(1.5)}$–$\text{(1.7)}$. Notice that if $u(x)$ is a solution of $\text{(1.10)}$–$\text{(1.11)}$ then $v(x) := u(1 - x)$ is a solution of $\text{(1.12)}$–$\text{(1.13)}$. Thus, it suffices to only consider $\text{(1.8)}$–$\text{(1.9)}$, $\text{(1.10)}$–$\text{(1.11)}$, and $\text{(1.14)}$–$\text{(1.15)}$. The structure of positive solutions for $\text{(1.8)}$–$\text{(1.9)}$ is known (see [4] and [7]) via the quadrature method introduced by Laetsch in [8]. We develop quadrature methods in Section 2 to completely determine the bifurcation diagram of $\text{(1.5)}$–$\text{(1.7)}$. In Section 3 we use Mathematica computations to show that for certain subsets of the parameter space, $\text{(1.5)}$–$\text{(1.7)}$ has up to exactly 8 positive solutions. For higher dimensional results, in the case when $\alpha(x, u) = 0$ on $\partial \Omega$ (Dirichlet boundary conditions) see [9], and for the case when $\alpha(x, u) = \frac{u}{u+1}$ on $\partial \Omega$ see recent work in [5].

2. Results via the quadrature method

2.1. Positive solutions of $\text{(1.8)}$–$\text{(1.9)}$. In this section we summarize the known results (see [9]) for positive solutions of $\text{(1.8)}$–$\text{(1.9)}$. Consider the boundary value problem:
\begin{align*}
-u'' &= au - bu^2 - c =: f(u), \quad x \in (0, 1), \quad (2.1) \\
u(0) &= 0, \quad u(1) = 0. \quad (2.2)
\end{align*}
It is easy to see that positive solutions of (2.1)–(2.2) must resemble Figure 1 where \( \ell_i \) for \( i = 1, 2 \) are the positive zeros of \( f(u) \). The following theorem details the structure of positive solutions of (2.1)–(2.2) for the case when \( b = 1 \):

**Theorem 2.1** ([4, 9]).

(1) If \( a < \lambda_1 \) then (2.1)–(2.2) has no positive solution for any \( c \geq 0 \).

(2) If \( \lambda_1 \leq a < \lambda_\ast \) (some \( \lambda_\ast > \lambda_1 \)) then there exists a \( c_0 > 0 \) such that if

- (a) \( 0 \leq c < c_0 \) then (2.1)–(2.2) has 2 positive solutions.
- (b) \( c = c_0 \) then (2.1)–(2.2) has a unique positive solution.
- (c) \( c > c_0 \) then (2.1)–(2.2) has no positive solution.

(3) If \( a > \lambda_\ast \) then there exist \( c_0, \tilde{c} > 0 \) such that if

- (a) \( \tilde{c} < c < c_0 \) then (2.1)–(2.2) has 2 positive solutions.
- (b) \( 0 \leq c < \tilde{c} \) or \( c = c_0 \) then (2.1)–(2.2) has a unique positive solution.
- (c) \( c > c_0 \) then (2.1)–(2.2) has no positive solution.

Figure 2 illustrates this theorem.

**2.2. Positive solutions of** (1.10)–(1.11). In this subsection, we adapt the quadrature method in [8] to study

\[-u'' = au - bu^2 - c =: f(u), \quad x \in (0, 1), \]

\[u(0) = 0, \quad u'(1) = -1.\]  

Now, define \( F(u) = \int_0^u f(s)ds \), the primitive of \( f(u) \). Since (2.3) is an autonomous differential equation, if \( u(x) \) is a positive solution of (2.3) with \( u'(x_0) = 0 \) for some
$x_0 \in (0, 1)$ then $v(x) := u(x_0 - x)$ and $w(x) := u(x_0 + x)$ both satisfy the initial value problem,

$$-z'' = f(z) \quad (2.5)$$
$$z(0) = u(x_0) \quad (2.6)$$
$$z'(0) = 0 \quad (2.7)$$

for all $x \in [0, d)$ where $d = \min\{x_0, 1 - x_0\}$. As a result of Picard’s existence and uniqueness theorem, $u(x_0 - x) \equiv u(x_0 + x)$. Thus, if we assume that $u(x)$ is a positive solution of (2.3)–(2.4) then it is symmetric around $x_0$ with $\rho := \|u\|_{\infty} = u(x_0)$. This implies that $u'(x_0) = 0$, $u'(x) > 0$; $[0, x_0]$, and $u'(x) < 0$; $(x_0, 1]$. Using symmetry about $x_0$, the boundary conditions (2.4), and the sign of $u''$ given by $f(u)$ we see that positive solutions of (2.3)–(2.4) must resemble Figure 3, where $\rho = \|u\|_{\infty}$ and $q = u(1)$. This implies that $\ell_1 < \rho < \ell_2$ and $0 \leq q < \rho$ where $\ell_i$, $i = 1, 2$ are the zeros of $f(u)$.

![Figure 3. Typical solution of (2.3)–(2.4)](image)

Multiplying (2.3) by $u'$ gives

$$-u'u'' = f(u)u' \quad (2.8)$$

Integration of (2.8) with respect to $x$ gives,

$$-\left(\frac{|u'(x)|^2}{2}\right) = [F(u(x))] + K. \quad (2.9)$$

Substituting $x = 1$ and $x = x_0$ into (2.9) yields,

$$-K = F(q) + \frac{1}{2} \quad (2.10)$$
$$K = -F(\rho). \quad (2.11)$$

Combining (2.10) and (2.11), we have

$$F(\rho) = F(q) + \frac{1}{2}. \quad (2.12)$$

Substituting (2.11) into (2.9) yields,

$$-\left(\frac{|u'(x)|^2}{2}\right) = [F(u(x))] - F(\rho). \quad (2.13)$$
Now, solving for $u'$ in (2.13) gives
\[ u'(x) = \sqrt{2} \sqrt{F(\rho) - F(u(x))}, \quad x \in [0, x_0], \] (2.14)
\[ u'(x) = -\sqrt{2} \sqrt{F(\rho) - F(u(x))}, \quad x \in [x_0, 1]. \] (2.15)

Integrating (2.14) and (2.15) with respect to $x$ and using a change of variables, we have
\[ \int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2}x, \quad x \in [0, x_0], \] (2.16)
\[ \int_\rho^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = -\sqrt{2}(x - x_0), \quad x \in [x_0, 1]. \] (2.17)

Substitution of $x = x_0$ into (2.16) and $x = 1$ into (2.17) gives
\[ \int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2}x_0 \] (2.18)
\[ \int_\rho^q \frac{ds}{\sqrt{F(\rho) - F(s)}} = -\sqrt{2}(1 - x_0). \] (2.19)

Finally, subtracting (2.19) from (2.18), yields
\[ 2 \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \int_0^q \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2}, \] (2.20)

or equivalently,
\[ 2 \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \int_0^q \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2}. \] (2.21)

We note that in order for $\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}}$ to be well defined, $F(\rho) > F(s)$ for all $s \in [0, \rho)$. Moreover, the improper integral is convergent if $f(\rho) > 0$. Thus, for such a positive solution to exist, $f(u)$ and $F(u)$ must resemble Figure 4, where $\mu_1$, $\ell_i$, and $\theta_i$ are the zeros of $f'(u)$, $f(u)$, and $F(u)$ respectively for $i = 1, 2$.

![Figure 4. Graph of $f(u)$ (left), and of $F(u)$ (right) (image)](image)

From Figure 4, we note that if $\rho \in (\theta_1, \ell_2)$ then both of these conditions hold and the integrals in (2.21) are well defined. From this and letting $c_1 := \frac{3\sigma^2}{105}$ and $c_2 := \frac{\sigma^2}{3}$, we can arrive at the following result.

**Theorem 2.2.** If $c > c^* (a, b)$ then (2.3)–(2.4) has no positive solution, where $c^* (a, b) = \min \{c_1, c_2\} = \frac{3\sigma^2}{105}$. 
Further, since \( x_0 \in (0, 1) \) is fixed for each \( \rho > 0 \), we need a unique \( q < \rho \) corresponding to each \( \rho \)-value such that (2.12) is satisfied. Otherwise, uniqueness of solutions to the initial value problem, (2.5)–(2.7), would be violated. Let
\[
H(x) := F(x) + \frac{1}{2}.
\]
It follows that \( H'(x) = -bx^2 + ax - c, \) \( H(0) = 1/2, \) and \( H'(0) = -c < 0. \) In order for a unique \( q < \rho \) to exist such that \( H(q) = F(\rho), \) \( H(x) \) must have the following structure in Figure 5, where \( H'(\ell_2) = 0. \) So, for such a unique \( q < \rho \) to exist \( F(\rho) > 1/2. \)

![Figure 5. Graph of \( H(x) \)](chart.png)

Since \( \rho \in (\theta_1, \ell_2), \) for this to be true we will need \( H(\ell_2) > 1/2. \) In fact, if
\[
F(\ell_2) > \frac{1}{2} \tag{2.22}
\]
then clearly for \( \rho \in (\theta_1, \ell_2) \) with \( \rho \approx \ell_2 \) we have \( F(\rho) > 1/2. \) It is easy to see that (2.22) will be satisfied if (solving using Mathematica)
\[
c < c_3 := \frac{9a^2}{144b} - \frac{9(a^4 - 96ab^2)}{144b \left( -a^6 - 240a^3b^2 + 16(72b^4 + 3\sqrt{b^2(a^3 + 12b^2)^3}) \right)^{1/3}} - \frac{9}{144b} \left( -a^6 - 240a^3b^2 + 16(72b^4 + 3\sqrt{b^2(a^3 + 12b^2)^3}) \right)
\]
and for \( c_3 \) to be positive (again using Mathematica)
\[
a > a_0 := \sqrt[3]{3b^2}
\]
both hold. This leads to the following results.

**Theorem 2.3.** If \( a \leq a_0 \) then (2.3)–(2.4) has no positive solution for any \( c \geq 0. \)

**Theorem 2.4.** If \( a > a_0 \) then there is a \( c^* (a, b) \leq \min\{c_1, c_2, c_3\} \) such that for \( c \geq c^* (2.3)–(2.4) \) has no positive solution.

We now state and prove the main theorem of this subsection.

**Theorem 2.5.** If \( a > a_0 \) and \( c < c^* (a, b) \) then there is a unique \( r(a, b) \in (\theta_1, \ell_2) \) such that \( F(r) = 1/2 \) and
\[
G(\rho) := 2 \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \int_0^q \frac{ds}{\sqrt{F(\rho) - F(s)}}
\]
is well defined for all \( \rho \in [r, \ell_2) \) where \( q < \rho \) is the unique solution of \( F(\rho) = H(q) \).
Moreover, \( [2.3] - [2.4] \) has a positive solution, \( u(x) \), with \( \rho = \|u\|_\infty \) if and only if \( G(\rho) = \sqrt{2} \) for some \( \rho \in [r, \ell_2) \).

Proof. Let \( a, b > 0 \) s.t. \( a > a_0 \) and \( c \in [0, c^*(a, b)) \). From the preceding discussion, it follows that if \( u \) is a positive solution to \( [2.3] - [2.4] \) with \( \rho = \|u\|_\infty \) then \( G(\rho) = \sqrt{2} \).

Next, suppose \( G(\rho) = \sqrt{2} \) for some \( \rho \in [r, \ell_2) \). Define \( u(x) : (0, 1) \to \mathbb{R} \) by

\[
\int_0^u \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2} x, \quad x \in [0, x_0],
\]

\[
\int_\rho^u \frac{ds}{\sqrt{F(\rho) - F(s)}} = -\sqrt{2}(x - x_0), \quad x \in [x_0, 1].
\]

Now, we show that \( u(x) \) is a positive solution to \( [2.3] - [2.4] \). It is easy to see that the turning point is given by \( x_0 = \frac{1}{\sqrt{2}} \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} \). The function, \( \int_0^u \frac{ds}{\sqrt{F(\rho) - F(s)}} \), is a differentiable function of \( u \) which is strictly increasing from 0 to \( x_0 \) as \( u \) increases from 0 to \( \rho \). Thus, for each \( x \in [0, x_0] \), there is a unique \( u(x) \) such that

\[
\int_0^u \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2} x
\]

Moreover, by the Implicit Function theorem, \( u \) is differentiable with respect to \( x \). Differentiating \( [2.25] \) gives

\[
u'(x) = \sqrt{2}[F(\rho) - F(u)], \quad x \in [0, x_0].
\]

Similarly, \( u \) is a decreasing function of \( x \) for \( x \in [x_0, 1] \) which yields,

\[
v'(x) = -\sqrt{2}[F(\rho) - F(u)], \quad x \in [x_0, 1].
\]

This implies

\[
\frac{-(u')^2}{2} = F(\rho) - F(u(x)).
\]

Differentiating again, we have \( -u''(x) = f(u(x)) \). Thus, \( u(x) \) satisfies \( [2.3] \). Now, from our assumption, \( G(\rho) = \sqrt{2} \), it follows that \( u(0) = 0 \) and \( u(1) = q(\rho) \). Since \( F(\rho) = H(q(\rho)) = F(q) + \frac{1}{4} \), we have that \( u''(1) = -\sqrt{2}[F(\rho) - F(q)] = -1 \). Hence, the boundary conditions \( [2.4] \) are both satisfied.

2.3. Positive solutions of \( [1.14] - [1.15] \). A similar quadrature method can be adapted to study

\[
-u'' = au - bu^2 - c =: f(u), \quad x \in (0, 1),
\]

\[
u'(0) = 1, \quad u'(1) = -1.
\]

Again, define \( F(u) = \int_0^u f(s)ds \), the primitive of \( f(u) \). Using a similar argument as before, symmetry about \( x_0 \), the boundary conditions \( [2.26] - [2.27] \), and the sign of \( u'' \) given by \( f(u) \) ensure that positive solutions of \( [2.26] - [2.27] \) must resemble Figure 6, where \( \rho = \|u\|_\infty \) and \( q = u(0) = u(1) \). Clearly, \( x_0 = 1/2 \) in this case.

Through an almost identical approach as the one in Section 2.2, we can prove the following results.

Theorem 2.6. If \( a \leq a_0 \) then \( [2.26] - [2.27] \) has no positive solution for any \( c \geq 0 \).

Theorem 2.7. If \( a > a_0 \) then there is a \( c^*(a, b) \leq \min\{c_1, c_2, c_3\} \) such that for \( c \geq c^* \) \( [2.26] - [2.27] \) has no positive solution.
We now state the main theorem of this subsection.

**Theorem 2.8.** If \( a > a_0 \) and \( c < c^*(a, b) \) then there is a unique \( r(a, b, c) \in (\theta, \ell_2) \) such that \( F(r) = \frac{1}{2} \) and

\[
G(\rho) := 2 \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - 2 \int_0^q \frac{ds}{\sqrt{F(\rho) - F(s)}}
\]

is well defined for all \( \rho \in [r, \ell_2) \) where \( q < \rho \) is the unique solution of \( F(\rho) = H(q) \).

Moreover, \((2.26) - (2.27)\) has a positive solution, \( u(x) \), with \( \rho = \|u\|_\infty \) if and only if \( G(\rho) = \sqrt{2} \) for some \( \rho \in [r, \ell_2) \).

**Remark.** See [7] where Ladner et al. adapted the quadrature method to study the case when \( \alpha(x, u) = \frac{u}{a} \) on \( \partial \Omega \). Also, see [6] where the quadrature method was adapted to study the case with a Strong Allee effect and \( \alpha(x, u) = \frac{u}{b} \) on \( \partial \Omega \).

3. **Computational results**

3.1. **Positive solutions of \((1.10) - (1.11)\) and \((1.12) - (1.13)\).** We are particularly interested in the case when \( b = 1 \). From Theorem 2.5, we plot the level sets of

\[
G(\rho) - \sqrt{2} = 0
\]

for \( a > \sqrt[3]{3} \) and \( \rho \in [r, \ell_2) \). By implementing a numerical root-finding algorithm in Mathematica we were able to solve equation \((3.1)\). Explicit formulas were used to calculate the unique \( r = r(a, b, c) \) and \( q = q(\rho) \) values. Note that these computations are expensive due to the natural of the improper integral equations involved. Figures 7-9 depict several level sets plotted within \([r, \ell_2) \times [0, c^*]\). In what follows, the green curve represents \( \rho \) vs \( c \) while the upper and lower branches of the dotted black curve represent \( \ell_2 \) and \( r \), respectively. The green curve’s lower branch begins to shrink for \( a \geq 10.1388 \). This is due to the fact that solutions of \((3.1)\) are outside of \([r, \ell_2) \). The bifurcation diagrams also indicate the following results.

**Theorem 3.1.** For \( b = 1 \), if \( a < a_4 \) (for \( a_4 \approx 5.0407 \)) then \((1.10) - (1.11)\) and \((1.12) - (1.13)\) have no positive solution for any \( c \geq 0 \).

**Theorem 3.2.** If \( b = 1 \) then \( c_0(a) \to c^*(a) \) as \( a \to \infty \). Furthermore, \( \rho \to \ell_2 \) as \( a \to \infty \) where \( u(x) \) is a positive solution to \((1.10) - (1.11)\) or \((1.12) - (1.13)\) with \( \|u\|_\infty = \rho \).
3.2. Positive solutions of (1.14)–(1.15). Again, we are particularly interested in the case when \( b = 1 \). Recalling Theorem 2.8, we plot the level sets of

\[
\tilde{G}(\rho) - \sqrt{2} = 0
\]

Using our numerical root-finding algorithm in Mathematica to solve equation (3.2) and explicit formulas to calculate the unique \( r = r(a, b, c) \) and \( q = q(\rho) \) values, level sets were plotted within \([r, \ell_2) \times [0, c^*]\). The blue curve breaks into two components somewhere around \( a = 4.39 \), with the lower component vanishing for \( a > 10.1387 \). This is due to the fact that the \( \rho \)-values, which are solutions of (3.2), are outside of \([r, \ell_2)\). These bifurcation diagrams also indicate the following results.

**Theorem 3.3.** For \( b = 1 \), if \( a < a_1 \) (for \( a_1 \approx 2.8324 \)) then (1.14)–(1.15) has no positive solution for any \( c \geq 0 \).

**Theorem 3.4.** If \( b = 1 \) then \( c_0(a) \to c^*(a) \) as \( a \to \infty \). Furthermore, \( \rho \to \ell_2 \) as \( a \to \infty \) where \( u(x) \) is a positive solution to (1.14)–(1.15) with \( \|u\|_{\infty} = \rho \).

3.3. Structure of Positive solutions to (1.5)–(1.7). Combining results from the three cases, (1.8)–(1.9), (1.10)–(1.11), and (1.14)–(1.15) while recalling that the
Theorem 3.5. If \( a \leq \min\left[\sqrt{3b^2}, \lambda_1\right] \) then (1.5)–(1.7) has no positive solution for any \( c \geq 0 \).

Moreover, our computational results for the case \( b = 1 \) provide the following nonexistence result.

Theorem 3.6. For \( b = 1 \), if \( a < a_1 \) (for \( a_1 \approx 2.8324 \)) then (1.5)–(1.7) has no positive solution for any \( c \geq 0 \).

Also, our computations indicate the following existence results for \( b = 1 \). For what follows, (1.8)–(1.9) is depicted in yellow, (1.10)–(1.11) and (1.12)–(1.13) both in green, and (1.14)–(1.15) in blue.
Theorem 3.7. For \( b = 1 \), if \( a \in [a_1, a_2) \) (for some \( a_2 > a_1 \)) (for \( a_2 \approx 4.39 \)) then there exists a \( C_0 > 0 \) such that if

1. \( 0 \leq c < C_0 \) then \((1.5) - (1.7)\) has exactly 2 positive solutions.
2. \( c = C_0 \) then \((1.5) - (1.7)\) has a unique positive solution.
3. \( c > C_0 \) then \((1.5) - (1.7)\) has no positive solution.

A bifurcation diagram of the case when \( b = 1 \) and \( a = 4 \) is shown in Figure 13.

Theorem 3.8. For \( b = 1 \), if \( a \in [a_2, a_3) \) (some \( a_3 \in (4.4, 5) \)) then there exist \( C_i > 0 \), \( i = 0, 1, 2 \), such that if

1. \( 0 \leq c \leq C_2 \) or \( C_1 \leq c < C_0 \) then \((1.5) - (1.7)\) has exactly 2 positive solutions.
2. \( C_2 < c < C_1 \) or \( c = C_0 \) then \((1.5) - (1.7)\) has a unique positive solution.
3. \( c > C_0 \) then \((1.5) - (1.7)\) has no positive solution.

Figure 14 illustrates Theorem 3.8.

Theorem 3.9. For \( b = 1 \), if \( a \in [a_3, a_4) \) (for \( a_4 \approx 5.0407 \)) then there exist \( C_i > 0 \), \( i = 0, 1, 2 \), such that if

1. \( 0 \leq c \leq C_1 \) then \((1.5) - (1.7)\) has exactly 2 positive solutions.
2. \( C_1 < c \leq C_0 \) then \((1.5) - (1.7)\) has a unique positive solution.
3. \( c > C_0 \) then \((1.5) - (1.7)\) has no positive solution.

Theorem 3.9 is illustrated in Figure 15.

Theorem 3.10. For \( b = 1 \), if \( a \in [a_4, a_5) \) (for \( a_5 = \pi^2 \)) then there exist \( C_i > 0 \), \( i = 0, 1, 2 \), such that if
Theorem 3.10 is depicted in Figure 16.

Theorem 3.11. For $b = 1$, if $a \in [a_5, a_6)$ (some $a_6 \in (10, 10.1388)$) then there exist $C_i > 0$, $i = 0, 1, 2, 3$, such that if

1. $0 \leq c \leq C_3$ then $(1.5) - (1.7)$ has exactly 8 positive solutions.
2. $c = C_3$ then $(1.5) - (1.7)$ has exactly 7 positive solutions.
3. $C_3 < c \leq C_2$ then $(1.5) - (1.7)$ has exactly 6 positive solutions.
4. $C_2 < c < C_1$ then $(1.5) - (1.7)$ has exactly 5 positive solutions.
5. $c = C_1$ then $(1.5) - (1.7)$ has exactly 3 positive solutions.
6. $C_1 < c \leq C_0$ then $(1.5) - (1.7)$ has a unique positive solution.
7. $c > C_0$ then $(1.5) - (1.7)$ has no positive solution.

Figure 17 shows the bifurcation diagram for $a = 10$, $b = 1$ along with Figure 18 which gives two small cross sections of the diagram.

Theorem 3.12. For $b = 1$, if $a \in [a_6, a_7)$ (for $a_7 \approx 10.1388$) then there exist $C_i > 0$, $i = 0, 1, 2, 3$, such that if

1. $0 \leq c \leq C_3$ then $(1.5) - (1.7)$ has exactly 8 positive solutions.
Figure 17. $\rho$ vs $c$ for $a = 10$, $b = 1$

Figure 18. $\rho$ vs $c$ cross-sections for $a = 10$, $b = 1$

1. $C_3 < c < C_2$ then $(1.5) - (1.7)$ has exactly 7 positive solutions.
2. $C_2 < c < C_1$ then $(1.5) - (1.7)$ has exactly 6 positive solutions.
3. $C_1 < c <= C_2$ then $(1.5) - (1.7)$ has exactly 5 positive solutions.
4. $C_1 < c < C_0$ then $(1.5) - (1.7)$ has a unique positive solution.
5. $c = C_1$ then $(1.5) - (1.7)$ has exactly 3 positive solutions.
6. $c > C_0$ then $(1.5) - (1.7)$ has no positive solution.

The bifurcation diagram for $a = 10.1, b = 1$ is depicted in Figures 19 and 20.

Figure 19. $\rho$ vs $c$ for $a = 10.1$, $b = 1$

Theorem 3.13. For $b = 1$, if $a \in [a_7, a_8]$ (for $a_8 = 4\pi^2$) then there exist $C_i > 0$, $i = 0, 1, 2, 3$, such that if
Figure 20. $\rho$ vs $c$ cross-sections for $a = 10.1$, $b = 1$

1. $0 \leq c < C_3$ or $C_2 \leq c < C_1$ then (1.5)–(1.7) has exactly 5 positive solutions.
2. $c = C_3$ then (1.5)–(1.7) has exactly 4 positive solutions.
3. $C_3 < c < C_2$ or $c = C_1$ then (1.5)–(1.7) has exactly 3 positive solutions.
4. $C_1 < c \leq C_0$ then (1.5)–(1.7) has a unique positive solution.
5. $c > C_0$ then (1.5)–(1.7) has no positive solution.

Figure 21 shows the bifurcation diagram for $a = 11$, $b = 1$.

Theorem 3.14. For $b = 1$, if $a \in (a_8, \infty)$ then there exist $C_i > 0$, $i = 0, 1, 2, 3$, such that if

1. $C_3 \leq c < C_2$ then (1.5)–(1.7) has exactly 5 positive solutions.
2. $0 \leq c < C_3$ or $c = C_3$ then (1.5)–(1.7) has exactly 4 positive solutions.
3. $C_2 < c \leq C_1$ then (1.5)–(1.7) has exactly 3 positive solutions.
4. $C_1 < c \leq C_0$ then (1.5)–(1.7) has a unique positive solution.
5. $c > C_0$ then (1.5)–(1.7) has no positive solution.

The bifurcation diagram for $a = 40$, $b = 1$ is shown in Figure 22

References

Figure 22. $\rho$ vs $c$ for $a = 40$, $b = 1$


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