Strongly nonlinear parabolic initial-boundary value problems in Orlicz spaces *

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Abstract

We prove existence and convergence theorems for nonlinear parabolic problems. We also prove some compactness results in inhomogeneous Orlicz-Sobolev spaces.

1 Introduction

Let Ω be a bounded domain in \( \mathbb{R}^N \), \( T > 0 \) and let

\[
A(u) = \sum_{|\alpha| \leq 1} (-1)^{|\alpha|} D^\alpha A_\alpha(x, t, u, \nabla u)
\]

be a Leray-Lions operator defined on \( L^p(0, T; W^{1,p}(\Omega)) \), \( 1 < p < \infty \). Boccardo and Murat [5] proved the existence of solutions for parabolic initial-boundary value problems of the form

\[
\frac{\partial u}{\partial t} + A(u) + g(x, t, u, \nabla u) = f \quad \text{in } \Omega \times (0, T),
\]

(1.1)

where \( g \) is a nonlinearity with the following growth condition

\[
g(x, t, s, \xi) \leq b(|s|) (c(x, t) + |\xi|^q), \quad q < p,
\]

(1.2)

and which satisfies the classical sign condition \( g(x, t, s, \xi)s \geq 0 \). The right hand side \( f \) is assumed (in [5]) to belong to \( L^p(0, T; W^{-1,p}(\Omega)) \). This result generalizes the analogous one of Landes-Mustonen [14] where the nonlinearity \( g \) depends only on \( x, t \) and \( u \). In [5] and [14], the functions \( A_\alpha \) are assumed to satisfy a polynomial growth condition with respect to \( u \) and \( \nabla u \). When trying to relax this restriction on the coefficients \( A_\alpha \), we are led to replace \( L^p(0, T; W^{1,p}(\Omega)) \) by an inhomogeneous Sobolev space \( W^{1,\phi} L_M \) built from an Orlicz space \( L_M \) instead of \( L^p \), where the N-function \( M \) which defines \( L_M \) is related to the actual growth of the \( A_\alpha \)'s. The solvability of (1.1) in this

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setting is proved by Donaldson [7] and Robert [16] in the case where \( g \equiv 0 \). It is our purpose in this paper, to prove existence theorems in the setting of the inhomogeneous Sobolev space \( W^{1,1}L_M \) by applying some new compactness results in Orlicz spaces obtained under the assumption that the N-function \( M(t) \) satisfies \( \Delta' \)-condition and which grows less rapidly than \( |t|^{N/(N-1)} \). These compactness results, which we are first established in [8], generalize those of Simon [17], Landes-Mustonen [14] and Boccardo-Murat [6]. It is not clear whether the present approach can be further adapted to obtain the same results for general N-functions.

For related topics in the elliptic case, the reader is referred to [2] and [3].

2 Preliminaries

Let \( M : \mathbb{R}^+ \to \mathbb{R}^+ \) be an N-function, i.e. \( M \) is continuous, convex, with \( M(t) > 0 \) for \( t > 0 \), \( \frac{M(t)}{t} \to 0 \) as \( t \to 0 \) and \( \frac{M(t)}{t} \to \infty \) as \( t \to \infty \). Equivalently, \( M \) admits the representation: \( M(t) = \int_0^t a(\tau) d\tau \) where \( a : \mathbb{R}^+ \to \mathbb{R}^+ \) is non-decreasing, right continuous, with \( a(0) = 0 \), \( a(t) > 0 \) for \( t > 0 \) and \( a(t) \to \infty \) as \( t \to \infty \). The N-function \( \overline{M} \) conjugate to \( M \) is defined by \( \overline{M}(t) = \int_0^t \overline{a}(\tau) d\tau \), where \( \overline{a} : \mathbb{R}^+ \to \mathbb{R}^+ \) is given by \( \overline{a}(t) = \sup\{s : a(s) \leq t\} \) [1, 11, 12].

The N-function \( M \) is said to satisfy the \( \Delta_2 \)-condition if, for some \( k > 0 \):

\[
M(2t) \leq k M(t) \quad \text{for all } t \geq 0, \tag{2.1}
\]

when this inequality holds only for \( t \geq t_0 > 0 \), \( M \) is said to satisfy the \( \Delta_2 \)-condition near infinity.

Let \( P \) and \( Q \) be two N-functions. \( P \ll Q \) means that \( P \) grows essentially less rapidly than \( Q \); i.e., for each \( \varepsilon > 0 \),

\[
\frac{P(t)}{Q(\varepsilon t)} \to 0 \quad \text{as } t \to \infty.
\]

This is the case if and only if

\[
\lim_{t \to \infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0.
\]

An N-function is said to satisfy the \( \Delta' \)-condition if, for some \( k_0 > 0 \) and some \( t_0 \geq 0 \):

\[
M(k_0 t t') \leq M(t) M(t'), \quad \text{for all } t, t' \geq t_0. \tag{2.2}
\]

It is easy to see that the \( \Delta' \)-condition is stronger than the \( \Delta_2 \)-condition. The following N-functions satisfy the \( \Delta' \)-condition: \( M(t) = t^p (\log t)^q \), where \( 1 < p < +\infty \), \( 0 \leq s < +\infty \) and \( q \geq 0 \) is an integer (\( \log \) being the iterated of order \( q \) of the function \( \log \)).

We will extend these N-functions into even functions on all \( \mathbb{R} \). Let \( \Omega \) be an open subset of \( \mathbb{R}^N \). The Orlicz class \( \mathcal{L}_M(\Omega) \) (resp. the Orlicz space \( L_M(\Omega) \)) is
defined as the set of (equivalence classes of) real-valued measurable functions \( u \) on \( \Omega \) such that:

\[
\int_\Omega M(u(x))\,dx < +\infty \quad \text{(resp. } \int_\Omega M\left(\frac{u(x)}{\lambda}\right)\,dx < +\infty \text{ for some } \lambda > 0).\]

Note that \( L_M(\Omega) \) is a Banach space under the norm

\[
\|u\|_{M,\Omega} = \inf \left\{ \lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right)\,dx \leq 1 \right\}
\]

and \( L_M(\Omega) \) is a convex subset of \( L_M(\Omega) \). The closure in \( L_M(\Omega) \) of the set of bounded measurable functions with compact support in \( \Omega \) is denoted by \( E_M(\Omega) \). The equality \( E_M(\Omega) = L_M(\Omega) \) holds if and only if \( M \) satisfies the \( \Delta_2 \) condition, for all \( t \) or for \( t \) large according to whether \( \Omega \) has infinite measure or not.

The dual of \( E_M(\Omega) \) can be identified with \( L_{1,M}(\Omega) \) by means of the pairing

\[
\int_\Omega u(x)v(x)\,dx,
\]

and the dual norm on \( L_{1,M}(\Omega) \) is equivalent to \( \|\cdot\|_{M,\Omega} \). The space \( L_M(\Omega) \) is reflexive if and only if \( M \) and \( \overline{M} \) satisfy the \( \Delta_2 \) condition, for all \( t \) or for \( t \) large, according to whether \( \Omega \) has infinite measure or not.

We now turn to the Orlicz-Sobolev space. \( W^{1,L_M}(\Omega) \) (resp. \( W^{1,E_M}(\Omega) \)) is the space of all functions \( u \) and its distributional derivatives up to order 1 lie in \( L_M(\Omega) \) (resp. \( E_M(\Omega) \)). This is a Banach space under the norm

\[
\|u\|_{1,M,\Omega} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_{M,\Omega}.
\]

Thus \( W^{1,L_M}(\Omega) \) and \( W^{1,E_M}(\Omega) \) can be identified with subspaces of the product of \( N+1 \) copies of \( L_M(\Omega) \). Denoting this product by \( \Pi L_M \), we will use the weak topologies \( \sigma(\Pi L_M,\Pi L_{1,M}) \) and \( \sigma(\Pi L_M,\Pi L_{1,M}) \).

Let \( W^{1,L_M}(\Omega) \) (resp. \( W^{1,E_M}(\Omega) \)) denote the space of distributions on \( \Omega \) which can be written as sums of derivatives of order \( \leq 1 \) of functions in \( L_{1,M}(\Omega) \) (resp. \( E_{1,M}(\Omega) \)). It is a Banach space under the usual quotient norm.

If the open set \( \Omega \) has the segment property, then the space \( D(\Omega) \) is dense in \( W^{1,L_M}(\Omega) \) for the modular convergence and for the topology \( \sigma(\Pi L_M,\Pi L_{1,M}) \) (cf. [9, 10]). Consequently, the action of a distribution in \( W^{-1,L_{1,M}}(\Omega) \) on an element of \( W^{1,L_M}(\Omega) \) is well defined.

For \( k > 0 \), we define the truncation at height \( k \), \( T_k : \mathbb{R} \to \mathbb{R} \) by

\[
T_k(s) = \begin{cases} 
s & \text{if } |s| \leq k \\
k s / |s| & \text{if } |s| > k. \end{cases}
\]

The following abstract lemmas will be applied to the truncation operators.
Lemma 2.1 Let \( F : \mathbb{R} \rightarrow \mathbb{R} \) be uniformly lipschitzian, with \( F(0) = 0 \). Let \( M \) be an \( N \)-function and let \( u \in W^1L_M(\Omega) \) (resp. \( W^1E_M(\Omega) \)). Then \( F(u) \in W^1L_M(\Omega) \) (resp. \( W^1E_M(\Omega) \)). Moreover, if the set of discontinuity points of \( F' \) is finite, then

\[
\frac{\partial}{\partial x_i} F(u) = \begin{cases} 
F'(u) \frac{\partial u}{\partial x_i} & \text{a.e. in } \{ x \in \Omega : u(x) \notin D \} \\
0 & \text{a.e. in } \{ x \in \Omega : u(x) \in D \}.
\end{cases}
\]

Proof By the previous lemma, \( F(u) \in W^1L_M(\Omega) \) for all \( u \in W^1L_M(\Omega) \) and

\[
\|F(u)\|_{1,M,\Omega} \leq C \|u\|_{1,M,\Omega},
\]

which gives easily the result. \( \square \)

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \), \( T > 0 \) and set \( Q = \Omega \times [0,T] \). Let \( m \geq 1 \) be an integer and let \( M \) be an \( N \)-function. For each \( \alpha \in \mathbb{N}^N \), denote by \( D^\alpha_x \) the distributional derivative on \( Q \) of order \( \alpha \) with respect to the variable \( x \in \mathbb{R}^N \). The inhomogeneous Orlicz-Sobolev spaces are defined as follows

\[
W^mL_M(Q) = \{ u \in L_M(Q) : D^\alpha_x u \in L_M(Q) \forall |\alpha| \leq m \}
\]

\[
W^mE_M(Q) = \{ u \in E_M(Q) : D^\alpha_x u \in E_M(Q) \forall |\alpha| \leq m \}
\]

The last space is a subspace of the first one, and both are Banach spaces under the norm

\[
\|u\| = \sum_{|\alpha| \leq m} \|D^\alpha_x u\|_{M,Q}.
\]

We can easily show that they form a complementary system when \( \Omega \) satisfies the segment property. These spaces are considered as subspaces of the product space \( \Pi L_M(Q) \) which have as many copies as there is \( \alpha \)-order derivatives, \( |\alpha| \leq m \). We shall also consider the weak topologies \( \sigma(\Pi L_M, \Pi E^\Omega) \) and \( \sigma(\Pi L_M, \Pi L^\Omega) \). If \( u \in W^mL_M(Q) \) then the function : \( t \mapsto u(t) = u(t,) \) is defined on \([0,T]\) with values in \( W^mL_M(\Omega) \). If, further, \( u \in W^mE_M(Q) \) then the concerned function is a \( W^mE_M(\Omega) \)-valued and is strongly measurable. Furthermore the following imbedding holds: \( W^mE_M(Q) \subset L^1(0,T;W^mE_M(\Omega)) \). The space \( W^mL_M(Q) \) is not in general separable, if \( u \in W^mL_M(Q) \), we cannot conclude that the function \( u(t) \) is measurable on \([0,T]\). However, the scalar function \( t \mapsto \|u(t)\|_{M,\Omega} \) is in \( L^1(0,T) \). The space \( W^mE_M(Q) \) is defined as the (norm) closure in \( W^mE_M(Q) \) of \( D(Q) \). We can easily show as in [10] that when \( \Omega \) has the segment property then each element \( u \) of the closure of \( D(Q) \) with respect of the weak * topology \( \sigma(\Pi L_M, \Pi E^\Omega) \) is limit, in \( W^mL_M(Q) \), of some subsequence \( (u_i) \subset D(Q) \) for the modular convergence; i.e., there exists \( \lambda > 0 \) such
that for all $|\alpha| \leq m$,

$$
\int_M \frac{D_2^\alpha u_i - D_2^\alpha u}{\lambda} \, dx \, dt \to 0 \text{ as } i \to \infty,
$$

this implies that $(u_i)$ converges to $u$ in $W^{m,x}L_M(Q)$ for the weak topology $\sigma(\Pi L_M, \Pi L_M^\infty)$. Consequently

$$
\frac{D}{D(\lambda)} \sigma(\Pi L_M, \Pi L_M^\infty) = \frac{D}{D(Q)} \sigma(\Pi L_M, \Pi L_M^\infty),
$$

this space will be denoted by $W^{m,x}_0L_M(Q)$. Furthermore, $W^{m,x}E_M(Q) = W^{m,x}_0L_M(Q) \cap \Pi E_M$. Poincaré’s inequality also holds in $W^{m,x}_0L_M(Q)$ i.e. there is a constant $C > 0$ such that for all $u \in W^{m,x}_0L_M(Q)$ one has

$$
\sum_{|\alpha| \leq m} \|D_2^\alpha u\|_{M,Q} \leq C \sum_{|\alpha| = m} \|D_2^\alpha u\|_{M,Q}.
$$

Thus both sides of the last inequality are equivalent norms on $W^{m,x}_0L_M(Q)$. We have then the following complementary system

$$
\begin{pmatrix}
W^{m,x}_0L_M(Q) \\
W^{m,x}_0E_M(Q)
\end{pmatrix} = \begin{pmatrix}
F \\
F_0
\end{pmatrix},
$$

$F$ being the dual space of $W^{m,x}_0E_M(Q)$. It is also, except for an isomorphism, the quotient of $\Pi L_M^\infty$ by the polar set $W^{m,x}_0E_M(Q) \perp$, and will be denoted by $F = W^{-m,x}L_M(Q)$ and it is shown that

$$
W^{-m,x}L_M(Q) = \left\{ f = \sum_{|\alpha| \leq m} D_2^\alpha f_\alpha : f_\alpha \in L_M(Q) \right\}.
$$

This space will be equipped with the usual quotient norm

$$
\|f\| = \inf \sum_{|\alpha| \leq m} \|f_\alpha\|_{M,Q}
$$

where the infimum is taken on all possible decompositions

$$
f = \sum_{|\alpha| \leq m} D_2^\alpha f_\alpha, \quad f_\alpha \in L_M(Q).
$$

The space $F_0$ is then given by

$$
F_0 = \left\{ f = \sum_{|\alpha| \leq m} D_2^\alpha f_\alpha : f_\alpha \in E_M(Q) \right\}
$$

and is denoted by $F_0 = W^{-m,x}E_M(Q)$.

**Remark 2.3** We can easily check, using [10, lemma 4.4], that each uniformly lipschitzian mapping $F$, with $F(0) = 0$, acts in inhomogeneous Orlicz-Sobolev spaces of order 1: $W^{1,x}L_M(Q)$ and $W^{1,x}_0L_M(Q)$. 
3 Galerkin solutions

In this section we shall define and state existence theorems of Galerkin solutions for some parabolic initial-boundary problem.

Let $\Omega$ be a bounded subset of $\mathbb{R}^N$, $T > 0$ and set $Q = \Omega \times [0, T]$. Let

$$A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha_x(A_\alpha(u))$$

be an operator such that

$$A_\alpha(x, t, \xi) : \Omega \times [0, T] \times \mathbb{R}^N_0 \rightarrow \mathbb{R} \text{ is continuous in } (t, \xi), \text{ for a.e. } x \in \Omega$$

and measurable in $x$, for all $(t, \xi) \in [0, T] \times \mathbb{R}^N_0$, where, $N_0$ is the number of all $\alpha$-order’s derivative, $|\alpha| \leq m$. (3.1)

$$|A_\alpha(x, s, \xi)| \leq \chi(x) \Phi(|\xi|) \text{ with } \chi(x) \in L^1(\Omega) \text{ and } \Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ increasing.} \quad (3.2)$$

$$\sum_{|\alpha| \leq m} A_\alpha(x, t, \xi) \xi_\alpha \geq -d(x, t) \text{ with } d(x, t) \in L^1(Q), \ d \geq 0. \quad (3.3)$$

Consider a function $\psi \in L^2(Q)$ and a function $\pi \in L^2(\Omega) \cap W^{m,1}_0(\Omega)$. We choose an orthonormal sequence $(\omega_i) \subset D(\Omega)$ with respect to the Hilbert space $L^2(\Omega)$ such that the closure of $(\omega_i)$ in $C^m(\Omega)$ contains $D(\Omega)$. $C^m(\Omega)$ being the space of functions which are $m$ times continuously differentiable on $\Omega$. For $V_n = \text{span} \langle \omega_1, \ldots, \omega_n \rangle$ and

$$\|u\|_{C^{1,m}(Q)} = \sup \{|D^\alpha_x u(x, t)|, \frac{\partial u}{\partial t}(x, t) : |\alpha| \leq m, (x, t) \in Q\}$$

we have

$$D(Q) \subset \left( \bigcup_{n=1}^\infty C^1([0, T], V_n) \right)^{C^{1,m}(Q)}$$

this implies that for $\psi$ and $\pi$, there exist two sequences $(\psi_n)$ and $(\pi_n)$ such that

$$\psi_n \in C^1([0, T], V_n), \ \psi_n \rightarrow \psi \text{ in } L^2(Q). \quad (3.4)$$

$$\pi_n \in V_n, \ \pi_n \rightarrow \pi \text{ in } L^2(\Omega) \cap W^{m,1}_0(\Omega). \quad (3.5)$$

Consider the parabolic initial-boundary value problem

$$\frac{\partial u}{\partial t} + A(u) = \psi \text{ in } Q,$$

$$D^\alpha_x u = 0 \text{ on } \partial\Omega \times [0, T], \text{ for all } |\alpha| \leq m - 1,$$

$$u(0) = \pi \text{ in } \Omega. \quad (3.6)$$

In the sequel we denote $A_\alpha(x, t, u, \nabla u, \ldots, \nabla^m u)$ by $A_\alpha(x, t, u)$ or simply by $A_\alpha(u)$. 
Definition 3.1 A function $u_n \in C^1([0,T], V_n)$ is called Galerkin solution of (3.6) if
\[
\int_\Omega \frac{\partial u_n}{\partial t} \varphi dx + \int_\Omega \sum_{|\alpha| \leq m} A_\alpha(u_n).D_\alpha^n \varphi dx = \int_\Omega \psi_n(t) \varphi dx
\]
for all $\varphi \in V_n$ and all $t \in [0,T]$; $u_n(0) = \tau_n$.

We have the following existence theorem.

Theorem 3.2 ([13]) Under conditions (3.1)-(3.3), there exists at least one Galerkin solution of (3.6).

Consider now the case of a more general operator
\[
A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D_\alpha^n (A_\alpha(u))
\]
where instead of (3.1) and (3.2) we only assume that
\[
A_\alpha(x, t, \xi) : \Omega \times [0,T] \times \mathbb{R}^{N_0} \to \mathbb{R}
\]
is continuous in $\xi$, for a.e. $(x, t) \in Q$
and measurable in $(x, t)$ for all $\xi \in \mathbb{R}^{N_0}$.
(3.7)

We have also the following existence theorem

Theorem 3.3 ([14]) There exists a function $u_n$ in $C([0,T], V_n)$ such that $\frac{\partial u_n}{\partial t}$ is in $L^1(0,T; V_n)$ and
\[
\int_{Q_\tau} \frac{\partial u_n}{\partial t} \varphi dx dt + \int_{Q_\tau} \sum_{|\alpha| \leq m} A_\alpha(x, t, u_n).D_\alpha^n \varphi dx dt = \int_{Q_\tau} \psi_n(t) \varphi dx dt
\]
for all $\tau \in [0,T]$ and all $\varphi \in C([0,T], V_n)$, where $Q_\tau = \Omega \times [0,\tau]$; $u_n(0) = \tau_n$.

4 Strong convergence of truncations

In this section we shall prove a convergence theorem for parabolic problems which allows us to deal with approximate equations of some parabolic initial-boundary problem in Orlicz spaces (see section 6). Let $\Omega$, be a bounded subset of $\mathbb{R}^N$ with the segment property and let $T > 0$, $Q = \Omega \times [0,T]$. Let $M$ be an N-function satisfying a $\Delta'$ condition and the growth condition
\[
M(t) \ll |t|^\frac{N}{N-1}
\]
and let $P$ be an N-function such that $P \ll M$. Let $A : W^{1,p} L_M(Q) \to W^{-1,p} L_M(Q)$ be a mapping given by
\[
A(u) = - \text{div} a(x, t, u, \nabla u)
\]

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where \( a(x, t, s, \xi) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \) is a Carathéodory function satisfying for a.e. \((x, t) \in \Omega \times [0, T]\) and for all \(s \in \mathbb{R}\) and all \(\xi, \xi^* \in \mathbb{R}^N:\)

\[
|a(x, t, s, \xi)| \leq c(x, t) + k_1 p^{-1} M(k_2 |s|) + k_3 p^{-1} M(k_4 |\xi|) \tag{4.1}
\]

\[
|a(x, t, s, \xi) - a(x, t, s, \xi^*)| |\xi - \xi^*| > 0 \quad \text{if} \ \xi \neq \xi^* \tag{4.2}
\]

\[
\alpha M(\frac{|\xi|}{\lambda}) - d(x, t) \leq a(x, t, s, \xi) \xi \tag{4.3}
\]

where \(c(x, t) \in E_{\overline{\mathcal{P}}}(Q),\) \(c \geq 0,\) \(d(x, t) \in L^1(Q),\) \(k_1, k_2, k_3, k_4 \in \mathbb{R}^+\) and \(\alpha, \lambda \in \mathbb{R}^*_+.\) Consider the nonlinear parabolic equations

\[
\frac{\partial u_n}{\partial t} - \text{div} a(x, t, u_n, \nabla u_n) = f_n + g_n \text{ in } \mathcal{D}'(Q) \tag{4.4}
\]

and assume that:

\[
u_n \rightharpoonup u \text{ weakly in } W^{1, \infty} L_M(Q) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{\mathcal{P}}}), \tag{4.5}
\]

\[
f_n \rightharpoonup f \text{ strongly in } W^{-1, \infty} E_{\overline{\mathcal{P}}}(Q), \tag{4.6}
\]

\[
g_n \rightharpoonup g \text{ weakly in } L^1(Q). \tag{4.7}
\]

We shall prove the following convergence theorem.

**Theorem 4.1** Assume that (4.1)-(4.7) hold. Then, for any \(k > 0,\) the truncation of \(u_n\) at height \(k\) (see (2.3) for the definition of the truncation) satisfies

\[
\nabla T_k(u_n) \rightharpoonup \nabla T_k(u) \text{ strongly in } (L^1_{\text{loc}}(Q))^N. \tag{4.8}
\]

**Remark 4.2** An elliptic analogous theorem is proved in Benkirane-Elmahi [2].

**Remark 4.3** Convergence (4.8) allows, in particular, to extract a subsequence \(n'\) such that:

\[
\nabla u_{n'} \rightharpoonup \nabla u \text{ a.e. in } Q.
\]

Then by lemma 4.4 of [9], we deduce that

\[
a(x, t, u_{n'}, \nabla u_{n'}) \rightharpoonup a(x, t, u, \nabla u) \text{ weakly in } L^1_{\overline{\mathcal{P}}}(Q))^N \text{ for } \sigma(\Pi L_{\overline{\mathcal{P}}}, \Pi E_M).
\]

**Proof of Theorem 4.1** Step 1: For each \(k > 0,\) define \(S_k(s) = \int_0^s T_k(\tau) d\tau.\) Since \(T_k\) is continuous, for all \(w \in W^{1, \infty} L_M(Q)\) we have \(S_k(w) \in W^{1, \infty} L_M(Q)\) and \(\nabla S_k(w) = T_k(w) \nabla w.\) So that, by mollifying as in [6], it is easy to see that for all \(\varphi \in \mathcal{D}(Q)\) and all \(v \in W^{1, \infty} L_M(Q)\) with \(\frac{\partial v}{\partial t} \in W^{-1, \infty} L_{\overline{\mathcal{P}}}(Q) + L^1(Q),\) we have

\[
\left( \frac{\partial v}{\partial t}, \varphi T_k(v) \right) = - \int_Q \frac{\partial \varphi}{\partial t} S_k(v) dx dt. \tag{4.9}
\]

where \(\left( \cdot, \cdot \right)\) means for the duality pairing between \(W^{1, \infty} L_M(Q) + L^1(Q)\) and \(W^{-1, \infty} L_{\overline{\mathcal{P}}}(Q) \cap L^\infty(Q).\) Fix now a compact set \(K\) with \(K \subset Q\) and a function
Step 2: Fix a real number \( r > 0 \) and set \( Q_r = \{ x \in Q : |\nabla T_k(u)| \leq r \} \) and
denote by $\chi_r$ the characteristic function of $Q_{(r)}$. Taking $s \geq r$ one has:

$$0 \leq \int_{Q_{(r)}} \varphi_K [a(x, t, u_n, \nabla T_k(u_n)) - a(x, t, u_n, \nabla T_k(u))]$$

\[
\times [\nabla T_k(u_n) - \nabla T_k(u)] \, dx \, dt
\]

$$\leq \int_{Q_{(r)}} \varphi_K [a(x, t, u_n, \nabla T_k(u_n)) - a(x, t, u_n, \nabla T_k(u))]$$

\[
\times [\nabla T_k(u_n) - \nabla T_k(u)] \, dx \, dt
\]

$$= \int_{Q_{(r)}} \varphi_K [a(x, t, u_n, \nabla T_k(u_n)) - a(x, t, u_n, \nabla T_k(u)\chi_s)]$$

\[
\times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] \, dx \, dt
\]

$$\leq \int_{Q} \varphi_K [a(x, t, u_n, \nabla T_k(u_n)) - a(x, t, u_n, \nabla T_k(u)\chi_s)]$$

\[
\times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] \, dx \, dt
\]

$$= \int_{Q} \varphi_K a(x, t, u_n, \nabla u_n) [\nabla T_k(u_n) - \nabla T_k(u)] \, dx \, dt$$

$$- \int_{Q} \varphi_K [a(x, t, u_n, \nabla u_n) - a(x, t, u_n, \nabla T_k(u_n)))]$$

\[
\times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] \, dx \, dt
\]

$$+ \int_{Q} \varphi_K a(x, t, u_n, \nabla u_n) [\nabla T_k(u) - \nabla T_k(u)\chi_s] \, dx \, dt$$

$$- \int_{Q} \varphi_K a(x, t, u_n, \nabla T_k(u)\chi_s) [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] \, dx \, dt.$$

Now pass to the limit in all terms of the right-hand side of the above equation.

By (4.11), the first one tends to 0. Denoting by $\chi_{G_n}$ the characteristic function of $G_n = \{(x, t) \in Q : |u_n(x, t)| > k\}$, the second term reads

$$\int_{Q} \varphi_K [a(x, t, u_n, \nabla u_n) - a(x, t, u_n, 0)] \chi_{G_n} \nabla T_k(u)\chi_s \, dx \, dt$$

which tends to 0 since $[a(x, t, u_n, \nabla u_n) - a(x, t, u_n, 0)]$ is bounded in $(L_{\infty}(Q))^N$, by (4.1) and (4.5) while $\chi_{G_n} \nabla T_k(u)\chi_s$ converges strongly in $(E_M(Q))^N$ to 0 by Lebesgue’s theorem. The fourth term of (4.12) tends to

$$- \int_{Q} \varphi_K a(x, t, u_n, \nabla T_k(u)\chi_s) [\nabla T_k(u) - \nabla T_k(u)\chi_s] \, dx \, dt$$

$$= \int_{Q_{(r)}} \varphi_K a(x, t, u, 0) \nabla T_k(u) \, dx \, dt$$

(4.14)

since $a(x, t, u_n, \nabla T_k(u)\chi_s)$ tends strongly to $a(x, t, u, \nabla T_k(u)\chi_s)$ in $(E_{\infty}(Q))^N$ while $\nabla T_k(u_n) - \nabla T_k(u)\chi_s$ converges weakly to $\nabla T_k(u) - \nabla T_k(u)\chi_s$ in $(L_M(Q))^N$ for $\sigma(\Pi L_M, \Pi E_{\infty})$. 

Strongly nonlinear parabolic initial-boundary value problems
Since \( a(x, t, u_n, \nabla u_n) \) is bounded in \( (L^\infty(Q))^N \) one has (for a subsequence still denoted by \( u_n \))
\[
a(x, t, u_n, \nabla u_n) \rightharpoonup h \quad \text{weakly in} \quad (L^\infty(Q))^N \quad \text{for} \quad \sigma(\Pi L^\infty, \Pi E_M).
\]
(4.15)

Finally, the third term of the right-hand side of (4.12) tends to
\[
\int_{Q \cap \{r\}} \varphi_K h \nabla T_k(u) \, dx \, dt.
\]
(4.16)

We have, then, proved that
\[
0 \leq \limsup_{n \to \infty} \int_{Q\setminus\{r\}} \varphi_K \left[ a(x, t, u_n, \nabla T_k(u_n)) - a(x, t, u_n, \nabla T_k(u)) \right]
\times \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] \, dx \, dt
\]
\[
\leq \int_{Q\setminus\{r\}} \varphi_K [h - a(x, t, u, 0)] \nabla T_k(u) \, dx \, dt.
\]
(4.17)

Using the fact that \([h - a(x, t, u, 0)] \nabla T_k(u) \in L^1(\Omega)\) and letting \( s \to +\infty \) we get, since \( |Q\setminus\{r\}| \to 0 \),
\[
\int_{Q\setminus\{r\}} \varphi_K \left[ a(x, t, u_n, \nabla T_k(u_n)) - a(x, t, u_n, \nabla T_k(u)) \right] \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] \, dx \, dt
\]
\[
\to 0 \quad \text{as} \quad n \to \infty.
\]
(4.18)

which approaches 0 as \( n \to \infty \). Consequently
\[
\int_{Q\setminus\{r\} \cap K} \left[ a(x, t, u_n, \nabla T_k(u_n)) - a(x, t, u_n, \nabla T_k(u)) \right] \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] \, dx \, dt 
\to 0
\]
as \( n \to \infty \). As in [2], we deduce that for some subsequence \( \nabla T_k(u_n) \rightharpoonup \nabla T_k(u) \) a.e. in \( Q\setminus\{r\} \cap K \). Since \( r, k \) and \( K \) are arbitrary, we can construct a subsequence (diagonal in \( r \), in \( k \) and in \( j \), where \( (K_j) \) is an increasing sequence of compacts sets covering \( Q \)), such that
\[
\nabla u_n \rightharpoonup \nabla u \quad \text{a.e. in} \quad Q.
\]
(4.19)

**Step 3:** As in [2] we deduce that
\[
\int_Q \varphi_K a(x, t, u_n, \nabla u_n) \nabla T_k(u) \, dx \, dt \to \int_Q \varphi_K a(x, t, u, \nabla u) \nabla T_k(u) \, dx \, dt
\]
as \( n \to \infty \), and that
\[
a(x, t, u_n, \nabla T_k(u_n)) \nabla T_k(u_n) \to a(x, t, u, \nabla T_k(u)) \nabla T_k(u) \quad \text{strongly in} \quad L^1(K).
\]
(4.20)

This implies that (see [2] if necessary): \( \nabla T_k(u_n) \to \nabla T_k(u) \) in \( (L_M(K))^N \) for the modular convergence and so strongly and convergence (4.8) follows.

Note that in convergence (4.8) the whole sequence (and not only for a subsequence) converges since the limit \( \nabla T_k(u) \) does not depend on the subsequence.
5 Nonlinear parabolic problems

Now, we are able to establish an existence theorem for a nonlinear parabolic initial-boundary value problems. This result which specially applies in Orlicz spaces generalizes analogous results in of Landes-Mustonen [14]. We start by giving the statement of the result.

Let \( \Omega \) be a bounded subset of \( \mathbb{R}^N \) with the segment property, \( T > 0 \), and \( Q = \Omega \times [0,T] \). Let \( M \) be an N-function satisfying the growth condition

\[
M(t) \ll |t|^\frac{N}{N-1},
\]

and the \( \triangle' \)-condition. Let \( P \) be an N-function such that \( P \ll M \). Consider an operator \( A : W^{1,x}_0 L_M(Q) \rightarrow W^{-1,x} L_M^\infty(Q) \) of the form

\[
A(u) = -\text{div} \ a(x,t,u,\nabla u) + a_0(x,t,u,\nabla u) (5.1)
\]

where \( a : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) and \( a_0 : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) are Carathéodory functions satisfying the following conditions, for a.e. \( (x,t) \in \Omega \times [0,T] \) for all \( s \in \mathbb{R} \) and \( \xi \neq \xi^* \in \mathbb{R}^N \):

\[
|a(x,t,s,\xi)| \leq c(x,t) + k_1 M^{-1}(k_2 |s|) + k_3 M^{-1}(k_4 |\xi|), \quad [a(x,t,s,\xi) - a(x,t,s,\xi^*)] |\xi - \xi^*| > 0, \quad a(x,t,s,\xi) + a_0(x,t,s,\xi) s \geq \alpha M(\frac{|\xi|}{\lambda}) - d(x,t) (5.2)
\]

where \( c(x,t) \in E_M^\infty(Q) \), \( c \geq 0 \), \( d(x,t) \in L^1(Q) \), \( k_1, k_2, k_3, k_4 \in \mathbb{R}^+ \) and \( \alpha, \lambda \in \mathbb{R}^+ \). Furthermore let

\[
f \in W^{-1,x} E_M^\infty(Q) (5.5)
\]

We shall use notations of section 3. Consider, then, the parabolic initial-boundary value problem

\[
\begin{align*}
\frac{\partial u}{\partial t} + A(u) &= f \quad \text{in } Q \\
u(x,t) &= 0 \quad \text{on } \partial \Omega \times [0,T] \\
u(x,0) &= \psi(x) \quad \text{in } \Omega.
\end{align*}
\]

where \( \psi \) is a given function in \( L^2(\Omega) \). We shall prove the following existence theorem.

**Theorem 5.1** Assume that (5.2)-(5.5) hold. Then there exists at least one weak solution \( u \in W^{1,x}_0 L_M(Q) \cap L^2(Q) \cap C([0,T],L^2(\Omega)) \) of (5.6), in the following sense:

\[
- \int_Q u \frac{\partial \varphi}{\partial t} \, dx \, dt + \int_Q u(t)\varphi(t)dx \biggr|_0^T + \int_Q a(x,t,u,\nabla u) \nabla \varphi \, dx \, dt \tag{5.7}
\]
for all $\varphi \in C^1([0,T], L^2(\Omega))$.

**Remark 5.2** In (5.6), we have $u \in W^{1,\infty}_0 L_M(Q) \subset L^1(0,T; W^{-1,1}(\Omega))$ and $\partial u \over \partial t \in W^{-1,1} L_{M}^{\infty}(Q) \subset L^1(0,T; W^{-1,1}(\Omega))$. Then $u \in W^{1,1}(0,T; W^{-1,1}(\Omega)) \subset C([0,T], W^{-1,1}(\Omega))$ with continuity of the imbedding. Consequently $u$ is, possibly after modification on a set of zero measure, continuous from $[0,T]$ into $W^{-1,1}(\Omega)$ in such a way that the third component of (5.6), which is the initial condition, has a sense.

**Proof of Theorem 4.1** It is easily adapted from the proof given in [14].

For convenience we suppose that $\psi = 0$. For each $n$, there exists at least one solution $u_n$ of the following problem (see Theorem 3.3 for the existence of $u_n$):

$$u_n \in C([0,T], V_n), \quad \frac{\partial u_n}{\partial t} \in L^1(0,T; V_n), \quad u_n(0) = \psi_n \equiv 0 \quad \text{and,}$$

for all $\tau \in [0,T]$,  

$$\int_{Q_\tau} \frac{\partial u_n}{\partial t} \varphi dx + \int_{Q_\tau} a(x,t,u_n,\nabla u_n) \cdot \nabla \varphi dx \; dt$$

$$+ \int_{Q_\tau} a_0(x,t,u_n,\nabla u_n) \cdot \varphi dx dt = \int_{Q_\tau} f_n \varphi dx dt, \quad \forall \varphi \in C([0,T], V_n).$$

where $f_k \subset \bigcup_{n=1}^{\infty} C([0,T], V_n)$ with $f_k \rightarrow f$ in $W^{-1,1}{E_{\infty}^\Delta} (Q)$. Putting $\varphi = u_n$ in (5.8), and using (5.2) and (5.4) yields

$$\|u_n\|_{W^{1,\infty}_0 L_M(Q)} \leq C, \quad \|u_n\|_{L^{\infty}(0,T; L^2(\Omega))} \leq C$$

$$\|a_0(x,t,u_n,\nabla u_n)\|_{L_{E_{\infty}^\Delta}(Q)} \leq C \quad \text{and} \quad \|a(x,t,u_n,\nabla u_n)\|_{L_{E_{\infty}^\Delta}(Q)} \leq C.$$

Hence, for a subsequence

$$u_n \rightarrow u \quad \text{weakly in} \quad W^{1,\infty}_0 L_M(Q) \quad \text{for} \quad \sigma(\Pi L_M, \Pi E_{\infty}^\Delta) \quad \text{and weakly in} \quad L^2(Q),$$

$$a_0(x,t,u_n,\nabla u_n) \rightharpoonup h_0, \quad a(x,t,u_n,\nabla u_n) \rightharpoonup h \quad \text{in} \quad L_{E_{\infty}^\Delta}(Q) \quad \text{for} \quad \sigma(\Pi L_{E_{\infty}^\Delta}, \Pi E_{M})$$

(5.10)

where $h_0 \in L_{E_{\infty}^\Delta}(Q)$ and $h \in (L_{E_{\infty}^\Delta}(Q))$. As in [14], we get that for some subsequence $u_n(x,t) \rightharpoonup u(x,t)$ a.e. in $Q$ (it suffices to apply Theorem 3.9 instead of Proposition 1 of [14]). Also we obtain

$$-\int_{Q} \frac{\partial \varphi}{\partial t} dx \; dt + \int_{Q_\tau} u(t) \varphi(t) dx \; dt + \int_{Q} h \nabla \varphi dx \; dt + \int_{Q} h_0 \varphi dx \; dt = \langle f, \varphi \rangle,$$

for all $\varphi \in C^1((0,T]; D(\Omega))$. The proof will be completed, if we can show that

$$\int_{Q} (h \nabla \varphi + h_0 \varphi) dx \; dt = \int_{Q} (a(x,t,u,\nabla u) \nabla \varphi + a_0(x,t,u,\nabla u) \varphi) dx \; dt$$

(5.11)

for all $\varphi \in C^1((0,T]; D(\Omega))$ and that $u \in C([0,T], L^2(\Omega))$. For that, it suffices to show that

$$\lim_{n \rightarrow \infty} \int_{Q} (a(x,t,u_n,\nabla u_n) [\nabla u_n - \nabla u] + a_0(x,t,u_n,\nabla u_n) [u_n - u]) \; dx \; dt \leq 0.$$
Indeed, suppose that (5.12) holds and let \( s > r > 0 \) and set \( Q^r = \{(x, t) \in Q : |\nabla u(x, t)| \leq r \} \). Denoting by \( \chi_s \) the characteristic function of \( Q^r \), one has

\[
0 \leq \int_{Q^r} [a(x, t, u_n, \nabla u_n) - a(x, t, u_n, \nabla u)] [\nabla u_n - \nabla u] \, dx \, dt
\]

\[
\leq \int_{Q^r} [a(x, t, u_n, \nabla u_n) - a(x, t, u_n, \nabla u)] [\nabla u_n - \nabla u] \, dx \, dt
\]

\[
= \int_{Q^r} [a(x, t, u_n, \nabla u_n) - a(x, t, u_n, \nabla u, \chi_s)] [\nabla u_n - \nabla u, \chi_s] \, dx \, dt
\]

\[
\leq \int_{Q} a_0(x, t, u_n, \nabla u_n) (u_n - u) - \int_{Q} a_0(x, t, u_n, \nabla u_n, \chi_s) [\nabla u_n - \nabla u, \chi_s] \, dx \, dt
\]

\[
+ \int_{Q^r \setminus Q^s} [a(x, t, u_n, \nabla u)(\nabla u_n - \nabla u) + a_0(x, t, u_n, \nabla u_n)(u_n - u)] \, dx \, dt
\]

\[
+ \int_{Q^r \setminus Q^s} a(x, t, u_n, \nabla u_n) \nabla u \, dx \, dt.
\]

(5.13)

The first term of the right-hand side tends to 0 since \( (a_0(x, t, u_n, \nabla u_n)) \) is bounded in \( L^{\infty}(Q) \) by (5.2) and \( u_n \to u \) strongly in \( L^M(Q) \). The second term tends to \( \int_{Q^r \setminus Q^s} h \nabla u \, dx \, dt \) since \( a(x, t, u_n, \nabla u_n, \chi_s) \) tends strongly in \( (E^{\infty}(Q))^N \) to \( a(x, t, u, \nabla u, \chi_s) \) and \( \nabla u_n \to \nabla u \) weakly in \( (L^M(Q))^N \) for \( \sigma(\Pi_{L^M}, \Pi_{E^{\infty}}) \). The third term satisfies (5.12) while the fourth term tends to \( h \nabla u \, dx \, dt \) since \( a(x, t, u_n, \nabla u_n) \to h \) weakly in \( (E^{\infty}(Q))^N \) for \( \sigma(\Pi_{L^M}, \Pi_{E^{\infty}}) \) and \( M \) satisfies the \( \Delta_2 \)-condition. We deduce then that

\[
0 \leq \limsup_{n \to \infty} \int_{Q^r} [a(x, t, u_n, \nabla u_n) - a(x, t, u_n, \nabla u)] [\nabla u_n - \nabla u] \, dx \, dt
\]

\[
\leq \int_{Q^r \setminus Q^s} [h - a(x, t, u, 0)] \nabla u \, dx \, dt \to 0 \quad \text{as} \quad s \to \infty.
\]

and so, by (5.3), we can construct as in [2] a subsequence such that \( \nabla u_n \to \nabla u \) a.e. in \( Q \). This implies that \( a(x, t, u_n, \nabla u_n) \to a(x, t, u, \nabla u) \) and that \( a_0(x, t, u_n, \nabla u_n) \to a_0(x, t, u, \nabla u) \) a.e. in \( Q \). Lemma 4.4 of [9] shows that \( h = a(x, t, u, \nabla u) \) and \( h_0 = a_0(x, t, u, \nabla u) \) and (5.11) follows. The remaining of the proof is exactly the same as in [14].

\[\square\]

**Corollary 5.3** The function \( u \) can be used as a testing function in (5.6) i.e.

\[
\frac{1}{2} \left[ \int_{\Omega} (u(t))^2 \, dx \right] + \int_{Q^r} [a(x, t, u, \nabla u) \nabla u + a_0(x, t, u, \nabla u) u] \, dx \, dt = \int_{Q^r} f u \, dx \, dt
\]

for all \( \tau \in [0, T] \).

The proof of this corollary is exactly the same as in [14].
6 Strongly nonlinear parabolic problems

In this last section we shall state and prove an existence theorem for strongly nonlinear parabolic initial-boundary problems with a nonlinearity \( g(x, t, s, \xi) \) having growth less than \( M(|\xi|) \). This result generalizes Theorem 2.1 in Boccardo-Murat [5]. The analogous elliptic one is proved in Benkirane-Elmahi [2].

The notation is the same as in section 5. Consider also assumptions (5.2)-(5.5) to which we will annex a Carathéodory function \( g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) satisfying, for a.e. \((x, t) \in \Omega \times [0, T]\) and for all \( s \in \mathbb{R} \) and all \( \xi \in \mathbb{R}^N \):

\[
  g(x, t, s, \xi)s \geq 0 \quad (6.1)
\]

\[
  |g(x, t, s, \xi)| \leq b(|s|)(c'(x, t) + R(|\xi|)) \quad (6.2)
\]

where \( c' \in L^1(Q) \) and \( b : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) and where \( R \) is a given \( N \)-function such that \( R \ll M \).

Consider the following nonlinear parabolic problem

\[
\begin{align*}
  \frac{\partial u}{\partial t} + A(u) + g(x, t, u, \nabla u) &= f \quad \text{in } Q, \\
  u(x, t) &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
  u(x, 0) &= \psi(x) \quad \text{in } \Omega.
\end{align*}
\]

(6.3)

We shall prove the following existence theorem.

**Theorem 6.1** Assume that (5.1)-(5.5), (6.1) and (6.2) hold. Then, there exists at least one distributional solution of (6.3).

**Proof** It is easily adapted from the proof of theorem 3.2 in [2] Consider first

\[
  g_n(x, t, s, \xi) = \frac{g(x, t, s, \xi)}{1 + \frac{x}{2} g(x, t, s, \xi)}
\]

and put \( A_n(u) = A(u) + g_n(x, t, u, \nabla u) \), we see that \( A_n \) satisfies conditions (5.2)-(5.4) so that, by Theorem 5.1, there exists at least one solution \( u_n \in W^{1, x}_{0} L_M(Q) \) of the approximate problem

\[
\begin{align*}
  \frac{\partial u_n}{\partial t} + A(u_n) + g_n(x, t, u_n, \nabla u_n) &= f \quad \text{in } Q, \\
  u_n(x, t) &= 0 \quad \text{on } \partial \Omega \times [0, T], \\
  u_n(x, 0) &= \psi(x) \quad \text{in } \Omega.
\end{align*}
\]

(6.4)

and, by Corollary 5.3, we can use \( u_n \) as testing function in (6.4). This gives

\[
\int_Q [a(x, t, u_n, \nabla u_n). \nabla u_n + a_0(x, t, u_n, \nabla u_n). u_n] \, dx \, dt \leq \langle f, u_n \rangle
\]

and thus \( (u_n) \) is a bounded sequence in \( W^{1, x}_0 L_M(Q) \). Passing to a subsequence if necessary, we assume that

\[
u_n \rightharpoonup u \quad \text{weakly in } W^{1, x}_0 L_M(Q) \text{ for } \sigma(\Pi L_M, \Pi E_M)
\]

(6.5)
for some $u \in W^{1,x}_0 L_M(Q)$. Going back to (6.4), we have

$$\int_Q g_n(x, t, u_n, \nabla u_n) u_n \, dx \, dt \leq C.$$ 

We shall prove that $g_n(x, t, u_n, \nabla u_n)$ are uniformly equi-integrable on $Q$. Fix $m > 0$. For each measurable subset $E \subset Q$, we have

$$\int_E |g_n(x, t, u_n, \nabla u_n)| \leq \int_{E \cap \{|u_n| \leq m\}} |g_n(x, t, u_n, \nabla u_n)| + \int_{E \cap \{|u_n| > m\}} |g_n(x, t, u_n, \nabla u_n)| \, dx \, dt$$

$$\leq b(m) \int_E (c'(x, t) + M(\nabla u_n) ) \, dx \, dt + \frac{1}{m} \int_{E \cap \{|u_n| > m\}} |g_n(x, t, u_n, \nabla u_n)| \, dx \, dt$$

$$\leq b(m) \int_E (c'(x, t) + M(\nabla u_n) ) \, dx \, dt + \frac{1}{m} \int_Q u_n g_n(x, t, u_n, \nabla u_n) \, dx \, dt$$

$$\leq b(m) \int_E c'(x, t) \, dx \, dt + b(m) \int_E R(\nabla u_n) \, dx \, dt + \frac{C}{m}$$

Let $\varepsilon > 0$, there is $m > 0$ such that $\frac{C}{m} < \frac{\varepsilon}{2}$. Furthermore, since $c'' \in L^1(Q)$ there exists $\delta_1 > 0$ such that $b(m) \int_E c''(x, t) \, dx \, dt < \frac{\varepsilon}{2}$. On the other hand, let $\mu > 0$ such that $\|\nabla u_n\|_{M, Q} \leq \mu, \forall n$. Since $R \ll M$, there exists a constant $K_\varepsilon > 0$ depending on $\varepsilon$ such that

$$b(m) R(s) \leq M\left(\frac{\varepsilon}{6}\frac{s}{\mu}\right) + K_\varepsilon$$

for all $s \geq 0$. Without loss of generality, we can assume that $\varepsilon < 1$. By convexity we deduce that

$$b(m) R(s) \leq \frac{\varepsilon}{6} M\left(\frac{s}{\mu}\right) + K_\varepsilon$$

for all $s \geq 0$. Hence

$$\int_E R\left(\frac{\nabla u_n}{\lambda}\right) \, dx \, dt \leq \frac{\varepsilon}{6} \int_E M\left(\frac{\nabla u_n}{\mu}\right) \, dx \, dt + K_\varepsilon |E|$$

$$\leq \frac{\varepsilon}{6} \int_E M\left(\frac{\nabla u_n}{\mu}\right) \, dx \, dt + K_\varepsilon |E|$$

$$\leq \frac{\varepsilon}{6} + K_\varepsilon |E|.$$ 

When $|E| \leq \varepsilon/(6K_\varepsilon)$, we have

$$b(m) \int_E R\left(\frac{\nabla u_n}{\lambda}\right) \, dx \, dt \leq \frac{\varepsilon}{3}, \quad \forall n.$$ 

Consequently, if $|E| < \delta = \inf(\delta_1, \frac{\varepsilon}{6K_\varepsilon})$ one has

$$\int_E |g_n(x, t, u_n, \nabla u_n)| \, dx \, dt \leq \varepsilon, \quad \forall n,$$
this shows that the $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable on $Q$. By Dunford-Pettis’s theorem, there exists $h \in L^1(Q)$ such that

$$g_n(x,t,u_n,\nabla u_n) \rightharpoonup h \text{ weakly in } L^1(Q).$$  \hspace{1cm} (6.6)

Applying then Theorem 4.1, we have for a subsequence, still denoted by $u_n$,

$$u_n \rightharpoonup u, \nabla u_n \rightarrow \nabla u \text{ a.e. in } Q \text{ and } u_n \rightarrow u \text{ strongly in } W^{1,\infty}_{0,M}(Q).$$ \hspace{1cm} (6.7)

We deduce that $a(x,t,u_n,\nabla u_n) \rightharpoonup a(x,t,u,\nabla u)$ weakly in $(L^\infty_M(Q))^N$ for $\sigma(\Pi L^\infty_M \Pi L^\infty_M)$ and since $\frac{\partial u_n}{\partial t} \rightarrow \frac{\partial u}{\partial t}$ in $\mathcal{D}'(Q)$ then passing to the limit in (6.4) as $n \rightarrow +\infty$, we obtain

$$\frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f \text{ in } \mathcal{D}'(Q).$$

This completes the proof of Theorem 6.1.

References


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