Stationary Solutions for a Schrödinger-Poisson System in $\mathbb{R}^3$ *

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Abstract

Under appropriate, almost optimal, assumptions on the data we prove existence of standing wave solutions for a nonlinear Schrödinger equation in the entire space $\mathbb{R}^3$ when the real electric potential satisfies a linear Poisson equation.

1 Introduction

Consider the time-dependent system which couples the Schrödinger equation

$$i\partial_t u = -\frac{1}{2} \Delta u + (V + \tilde{V})u$$ \hspace{1cm} (1.1)

with initial value $u(x, 0) = u(x)$, and the Poisson equation

$$-\Delta V = |u|^2 - n^*.$$ \hspace{1cm} (1.2)

The dopant-density $n^*$ and the effective potential $\tilde{V}$ are given time-independent real functions. There are many papers dealing with the physical problem modelled by this system from which we mention Markowich, Ringhofer & Schmeiser [8]; Illner, Kavian & Lange [3]; Nier [9]; Illner, Lange, Toomire & Zweifel [4], and references therein.

In this work we are mainly concerned with the proof of standing waves (actually ground states) of (1.1)–(1.2) in the entire space $\mathbb{R}^3$, i.e. solutions of the form

$$u(x, t) = e^{i\omega t} u(x)$$

with real number $\omega$ (frequency) and real wave function $u$. Hence we are interested in the stationary system

$$-\frac{1}{2} \Delta u + (V + \tilde{V})u + \omega u = 0 \quad \text{in} \ \mathbb{R}^3$$ \hspace{1cm} (1.3)

$$-\Delta V = |u|^2 - n^* \quad \text{in} \ \mathbb{R}^3$$ \hspace{1cm} (1.4)

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under appropriate, almost optimal, assumptions on $\tilde{V}$ and $n^*$. We suppose first that $\tilde{V} \in L_{\text{loc}}^1(\mathbb{R}^3)$ and $n^* \in L^{6/5}(\mathbb{R}^3)$.

Let us remark that if $V_0$ is such that $-\Delta V_0 = -n^*$ then $(0, V_0)$ is a solution of the system (1.3)-(1.4). But here, we deal with solutions $(u, V)$ in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ such that $u \not\equiv 0$.

F. Nier [9] has studied the system (1.3)-(1.4). He has showed the existence of a solution for small data i.e. when $\|\tilde{V}\|_{L^2}$ and $\|n^*\|_{L^2}$ are small enough. Conversely to our approach here, he has began by solving (1.3) for a fixed $V$ and investigate the Poisson equation then obtained.

In this paper we solve first explicitly the Poisson equation (1.4) for a fixed $u$ in $H^1(\mathbb{R}^3)$. Next we substitute this solution $V = V(u)$ in the Schrödinger equation (1.3) and look into the solvability of

$$-rac{1}{2} \Delta u + (V(u) + \tilde{V}) u + \omega u = 0 \quad \text{in } \mathbb{R}^3. \quad (1.5)$$

Using the explicit formula of $V(u)$, this equation appears as a Hartree equation studied by P.L. Lions [6] in the case where $n^* \equiv 0$ and $\tilde{V}(x) := -2/|x|$. The fact that $\tilde{V}$ in [6] converges to zero at infinity plays a crucial role to prove existence of solutions. However, in this paper we show that a slight modification of the arguments used in that paper allows us to prove existence of a ground state in the case $\tilde{V}$ satisfying (1.7), (1.9) and $n^*$ not necessarily zero (but satisfying (1.8) and (1.9) as below).

Before giving our hypotheses on $\tilde{V}$ and $n^*$ let us define a decomposition which will be useful in the sequel.

**Definition 1.1** We say that $g$ satisfies the decomposition (1.6) if:

(i) $g \in L_{\text{loc}}^1(\mathbb{R}^3)$,

(ii) $g \geq 0$, and

(iii) There exists $q_0 \in [3/2, \infty]$ : $\forall \lambda > 0 \exists g_{1\lambda} \in L^{q_0}(\mathbb{R}^3), q_{\lambda} \in [3/2, \infty]$ and $g_{2\lambda} \in L^{q_{\lambda}}(\mathbb{R}^3)$ such that

$$g = g_{1\lambda} + g_{2\lambda} \quad \text{and} \quad \lim_{\lambda \to 0} \|g_{1\lambda}\|_{L^{q_0}} = 0. \quad (1.6)$$

For convenience, we use throughout this paper the following notations:

- $\|\cdot\|$ denotes the norm $\|\cdot\|_{L^2}$ on $L^2(\mathbb{R}^3)$,
- $I_A$ denotes the characteristic function of the set $A \subset \mathbb{R}^3$,
- $[F \leq \lambda]$ denotes the set $\{x; F(x) \leq \lambda\}$ for a function $F$ and $\lambda \in \mathbb{R}$.

Let us give now two examples of functions satisfying the conditions in Definition 1.1.

**Example 1.2** The following two functions satisfy the decomposition (1.6):

(i) $g(x) := 1/|x|^{\alpha}$ for some $0 < \alpha < 2$.

(ii) $|g|$ where $g$ is a function in $L^r(\mathbb{R}^3)$ for some $r > 3/2$. 

Proof. To prove (i) we write, for $\lambda > 0$,

$$\frac{1}{|x|^\alpha} := \frac{1}{|x|^\alpha} I_{|x|>1/\lambda} + \frac{1}{|x|^\alpha} I_{|x|\leq 1/\lambda}.$$ 

Elementary calculations give

$$\|g_{1\lambda}\|_{L^{q_0}} = \frac{4\pi}{\alpha q_0 - 3} (\lambda)^{\alpha q_0 - 3}$$

and

$$\|g_{2\lambda}\|_{L^q} = \frac{4\pi}{3 - \alpha q} (\lambda)^{3 - \alpha q}.$$ 

Hence it suffices to choose any finite numbers $q_0, q$ such that $3/2 < q < 3/\alpha < q_0$. To show (ii) write, as above,

$$|g| := |g| I_{|g|\leq \lambda} + |g| I_{|g|>\lambda}.$$ 

It is clear that $\|g_{1\lambda}\|_{L^\infty} \leq \lambda (q_0 = \infty)$ and $\|g_{2\lambda}\|_{L^r} \leq \|g\|_{L^r} (q_\lambda = r)$. □

Hypotheses. In what follows we assume that

$$\tilde{V}^+ \in L^1_{loc}(\mathbb{R}^3) \quad \text{and} \quad \tilde{V}^- \quad \text{satisfies the decomposition} \quad (1.6) , \quad (1.7)$$

where $\tilde{V}^+(x) := \max(\tilde{V}(x), 0)$ and $\tilde{V}^-(x) := \max(-\tilde{V}(x), 0)$. We suppose also that

$$n^* \in L^1 \cap L^{6/5}(\mathbb{R}^3) \quad (1.8)$$

and finally if we denote by

$$g(x) := 2\tilde{V}(x) - \frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{n^*(y)}{|x-y|} \, dy$$

we assume that

$$\inf \left\{ \int_{\mathbb{R}^3} (|\nabla \varphi|^2 + g(x)\varphi^2) \, dx, \int_{\mathbb{R}^3} |\varphi|^2 = 1 \right\} < 0. \quad (1.9)$$

Remark that in the case of [6] (where $n^* \equiv 0$ and $\tilde{V}(x) := -2/|x|$), all the three hypotheses above are satisfied. Indeed, (1.7) and (1.8) follow from (i) of Example 1.2. Moreover, if we consider $\Phi(x) := e^{-2|x|}$ then it verifies

$$-\Delta \Phi - 4 \frac{\Phi}{|x|} = -4\Phi,$$

and consequently

$$\inf \left\{ \int_{\mathbb{R}^3} |\nabla \varphi|^2 - 4 \int_{\mathbb{R}^3} \frac{\varphi^2}{|x|} \, dx, \int_{\mathbb{R}^3} |\varphi|^2 = 1 \right\} < 0$$
i.e. (1.9) is satisfied also.

Our main result is the following. We prove that the Schrödinger–Poisson system (1.3)–(1.4) has a ground state, minimizing the energy functional corresponding to (1.5), given by (see Lemma 2.2):

\[
E(\varphi) := \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \varphi|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} |\nabla V(\varphi)|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} \tilde{V} \varphi^2 \, dx + \frac{\omega}{2} \int_{\mathbb{R}^3} \varphi^2 \, dx
\]  

(1.10)

Theorem 1.3 Under the assumptions (1.7), (1.8), and (1.9) there exists \( \omega_\ast > 0 \) such that for all \( 0 < \omega < \omega_\ast \) the equation (1.5) has a nonnegative solution \( u \not\equiv 0 \) which minimizes the functional \( E \):

\[
E(u) = \min_{\varphi \in H^1(\mathbb{R}^3)} E(\varphi).
\]

The remainder of this paper is organized as follows: In section 2 we present some preliminary lemmas which will be useful in the sequel. In section 3, we conclude by proving our main result.

2 Preliminary results

In this section we present a few preliminary lemmas which shall be required in several proofs. Recall (cf. [7, Theorem I.1] or [10, p.151]) that \( \mathcal{D}^{1,2}(\mathbb{R}^3) \) is the completion of \( C^\infty_0(\mathbb{R}^3) \) for the norm

\[
||\varphi||_{\mathcal{D}^{1,2}} = \left( \int_{\mathbb{R}^3} |\nabla \varphi|^2 \, dx \right)^{1/2}.
\]

By a Sobolev inequality, \( \mathcal{D}^{1,2}(\mathbb{R}^3) \) is continuously embedded in \( L^6(\mathbb{R}^3) \), an equivalent characterization is

\[
\mathcal{D}^{1,2}(\mathbb{R}^3) := \{ \varphi \in L^6(\mathbb{R}^3); |\nabla \varphi| \in L^2(\mathbb{R}^3) \}.
\]

For the solvability of the Poisson equation (1.3) we state the following lemma.

Lemma 2.1 For all \( f \in L^{6/5}(\mathbb{R}^3) \), the equation

\[
-\Delta W = f \quad \text{in} \quad \mathbb{R}^3
\]

(2.1)

has a unique solution \( W \in \mathcal{D}^{1,2}(\mathbb{R}^3) \) given by

\[
W(f)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} \, dy.
\]

(2.2)
Proof. The existence and the uniqueness of the solution of (2.1) follow from corollary 3.1.4 of reference [5], by minimizing on $D^{1,2}(\mathbb{R}^3)$ the functional

$$J(v) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla v|^2 dx - \int_{\mathbb{R}^3} f v dx.$$ 

For this, using Hölder’s and Sobolev’s inequalities we check easily that $J$ is coercive (that is $J(v_n) \to +\infty$ as $\|v_n\|_{H^1} \to \infty$), strictly convex, lower semi-continuous and $C^1$ on $D^{1,2}(\mathbb{R}^3)$. Hence $J$ attains its minimum at $W \in D^{1,2}(\mathbb{R}^3)$ which is the unique solution of (2.1).

By uniqueness, $W$ is the Newtonian potential of $f$ and has (cf. [1, p.235]) an explicit formula given by (2.2). Furthermore, multiplying (2.1) by $W$ and integrating we obtain

$$\|\nabla W\|^2 = \int_{\mathbb{R}^3} f(x)W(x)dx.$$ 

After using Hölder and Sobolev inequalities we get

$$\|\nabla W\| \leq S_\ast^{1/2} \|f\|_{L^{6/5}}$$

(2.3)

where $S_\ast$ is the best Sobolev constant in

$$\|v\|^2_{L^{6/5}(\mathbb{R}^3)} \leq S_\ast \|\nabla v\|^2_{L^2(\mathbb{R}^3)}.$$

(2.4)

Hence the linear mapping $f \mapsto W$ is continuous from $L^{6/5}(\mathbb{R}^3)$ into $D^{1,2}(\mathbb{R}^3)$.

□

Now in order to find a solution of equation (1.5), we are going to show that the operator

$$v \mapsto -\frac{1}{2}\Delta v + (W(|v|^2 - n^*) + \tilde{V})v + \omega v$$

is the derivative of a functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ and hence equation (1.5) has a variational structure. To this end, we have the following lemma (see also [3])

**Lemma 2.2** Let $n^* \in L^{6/5}(\mathbb{R}^3)$. For $\varphi \in H^1(\mathbb{R}^3)$ we denote by $V(\varphi) := W(|\varphi|^2 - n^*)$ the unique solution of (2.1) when $f := |\varphi|^2 - n^*$. Define

$$I(\varphi) := \frac{1}{4} \int_{\mathbb{R}^3} |\nabla V(\varphi)|^2 dx.$$ 

Then $I$ is $C^1$ on $H^1(\mathbb{R}^3)$ and its derivative is given by

$$\langle I'(\varphi), \psi \rangle = \int_{\mathbb{R}^3} V(\varphi) \varphi \psi dx \quad \forall \psi \in H^1(\mathbb{R}^3).$$

(2.5)
Proof. Note that if \( \varphi \in H^1(\mathbb{R}^3) \) then, by interpolation, \( |\varphi|^2 \in L^{6/5}(\mathbb{R}^3) \). So taking \( f = |\varphi|^2 - n^* \) and multiplying the equation (2.1) by \( V(\varphi) := W(|\varphi|^2 - n^*) \) we deduce that \( \| \nabla V(\varphi) \|^2 = \int f(x)V(\varphi)(x)dx \), and hence in view of (2.2) we get

\[
I(\varphi) = \frac{1}{16\pi} \int \int \frac{(|\varphi|^2 - n^*)(x)(|\varphi|^2 - n^*)(y)}{|x-y|} dx dy.
\]  

(2.6)

Using this expression, we show easily that (2.5) holds for the Gâteaux differential of \( I \) i.e. for all \( \varphi, \psi \in H^1(\mathbb{R}^3) \)

\[
\lim_{t \to 0^+} \frac{I(\varphi + t\psi) - I(\varphi)}{t} = \int_{\mathbb{R}^3} V(\varphi)\psi dx,
\]

and that the mapping \( \varphi \mapsto \varphi V(\varphi) \) is continuous on \( H^1(\mathbb{R}^3) \). Thus \( I \) is Frechet differentiable and \( C^1 \) on \( H^1(\mathbb{R}^3) \) and its derivative satisfies (2.5). \( \square \)

At certain steps of our proof of Theorem 1.3, we need some estimates for which we will use the next inequalities.

Lemma 2.3 (i) If \( \theta \in L^r(\mathbb{R}^3) \) for some \( r \geq 3/2 \) then \( \forall \delta > 0, \exists C_\delta > 0 \) such that

\[
\int_{\mathbb{R}^3} \theta(x)|\varphi(x)|^2 dx \leq \delta \| \nabla \varphi \|^2 + C_\delta \| \varphi \|^2 \quad \forall \varphi \in H^1(\mathbb{R}^3)
\]

(2.7)

(ii) For all \( \varphi \in D^{1,2}(\mathbb{R}^3) \) and \( y \in \mathbb{R}^3 \) one has

\[
\int_{\mathbb{R}^3} \frac{|\varphi(x)|^2}{|x-y|^2} dx \leq 4\| \nabla \varphi \|^2
\]

(2.8)

(iii) For any \( \delta > 0 \) and all \( y \in \mathbb{R}^3 \)

\[
\int_{\mathbb{R}^3} \frac{|\varphi(x)|^2}{|x-y|} dx \leq \delta \| \nabla \varphi \|^2 + \frac{4}{\delta} \| \varphi \|^2 \quad \forall \varphi \in H^1(\mathbb{R}^3)
\]

(2.9)

Proof. In order to prove (i) we show first that (2.7) holds for any \( \theta \in L^{\infty} + L^{3/2} \) and conclude since \( L^r(\mathbb{R}^3) \subset L^{\infty}(\mathbb{R}^3) + L^{3/2}(\mathbb{R}^3) \) for all \( r \geq 3/2 \). Let \( \theta = \theta_1 + \theta_2 \) with \( \theta_1 \in L^{\infty} \) and \( \theta_2 \in L^{3/2} \). Then for each \( \lambda > 0 \) we have

\[
\int_{\mathbb{R}^3} \theta(x)|\varphi(x)|^2 dx \leq \| \theta_1 \|_{L^{\infty}} \| \varphi \|^2 + \lambda \int_{|\theta_2| \leq \lambda} |\varphi|^2 dx + \int_{|\theta_2| > \lambda} |\theta_2| \| \varphi \|^2 dx
\]

\[
\leq \left( \| \theta_1 \|_{L^{\infty}} + \lambda \right) \| \varphi \|^2 + \| \theta_2 \|_{L^{3/2}(|\theta_2| > \lambda)} \| \varphi \|^2_2
\]

\[
\leq \left( \| \theta_1 \|_{L^{\infty}} + \lambda \right) \| \varphi \|^2 + S_* \| \theta_2 \|_{L^{3/2}(|\theta_2| > \lambda)} \| \nabla \varphi \|^2
\]

where \( S_* \) is the best Sobolev constant in (2.4) and \( \theta_2^2 := \theta_2 \| \theta_2 \|_{L^{3/2}} \). It is clear that \( |\theta_2| \leq |\theta_2| \) for all \( \lambda > 0 \) and that \( \theta_2^2 \to 0 \) pointwise a.e. when \( \lambda \to +\infty \). Since \( \theta_2 \in L^{3/2} \) then by Lebesgue convergence theorem we infer that \( \| \theta_2^2 \|_{L^{3/2}} \) converges to zero. Hence for any \( \delta > 0 \) there exists \( K_\delta > 0 \) such that if \( \lambda \geq K_\delta \) one has \( S_* \| \theta_2^2 \|_{L^{3/2}} \leq \delta \). Choosing \( C_\delta := \| \theta_1 \|_{L^{\infty}} + K_\delta \) we deduce that (2.7) holds for all \( \theta \in L^{\infty}(\mathbb{R}^3) + L^{3/2}(\mathbb{R}^3) \).
Regarding \((ii)\), \((2.8)\) is the classical Hardy inequality (see [2]).

Finally, to show \((iii)\) for all \(\delta > 0\) and any \(y \in \mathbb{R}\), we write

\[
\int_{\mathbb{R}^3} \frac{\lvert \varphi(x) \rvert^2}{\lvert x - y \rvert} \, dx = \int_{\{x-y| < \frac{\delta}{4}\}} \frac{\lvert \varphi(x) \rvert^2}{\lvert x - y \rvert^2} \, dx + \int_{\{x-y| \geq \frac{\delta}{4}\}} \frac{\lvert \varphi(x) \rvert^2}{\lvert x - y \rvert} \, dx
\]

\[
\leq \frac{\delta}{4} \int_{\mathbb{R}^3} \frac{\lvert \varphi(x) \rvert^2}{\lvert x - y \rvert^2} \, dx + \frac{4}{\delta} \int_{\mathbb{R}^3} \frac{\lvert \varphi(x) \rvert^2}{\lvert x - y \rvert} \, dx
\]

and \((2.9)\) holds by using Hardy inequality \((2.8)\). \(\square\)

**Remark 2.4** Note that \(\tilde{V}^-\) satisfies the inequality \((2.7)\) i.e. \(\forall \delta > 0 \exists C_\delta > 0\) such that

\[
\int_{\mathbb{R}^3} \tilde{V}^- (x) \lvert \varphi(x) \rvert^2 \, dx \leq \delta \lVert \nabla \varphi \rVert^2 + C_\delta \lVert \varphi \rVert^2 \quad \forall \varphi \in H^1(\mathbb{R}^3).
\] \((2.10)\)

Indeed, by \((1.7)\) \(\tilde{V}^-\) satisfies the decomposition \((1.6)\). Then for a fixed \(\lambda > 0\) we have

\[
\tilde{V}^- = \tilde{V}_{1\lambda}^- + \tilde{V}_{2\lambda}^-
\]

where for \(i = 1, 2\), \(\tilde{V}_{i\lambda}^- \in L^s(\mathbb{R}^3)\) for some \(s \in [3/2, \infty)\) \((s = q_0, s = q_\lambda)\). Hence by Lemma 2.3 each \(\tilde{V}_{i\lambda}^-\) satisfies the inequality \((2.7)\) and consequently \(\tilde{V}^-\) also.

To finish this section we state the following convergence Lemma.

**Lemma 2.5** Let \(\psi \in L^r(\mathbb{R}^3)\) for some \(r > 3/2\). If \(v_n \rightharpoonup 0\) weakly in \(H^1(\mathbb{R}^3)\) then

\[
\int_{\mathbb{R}^3} \psi(x) v_n^2(x) \, dx \to 0 \quad as \quad n \to +\infty
\]

**Proof.** Consider the subset of \(\mathbb{R}^3\), \(A_\lambda := \{\lvert \psi \rvert > \lambda\}\) and a compact subset \(K\) of \(A_\lambda\) suitably chosen later. We write

\[
\int_{\mathbb{R}^3} \psi |v|^2 \, dx = \int_{\mathbb{R}^3 - A_\lambda} \psi |v|^2 \, dx + \int_{A_\lambda - K} \psi |v|^2 \, dx + \int_{K} \psi |v|^2 \, dx
\]

\[
\leq \lambda \lVert v \rVert^2 + \lVert \psi \rVert_{L^r(\mathbb{R}^3)} \lVert v \rVert_{L^{2r'}(\mathbb{R}^3)}^2 + \lVert \psi \rVert_{L^r(\mathbb{R}^3)} \lVert v_n \rVert_{L^{2r'}(K)}^2 + \lVert \psi \rVert_{L^r(\mathbb{R}^3)} \lVert v_n \rVert_{L^{2r'}(K)}^2
\]

where \(\frac{1}{r} + \frac{1}{r'} = 1\). In the last inequality we used that \((v_n)_n\) is bounded in \(H^1(\mathbb{R}^3)\) (note that \(2 < 2r' < 6\)). For a given arbitrary \(\delta > 0\), we fix first \(\lambda\) such that \(\lambda C_0 \leq \frac{\delta}{3}\). Next we choose a compact subset \(K \subset A_\lambda\) such that

\[
C_1 \lVert \psi \rVert_{L^r(\mathbb{R}^3)} \leq \frac{\delta}{3}
\]
and finally since \( v_n \to 0 \) in \( H^1(\mathbb{R}^3) \) and \( 2 < 2r' < 6 \) then up a subsequence \( \|v_n\|_{L^{2r'}(K)}^2 \) converges to 0 and therefore there exists \( N_\delta \in \mathbb{N} \) such that for all \( n \geq N_\delta \) we get
\[
\|\psi\|_{L^r(K)}^2 \leq \frac{\delta}{3}
\]
which completes the proof. \( \square \)

3 Proof of Theorem 1.3

Now we are in position to prove our main result. To this end, we shall minimize the energy functional
\[
E(\varphi) := \frac{1}{4} \int |\nabla \varphi|^2 dx + I(\varphi) + \frac{1}{2} \int \tilde{V} \varphi^2 dx + \frac{\omega}{2} \int \varphi^2 dx
\]
whose critical points correspond, on account of Lemma 2.2, to solutions of (1.5). Using (2.6), we may decompose \( E(\varphi) \) as
\[
E(\varphi) = E_1(\varphi) - E_2(\varphi) + E_3(\varphi) + E(0) \quad (3.1)
\]
where
\[
E_1(\varphi) := \frac{1}{4} \int |\nabla \varphi|^2 dx + \frac{1}{2} \int \tilde{V}^+ \varphi^2 dx + \frac{\omega}{2} \int \varphi^2 dx
\]
\[
E_2(\varphi) := \frac{1}{2} \int \tilde{V}^- \varphi^2 dx + \frac{1}{8\pi} \int \int \frac{n^*(y)}{|x-y|} \varphi^2(x) dx dy
\]
\[
E_3(\varphi) := \frac{1}{16\pi} \int \int \frac{\varphi^2(x)\varphi^2(y)}{|x-y|} dx dy
\]
\[
E(0) := \frac{1}{16\pi} \int \int \frac{n^*(x)n^*(y)}{|x-y|} dx dy.
\]
The proof of Theorem 1.3 is divided into the four following Lemmas:

**Lemma 3.1** Let \( \omega > 0 \) and \( c \in \mathbb{R} \). If the set \([E \leq c]\) is bounded in \( L^2(\mathbb{R}^3) \) then it is also bounded in \( H^1(\mathbb{R}^3) \).

**Proof.** By the expression (3.1), \( E(\varphi) \leq c \) implies in particular
\[
\frac{1}{4} \|\nabla \varphi\|^2 - E_2(\varphi) \leq c_0 \quad (3.2)
\]
where \( c_0 := c - E(0) \) and since the other terms are nonnegative. To estimate \( E_2(\varphi) \) we use (2.9) which gives for any \( \delta > 0 \),
\[
\int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{n^*(y)}{|x-y|} \varphi^2(x) dx dy \leq \left( \delta \|\nabla \varphi\|^2 + \frac{4}{\delta} \|\varphi\|^2 \right) \|n^*\|_{L^1}.
\]
Using this inequality, Remark 2.4 and choosing \( \delta \) such that 
\[
\delta \left( \frac{1}{2} + \frac{||\nabla \omega||_{L^1}}{8\pi} \right) < \frac{1}{8}
\]
we obtain 
\[
E_2(\varphi) \leq \frac{1}{8} \||\nabla \varphi||^2 + K_0||\varphi||^2
\]
where \( K_0 \) is a positive constant. In Consequence (3.2) gives
\[
\frac{1}{8} \||\nabla \varphi||^2 \leq K_0||\varphi||^2 + c_0.
\]

**Lemma 3.2** For all \( \omega > 0 \) and \( c \in \mathbb{R} \) the set \( [E \leq c] \) is bounded in \( L^2(\mathbb{R}^3) \).

**Proof.** Assume by contradiction that there exists a sequence \((u_j)_{j} \subset H^1(\mathbb{R}^3)\) such that \( E(u_j) \leq c \) and \( \|u_j\| \to +\infty \). Let \( v_j := u_j/\|u_j\| \) then \( \|v_j\| = 1 \) and from \( E(u_j) \leq c \) we get
\[
\frac{1}{4} \int |\nabla v_j|^2 dx - E_2(v_j) + E_3(v_j)\|u_j\|^2 + \frac{\omega}{2} \leq \frac{c_0}{\|u_j\|^2}.
\]
By using the estimate (3.3) for \( \varphi := v_j \) we obtain
\[
\frac{1}{8} \||\nabla v_j||^2 + E_3(v_j)\|u_j||^2 + \frac{\omega}{2} \leq \frac{c_0}{\|u_j||^2} + K_0.
\]
Since \( \omega \) and \( E_3(v_j) \) are nonnegative, this inequality implies that \((v_j)_j\) is bounded in \( H^1(\mathbb{R}^3) \) and that \( E_3(v_j)\|u_j||^2 \) is also bounded; i.e.
\[
\left( \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{v_j^2(x)v_j^2(y)}{|x-y|} dxdy \right)\|u_j||^2 \leq c_1.
\]
Let then \( v \in H^1(\mathbb{R}^3) \) be such that for a subsequence of \( v_j \), noted again \( v_j \), we have \( v_j \rightharpoonup v \) weakly in \( H^1(\mathbb{R}^3) \), \( v_j \to v \) pointwise almost everywhere and \( v_j \) converging to \( v \) strongly in \( L^p_{\text{loc}}(\mathbb{R}^3) \) for any \( 1 \leq p < 3 \). By Fatou’s Lemma we deduce that
\[
\int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|} dxdy \leq \liminf_{j \to +\infty} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{v_j^2(x)v_j^2(y)}{|x-y|} dxdy
\]
\[
\leq \liminf_{j \to +\infty} \frac{c_1}{\|u_j||^2} = 0
\]
and therefore \( v \equiv 0 \). On the other hand, it follows from (3.4) that
\[
\frac{\omega}{2} - E_2(v_j) \leq \frac{c_0}{\|u_j||^2}.
\]
Set
\[
h(x) := \tilde{V}^-(x) + V^*(x)
\]
where \( V^*(x) := \frac{1}{4\pi} \int \frac{n^*(y)}{|x-y|} dy \) is the Newtonian potential of \( n^* \) given by Lemma 2.1. Then (3.6) is equivalent to
\[
\omega - \int_{\mathbb{R}^3} h(x)v_j^2(x)dx \leq \frac{2c_0}{\|u_j||^2}.
\]
Using successively the hypothesis (1.7) and Lemma 2.5 we may show that
\[ \int_{\mathbb{R}^3} h(x)v_j^2(x)dx \to 0 \quad \text{as } j \to +\infty. \tag{3.9} \]

Passing to the limit in (3.8) we infer that \( \omega \leq 0 \) which is a contradiction. In conclusion, any \( (u_j)_j \subset H^1(\mathbb{R}^3) \) such that \( E(u_j) \leq c \) is bounded in \( L^2(\mathbb{R}^3) \). \( \square \)

**Lemma 3.3** For any \( \omega > 0 \) the functional \( E \) is weakly lower semi-continuous on \( H^1(\mathbb{R}^3) \) and attains its minimum on \( H^1(\mathbb{R}^3) \) at \( u \geq 0 \).

**Proof.** First, to show that the functional \( E \) is weakly lower semi-continuous, remark that in the expression (3.1) the term \( E_1 \) and \( E_3 \) are continuous and convex (therefore weakly lower semi-continuous). Then we just have to prove that \( u \mapsto \int_{\mathbb{R}^3} h(x)u^2(x)dx \) is weakly sequentially continuous on \( H^1(\mathbb{R}^3) \) where \( h \) is defined by (3.7). Consider \( u_j \to u \) weakly in \( H^1(\mathbb{R}^3) \) and write
\[ \int_{\mathbb{R}^3} h(x)u_j^2(x)dx = \int_{\mathbb{R}^3} h(x)(u_j - u)^2dx + 2\int_{\mathbb{R}^3} h(x)u_j - u)dx + \int_{\mathbb{R}^3} h(x)u^2dx. \]
Taking \( (u_j - u) \) instead of \( v_j \) in (3.9) we infer that
\[ \int_{\mathbb{R}^3} h(x)(u_j - u)^2dx \to 0 \quad \text{as } j \to \infty. \]
Moreover, similarly to the proof of (3.9) we show that
\[ \int_{\mathbb{R}^3} h(x)u_jdx \to 0 \quad \text{as } j \to \infty, \]
and consequently
\[ \int_{\mathbb{R}^3} h(x)u_j^2(x)dx \to \int_{\mathbb{R}^3} h(x)u^2(x)dx \quad \text{as } j \to \infty. \]
This means that \( u \mapsto \int_{\mathbb{R}^3} h(x)u^2(x)dx \) is weakly sequentially continuous on \( H^1(\mathbb{R}^3) \) and therefore \( E \) is weakly lower semi-continuous on \( H^1(\mathbb{R}^3) \).

Next, if we denote \( \mu := \inf \{ E(\varphi); \varphi \in H^1(\mathbb{R}^3) \} \) and \( (u_n)_n \subset H^1(\mathbb{R}^3) \) a minimizing sequence then by Lemmas 3.1 and 3.2, \( (u_n)_n \) is bounded in \( H^1(\mathbb{R}^3) \) and therefore there exists \( u \in H^1(\mathbb{R}^3) \) such that \( u_n \to u \) weakly in \( H^1(\mathbb{R}^3) \). The functional \( E \) being weakly lower semi-continuous on \( H^1(\mathbb{R}^3) \) we have
\[ E(u) \leq \liminf_{n \to +\infty} E(u_n) = \mu \]
and consequently \( E(u) = \mu \). Since \( E \) is \( C^1 \) on \( H^1(\mathbb{R}^3) \) then \( E'(u) = 0 \) and in view of Lemma 2.2, \( u \) is a solution of the equation (1.5).

Let us remark finally that by a simple inspection we have \( E(|u|) \leq E(u) \) and therefore we may assume that \( u \geq 0 \). \( \square \)

**Lemma 3.4** There exists \( \omega_0 > 0 \) such that if \( 0 < \omega < \omega_0 \) then \( E(u) < E(0) \) and thus \( u \neq 0 \).
Proof. Assuming (1.9), there exist $\mu_1 < 0$ and $\varphi_1 \in H^1(\mathbb{R}^3)$ such that $\int |\varphi_1|^2 = 1$ and

$$\int_{\mathbb{R}^3} |\nabla \varphi_1|^2 dx + \int_{\mathbb{R}^3} \varphi(x)\varphi_1^2(x) dx < \mu_1.$$  

From (3.1) we observe that

$$\int_{\mathbb{R}^3} |\nabla \varphi|^2 dx + \int_{\mathbb{R}^3} \varrho(x)\varphi^2(x) dx = 4E_1(\varphi) - 4E_2(\varphi) - 2\omega \int_{\mathbb{R}^3} \varphi^2(x) dx.$$  

Then the last inequality gives

$$E_1(\varphi) - E_2(\varphi) - \frac{\omega}{2} < \frac{\mu_1}{4}.$$  

Now, for $t > 0$ and using again (3.1) we compute easily

$$E(t\varphi_1) - E(0) = t^2E_1(\varphi_1) - t^2E_2(\varphi_1) + t^4E_3(\varphi_1)$$

$$< \frac{t^2}{4} \left[(\mu_1 + 2\omega) + 4t^2E_3(\varphi_1)\right].$$  

Hence, if $(\mu_1 + 2\omega) < 0$ there exists $t_0 > 0$ small enough such that for all $0 < t \leq t_0$, 

$$(\mu_1 + 2\omega) + 4t^2E_3(\varphi_1) < 0.$$  

In other words, setting $\omega_* := \frac{\mu_1}{2}$ if $0 < \omega < \omega_*$ we have $E(t\varphi_1) < E(0)$ for $0 < t \leq t_0$. Since $E(u) := \inf\{E(\varphi); \varphi \in H^1(\mathbb{R}^3)\}$, this implies that $E(u) < E(0)$ and consequently $u \not\equiv 0$. The proof of Theorem 1.3 is thus complete. □

Remark 3.5 If $n^*$ is nonnegative then we may replace the assumption (1.9) by the next one

$$\inf\left\{ \int |\nabla \varphi|^2 dx + 2 \int \tilde{V}(x)\varphi^2 dx; \int |\varphi|^2 = 1 \right\} < 0$$

which does not depend on $n^*$ and implies obviously (1.9).

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References


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