UPPER AND LOWER SOLUTIONS FOR A SECOND-ORDER THREE-POINT SINGULAR BOUNDARY-VALUE PROBLEM

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Abstract. We study the singular boundary-value problem
\[ u'' + q(t)g(t, u) = 0, \quad t \in (0, 1), \quad \gamma > 0 \]
\[ u(0) = 0, \quad u(1) = \gamma u(\eta). \]
The singularity may appear at \( t = 0 \) and the function \( g \) may be superlinear at infinity and may change sign. The existence of solutions is obtained via an upper and lower solutions method.

1. Introduction

Motivated by the study of multi-point boundary-value problems for linear second order ordinary differential equations, Gupta [7] studied certain three point boundary-value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multi-point boundary-value problems have been studied by several authors using the Leray-Schauder theorem, nonlinear alternative of Leray-Schauder or coincidence degree theory. We refer the reader to [3, 4, 5, 9, 12, 13, 14, 15] for some existence results of nonlinear multi-point boundary-value problems. Recently, Ma [14] proved the existence of positive solutions for the three point boundary-value problem
\[ u'' + b(t)g(u) = 0, \quad t \in (0, 1) \]
\[ u(0) = 0, \quad u(1) = \alpha u(\eta), \]
where \( \eta \in (0, 1), \quad 0 < \alpha < 1/\eta, \quad b \geq 0 \) and \( g \geq 0 \) is either superlinear or sublinear. He applied a fixed point theorem in cones.

In this paper, we study the singular three-point boundary-value problem
\[ u'' + q(t)g(t, u) = 0, \quad t \in (0, 1), \quad \gamma > 0 \]
\[ u(0) = 0, \quad u(1) = \gamma u(\eta). \]

The singularity may appear at \( t = 0 \), and the function \( g \) may be superlinear at \( u = \infty \) and may change sign.
Some basic results on the singular two point boundary-value problems were obtain in [1, 11, 17], in all these papers the arguments rely on the assumption that \( g(t, u) \) is positive. This implies that the solutions are concave. Recently, some authors have studied the case when \( g \) is allowed to change sign by applying the modified upper and lower solutions method; see for example [11].

The present work is a direct extension of some results on the singular two-point boundary-value problems. As in [11], our technique relies essentially on a modified method of upper and lower solutions method for singular three-point boundary-value problems which we believe is well adapted to this type of problems.

2. Upper and lower solutions

Consider the three-point boundary-value problem

\[
\begin{align*}
    u'' + f(t, u) &= 0, & t \in (0, 1), & \eta \in (0, 1), & \gamma \in (0, 1/\eta) \\
    u(0) &= A, & u(1) - \gamma u(\eta) &= B.
\end{align*}
\]  

(2.1)

We use the following assumption:

\begin{enumerate}
    \item[(A1)] \( f : (0, 1] \times \mathbb{R} \to \mathbb{R} \) is a continuous function, there exist two functions \( \alpha, \beta \in C([0, 1], \mathbb{R}) \) and \( \alpha(t) \leq \beta(t) \), for all \( t \in [0, 1] \), if there exist a function \( h \in C((0, 1], (0, \infty)) \), such that
        \[
        |f(t, u)| \leq h(t) \quad \text{for} \quad \alpha(t) \leq u \leq \beta(t),
        \]  
        \[
        \lim_{t \to 0^+} t^2 h(t) = 0, \quad \int_0^1 th(t)dt < \infty.
        \]  
        \[
        (2.2)
        \]
    \item[(2.3)] \end{enumerate}

We call a function \( \alpha(t) \) a lower solution for (2.1), if \( \alpha \in C([0, 1], \mathbb{R}) \cap C^2((0, 1), \mathbb{R}) \), and

\[
\alpha'' + f(t, \alpha) \geq 0, \quad \text{for} \quad t \in (0, 1),
\]

\[
\alpha(0) \leq A, \quad \alpha(1) - \gamma \alpha(\eta) \leq B.
\]

Similarly, we call a function \( \beta(t) \) an upper solution for (2.1), if \( \beta \in C([0, 1], \mathbb{R}) \cap C^2((0, 1), \mathbb{R}) \), and

\[
\beta'' + f(t, \beta) \leq 0, \quad \text{for} \quad t \in (0, 1),
\]

\[
\beta(0) \geq A, \quad \beta(1) - \gamma \beta(\eta) \geq B.
\]

A function \( u(t) \) is said to be a solution to (2.1), if it is both a lower and an upper solution to (2.1).

Our first result reads as follows.

**Theorem 2.1.** Assume (A1) and let \( \alpha, \beta \) be, respectively, a lower solution and an upper solution for (2.1) such that \( \alpha(t) \leq \beta(t) \) on \([0, 1]\). Then (2.1) has at least one solution \( u(t) \) such that

\[
\alpha(t) \leq u(t) \leq \beta(t), \quad \text{for} \quad t \in [0, 1].
\]

Consider now the modified boundary-value problem

\[
\begin{align*}
    u'' + f_1(t, u) &= 0, & t \in (0, 1), \\
    u(0) &= A, & u(1) - \gamma u(\eta) &= B,
\end{align*}
\]  

(2.4)
where
\[ f_1(t,u) = \begin{cases} f(t,\alpha(t)), & \text{if } u < \alpha(t), \\ f(t,u), & \text{if } \alpha(t) \leq u \leq \beta(t), \\ f(t,\beta(t)), & \text{if } u > \beta(t). \end{cases} \]

**Lemma 2.2.** Assume that (2.3) holds. Then the boundary-value problem
\[ y'' = -h(t), \quad 0 < t < 1, \]
\[ y(0) = A, \quad y(1) = \gamma y(\eta) = B \tag{2.5} \]
has a unique solution \( y(t) \) in \( C([0,1],[0,\infty]) \cap C^2((0,1),\mathbb{R}) \), which can be written as
\[ y(t) = A + \frac{B - A(1 - \gamma)}{1 - \gamma \eta} t + \int_0^1 G(t,s)h(s)ds, \quad 0 \leq t \leq 1, \]
where \( G(t,s) \) is Green’s function of the boundary-value problem \(-y'' = 0, y(0) = 0, y(1) = \gamma y(\eta)\). The function \( G \) is explicitly given by: when \( 0 \leq s \leq \eta, \)
\[ G(t,s) = \begin{cases} \frac{t(1-t)(\eta-s)}{1-\gamma \eta}, & s \leq t, \\ \frac{1-t}{1-\gamma \eta}, & s > t; \end{cases} \]
when \( \eta < s \leq 1, \)
\[ G(t,s) = \begin{cases} \frac{t(1-t)(\eta-s)}{1-\gamma \eta}, & s \leq t, \\ \frac{1-t}{1-\gamma \eta}, & s > t. \end{cases} \]

**Proof.** Uniqueness. The proof of the uniqueness of a solution is standard and hence omitted. Existence. Let
\[ y(t) := A + \frac{B - A(1 - \gamma)}{1 - \gamma \eta} t + \int_0^1 G(t,s)h(s)ds, \quad 0 \leq t \leq 1; \]
i.e.,
\[ y(t) = \begin{cases} A + \frac{B - A(1 - \gamma)}{1 - \gamma \eta} t + \int_0^t s(1-t)(\eta-s)\frac{1}{1-\gamma \eta} h(s)ds \\ + \int_0^\eta \frac{1}{1-\gamma \eta} h(s)ds + \int_0^{\eta} t(1-s)\frac{1}{1-\gamma \eta} h(s)ds, \quad 0 \leq t \leq \eta, \\ A + \frac{B - A(1 - \gamma)}{1 - \gamma \eta} t + \int_\eta^\eta s(1-t)(\eta-s)\frac{1}{1-\gamma \eta} h(s)ds \\ + \int_\eta^t \frac{1}{1-\gamma \eta} h(s)ds + \int_t^{\eta} t(1-s)\frac{1}{1-\gamma \eta} h(s)ds, \quad \eta < t \leq 1. \end{cases} \]

Then we have
\[ y'(t) = \begin{cases} \frac{B - A(1 - \gamma)}{1 - \gamma \eta} + \int_0^t s\frac{1}{1-\gamma \eta} h(s)ds \\ + \int_\eta^t \frac{1}{1-\gamma \eta} h(s)ds, \quad 0 < t \leq \eta, \\ \frac{B - A(1 - \gamma)}{1 - \gamma \eta} + \int_0^{\eta} s\frac{1}{1-\gamma \eta} h(s)ds \\ + \int_\eta^t \frac{1}{1-\gamma \eta} h(s)ds + \int_t^{\eta} \frac{1}{1-\gamma \eta} h(s)ds, \quad \eta < t \leq 1. \end{cases} \]
and \( y''(t) = -h(t) \) for all \( t \in (0,1) \). Since \( \int_0^1 th(t)dt < \infty, \lim_{\eta \rightarrow 0^+} \int_0^\eta sh(s)ds = 0; \) so we have
\[ y(0) = A + \lim_{\eta \rightarrow 0^+} \int_0^\eta \frac{1}{1-\gamma \eta} h(s)ds. \]
If \( \int_0^1 \frac{1-s-\gamma(\eta-s)}{1-\gamma \eta} h(s) ds < \infty \), then \( y(0) = A \). If \( \int_0^1 \frac{1-s-\gamma(\eta-s)}{1-\gamma \eta} h(s) ds = \infty \), then by (2.3) we obtain
\[
y(0) = A + \lim_{t \to 0^+} \int_0^1 \frac{1-s-\gamma(\eta-s)}{1-\gamma \eta} h(s) ds = A + \lim_{t \to 0^+} t^2 h(t) \frac{1-\gamma + t(\gamma - 1)}{1-\gamma \eta} = A.
\]
We have also
\[
y(1) - \gamma y(\eta) = \frac{B-A(1-\gamma)}{1-\gamma \eta} + \int_0^\eta \frac{s \gamma (1-\eta)}{1-\gamma \eta} h(s) ds + \int_\eta^1 \frac{\gamma \eta (1-s)}{1-\gamma \eta} h(s) ds
\]
\[
- \gamma \frac{B-A(1-\gamma)}{1-\gamma \eta} \eta + \int_0^\eta \frac{s (1-\eta)}{1-\gamma \eta} h(s) ds + \int_\eta^1 \frac{\eta (1-s)}{1-\gamma \eta} h(s) ds = B.
\]
This shows that \( y(t) \) is a positive solution of (2.5), and \( y \in C([0,1], [0, \infty)) \cap C^2((0,1), \mathbb{R}) \). \( \square \)

Let us define an operator \( \Phi : X \to X \) by
\[
(\Phi u)(t) = A + \frac{B-A(1-\gamma)}{1-\gamma \eta} t + \int_0^t G(t,s) f_1(s,u(s)) ds,
\]
where \( X = \{ u \in C([0,1], \mathbb{R}) \text{ with the norm } \| u \| \} \) is a Banach space, with
\[
\| u \| := \sup \{ |u(t)| : 0 \leq t \leq 1 \}.
\]
Without loss of generality, we assume that \( A = B = 0 \).

To prove the existence of a solution to (2.4), we need the following Lemma.

**Lemma 2.3.** The function \( \Phi \) is continuous from \( X \) to \( X \) and \( \Phi(X) \) is a compact subset of \( X \).

**Proof.** As in the proof of Lemma 2.2 from the definition of \( f_1 \) and from (2.6), we have
\[
| (\Phi u)(t) | \leq \int_0^1 G(t,s) f_1(s,u(s)) ds \leq \int_0^1 G(t,s) h(s) ds = y(t), \quad t \in [0,1].
\]
So we have \( \Phi u \in C([0,1], \mathbb{R}) \cap C^2((0,1), \mathbb{R}) \), and
\[
\| \Phi u \| \leq \| y \|.
\]
This shows that \( \Phi(X) \) is a bounded subset of \( X \).

Noting the facts that \( y(0) = 0 \) and the continuity of \( y(t) \) on \([0,1] \), we have from (2.7) that for any \( \epsilon > 0 \), one can find a \( \delta_1 > 0 \) (independent with \( u \)) such that
\[
0 < \delta_1 < 1/8 \text{ and } (\Phi u)(t) < \frac{\epsilon}{2} \quad t \in [0, 2\delta_1].
\]
On the other hand, from (2.6), since \( |f_1(s,u(s))| \leq h(s), s \in (0,1) \), we can obtain
\[
| (\Phi u)'(t) | \leq L, \quad t \in [\delta_1,1].
\]
Let \( \delta_2 = \frac{\epsilon}{2L}, \) then for \( t_1, t_2 \in [\delta_1,1], | t_2 - t_1 | < \delta_2, \) we have
\[
| (\Phi u)(t_1) - (\Phi u)(t_2) | \leq L | t_1 - t_2 | < \frac{\epsilon}{2}.
\]
Define \( \delta = \min \{ \delta_1, \delta_2 \}, \) then using (2.9), (2.10), we obtain
\[
| (\Phi u)(t_1) - (\Phi u)(t_2) | < \epsilon,
\]
for $t_1, t_2 \in [0,1], \ |t_1 - t_2| < \delta$. This shows that $\{(\Phi u)(t) : u \in X\}$ is equicontinuous on $[0,1]$.

We can obtain the continuity of $\Phi$ in a similar way as above. In fact, if $u_n, u \in X$ and $\|u_n - u\| \to 0$ as $n \to \infty$, then we have

$$| (\Phi u_n)(t) - (\Phi u)(t) | \leq 2 \int_0^1 G(t,s)h(s)ds = 2y(t), \quad t \in [0,1],$$

(2.12)

Noting the facts that $y(0) = 0$ and the continuity of $y(t)$ on $[0,1]$, then for any $\epsilon > 0$, one can find a $\delta_1 > 0$ (independent of $u_n$) such that $0 < \delta_1 < 1/8$ and

$$| (\Phi u_n)(t) - (\Phi u)(t) | < \epsilon, \quad t \in [0,\delta_1].$$

(2.13)

On the other hand, from the continuity of $f_1$, one has

$$| (\Phi u_n)(t) - (\Phi u)(t) | \to 0, \quad t \in [\delta_1,1],$$

(2.14)

as $n \to \infty$. This together with (2.13) implies that $\|\Phi u_n - \Phi u\| \to 0$ as $n \to \infty$. Therefore, $\Phi : X \to X$ is completely continuous. The proof is complete.

\textbf{Lemma 2.4.} Let $u(t)$ be a solution to (2.4). Then $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0,1]$; i.e., $u(t)$ is a solution to (2.1).

\textit{Proof.} We first prove that $u(t) \leq \beta(t)$ on $[0,1]$. Let $x(t) := u(t) - \beta(t)$. Assume that $u(t) > \beta(t)$ for some $t \in [0,1]$. Since $u(0) = 0 \leq \beta(0)$, it follows that

$$x(0) \leq 0, \quad x(1) = u(1) - \beta(1) \leq \gamma u(\eta) - \gamma\beta(\eta) = \gamma x(\eta).$$

Let $\sigma \in (0,1]$ be such that $x(\sigma) = \max_{t \in [0,1]} x(t)$. Then $x(\sigma) > 0$.

Case(i): $\sigma \in (0,1)$. So there exists an interval $(a, \sigma] \subset (0,1)$ such that $x(t) > 0$ in $(a, \sigma]$, and

$$x(a) = 0, \quad x(\sigma) = \max_{t \in [0,1]} x(t) > 0, \quad x'(\sigma) = 0.$$

For $t \in (a, \sigma]$ we have that $f_1(t, u(t)) = f(t, \beta(t))$ and therefore

$$u''(t) + f_1(t, u(t)) = u''(t) + f(t, \beta(t)) = 0 \quad \text{for all } t \in (a, \sigma].$$

On the other hand, as $\beta$ is an upper solution for (2.1), we have

$$\beta''(t) + f(t, \beta(t)) \leq 0 \quad \text{for all } t \in (a, \sigma].$$

Thus, we obtain $u''(t) \geq \beta''(t)$ for all $t \in (a, \sigma]$, and hence, $x''(t) \geq 0$. Then

$$x'(t) \leq 0 \quad \text{on } (a, \sigma] \text{ which is a contradiction.}$$

Case(ii): $\sigma = 1$. So there exists $(a, 1] \subset (0,1]$ such that

$$x(a) = 0, \quad x(1) = \max_{t \in [0,1]} x(t), x(1) - \gamma x(\eta) \leq 0.$$

In the same way as in Case(i), we can obtain that $x(t) > 0, x''(t) \geq 0, t \in (a,1]$. Since $x(\eta) \geq \frac{1}{\gamma}x(1) > 0$, then $\eta > a$.

\textbf{Consider the three-point boundary-value problem}

$$x'' = h(t) > 0, \quad a < t < 1,$$

$$x(a) = 0, \quad x(1) - \gamma x(\eta) = b_1 \leq 0.$$  

(2.15)

Then this equation has a unique solution $x(t) \in C([a, \sigma], [0,\infty)) \cap C^2((a,1), \mathbb{R})$, which can be represented as

$$x(t) = \frac{b_1(t-a)}{1-a-\gamma(\eta-a)} - \int_a^t G_{[a,1]}(t,s)h(s)ds, \quad a \leq t \leq 1,$$
where $G_{[a,1]}(t, s)$ is the Green’s function of the boundary-value problem $-y'' = 0$, $y(a) = 0$, $y(1) = \gamma y(\eta)$, which is explicitly given by: when $a \leq s \leq \eta$,

$$G_{[a,1]}(t, s) = \begin{cases} \frac{(s-a)(1-t-\gamma(\eta-a))}{1-\alpha-\gamma(\eta-a)}, & s \leq t, \\ \frac{(t-a)(1-s-\gamma(\eta-a))}{1-\alpha-\gamma(\eta-a)}, & s > t; \end{cases}$$

when $\eta < s \leq 1$,

$$G_{[a,1]}(t, s) = \begin{cases} \frac{(s-a)(1-t+\gamma(t-s)(\eta-a))}{1-\alpha-\gamma(\eta-a)}, & s \leq t, \\ \frac{(t-a)(1-s-\gamma(\eta-a))}{1-\alpha-\gamma(\eta-a)}, & s > t. \end{cases}$$

Since $0 < \gamma < \frac{1}{\eta} < \frac{1-a}{\eta-a}$, then $G_{[a,1]}(t, s) \geq 0$, and hence $x(t) \leq 0$ on $[a, 1]$, which is a contradiction. In very much the same way, we can prove that $u(t) \geq \alpha(t)$ on $[0, 1]$.

### 3. Main results

Let $g : [0, 1] \times (0, \infty) \to \mathbb{R}$ be a continuous function and $q \in C((0, 1], \mathbb{R}^+_0)$. Consider the three-point boundary-value problem

$$u'' + g(t)u(t) = 0, \quad t \in (0, 1), \quad \eta \in (0, 1), \quad \gamma \in (0, 1]$$

$$u(0) = 0, \quad u(1) = \gamma u(\eta).$$

**Theorem 3.1.** Assume that

- (H1) $|g(t, x)| \leq F(x) + Q(x)$ on $[0, 1] \times (0, \infty)$ with $F > 0$ continuous and non-increasing on $(0, \infty)$, $Q \geq 0$ continuous on $[0, \infty)$, and $\frac{x}{t}$ nondecreasing on $(0, \infty)$;
- (H2) there exist constants $L > 0$ and $\varepsilon > 0$ such that $g(t, x) > L$ for all $(t, x) \in [0, 1] \times (0, \varepsilon]$, and $F(x) > L$, $x \in (0, \varepsilon]$;
- (H3)

$$\lim_{t \to 0^+} t^2q(t) = 0, \quad \int_0^1 tq(t)dt < \infty,$$

$$\sup_{0 < c \leq (0, \infty)} \left( \frac{1}{1 + \frac{Q(c)}{F(c)}} \int_0^c \frac{du}{F(u)} \right) > b_0,$$

where $b_0 = \int_0^1 r q(r)dr$.

Then [3.1] has at least one solution $u \in C([0, 1], [0, \infty)) \cap C^2((0, 1], \mathbb{R})$ with $u(t) > 0$ on $(0, 1]$.

From Lemma [2.2] we obtain the following result.

**Lemma 3.2.** There exists an unique solution $W \in C([0, 1], [0, \infty)) \cap C^2((0, 1], \mathbb{R})$, with $W(t) > 0$ on $(0, 1]$ to the problem

$$W'' + g(t) = 0, \quad 0 < t < 1,$$

$$W(0) = 0, \quad W(1) = \gamma W(\eta).$$

Choose $M > 0$, $\delta > 0$ $(\delta < M)$ such that

$$\frac{1}{1 + \frac{Q(M)}{F(M)}} \int_\delta^M \frac{du}{F(u)} > b_0.$$
Let \( n_0 \in \{1, 2, \ldots \} \) be chosen so that \( 1/n_0 < \min\{\varepsilon - m\|W\|, \delta\} \), where \( W \) is the solution of (3.4), and \( 0 < m < \min\{L, \varepsilon/\|W\|, 1\} \) is chosen and fixed. Let \( N^+ = \{n_0, n_0 + 1, \ldots \} \).

We first show that the boundary-value problem
\[
\begin{align*}
    u'' + q(t)g(t, u) &= 0, \quad 0 < t < 1, \\
    u(0) &= \frac{1}{n}, \quad u(1) - \gamma u(\eta) = \frac{1 - \gamma}{n}, \quad n \in N^+
\end{align*}
\]  
(3.6)

has a solution \( u_n \) for each \( n \in N^+ \) with \( u_n(t) \geq 1/n \) for \( t \in [0, 1] \) and \( \|u_n\| < M \).

We have the following Claim

Claim: Let \( \alpha_n(t) = mW(t) + \frac{1}{n}, \quad t \in [0, 1] \), then \( \alpha_n(t) \) is a (strict) lower solution for problem (3.6).

Proof. For the choice of \( m \) and \( n \), we have \( mW(t) + \frac{1}{n} \leq m\|W\| + \frac{1}{n_0} < \varepsilon \), then from (H2),
\[
g(t, mW(t) + \frac{1}{n}) > L > m \quad \text{for all} \quad t \in [0, 1].
\]

Then we obtain
\[
\begin{align*}
    \alpha_n''(t) + q(t)g(t, \alpha_n(t)) &= (mW(t) + \frac{1}{n})'' + q(t)g(t, mW(t) + \frac{1}{n}) \\
    &= mW''(t) + q(t)g(t, mW(t) + \frac{1}{n}) \\
    &= q(t)(g(t, mW(t) + \frac{1}{n}) - m) > 0, \quad 0 < t < 1.
\end{align*}
\]
We obtain \( \alpha_n(0) = mW(0) + \frac{1}{n} = \frac{1}{n} \), and
\[
\alpha_n(1) - \gamma \alpha_n(\eta) = mW(1) + \frac{1}{n} - \gamma (mW(\eta) + \frac{1}{n}) \\
    = m(W(1) - \gamma W(\eta)) + \frac{1 - \gamma}{n} = \frac{1 - \gamma}{n}.
\]
Thus the proof of Claim is complete. \( \square \)

To find the upper solution of (3.6), we consider the problem
\[
\begin{align*}
    u'' + q(t)F(u)(1 + \frac{Q(M)}{F(M)}) &= 0, \quad 0 < t < 1, \\
    u(0) &= \frac{1}{n}, \quad u(1) - \gamma u(\eta) = \frac{1 - \gamma}{n}.
\end{align*}
\]  
(3.7)

To show that this problem has a solution we study
\[
\begin{align*}
    u'' + q(t)F^*(u)(1 + \frac{Q(M)}{F(M)}) &= 0, \quad 0 < t < 1, \\
    u(0) &= \frac{1}{n}, \quad u(1) - \gamma u(\eta) = \frac{1 - \gamma}{n},
\end{align*}
\]  
(3.8)

where
\[
F^*(u) = \begin{cases}
F(u), & u \geq 1/n, \\
F(\frac{1}{n}), & u < 1/n.
\end{cases}
\]
Then \( F^*(u) \leq F(u) \) for \( u > 0 \).

In the same way as in the Claim, we can easily prove \( \alpha_n(t) = \frac{1}{n} + mW(t) \) is also a (strict) lower solution of (3.8).
By Lemma 2.2 let $\beta^0_n \in C([0,1], \mathbb{R}) \cap C^2((0,1], \mathbb{R})$ be the unique solution of the boundary-value problem

$$u'' + q(t)F(\alpha_n(t))(1 + \frac{Q(M)}{F(M)}) = 0, \quad 0 < t < 1,$$

$$u(0) = \frac{1}{n}, \quad u(1) - \gamma u(\eta) = \frac{1 - \gamma}{n}. \tag{3.9}$$

Since $\beta^0_n$ is a solution of this equation,

$$\beta^{0''}_n + q(t)F(\alpha_n(t))(1 + \frac{Q(M)}{F(M)}) = 0, \quad 0 < t < 1,$$

$$\beta^0_n(0) = \frac{1}{n}, \quad \beta^0_n(1) - \gamma \beta^0_n(\eta) = \frac{1 - \gamma}{n}.$$

On the other hand, as $\alpha_n$ is a lower solution of (3.8), and $\alpha_n \geq 1/n$, we have

$$\alpha^{n''}_n + q(t)F(\alpha_n(t))(1 + \frac{Q(M)}{F(M)}) \geq 0, \quad 0 < t < 1,$$

$$\alpha_n(0) = \frac{1}{n}, \quad \alpha_n(1) - \gamma \alpha_n(\eta) = \frac{1 - \gamma}{n}.$$

So we obtain $\alpha_n(t) \leq \beta^0_n(t)$ for $t \in [0, 1]$. Thus

$$\beta^{0''}_n + q(t)F^*(\beta^0_n)(1 + \frac{Q(M)}{F(M)})$$

$$= -q(t)F(\alpha_n)(1 + \frac{Q(M)}{F(M)}) + q(t)F(\beta^0_n)(1 + \frac{Q(M)}{F(M)})$$

$$= q(t)(1 + \frac{Q(M)}{F(M)})(F(\beta^0_n) - F(\alpha_n)) \leq 0,$$

so that $\beta^0_n$ is an upper solution for problem (3.8).

If we now take $\alpha^0_n = \alpha_n$, we have that $\alpha^0_n$ and $\beta^0_n$ are, respectively, a lower and an upper solution of (3.8) with $\alpha^0_n(t) \leq \beta^0_n(t)$, for all $t \in [0, 1]$. So by the Lemma 2.4, we know that there exists a solution $\beta_n \in C([0,1], \mathbb{R}) \cap C^2((0,1], \mathbb{R})$ of (3.8) such that

$$\alpha_n(t) = \alpha^0_n(t) \leq \beta_n \leq \beta^0_n(t), \quad \forall t \in [0, 1].$$

Now we claim that $\|\beta_n\| < M$. Suppose this is false; i.e., suppose $\|\beta_n\| \geq M$. Since $\beta_n(1) - \frac{1}{n} = \gamma(\beta_n(\eta) - \frac{1}{n}) \leq \beta_n(\eta) - \frac{1}{n}$, $\beta''_n(t) \leq 0$ on $(0,1)$ and $\beta_n \geq \frac{1}{n}$ on $[0,1]$, there exists $\sigma \in (0,1)$ with $\beta''_n(\sigma) \geq 0$ on $(0,\sigma)$, $\beta_n(\sigma) \leq 0$ on $(\sigma,1)$ and $\beta_n(\sigma) = \|\beta_n\|$.

Then for $z \in (0,1)$, we have

$$-\beta''_n(z) \leq F(\beta_n(z))(1 + \frac{Q(M)}{F(M)})q(z). \tag{3.10}$$

Integrate from $t(0 < t \leq \sigma)$ to $\sigma$ to obtain

$$\beta'_n(t) \leq (1 + \frac{Q(M)}{F(M)}) \int_t^\sigma F(\beta_n(z))q(z)dz;$$

so we have

$$\frac{\beta'_n(t)}{F(\beta_n(t))} \leq (1 + \frac{Q(M)}{F(M)}) \int_t^\sigma q(z)dz.$$
Then integrate from 0 to $\sigma$ to obtain
\[
\int_{\frac{\beta_n(t)}{\sigma}}^{\beta_n(t)} \frac{dy}{F(y)} \leq (1 + \frac{Q(M)}{F(M)}) \int_0^\sigma \left( \int_0^\sigma q(z)dz \right) dt = (1 + \frac{Q(M)}{F(M)}) \int_0^\sigma t q(t) dt.
\]
Consequently
\[
\int_{\frac{\beta_n(t)}{M}}^{\beta_n(t)} \frac{dy}{F(y)} \leq (1 + \frac{Q(M)}{F(M)}) \int_0^1 t q(t) dt. \tag{3.11}
\]
This contradicts (3.5) and consequently $\|\beta_n\| < M$.

It follows from the fact $\beta_n \geq 1/n$, we can obtain $\beta_n$ is a solution of (3.7) also. Since $F$ is nonincreasing on $(0, \infty)$, similar to the proof of Lemma 2.4, we can obtain the uniqueness of solutions to (3.7).

Next we show that $\beta_n$ is an upper solution of (3.6). Observe that
\[
|g(t, x)| \leq F(x) + Q(x) \quad \text{on } [0, 1] \times (0, \infty).
\]
We have
\[
\beta''_n(t) + q(t)g(t, \beta_n(t)) \leq -q(t)F(\beta_n(t)) \left( 1 + \frac{Q(M)}{F(M)} \right) + q(t)|g(t, \beta_n(t))| \\
\leq q(t)F(\beta_n(t)) \left( \frac{Q(\beta_n(t))}{F(\beta_n(t))} - \frac{Q(M)}{F(M)} \right) \leq 0, \quad t \in (0, 1).
\]
Thus $\beta_n$ is an upper solution for problem (3.6).

This together with the Claim yields that $\alpha_n$ and $\beta_n$ are, respectively, a lower and an upper solution for (3.6) with $\alpha_n \leq \beta_n$ for all $t \in [0, 1]$. So we conclude (3.6) has a solution $u_n \in C([0, 1], \mathbb{R}) \cap C^2((0, 1), \mathbb{R})$ such that
\[
mW(t) + \frac{1}{n} = \alpha_n(t) \leq u_n(t) \leq \beta_n(t) \leq M, \forall t \in [0, 1].
\]
Consider now the pointwise limit
\[
z(t) := \lim_{n \to +\infty} u_n(t), \quad \forall t \in [0, 1]. \tag{3.12}
\]
Let $e = [a, 1] \subset (0, 1]$. Let $t_n \in (a, 1)$ such that $u_n'(t_n) = (u_n(1) - u_n(a))/(1 - a)$. We obtain
\[
u_n'(t) = \frac{u_n(1) - u_n(a)}{1 - a} + \int_{t_n}^t q(s)g(s, u_n(s)) ds, \quad t \in e.
\]
Since $mW(t) \leq u_n(t) \leq M$, then we have
\[
|u_n'(t)| \leq 2M + \frac{1}{1 - a} + \left( 1 + \frac{Q(M)}{F(M)} \right) \int_a^1 q(t)F(mW(t)) dt := C(a, 1), \quad t \in e. \tag{3.13}
\]
Set $v_n = \max_{t \in e} |u_n'(t)|$, which implies $v_n$ is bounded. That means $u_n'(t)$ is bounded on $e$.

Then, by the Ascoli-Arzela theorem, it is standard to conclude that $z(t)$ is a solution of (3.1) on the interval $e = [a, 1]$. Since $e$ is arbitrary, we find that $z \in C((0, 1], [0, \infty)) \cap C^2((0, 1), \mathbb{R})$, and $z''(t) + q(t)g(t, z(t)) = 0, \quad t \in (0, 1)$.

Also, we have
\[
z(0) = \lim_{n \to +\infty} \frac{1}{n} = 0, \quad z(1) - \gamma z(\eta) = \lim_{n \to +\infty} \frac{1 - \gamma}{n} = 0.
\]
The same argument as in [11] works, we can prove the continuity of $z(t)$ at $t = 0$ and $t = 1$. The proof is complete.
By essentially the same argument as in Theorem 3.1 and [2, Theorem 4.2], we have the following result.

**Theorem 3.3.** Assume that

\((H1^*)\) for any \(r > 0\) there is \(h_r \in C((0, 1], (0, \infty))\): \(|q(t)g(t, x)| \leq h_r(t)\) for all \((t, x) \in (0, 1] \times [r, \infty)\), such that

\[
\lim_{t \to 0^+} t^2h_r(t) = 0, \quad \int_0^1 th_r(t)dt < +\infty;
\]

\((H2^*)\) there exist constants \(L > 0\) and \(\varepsilon > 0\) such that \(g(t, x) > L\) for all \((t, x) \in [0, 1] \times (0, \varepsilon]\).

Then (3.1) has at least one solution \(u \in C([0, 1], [0, \infty]) \cap C^2((0, 1), \mathbb{R})\). Moreover, if \(g(t, x)\) is non-increasing in \(x > 0\), then the solution is unique.

4. AN EXAMPLE

Consider the singular boundary-value problem

\[
u'' + \sigma t^{-m}(u^{-\alpha} + u^\beta - T \sin(8\pi t)) = 0, \quad t \in (0, 1)\\u(0) = 0, \quad zu(1) = \gamma u(\eta), \quad \eta \in (0, 1), \quad \gamma \in (0, 1]
\]

with \(0 \leq m < 2, \sigma > 0, \alpha > 0, \beta \geq 0\). Set

\[
F(u) = u^{-\alpha}, \quad Q(u) = u^\beta + 1, \quad q(t) = \sigma t^{-m},
\]

\[
b_0 = \int_0^1 q(r)dr = \frac{\sigma}{2 - m}.
\]

Applying Theorem 3.1, we find that (4.1) has a positive solutions if

\[
\sigma < (2 - m) \sup_{x \in (0, \infty)} \frac{x^{\alpha+1}}{(\alpha + 1)(1 + x^\alpha + x^{\alpha+\beta})}.
\]

(4.2)

Obviously, \((H1)-(H3)\) in Theorem 3.1 are satisfied. Thus, (4.1) has a solution \(u \in C([0, 1], [0, \infty) \cap C^2((0, 1), \mathbb{R})\) with \(u > 0\) on \((0, 1]\).

We remark that if \(0 \leq \beta < 1\), then (4.1) has at least one positive solution for all \(\sigma > 0\), since the right-hand side of (4.2) is infinity.

REFERENCES


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