MULTIPLE POSITIVE SOLUTIONS FOR SINGULAR \( m \)-POINT BOUNDARY-VALUE PROBLEMS WITH NONLINEARITIES DEPENDING ON THE DERIVATIVE

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Abstract. Using the fixed point theorem in cones, this paper shows the existence of multiple positive solutions for the singular \( m \)-point boundary-value problem

\[
\begin{align*}
  x''(t) + a(t)f(t, x(t), x'(t)) &= 0, \quad 0 < t < 1, \\
  x'(0) &= 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i),
\end{align*}
\]

where \( 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1 \), \( a_i \in [0, 1) \), \( i = 1, 2, \ldots, m - 2 \), with \( 0 < \sum_{i=1}^{m-2} a_i < 1 \) and \( f \) maybe singular at \( x = 0 \) and \( x' = 0 \).

1. Introduction

The study of multi-point boundary-value problems (BVP) for linear second-order ordinary differential equations was initiated by Il’in and Moiseev \[5, 6\]. Since then, many authors have studied general nonlinear multi-point BVP; see for examples \[4, 17\], and references therein. Gupta, Ntouyas and Tsamatos \[4\] considered the existence of a solution in \( C^1[0, 1] \) for the \( m \)-point boundary-value problem

\[
\begin{align*}
  x''(t) &= f(t, x(t), x'(t)) + e(t), \quad 0 < t < 1, \\
  x'(0) &= 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i),
\end{align*}
\]

where \( \xi_i \in (0, 1) \), \( i = 1, 2, \ldots, m - 2 \), \( 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1 \), \( a_i \in \mathbb{R} \), \( i = 1, 2, \ldots, m - 2 \), have the same sign, \( \sum_{i=1}^{m-2} a_i \neq 1 \), \( e \in L^1[0, 1] \), \( f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R} \) is a function satisfying Carathéodory conditions and a growth condition of the form

\[
|f(t, u, v)| \leq p_1(t)|u| + q_1(t)|v| + r_1(t),
\]

where \( p_1, q_1, r_1 \in L^1[0, 1] \). Recently, using Leray-Schauder continuation theorem, Ma and O’Regan proved the existence of positive solutions of \( C^1[0, 1] \) solutions for the above BVP, where \( f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R} \) satisfies the Carathéodory conditions (see \[17\]). Khan and Webb \[10\] obtained

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a interesting result which presents the multiplicity of existence of at least three solutions of a second-order three-point boundary-value problem. However, up to now, there are a fewer results on the existence of multiple solutions to (1.1). In view of the importance of the research on the multiplicity of positive solutions for differential equations [1, 2, 9, 10, 11, 12, 14, 15], the goal of this paper is to fill this gap in the literature.

There are main four sections in this paper. In section 2, we give a special cone and its properties. In section 3, using the theory of fixed point index on a cone, we present the existence of multiple positive solutions to (1.1) with \( f \) and its properties. In section 4, under the condition \( f \) is singular at \( x' = 0 \) but not at \( x = 0 \). In section 5, under the condition \( f \) is singular at \( x = 0 \) but not at \( x' = 0 \), we present the existence of multiple positive solutions to (1.1).

2. Preliminaries

Let \( C^1[0, 1] = \{ x : [0, 1] \to \mathbb{R} \text{ such that } x(t) \text{ be continuous on } [0, 1] \text{ and } x'(t) \text{ continuous on } [0, 1] \} \) with norm \( \| x \| = \max \{ \gamma \| x \|_1, \gamma \| x \|_2 \} \), where

\[
\| x \|_1 = \max_{t \in [0, 1]} |x(t)|, \quad \| x \|_2 = \max_{t \in [0, 1]} |x'(t)|, \\
\gamma = \sum_{i=1}^{m-2} a_i (1 - \xi_i), \quad \delta = \sum_{i=1}^{m-2} a_i (1 - \xi_i).
\]

Let

\[ P = \{ x \in C^1[0, 1] : x(t) \geq \gamma \| x \|_1, \forall t \in [0, 1], x(0) \geq \delta \| x \|_2 \}. \]

Obviously, \( C^1[0, 1] \) is a Banach space and \( P \) is a cone in \( C^1[0, 1] \).

**Lemma 2.1.** Let \( \Omega \) be a bounded open set in real Banach space \( E \), \( \theta \in \Omega \), \( P \) be a cone in \( E \) and \( A : \Omega \cap P \to P \) be continuous and completely continuous. Suppose

\[
\lambda Ax \neq x, \quad \forall x \in \partial \Omega \cap P, \lambda \in (0, 1).
\]

Then \( i(A, \Omega \cap P, P) = 1 \).

**Lemma 2.2.** Let \( \Omega \) be a bounded open set in real Banach space \( E \), \( \theta \in \Omega \), \( P \) be a cone in \( E \) and \( A : \Omega \cap P \to P \) be continuous and completely continuous. Suppose

\[
Ax \not\leq x, \quad \forall x \in \partial \Omega \cap P.
\]

Then \( i(A, \Omega \cap P, P) = 0 \).

Let \( \mathbb{R}_+ = (0, +\infty), \mathbb{R}_- = (-\infty, 0), \mathbb{R} = (-\infty, +\infty) \). The following conditions will be used in this article.

\[
a(t) \in C((0, 1) \cap L^1[0, 1], \quad a(t) > 0, \quad t \in (0, 1); \\
f \in C((0, 1] \times \mathbb{R}_+ \times \mathbb{R}_-, [0, +\infty)).
\]

There exists \( g \in C([0, +\infty) \times (-\infty, 0], [0, +\infty)) \) such that

\[
f(t, x, y) \leq g(x, y), \forall (t, x, y) \in [0, 1] \times \mathbb{R}_+ \times \mathbb{R}_-.
\]
For $x \in P$ and $t \in [0, 1]$, define operator

$$(Ax)(t) = - \int_0^t (t - s)a(s)f(s, x(s)x'(s))ds$$

$$+ \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1 - s)a(s)f(s, x(s), x'(s))ds - \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 (\xi_i - s)a(s)f(s, x(s), x'(s))ds \right).$$

(2.6)

**Lemma 2.3** ([17]). Assume (2.1). Then for $y \in C[0, 1]$ the problem

$$x'' + y(t) = 0, t \in (0, 1)$$

$$x'(0) = \sum_{i=1}^{m-2} b_i x'(\xi_i), \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$$

(2.7)

has a unique solution

$$x(t) = - \int_0^t (t - s)y(s)ds + Mt + N,$$

(2.8)

where,

$$M = \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)y(s)ds}{\sum_{i=1}^{m-2} b_i - 1},$$

$$N = \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1 - s)a(s)y(s)ds - \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 (\xi_i - s)a(s)y(s)ds \right)$$

$$- \sum_{i=1}^{m-2} b_i \int_{\xi_i}^1 a(s)y(s)ds \left( 1 - \sum_{i=1}^{m-2} a_i \xi_i \right).$$

Further, if $y \geq 0$, for all $t \in [0, 1]$, $x$ satisfies

$$\inf_{t \in [0, 1]} x(t) \geq \gamma \|x\|_1,$$

(2.9)

where $\gamma = (\sum_{i=1}^{m-2} a_i (1 - \xi_i))/(1 - \sum_{i=1}^{m-2} a_i \xi_i)$.

**Lemma 2.4.** Suppose (2.3)–(2.5) hold. Then $A : P \to P$ is a completely continuous operator.
Proof. For \( x \in P \), from (2.6), one has

\[
(Ax)(t) \geq - \int_0^1 (1 - s)a(s)f(s, x(s), x'(s))ds + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1 - s)a(s)f(s, x(s), x'(s))ds \right) \\
- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s, x(s), x'(s))ds \\
\geq \frac{\sum_{i=1}^{m-2} a_i (1 - \xi)}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 a(s)f(s, x(s), x'(s))ds \geq 0, \quad t \in [0, 1],
\]

(2.10)

\[
|(Ax)(t)| = - \int_0^1 (t - s)a(s)f(s, x(s), x'(s))ds + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1 - s)a(s)f(s, x(s), x'(s))ds \right) \\
- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s, x(s), x'(s))ds \\
\leq \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 a(s)f(s, x(s), x'(s))ds \\
\leq \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 a(s)ds \max_{0 \leq c \leq \|x\|, -\|x\| \leq \|x'\| \leq 0} g(c, c') < +\infty, \quad t \in [0, 1],
\]

(2.11)

\[
|(Ax)'(t)| = \left| - \int_0^t a(s)f(s, x(s), x'(s))ds \right| \\
= \int_0^t a(s)f(s, x(s), x'(s))ds \\
\leq \int_0^1 a(s)f(s, x(s), x'(s))ds \\
\leq \int_0^1 a(s)ds \max_{0 \leq c \leq \|x\|, -\|x\| \leq \|x'\| \leq 0} g(c, c') < +\infty,
\]

(2.12)

which implies that \( A \) is well defined.

\[
(Ax)(0) = \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1 - s)a(s)f(s, x(s), x'(s))ds \right)
\]
On the other hand, then
\[
Ax \subseteq \mathbb{R}^P
\]
By Lemma 2.3, we have \((\xi_i - s)a(s)f(s, x(s), x'(s))ds)\)

\[\frac{1}{1 - \sum_{i=1}^{m-2} a_i \int_0^1 (1 - s)a(s)f(s, x(s), x'(s))ds} \geq \sum_{i=1}^{m-2} a_i (1 - \xi_i) \int_0^1 a(s)f(s, x(s), x'(s))ds.\]

On the other hand,
\[
\|Ax\|_2 = \max_{t \in [0,1]} |(Ax)'(t)| = \max_{t \in [0,1]} \left| - \int_0^t a(s)f(s, x(s)x'(s))ds \right| = \int_0^1 a(s)f(s, x(s)x'(s))ds.
\]

Then
\[
(Ax)(0) \geq \delta \|Ax\|_2. \tag{2.13}
\]

By Lemma 2.3 we have \((Ax)(t) \geq \gamma \|Ax\|_1\). As a result, \(Ax \in P\), which implies \(AP \subseteq P\). By a standard argument, we know that \(A : P \to P\) is continuous and completely continuous.

3. Singularities at \(x' = 0\) but not at \(x = 0\)

In this section the nonlinearity \(f\) may be singular at \(x' = 0\) but not at \(x = 0\). We will assume that the following conditions hold.

(H1) \(a(t) \in C(0,1) \cap L^1[0,1], \ a(t) > 0, \ t \in (0,1)\)

(H2) \(f(t, u, z) \leq h(u)[g(z) + r(z)], \) where \(f \in C([0,1] \times \mathbb{R}_+ \times \mathbb{R}_-, \mathbb{R}_+), \ g(z) > 0\) continuous and nondecreasing on \(\mathbb{R}_-, \ h(u) \geq 0\) continuous and nondecreasing on \(\mathbb{R}_+, \ r(z) > 0\) continuous and non-increasing on \((−∞, 0]\);

(H3)
\[
\sup_{c \in \mathbb{R}_+} \frac{c}{\frac{1}{1-\sum_{i=1}^{m-2} a_i \xi_i + 1} \int_0^1 h(c) \int_0^1 a(s)ds} > 1,
\]

where \(I(z) = \int_0^z g(u)du = r(u)\), \(z \in \mathbb{R}_-\);

(H4) There exists a function \(g_1 \in C([0,+∞) \times (−∞, 0], [0,0] \times \mathbb{R}_+)\), such that \(f(t, u, z) \geq g_1(t, u, z), \forall (t, u, z) \in [0,1] \times \mathbb{R}_+ \times \mathbb{R}_-\), and \(\lim_{u \to -∞} \frac{g_1(u, z)}{a} = +∞\), uniformly for \(z \in \mathbb{R}_-\).

(H5) There exists a function \(\Psi_H \in C([0,1], \mathbb{R}_+)\) and a constant \(0 \leq \delta < 1\) such that \(f(t, u, z) \geq \Psi_H(t)u^\delta\), for all \((t, u, z) \in [0,1] \times [0,H] \times \mathbb{R}_-\).
For \( n \in \{1, 2, \ldots \} \) and \( x \in P \), define operator
\[
(A_n x)(t) = - \int_0^t (t - s)a(s)f(s, x(s), -|x'(s)| - \frac{1}{n}) ds 
+ \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1 - s)a(s)f(s, x(s), -|x'(s)| - \frac{1}{n}) ds \right) 
- \sum_{i=1}^{m-2} a_i \int_0^t (\xi_i - s)a(s)f(s, x(s), -|x'(s)| - \frac{1}{n}) ds, \quad t \in [0, 1].
\] (3.1)

**Theorem 3.1.** Suppose (H1)–(H5) hold. Then \([1, 1]\) has at least two positive solutions \( x_{0,1}, x_{0,2} \in C^1[0, 1] \cap C^2(0, 1) \) with \( x_{0,1}(t), x_{0,2}(t) > 0, \ t \in [0, 1] \).

**Proof.** Choose \( R_1 > 0 \) such that
\[
\frac{R_1}{1 - \sum_{i=1}^{m-2} a_i} > 1.
\] (3.2)

From the continuity of \( I^{-1} \) and \( h \), we can choose \( \varepsilon > 0 \) and \( \varepsilon < R_1 \) with
\[
\frac{R_1}{1 - \sum_{i=1}^{m-2} a_i} > 1,
\] (3.3)

\( n_0 \in \{1, 2, \ldots \} \) with \( \frac{1}{n_0} < \min\{\varepsilon, \delta/2\} \) and let \( N_0 = \{n_0, n_0 + 1, \ldots \} \).

Lemma 2.4 guarantees that for \( n \in N_0 \), \( A_n : P \to P \) is a completely continuous operator. Let
\[
\Omega_1 = \{x \in C^1[0, 1] : \|x\| < R_1\}.
\]

We show that
\[
x \neq \mu A_n x, \quad \forall x \in P \cap \partial \Omega_1, \quad \mu \in (0, 1), \ n \in N_0.
\] (3.4)

In fact, if there exists an \( x_0 \in P \cap \partial \Omega_1 \) and \( \mu_0 \in (0, 1) \) such that \( x_0 = \mu_0 A_n x_0 \),
\[
x_0(t) = -\mu_0 \int_0^t (t - s)a(s)f(s, x_0(s), -|x_0'(s)| - \frac{1}{n}) ds 
+ \frac{\mu_0}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1 - s)a(s)f(s, x_0(s), -|x_0'(s)| - \frac{1}{n}) ds \right) 
- \sum_{i=1}^{m-2} a_i \int_0^t (\xi_i - s)a(s)f(s, x_0(s), -|x_0'(s)| - \frac{1}{n}) ds, \quad t \in [0, 1].
\]

Then
\[
x_0'(t) = -\mu_0 \int_0^t a(s)f(s, x_0(s), -|x_0'(s)| - \frac{1}{n}) ds, \quad \forall t \in [0, 1].
\] (3.5)

Obviously, \( x_0'(t) \leq 0, \ t \in (0, 1) \), and since \( x_0(1) > 0 \), \( x_0(t) > 0, \ t \in [0, 1] \). Differentiating (3.5), we have
\[
x_0''(t) + \mu_0 a(t)f(t, x_0(t), x_0'(t) - \frac{1}{n}) = 0, \quad 0 < t < 1,
\]
\[
x_0''(0) = 0, \quad x_0(1) = \sum_{i=1}^{m-2} a_i x_0(\xi_i).
\] (3.6)
Then
\[-x_0''(t) = \mu_0 a(t) f(t, x_0(t), x_0'(t) - \frac{1}{n})\]
\[\leq a(t) h(x_0(t)) |g(x_0'(t) - \frac{1}{n}) + r(x_0'(t) - \frac{1}{n})|, \quad \forall t \in (0, 1),\]
\[-x_0''(t) \leq a(t) h(x_0(t)), \quad \forall t \in (0, 1).\]

Integrating from 0 to \(t\), we have
\[I(x_0'(t) - \frac{1}{n}) - I(-\frac{1}{n}) \leq \int_0^t a(s) h(x_0(s)) ds \leq h(R_1) \int_0^t a(s) ds,\]
and
\[I(x_0'(t) - \frac{1}{n}) \leq h(R_1) \int_0^t a(s) ds + I(-\varepsilon).\]

Then
\[x_0'(t) \geq I^{-1}(h(R_1) \int_0^t a(s) ds + I(-\varepsilon));\]
that is,
\[-x_0'(t) \leq -I^{-1}(h(R_1) \int_0^t a(s) ds + I(-\varepsilon)), \quad t \in (0, 1).\] (3.7)

Then
\[x_0(0) = \frac{\mu_0}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 -x_0'(s) ds - \frac{\mu_0 \sum_{i=1}^{m-2} a_i \int_0^\xi a(s) ds}{1 - \sum_{i=1}^{m-2} a_i},\]
\[\leq \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 -I^{-1}\left(h(R_1) \int_0^\xi a(\tau) d\tau + I(-\varepsilon)\right) ds\]
\[+ \frac{\sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i} \int_0^\xi -I^{-1}(h(R_1) \int_0^\tau a(\tau) d\tau + I(-\varepsilon)) ds\]
\[\leq \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 -I^{-1}\left(h(R_1) \int_0^1 a(\tau) d\tau + I(-\varepsilon)\right) ds\]
\[+ \frac{\sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i} \int_0^\xi -I^{-1}(h(R_1) \int_0^1 a(\tau) d\tau + I(-\varepsilon)) ds\]
\[= \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 a(s) ds + I(-\varepsilon)).\]

Since \(x_0(0) \geq x_0(t) \geq \gamma \|x_0\|_1 \geq \gamma \delta \|x_0\|_2, x_0(0) \geq \|x_0\| = R_1,\)
\[\frac{R_1}{1 + \sum_{i=1}^{m-2} a_i \xi I^{-1}(h(R_1) \int_0^1 a(s) ds + I(-\varepsilon)}) \leq 1,\] (3.8)

which is a contradiction to \(3.3\). Then \(3.4\) holds.

From Lemma \([2.1]\) for \(n \in \mathbb{N}_0,\)
\[i(A_n, \Omega_1 \cap P, P) = 1.\] (3.9)

Now we show that there exists a set \(\Omega_2\) such that
\[A_n x \not\leq x, \quad \forall x \in \partial \Omega_2 \cap P.\] (3.10)
Choose \( a^* \) with \( 0 < a^* < 1 \). Let
\[
N^* = \left( \frac{1}{\gamma a^* \sum_{i=1}^{m-2} a_i (1 - \xi_i) \int_0^1 a(s)ds} \right)^{-1} + 1.
\]

From (H4), there exists \( R_2 > R_1 \) such that
\[
g_1(x, y) \geq N^* x, \quad \forall x \geq R_2, \ y \in \mathbb{R}. \tag{3.11}
\]

Let \( \Omega_2 = \{ x \in C^1[0, 1] : \|x\| < \frac{R_2}{a^*} \} \). Then
\[
Ax \not\in x, \quad \forall x \in \partial \Omega_2 \cap P.
\]

In fact, if there exists \( x_0 \in \partial \Omega \cap P \) with \( x_0 \geq A_n x_0 \). By the definition of the cone and Lemma 2.3, one has
\[
x_0(t) \geq \gamma \|x_0\|_1 \geq \gamma x(0) \geq \gamma \delta \|x_0\|_2, \quad x_0(t) \geq \frac{R_2}{a^*} > R_2, \quad \forall t \in [0, 1],
\]
from (3.11),
\[
\gamma x_0(t) \geq \gamma A_n x_0(t)
\]
\[
\gamma \left( - \int_0^t (t - s)a(s)f(s, x_0(s), -|x'_0(s)| - \frac{1}{n})ds + \frac{1}{1 - \sum_{i=1}^{m-2} a_i \left( \int_0^1 (1 - s)a(s)f(s, x_0(s), -|x'_0(s)| - \frac{1}{n})ds - \sum_{i=1}^{m-2} a_i \int_0^\xi (\xi - s)a(s)f(s, x_0(s), -|x'_0(s)| - \frac{1}{n})ds \right) \right)
\]
\[
\geq \gamma \left( - \int_0^t (t - s)a(s)f(s, x_0(s), -|x'_0(s)| - \frac{1}{n})ds + \frac{1}{1 - \sum_{i=1}^{m-2} a_i \left( \int_0^1 (1 - s)a(s)f(s, x_0(s), -|x'_0(s)| - \frac{1}{n})ds - \sum_{i=1}^{m-2} a_i \int_0^\xi (\xi - s)a(s)f(s, x_0(s), -|x'_0(s)| - \frac{1}{n})ds \right) \right)
\]
\[
= \frac{\gamma \sum_{i=1}^{m-2} a_i (1 - \xi_i) \int_0^1 a(s)f(s, x_0(s), -|x'_0(s)| - \frac{1}{n})ds}{1 - \sum_{i=1}^{m-2} a_i}
\]
\[
\geq \frac{\gamma \sum_{i=1}^{m-2} a_i (1 - \xi_i) \int_0^1 a(s)g_1(x_0(s), -|x'_0(s)| - \frac{1}{n})ds}{1 - \sum_{i=1}^{m-2} a_i}
\]
\[
\geq \frac{\gamma \sum_{i=1}^{m-2} a_i (1 - \xi_i) \int_0^1 a(s)dsN^* x_0(s)}{1 - \sum_{i=1}^{m-2} a_i}
\]
\[
\geq a^* \frac{\gamma \sum_{i=1}^{m-2} a_i (1 - \xi_i) \int_0^1 a(s)dsN^* \frac{R_2}{a^*}}{1 - \sum_{i=1}^{m-2} a_i} \geq \frac{R_2}{a^*}.
\]
Then \( \| x_0 \| \geq \gamma \| x_0 \|_1 > \frac{B_2}{a} \), which is a contradiction to \( x_0 \in \partial \Omega_2 \cap P \). Then \( (3.10) \) holds. From Lemma 2.2

\[ i(A_n, \Omega_2 \cap P, P) = 0. \] (3.12)

which with \( (3.9) \) guarantee that

\[ i(A_n, (\Omega_2 - \bar{\Omega}_1) \cap P, P) = -1. \] (3.13)

From this equality and \( (3.9) \), \( A_n \) has two fixed points with \( x_{n,1} \in \Omega_1 \cap P, x_{n,2} \in (\Omega_2 - \bar{\Omega}_1) \cap P \).

For each \( n \in N_0 \), there exists \( x_{n,1} \in \Omega_1 \cap P \) such that \( x_{n,1} = A_n x_{n,1} \); that is,

\[ x_{n,1}(t) = -\int_0^t (t - s)a(s) f(s, x_{n,1}(s)) ds - \frac{1}{n}ds, \quad n \in N_0, \quad t \in (0,1). \]

Now we consider \( \{x_{n,1}(t)\}_{n \in N_0} \) and \( \{x'_{n,1}(t)\}_{n \in N_0} \). Since \( \|x_{n,1}\| \leq R_1 \), it follows that

\[ \{x_{n,1}(t)\} \text{ is uniformly bounded on } [0,1]. \] (3.15)

\[ \{x'_{n,1}(t)\} \text{ is uniformly bounded on } [0,1]. \] (3.16)

Then

\[ \{x_{n,1}(t)\} \text{ is equicontinuous on } [0,1]. \] (3.17)

As in the proof as \( (3.6) \),

\[ x_{n,1}''(t) + a(t)f(t, x_{n,1}(t), x'_{n,1}(t)) - \frac{1}{n} = 0, \quad 0 < t < 1, \]

\[ x'_{n,1}(0) = 0, x_{n,1}(1) = \sum_{i=1}^{m-2} a_i x_{n,1}(\xi_i). \] (3.18)

Now we show that for all \( t_1, t_2 \in [0,1] \),

\[ |I(x'_{n,1}(t_2) - \frac{1}{n}) - I(x'_{n,1}(t_1) - \frac{1}{n})| \leq h(R_1) \int_{t_1}^{t_2} a(t)dt. \] (3.19)

From \( (3.18) \),

\[ -x_{n,1}''(t) = a(t)f(t, x_{n,1}(t), x'_{n,1}(t) - \frac{1}{n}) \]

\[ \leq a(t)h(x_{n,1}(t)) [g(x'_{n,1}(t) - \frac{1}{n}) + r(x_{n,1}(t) - \frac{1}{n})], \quad \forall t \in (0,1), \]

and

\[ x_{n,1}''(t) = -a(t)f(t, x_{n,1}(t), x'_{n,1}(t) - \frac{1}{n}) \]

\[ \geq -a(t)h(x_{n,1}(t)) [g(x'_{n,1}(t) - \frac{1}{n}) + r(x_{n,1}(t) - \frac{1}{n})], \quad \forall t \in (0,1), \]
From (3.15)–(3.17), (3.25) and the Arzela-Ascoli Theorem, 

\[ \{ \Psi \} \] 

are relatively compact on 

\[ C_{\epsilon} \] 

Then, for all 

\[ t_1, t_2 \in [0, 1] \text{ and } t_1 < t_2, \]

\[
\left| - \int_{t_1}^{t_2} \frac{1}{g(x'_{n,1}(s) - \frac{1}{n}) + r(x'_{n,1}(s) - \frac{1}{n}} d(x_{n,1}(s) - \frac{1}{n}) \right| \leq h(R_1) \int_{t_1}^{t_2} a(t) dt \\
= h(R_1) | \int_{t_1}^{t_2} a(t) dt |,
\]

Inequality (3.19) holds.

Since \( I^{-1} \) is uniformly continuous on \([0, I(-R_1 - \epsilon)]\), for all \( \epsilon > 0 \), there exists \( \epsilon' > 0 \) such that

\[
|I^{-1}(s_1) - I^{-1}(s_2)| < \epsilon, \quad \forall |s_1 - s_2| < \epsilon', \quad s_1, s_2 \in [0, I(-R_1 - \epsilon)].
\]

And (3.19) guarantees that for \( \epsilon' > 0 \), there exists \( \delta' > 0 \) such that

\[
|I(x'_{n,1}(t_2) - \frac{1}{n}) - I(x'_{n,1}(t_1) - \frac{1}{n})| < \epsilon', \quad \forall |t_1 - t_2| < \delta', \quad t_1, t_2 \in [0, 1].
\]

From this inequality and (3.22),

\[
|x'_{n,1}(t_2) - x'_{n,1}(t_1)| = |x'_{n,1}(t_2) - \frac{1}{n} - (x'_{n,1}(t_1) - \frac{1}{n})| \\
= |I^{-1}(I(x'_{n,1}(t_2) - \frac{1}{n})) - I^{-1}(I(x'_{n,1}(t_1) - \frac{1}{n}))| \\
< \epsilon, \quad \forall |t_1 - t_2| < \delta', \quad t_1, t_2 \in [0, 1];
\]

that is,

\[
\{ x'_{n,1}(t) \} \text{ is equicontinuous on } [0, 1].
\]

From (3.15)–(3.17), (3.25) and the Arzela-Ascoli Theorem, \( \{ x_{n,1}(t) \} \) and \( \{ x'_{n,1}(t) \} \) are relatively compact on \( C[0, 1] \), which implies there exists a subsequence \( \{ x_{n,j,1} \} \) of \( \{ x_{n,1} \} \) and function \( x_{0,1}(t) \in C[0, 1] \) such that

\[
\lim_{j \to +\infty} \max_{t \in [0, 1]} |x_{n,j,1}(t) - x_{0,1}(t)| = 0, \quad \lim_{j \to +\infty} \max_{t \in [0, 1]} |x'_{n,j,1}(t) - x'_{0,1}(t)| = 0.
\]

Since \( x'_{n,j,1}(0) = 0, \ x_{n,j,1}(1) = \sum_{i=1}^{m-2} a_i x_{n,j,1}(\xi_i), \ x'_{n,j,1}(t) < 0, \ x_{n,j,1}(t) > 0, \ t \in (0, 1), \ j \in \{1, 2, \ldots\}, \)

\[
x'_{0,1}(0) = 0, \quad x_{0,1}(1) = \sum_{i=1}^{m-2} a_i x_{0,1}(\xi_i), \quad x'_{0,1}(t) \leq 0, \quad x_{0,1}(t) \geq 0, \quad t \in (0, 1).
\]

For \( (t, x_{n,j,1}(t), x'_{n,j,1}(t) - \frac{1}{n_j}) \in [0, 1] \times [0, R_1 + \epsilon] \times (-\infty, 0) \), from (H5) there exists a function \( \Psi_{R_1} \in C([0, 1], \mathbb{R}_+) \) such that

\[
f(t, x_{n,j,1}(t), x'_{n,j,1}(t) - \frac{1}{n_j}) ds \geq \Psi_{R_1}(t)(x_{n,j,1}(t))^{\delta}, \quad 0 \leq \delta < 1.
\]
Then, for $n \in N_0$,
\[
x_{n_j,1}(t) = -\int_0^t (t-s)a(s)f(s,x_{n_j,1}(s),x'_{n_j,1}(s) - \frac{1}{n_j})\,ds
\]
\[
+ \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1-s)a(s)f(s,x_{n_j,1}(s),x'_{n_j,1}(s) - \frac{1}{n_j})\,ds \right)
\]
\[
- \sum_{i=1}^{m-2} a_i \int_0^1 (\xi_i - s)a(s)f(s,x_{n_j,1}(s),x'_{n_j,1}(s) - \frac{1}{n_j})\,ds
\]
\[
\geq -\int_0^1 (1-s)a(s)f(s,x_{n_j,1}(s),x'_{n_j,1}(s) - \frac{1}{n_j})\,ds
\]
\[
+ \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1-s)a(s)f(s,x_{n_j,1}(s),x'_{n_j,1}(s) - \frac{1}{n_j})\,ds \right)
\]
\[
- \sum_{i=1}^{m-2} a_i \int_0^1 (\xi_i - s)a(s)f(s,x_{n_j,1}(s),x'_{n_j,1}(s) - \frac{1}{n_j})\,ds
\]
\[
\geq \frac{\sum_{i=1}^{m-2} a_i (1 - \xi_i)}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 a(s)f(s,x_{n_j,1}(s),x'_{n_j,1}(s) - \frac{1}{n_j})\,ds
\]
\[
\geq \frac{\sum_{i=1}^{m-2} a_i (1 - \xi_i)}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 a(s)\Psi_{R_1}(s)(x_{n_j,1}(s))^{\gamma}\,ds
\]
\[
\geq \frac{\sum_{i=1}^{m-2} a_i (1 - \xi_i)}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 a(s)\Psi_{R_1}(s)\gamma^{\delta} \,ds \Vert x_{n_j,1} \Vert_{L_1}^{\delta},
\]
which implies
\[
\Vert x_{n_j,1} \Vert_1 \geq \left( \frac{\sum_{i=1}^{m-2} a_i (1 - \xi_i)}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 a(s)\Psi_{R_1}(s)\gamma^{\delta} \,ds \right)^{\frac{1}{\delta}},
\]
and
\[
x_{n_j,1}(t) \geq \left( \frac{\sum_{i=1}^{m-2} a_i (1 - \xi_i)}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 a(s)\Psi_{R_1}(s)\gamma^{\delta} \,ds \right)^{\frac{1}{\delta}} = a_0 > 0.
\]
Thus
\[
x'_{n_j,1}(t) = -\int_0^t a(s)f(s,x_{n_j,1}(s),x'_{n_j,1}(s) - \frac{1}{n_j})\,ds
\]
\[
\leq -\int_0^t a(s)\Psi_{R_1}(s)(x_{n_j,1}(s))^{\delta}\,ds
\]
\[
\leq -\int_0^t a(s)\Psi_{R_1}(s)d\sigma_0^\delta, \quad t \in [0, 1], n \in N_0.
\]
Consequently,
\[
\inf_{j \geq 1} \min_{s \in \left[\frac{1}{2}, t\right]} |x_{n,j,1}'(s)| > 0, \quad t \in \left[\frac{1}{2}, 1\right),
\]
\[
\inf_{j \geq 1} \min_{s \in [0, \frac{1}{2}]} |x_{n,j,1}'(s)| > 0, \quad t \in (0, \frac{1}{2}].
\]

Since
\[
x_{n,j,1}'(t) - x_{n,j,1}'\left(\frac{1}{2}\right) = -\int_{1/2}^{t} a(s)f(s, x_{n,j,1}(s), x_{n,j,1}'(s) - \frac{1}{n_j}) ds, \quad t \in (0, 1),
\]
and
\[
f(t, x_{n,j,1}(t), x_{n,j,1}'(t) - \frac{1}{n_j}) \leq h(x_{n,j,1}(t))[g(x_{n,j,1}'(t) - \frac{1}{n_j}) + r(x_{n,j,1}'(t) - \frac{1}{n_j})]
\leq h\left(\frac{R_1}{\gamma}\right)[g(-\int_{0}^{t} a(s)\Psi R_{i}(s) dsa_{0}^{c}) + r\left(\frac{R_1}{\gamma\delta} - \varepsilon\right)],
\]
letting \(j \to +\infty\), the Lebesgue Dominated Convergence Theorem guarantees that
\[
x_{0,1}'(t) - x_{0,1}'\left(\frac{1}{2}\right) = -\int_{1/2}^{t} a(s)f(s, x_{0,1}(s), x_{0,1}'(s)) ds, \quad t \in (0, 1). \tag{3.27}
\]

Differentiating, we have
\[
x_{0,1}''(t) + a(t)f(t, x_{0,1}(t), x_{0,1}'(t)) = 0, \quad 0 < t < 1,
\]
and from (3.26) \(x_{0,1}(t)\) is a positive solution of (1.1) with \(x_{0,1} \in C^4[0, 1] \cap C^2(0, 1)\).

For the set \(\{x_{n,2}\}_{n \in \mathbb{N}_0} \subseteq \left(\Omega_2 - \Omega_1\right) \cap P\), the proof is as that for the set \(\{x_{n,1}\}_{n \in \mathbb{N}_0}\) with \(\lim_{j \to +\infty} x_{n,2} = x_{0,2} \in C^4[0, 1] \cap C^2(0, 1)\). Moreover, \(x_{0,2}\) is a positive solution to (1.1).

\[\square\]

**Example 3.1** In (1.1), let \(f(t, u, z) = \mu[1 + (-z)^{-a}][1 + u^b + u^d]\) and \(a(t) \equiv 1\) with \(0 < a < 1, b > 1, 0 < d < 1\) and \(\mu > 0\). If
\[
\mu < \sup_{c \in \mathbb{R}^+} I\left(-\frac{c(1 - \sum_{i=1}^{m-2} a_i)}{1 + \sum_{i=1}^{m-2} a_i} \right) \left(\frac{1}{1 + c^b + c^d}\right). \tag{3.28}
\]

Then (1.1) has at least two positive solutions \(x_{0,1}, x_{0,2} \in C^4[0, 1] \cap C^2(0, 1)\).

We apply Theorem 3.1 with \(g(z) = (-z)^{-a}, r(z) = 1, h(u) = \mu(1 + u^b + u^d), \Psi(t) = \mu, g_1(u, z) = \mu u^b\). (H1), (H2), (H4), (H5) hold. Also
\[
\sup_{c \in \mathbb{R}^+} c \left(\frac{\sum_{i=1}^{m-2} a_i^{1+c} + 1}{1 + \sum_{i=1}^{m-2} a_i} \right) I^{-1}\left(\frac{c}{1 + c^b + c^d}\right)
\leq \sup_{c \in \mathbb{R}^+} c \left(\frac{\sum_{i=1}^{m-2} a_i^{1+c} + 1}{1 + \sum_{i=1}^{m-2} a_i} \right) I^{-1}\left(\mu (1 + c^b + c^d)\right),
\]
and (3.28) guarantees that (H3) holds.
4. Singularities at $x' = 0$ and $x = 0$

In this section the nonlinearity $f$ may be singular at $x' = 0$ and $x = 0$. We assume that the following conditions hold.

(P1) $a(t) \in C[0, 1], a(t) > 0, t \in (0, 1)$;

(P2) $f(t, u, z) \leq [h(u) + \omega(u)]g(z) + r(z)$, where $f \in C([0, 1] \times \mathbb{R}_+ \times \mathbb{R}_-, \mathbb{R}_+), g(z) > 0$ continuous and non-increasing on $(-\infty, 0], \omega(u) > 0$ continuous and non-increasing on $[0, +\infty), h(u) \geq 0$ continuous and nondecreasing on $\mathbb{R}_+, r(z) > 0$ continuous and nondecreasing on $\mathbb{R}_-$;

(P3) \[
\sup_{c \in \mathbb{R}_+} \frac{c}{1 - \sum_{i=1}^{m-2} a_i \xi_i + 1} (I^{-1}[-\max_{t \in [0, 1]} a(t) h(R_1 + R_1 \omega(s)ds)]) > 1,
\]

where $I(z) = \int_0^z \frac{udz}{g(u) + r(u)}, z \in \mathbb{R}_-, \int_0^a \omega(s)ds < +\infty$;

(P4) There exists a function $g_1 \in C([0, +\infty) \times (-\infty, 0], [0, +\infty))$, such that $f(t, u, z) \geq g_1(u, z), \forall (t, u, z) \in [0, 1] \times \mathbb{R}_+ \times \mathbb{R}_-$, and $\lim_{u \to +\infty} \frac{g_1(u, z)}{u} = +\infty$, uniformly for $z \in \mathbb{R}_-$;

(P5) There exists a function $\Psi_H \in C([0, 1], \mathbb{R}_+]$ with $f(t, u, z) \geq \Psi_H(t)$, for all $(t, u, z) \in [0, 1] \times [0, H] \times [-H, 0]$.

For $n \in \{1, 2, \ldots\}, x \in P, t \in [0, 1]$, define operator

\[
(A_n x)(t) = -\int_0^t (t - s) a(s) f(s, x(s) + \frac{1}{n}, -|x'(s)| - \frac{1}{n})ds + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1 - s) a(s) f(s, x(s) + \frac{1}{n}, -|x'(s)| - \frac{1}{n})ds \right. \\
- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s) a(s) f(s, x(s) + \frac{1}{n}, -|x'(s)| - \frac{1}{n})ds \bigg).
\]

**Theorem 4.1.** Suppose (P1)–(P5) hold. Then (1.1) has at least two positive solutions $x_{0,1}, x_{0,2} \in C^1[0, 1] \cap C^2(0, 1)$ and $x_{0,1}(t), x_{0,2}(t) > 0, t \in [0, 1]$.

**Proof.** Choose $R_1 > 0$ such that

\[
\frac{1}{1 - \sum_{i=1}^{m-2} a_i \xi_i + 1} (I^{-1}[-\max_{t \in [0, 1]} a(t) h(R_1 + R_1 \omega(s)ds)]) > 1.
\]

From the continuity of $I^{-1}$ and $h$, we can choose $\epsilon > 0$ and $\epsilon < R_1$ such that

\[
\frac{1}{1 - \sum_{i=1}^{m-2} a_i \xi_i + 1} (I^{-1}[I(-\epsilon) - \max_{t \in [0, 1]} a(t) h(R_1 + \epsilon) + \int_0^{R_1 + \epsilon} \omega(s)ds)]) > 1
\]

is greater than $1, \forall n \in \{1, 2, \ldots\}$ with $\frac{1}{n_0} < \min(\epsilon, \delta/2)$ and let $N_0 = \{n_0, n_0 + 1, \ldots\}$. Then Lemma 2.4 guarantees that for $n \in N_0, A_n : P \to P$ is a completely continuous operator. Let

\[
\Omega_1 = \{x \in C^1[0, 1] : \|x\| < R_1\}.
\]

We show that

\[
x \neq \mu A_n x, \quad \forall x \in P \cap \partial \Omega_1, \mu \in (0, 1], n \in N_0.
\]
In fact, if there exists an \( x_0 \in P \cap \partial \Omega_1 \) and \( \mu_0 \in (0, 1] \) with \( x_0 = \mu_0 A_n x_0 \).

\[
x_0(t) = -\mu_0 \int_0^t (t-s)a(s)f(s, x_0(s) + \frac{1}{n}, -|x_0'(s)| - \frac{1}{n})ds
\]

\[
+ \frac{\mu_0}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1-s)a(s)f(s, x_0(s) + \frac{1}{n}, -|x_0'(s)| - \frac{1}{n})ds
\right)
\]

\[
- \sum_{i=1}^{m-2} a_i \int_0^t (\xi_i - s)a(s)f(s, x_0(s) + \frac{1}{n}, -|x_0'(s)| - \frac{1}{n})ds, \quad t \in [0, 1].
\]

Then

\[
x_0'(t) = -\mu_0 \int_0^t a(s)f(s, x_0(s) + \frac{1}{n}, -|x_0'(s)| - \frac{1}{n})ds, \quad \forall t \in [0, 1]. \tag{4.5}
\]

Obviously, \( x_0'(t) \leq 0, \quad t \in (0, 1) \), and since \( x_0(1) > 0, \quad x_0(t) > 0, \quad t \in [0, 1] \). Differentiating (4.5), we have

\[
x_0''(t) + \mu_0 a(t)f(t, x_0(t) + \frac{1}{n}, x_0'(t) - \frac{1}{n}) = 0, \quad 0 < t < 1,
\]

\[
x_0'(0) = 0, \quad x_0(1) = \sum_{i=1}^{m-2} a_i x_0(\xi_i). \tag{4.6}
\]

Then, for \( t \in (0, 1) \),

\[
-x_0''(t) = \mu_0 a(t)f(t, x_0(t) + \frac{1}{n}, x_0'(t) - \frac{1}{n})
\]

\[
\leq a(t) [h(x_0(t) + \frac{1}{n}) + \omega(x_0(t) + \frac{1}{n})] [g(x_0'(t) - \frac{1}{n}) + r(x_0'(t) - \frac{1}{n})],
\]

and

\[
\frac{-x_0''(t)}{g(x_0(t) - \frac{1}{n}) + r(x_0'(t) - \frac{1}{n})} \leq a(t) [h(x_0(t) + \frac{1}{n}) + \omega(x_0(t) + \frac{1}{n})], \quad \forall t \in (0, 1),
\]

and

\[
\frac{-x_0''(t)(x_0'(t) - \frac{1}{n})}{g(x_0(t) - \frac{1}{n}) + r(x_0'(t) - \frac{1}{n})} \geq a(t) [h(R_1 + \epsilon) + \omega(x_0(t) + \frac{1}{n}(1-t))] (x_0'(t) - \frac{1}{n})
\]

\[
\geq a(t) [h(R_1 + \epsilon) + \omega(x_0(t) + \frac{1}{n}(1-t))] (x_0'(t) - \frac{1}{n}) \tag{4.7}
\]

Integrating from 0 to \( t \), we have

\[
I(x_0'(t) - \frac{1}{n}) - I(-\frac{1}{n})
\]

\[
\geq \int_0^t a(s) [h(R_1 + \epsilon)(x_0'(s) - \frac{1}{n}) + \omega(x_0(s) + \frac{1}{n}(1-s))(x_0'(s) - \frac{1}{n})]ds
\]

\[
\geq \max_{t \in [0,1]} a(t) [h(R_1 + \epsilon) \left( \int_0^t x_0'(s)ds - \int_0^t \frac{1}{n}ds \right) + \int_0^t \omega(x_0(s) + \frac{1}{n}(1-s))dx_0(s)
\]

\[
+ \frac{1}{n}(1-s)).
\]
\[ \geq - \max_{t \in [0,1]} a(t)(h(R_1 + \epsilon)(R_1 + \epsilon) + \int_0^{R_1+\epsilon} \omega(s)ds, \]

\[ I(x'_0(t) - \frac{1}{n}) \geq I(-\epsilon) - \max_{t \in [0,1]} a(t)(h(R_1 + \epsilon)(R_1 + \epsilon) + \int_0^{R_1+\epsilon} \omega(s)ds). \]

Then

\[ x'_0(t) \geq I^{-1}\left( I(-\epsilon) - \max_{t \in [0,1]} a(t) \{ h(R_1 + \epsilon)(R_1 + \epsilon) + \int_0^{R_1+\epsilon} \omega(s)ds \} \right); \]

that is,

\[ -x'_0(t) \leq -I^{-1}\left( I(-\epsilon) - \max_{t \in [0,1]} a(t)(h(R_1 + \epsilon)(R_1 + \epsilon) + \int_0^{R_1+\epsilon} \omega(s)ds) \right), \quad t \in (0,1). \]

Since

\[ x_0(0) = \frac{\mu_0}{1 - \sum_{i=1}^{m-2} a_i} \int_0^{1/n} -x'_0(s)ds - \frac{\mu_0}{1 - \sum_{i=1}^{m-2} a_i} \int_0^{\epsilon} -x'_0(s)ds \]

\[ \leq \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \int_0^{1/n} -I^{-1}\left( I(-\epsilon) - \max_{t \in [0,1]} a(t) \{ h(R_1 + \epsilon)(R_1 + \epsilon) \right.

\[ + \int_0^{R_1+\epsilon} \omega(s)ds \left. \right)ds + \frac{\mu_0}{1 - \sum_{i=1}^{m-2} a_i} \int_0^{\epsilon} -I^{-1}\left( I(-\epsilon) \right. \]

\[ - \max_{t \in [0,1]} a(t)(h(R_1 + \epsilon)(R_1 + \epsilon) + \int_0^{R_1+\epsilon} \omega(s)ds) \right)ds \]

\[ = \frac{1 + \sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} a_i} \left( -I^{-1}\left( I(-\epsilon) - \max_{t \in [0,1]} a(t)(h(R_1 + \epsilon)(R_1 + \epsilon) \right. \]

\[ + \int_0^{R_1+\epsilon} \omega(s)ds \right) \right). \]

Since \( x_0(0) \geq x_0(t) \geq \gamma \| x_0 \|_1 \geq \gamma \delta \| x_0 \|_2, \) \( x_0(0) \geq \| x_0 \| = R_1. \) So

\[ \frac{R_1}{1 - \sum_{i=1}^{m-2} a_i (I^{-1}[I(-\epsilon) - \max_{t \in [0,1]} a(t)](h(R_1 + \epsilon)(R_1 + \epsilon) + \int_0^{R_1+\epsilon} \omega(s)ds))} \leq 1, \]

which is a contradiction to (4.3). Then (4.4) holds.

From Lemma 2.1, for \( n \in N_0, \)

\[ i(A_n, \Omega_1 \cap P, P) = 1. \] (4.10)

Now we show that there exists a set \( \Omega_2 \) such that

\[ A_n x \not\leq x, \quad \forall x \in \partial \Omega_2 \cap P. \] (4.11)

Choose \( a^*, N^* \) as in section 3. Let

\[ \Omega_2 = \{ x \in C^1[0,1] : \| x \| < \frac{R_2}{a^*} \}. \]

Then

\[ A_n x \not\leq x, \quad \forall x \in \partial \Omega_2 \cap P. \]
In fact, if there exists \( x_0 \in \partial \Omega_2 \cap P \) with \( x_0 \geq A_n x_0 \). By the definition of the cone and Lemma 2.3 one has

\[
x_0(t) \geq \gamma \|x_0\|_1 \geq \gamma x(0) \geq \gamma \delta \|x_0\|_2,
\]

and so \( x_0(t) \geq \frac{R_2}{a^*} > R_2 \), for all \( t \in [0, 1] \), \( x_0(t) + \frac{1}{a} > R_2 \), from (3.11),

\[
\gamma x_0(t) \geq \gamma A_n x_0(t)
\]

\[
= \gamma \left( - \int_0^t (t - s) a(s) f(s, x_0(s) + \frac{1}{n} - |x'_0(s)| - \frac{1}{n}) ds 
\right.
\]

\[
+ \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1 - s) a(s) f(s, x_0(s) + \frac{1}{n} - |x'_0(s)| - \frac{1}{n}) ds 
\right.
\]

\[
- \sum_{i=1}^{m-2} a \int_0^t (\xi - s) a(s) f(s, x_0(s) + \frac{1}{n} - |x'_0(s)| - \frac{1}{n}) ds 
\left) \right)
\]

\[
\geq \gamma \left( - \int_0^t (t - s) a(s) f(s, x_0(s) + \frac{1}{n} - |x'_0(s)| - \frac{1}{n}) ds 
\right.
\]

\[
+ \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1 - s) a(s) f(s, x_0(s) + \frac{1}{n} - |x'_0(s)| - \frac{1}{n}) ds 
\right.
\]

\[
- \sum_{i=1}^{m-2} a \int_0^t (\xi - s) a(s) f(s, x_0(s) + \frac{1}{n} - |x'_0(s)| - \frac{1}{n}) ds 
\left) \right)
\]

\[
\geq \gamma \left( \sum_{i=1}^{m-2} a_i \int_0^1 a(s) f(s, x_0(s) + \frac{1}{n} - |x'_0(s)| - \frac{1}{n}) ds 
\right.
\]

\[
- \sum_{i=1}^{m-2} a \int_0^t (\xi - s) a(s) f(s, x_0(s) + \frac{1}{n} - |x'_0(s)| - \frac{1}{n}) ds 
\left) \right)
\]

\[
= \gamma \sum_{i=1}^{m-2} a_i (1 - \xi) \int_0^1 a(s) f(s, x_0(s) + \frac{1}{n} - |x'_0(s)| - \frac{1}{n}) ds 
\]

\[
\geq \gamma \sum_{i=1}^{m-2} a_i (1 - \xi) \int_0^1 a(s) g(x_0(s) + \frac{1}{n} - |x'_0(s)| - \frac{1}{n}) ds 
\]

\[
\geq \gamma \sum_{i=1}^{m-2} a_i (1 - \xi) \int_0^1 a(s) ds N^* x_0(s) 
\]

\[
\geq a^* \gamma \sum_{i=1}^{m-2} a_i (1 - \xi) \int_0^1 a(s) ds N^* \frac{R_2}{a^*} > \frac{R_2}{a^*}.
\]

Then \( \|x_0\| \geq \|x_0\|_1 > \frac{R_2}{a^*} \), which is a contradiction to \( x_0 \in \partial \Omega_2 \cap P \). Then (4.11) holds.

From Lemma 2.2

\[
i(A_n, \Omega_2 \cap P, P) = 0. \quad (4.12)
\]

This equality and (4.10) guarantee,

\[
i(A_n, (\Omega_2 - \bar{\Omega}_1) \cap P, P) = -1. \quad (4.13)
\]

From this equality and (4.10), \( A_n \) has two fixed points with \( x_{n,1} \in \Omega_1 \cap P, x_{n,2} \in (\Omega_2 - \bar{\Omega}_1) \cap P \). For each \( n \in N_0 \), there exists \( x_{n,1} \in \Omega_1 \cap P \) with \( x_{n,1} = A_n x_{n,1} \);
that is,

\[ x_{n,1}(t) = - \int_0^t (t-s)a(s)f(s,x_{n,1}(s) + \frac{1}{n},-x'_{n,1}(s)) - \frac{1}{n} ds \]

\[ + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1-s)a(s)f(s,x_{n,1}(s) + \frac{1}{n},-x'_{n,1}(s)) - \frac{1}{n} ds \right) \]

\[ - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s,x_{n,1}(s) + \frac{1}{n},-x'_{n,1}(s)) - \frac{1}{n} ds \].

(4.14)

As in the same proof of (3.5), \( x'_n(t) \leq 0, t \in (0,1) \) and

\[ x'_n(t) = - \int_0^t a(s)f(s,x_{n,1}(s) + \frac{1}{n},x'_{n,1}(s)) - \frac{1}{n} ds, \quad n \in N_0, \ t \in (0,1). \]

Now we consider \( \{x_n(t)\}_{n \in N_0} \) and \( \{x'_n(t)\}_{n \in N_0} \), since \( \|x_n\| \leq R_1 \), it follows that

\[ \{x_n(t)\} \text{ is uniformly bounded on } [0,1], \quad (4.15) \]

\[ \{x'_n(t)\} \text{ is uniformly bounded on } [0,1]. \quad (4.16) \]

Then

\[ \{x_n(t)\} \text{ is equicontinuous on } [0,1]. \quad (4.17) \]

As in the same proof of (3.6),

\[ x''_n(t) + a(t)f(t,x_{n,1}(t) + \frac{1}{n},x'_{n,1}(t)) - \frac{1}{n} = 0, \quad 0 < t < 1, \]

\[ x'_n(0) = 0, \quad x_{n,1}(1) = \sum_{i=1}^{m-2} a_i \xi_i. \quad (4.18) \]

Now we show for all \( t_1, \ t_2 \) in \([0,1]\),

\[ |I(x'_{n,1}(t_2) - \frac{1}{n}) - I(x'_{n,1}(t_1) - \frac{1}{n})| \]

\[ \leq \max_{t \in [0,1]} a(t) [h(R_1 + \epsilon)(|x_{n,1}(t_2) - x_{n,1}(t_1)| + |t_2 - t_1|) \]

\[ + | \int_{x_{n,1}(t_1) + \frac{1}{n}(1-t_1)}^{x_{n,1}(t_2) + \frac{1}{n}(1-t_2)} \omega(s) ds|]. \quad (4.19) \]

From (4.18), it follows that for \( t \in (0,1), \)

\[ -x''_n(t) = a(t)f(t,x_{n,1}(t) + \frac{1}{n},x'_{n,1}(t) - \frac{1}{n}) \]

\[ \leq a(t)[h(x_{n,1}(t) + \frac{1}{n}) + \omega(x_{n,1}(t) + \frac{1}{n})][g(x'_{n,1}(t) - \frac{1}{n}) + r(x'_{n,1}(t) - \frac{1}{n})], \]

\[ x''_n(t) = -a(t)f(t,x_{n,1}(t) + \frac{1}{n},x'_{n,1}(t) - \frac{1}{n}) \]

\[ \geq -a(t)[h(x_{n,1}(t) + \frac{1}{n}) + \omega(x_{n,1}(t) + \frac{1}{n})][g(x'_{n,1}(t) - \frac{1}{n}) + r(x'_{n,1}(t) - \frac{1}{n})], \]
and so for \( t \in (0, 1) \),

\[
-x''_{n,1}(t)(x'_{n,1}(t) - \frac{1}{n}) \frac{g(x'_{n,1}(t) - \frac{1}{n})}{g(x'_{n,1}(t) \frac{1}{n})} + r(x'_{n,1}(t) - \frac{1}{n}) \\
\geq a(t)[h(x_{n,1}(t) + \frac{1}{n}) + \omega(x_{n,1}(t) + \frac{1}{n})](x'_{n,1}(t) - \frac{1}{n}) \\
\geq a(t)[h(R_1 + \epsilon)(x'_{n,1}(t) - \frac{1}{n}) + \omega(x_{n,1}(t) + \frac{1}{n}(1 - t))(x_{n,1}(t) + \frac{1}{n}(1 - t))'].
\]

(4.20)

Then, for all \( t_1, t_2 \in [0, 1] \) and \( t_1 < t_2 \),

\[
I(x'_{n,1}(t_2) - \frac{1}{n}) - I(x'_{n,1}(t_1) - \frac{1}{n}) \\
\geq \max_{t \in [0,1]} a(t)[h(R_1 + \epsilon)(\int_{t_1}^{t_2} x'_{n,1}(s)ds - \frac{1}{n}(t_2 - t_1)) + \int_{x_{n,1}(t_1) + \frac{1}{n}(1 - t_2)}^{x_{n,1}(t_2) + \frac{1}{n}(1 - t_1)} \omega(s)ds] \\
\geq - \max_{t \in [0,1]} a(t)[h(R_1 + \epsilon)(|x_{n,1}(t_2) - x_{n,1}(t_1)| + |t_2 - t_1|) \\
+ \int_{x_{n,1}(t_1) + \frac{1}{n}(1 - t_1)}^{x_{n,1}(t_2) + \frac{1}{n}(1 - t_2)} \omega(s)ds].
\]

(4.21)

Therefore, (4.19) holds. Since \( I^{-1} \) is uniformly continuous on \([0, I(-R_1 - \epsilon)]\), for all \( \epsilon > 0 \), there exists \( \epsilon' > 0 \) such that

\[
|I^{-1}(s_1) - I^{-1}(s_2)| < \epsilon', \forall |s_1 - s_2| < \epsilon', s_1, s_2 \in [0, I(-R_1 - \epsilon)].
\]

(4.22)

Then (4.19) guarantees that for \( \epsilon' > 0 \), there exists \( \delta' > 0 \) such that

\[
|x'_{n,1}(t_2) - \frac{1}{n}) - I(x'_{n,1}(t_1) - \frac{1}{n})| < \epsilon', \ \forall |t_1 - t_2| < \delta', t_1, t_2 \in [0, 1].
\]

(4.23)

From this inequality and (4.22),

\[
|x'_{n,1}(t_2) - x'_{n,1}(t_1)| = |x'_{n,1}(t_2) - \frac{1}{n} - (x'_{n,1}(t_1) - \frac{1}{n})| \\
= |I^{-1}(I(x'_{n,1}(t_2) - \frac{1}{n})) - I^{-1}(I(x'_{n,1}(t_1) - \frac{1}{n}))| \\
< \epsilon', \ \forall |t_1 - t_2| < \delta', t_1, t_2 \in [0, 1];
\]

(4.24)
From (4.15)–(4.17), (4.25) and the Arzela-Ascoli Theorem, \( \{x_{n,1}(t)\} \) and \( \{x'_{n,1}(t)\} \) are relatively compact on \( C[0,1] \), which implies, there exists a subsequence \( \{x_{n,j}\} \) of \( \{x_{n,1}\} \) and function \( x_{0,1}(t) \in C[0,1] \) such that

\[
\lim_{j \to +\infty} \max_{t \in [0,1]} |x_{n,j}(t) - x_{0,1}(t)| = 0, \quad \lim_{j \to +\infty} \max_{t \in [0,1]} |x'_{n,j}(t) - x'_{0,1}(t)| = 0.
\]

Since \( x'_{n,1}(0) = 0 \), \( x_{n,j}(1) = \sum_{i=1}^{m-2} a_i x_{n,j}(\xi_i) \), \( x'_{n,1}(t) < 0 \), \( x_{n,j}(t) > 0 \), \( t \in (0,1), j \in \{1,2,\ldots\} \),

\[
x'_{0,1}(0) = 0, x_{0,1}(1) = \sum_{i=1}^{m-2} a_i x_{0,1}(\xi_i), x'_{0,1}(t) \leq 0, x_{0,1}(t) \geq 0, \quad t \in (0,1).
\]

For \( (t,x_{n,j}(t) + \frac{1}{n_j}, x'_{n,j}(t) - \frac{1}{n_j}) \in [0,1] \times [0,R_1 + \epsilon] \times (-\infty,0) \), from (P5) there exists a function \( \Psi_{R_1} \in C([0,1],\mathbb{R}_+) \) such that

\[
f(t,x_{n,j}(t) + \frac{1}{n_j}, x'_{n,j}(t) - \frac{1}{n_j})ds \geq \Psi_{R_1}(t).
\]

Then, for \( n \in \mathbb{N}_0 \),

\[
x_{n,j}(t) = -\int_0^t (t-s)a(s)f(s,x_{n,j}(s) + \frac{1}{n_j}, x'_{n,j}(s) - \frac{1}{n_j})ds \\
+ \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1-s)a(s)f(s,x_{n,j}(s) + \frac{1}{n_j}, x'_{n,j}(s) - \frac{1}{n_j})ds \\
- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s,x_{n,j}(s) + \frac{1}{n_j}, x'_{n,j}(s) - \frac{1}{n_j})ds \right) \\
\geq -\int_0^1 (1-s)a(s)f(s,x_{n,j}(s) + \frac{1}{n_j}, x'_{n,j}(s) - \frac{1}{n_j})ds \\
+ \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1-s)a(s)f(s,x_{n,j}(s) + \frac{1}{n_j}, x'_{n,j}(s) - \frac{1}{n_j})ds \\
- \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s,x_{n,j}(s) + \frac{1}{n_j}, x'_{n,j}(s) - \frac{1}{n_j})ds \right) \\
\geq \sum_{i=1}^{m-2} a_i \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 (1-s)a(s)f(s,x_{n,j}(s) + \frac{1}{n_j}, x'_{n,j}(s) - \frac{1}{n_j})ds \\
- \sum_{i=1}^{m-2} a_i \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \int_0^{\xi_i} (\xi_i - s)a(s)f(s,x_{n,j}(s) + \frac{1}{n_j}, x'_{n,j}(s) - \frac{1}{n_j})ds \\
= \sum_{i=1}^{m-2} a_i \frac{(1 - \xi_i)}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 a(s)f(s,x_{n,j}(s) + \frac{1}{n_j}, x'_{n,j}(s) - \frac{1}{n_j})ds \\
\geq \sum_{i=1}^{m-2} a_i \frac{(1 - \xi_i)}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 a(s)\Psi_{R_1}(s)ds = a_0,
\]

and

\[
x'_{n,j}(t) = -\int_0^t a(s)f(s,x_{n,j}(s) + \frac{1}{n_j}, x'_{n,j}(s) - \frac{1}{n_j})ds.
\]
Then (4.28) guarantees that (P3) holds. 

We apply Theorem 4.1 with $\mu < $ sup $a(t)(c + c^{1-d} + e^{1+b})$. Then equation (1.1) has at least two positive solutions $x_{0,1}, x_{0,2} \in C^1[0,1] \cap C^2(0,1).$
5. Singularities at \(x = 0\) but not \(x' = 0\)

In this section the nonlinearity \(f\) may be singular at \(x = 0\), but not at \(x' = 0\). We assume that the following conditions hold.

\(\text{(S1)}\) \(a(t) \in C([0, 1])\), \(a(t) > 0\), \(t \in (0, 1)\);

\(\text{(S2)}\) \(f(t, u, z) \leq \frac{1}{2} [h(u) + \omega(u)] r(z)\), where \(f \in C([0, 1] \times \mathbb{R}_+ \times \mathbb{R}_-\), \(\omega(u) > 0\) continuous and non-increasing on \([0, +\infty)\), \(h(u) \geq 0\) continuous and non-decreasing on \(\mathbb{R}_+\), \(r(z) > 0\) continuous and non-decreasing on \(\mathbb{R}_-\);

\(\text{(S3)}\) 
\[
\sup_{c \in \mathbb{R}_+} \frac{c}{\sum_{i=1}^{m-2} a_i \xi_i + 1} (I^{-1} [- \max_{t \in [0, 1]} a(t)(ch(c) + \int_0^t \omega(s)ds)]) > 1,
\]
where \(I(z) = \int_0^z \frac{adu}{r(u)}\), \(z \in \mathbb{R}_-\), \(\int_0^z \omega(s)ds < +\infty\), \(a \in \mathbb{R}_+\);

\(\text{(S4)}\) There exists a function \(g_1 \in C([0, +\infty) \times (-\infty, 0], [0, +\infty))\) such that \(f(t, u, z) \geq g_1(u, z, \forall (t, u, z) \in [0, 1] \times \mathbb{R}_+ \times \mathbb{R}_-\), and \(\lim_{u \to +\infty} \frac{g_1(u, z)}{u} = +\infty\), uniformly for \(z \in \mathbb{R}_-\);

\(\text{(S5)}\) There exists a function \(\Psi_H \in C([0, 1], \mathbb{R}_+\) and a constant \(0 \leq \delta\) such that \(f(t, u, z) \geq \Psi_H(t)(-\delta)^2\), for all \((t, u, z) \in [0, 1] \times [0, H] \times [-H, 0]\).

For \(n \in \{1, 2, \ldots\}\), \(x \in P\), define operator

\[
(A_n)x(t) = - \int_0^t (t-s) a(s) f(s, x(s) + \frac{1}{n}, -|x'(s)|)ds + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1-s) a(s) f(s, x(s) + \frac{1}{n}, -|x'(s)|)ds \right) - \sum_{i=1}^{m-2} a_i \int_0^1 (\xi_i - s) a(s) f(s, x(s) + \frac{1}{n}, -|x'(s)|)ds, \quad t \in [0, 1].
\]

**Theorem 5.1.** Suppose \((\text{S1})\)–\((\text{S5})\) hold. Then \((1.1)\) has at least two positive solutions \(x_{0,1}, x_{0,2}\) in \(C^1([0, 1])\) \(\cap C^2(0, 1)\) with \(x_{0,1}(t), x_{0,2}(t) > 0\), \(t \in [0, 1]\).

**Proof.** Choose \(R_1 > 0\) such that

\[
R_1 \left( \frac{\sum_{i=1}^{m-2} a_i \xi_i + 1}{1 - \sum_{i=1}^{m-2} a_i} (I^{-1} [- \max_{t \in [0, 1]} a(t)(R_1 h(R_1) + \int_0^{R_1} \omega(s)ds)]) \right) > 1.
\]
From the continuity of \(I^{-1}\) and \(h\), we can choose \(\epsilon > 0\) and \(\epsilon < R_1\) such that

\[
R_1 \left( \frac{\sum_{i=1}^{m-2} a_i \xi_i + 1}{1 - \sum_{i=1}^{m-2} a_i} (I^{-1} [- \max_{t \in [0, 1]} a(t)(R_1 h(R_1 + \epsilon) + \int_0^{R_1+\epsilon} \omega(s)ds)]) \right) > 1.
\]
Let \(n_0 \in \{1, 2, \ldots\}\) with \(\frac{1}{n_0} < \min(\epsilon, \delta/2)\) and let \(N_0 = \{n_0, n_0 + 1, \ldots\}\). Then Lemma 2.4 guarantees that for \(n \in N_0\), \(A_n : P \to P\) is a completely continuous operator. Let \(\Omega_1 = \{x \in C^1([0, 1]) : \|x\| < R_1\}\).

We show that

\[
x \neq \mu A_n x, \quad \forall x \in P \cap \partial \Omega_1, \quad \mu \in (0, 1], \quad n \in N_0.
\]
In fact, if there exists an \( x_0 \in P \cap \partial \Omega_1 \) and \( \mu_0 \in (0, 1] \) with \( x_0 = \mu_0 A_n x_0 \).

\[
x_0(t) = -\mu_0 \int_0^t (t - s)a(s)f(s, x_0(s)) + \frac{1}{n}, -|x_0'(s)|)ds
\]
\[
+ \frac{\mu_0}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1 - s)a(s)f(s, x_0(s)) + \frac{1}{n}, -|x_0'(s)|)ds
\]
\[
- \sum_{i=1}^{m-2} a_i \int_0^1 (\xi_i - s)a(s)f(s, x_0(s)) + \frac{1}{n}, -|x_0'(s)|)ds \right).
\]

Then

\[
x_0'(t) = -\mu_0 \int_0^t a(s)f(s, x_0(s)) + \frac{1}{n}, -|x_0'(s)|)ds, \quad \forall t \in [0, 1]. \tag{5.5}
\]

Obviously, \( x_0'(t) \leq 0, \) \( t \in (0, 1) \), and since \( x_0(1) > 0 \), \( x_0(t) > 0, t \in [0, 1] \). Differentiating (5.5), we have

\[
x_0''(t) + \mu_0 a(t)f(t, x_0(t)) + \frac{1}{n}, x_0'(t)) = 0, 0 < t < 1,
\]
\[
x_0'(0) = 0, \quad x_0(1) = \sum_{i=1}^{m-2} a_i x_0(\xi_i). \tag{5.6}
\]

and

\[
-x_0''(t) = \mu_0 a(t)f(t, x_0(t)) + \frac{1}{n}, x_0'(t)) \leq a(t)[h(x_0(t) + \frac{1}{n}) + \omega(x_0(t) + \frac{1}{n})]r(x_0'(t)), \quad \forall t \in (0, 1).
\]

Then

\[
\frac{-x_0''(t)}{r(x_0'(t))} \leq a(t)[h(x_0(t) + \frac{1}{n}) + \omega(x_0(t) + \frac{1}{n})], \quad \forall t \in (0, 1),
\]

and

\[
\frac{-x_0''(t)x_0'(t)}{r(x_0'(t))} \geq a(t)[h(R_1 + \epsilon) + \omega(x_0(t) + \frac{1}{n})]x_0'(t)
\]
\[
\geq a(t)[h(R_1 + \epsilon) + \omega(x_0(t) + \frac{1}{n})]x_0'(t).
\]

Integrating from 0 to \( t \), we have

\[
I(x_0'(t)) \geq \int_0^t a(s)[h(R_1 + \epsilon)x_0'(s)]ds + \omega(x_0(s) + \frac{1}{n})x_0'(s)|ds
\]
\[
\geq \max_{t \in [0, 1]} a(t) \int_0^t [h(R_1 + \epsilon)x_0'(s)\omega(x_0(s) + \frac{1}{n})]ds
\]
\[
\geq - \max_{t \in [0, 1]} a(t)(h(R_1 + \epsilon)R_1 + \int_0^t \omega(x_0(s) + \frac{1}{n})d(x_0(s) + \frac{1}{n}))
\]
\[
= - \max_{t \in [0, 1]} a(t)(h(R_1 + \epsilon)R_1 + \frac{1}{x_0(t)+\frac{1}{n}})\omega(s)ds
\]
\[
\geq - \max_{t \in [0, 1]} a(t)(h(R_1 + \epsilon)R_1 + \int_{(0, t)+\frac{1}{n}}^R \omega(s)ds)
\]
and
\[ I(x_0'(t)) \geq - \max_{t \in [0,1]} a(t)(h(R_1 + \epsilon)R_1 + \int_0^{R_1 + \epsilon} \omega(s)ds). \]

Then
\[ x_0'(t) \geq I^{-1}( - \max_{t \in [0,1]} a(t)(h(R_1 + \epsilon)R_1 + \int_0^{R_1 + \epsilon} \omega(s)ds) ); \]
that is,
\[ -x_0'(t) \leq -I^{-1}( - \max_{t \in [0,1]} a(t)(h(R_1 + \epsilon)R_1 + \int_0^{R_1 + \epsilon} \omega(s)ds) ), \quad t \in (0,1). \] (5.7)

Since \( x_0(0) \)
\[ = \frac{\mu_0}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 -x_0'(s)ds - \frac{\mu_0 \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} -x_0'(s)ds}{1 - \sum_{i=1}^{m-2} a_i} \]
\[ \leq \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 -I^{-1}( - \max_{t \in [0,1]} a(t)(h(R_1 + \epsilon)R_1 + \int_0^{R_1 + \epsilon} \omega(s)ds) )ds \]
\[ + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} -I^{-1}( - \max_{t \in [0,1]} a(t)(h(R_1 + \epsilon)R_1 + \int_0^{R_1 + \epsilon} \omega(s)ds) )ds}{1 - \sum_{i=1}^{m-2} a_i} \]
\[ = \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \xi_i (I^{-1}( - \max_{t \in [0,1]} a(t)(h(R_1 + \epsilon)R_1 + \int_0^{R_1 + \epsilon} \omega(s)ds) ), \]
\[ x_0(0) \geq x_0(t) \geq \gamma_1 \| x_0 \| \geq \gamma_2 \| x_0 \|_2 \geq x_0(0) \geq \| x_0 \| = R_1. \] (5.8)

So
\[ R_1 \]
\[ \leq \frac{1 - \sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i} (I^{-1}[ - \max_{t \in [0,1]} a(t)(R_1 h(R_1 + \epsilon) + \int_0^{R_1 + \epsilon} \omega(s)ds)] \]
which is a contradiction to (5.3). Then (5.4) holds. From Lemma 2.1 for \( n \in N_0 \),
\[ i(A_n, \Omega_1 \cap P, P) = 1. \] (5.10)

Now we show that there exists a set \( \Omega_2 \) such that
\[ A_n x \not\subset x, \quad \forall x \in \partial \Omega_2 \cap P. \] (5.11)

Choose \( a^*, N^* \) as in section 3. Let
\[ \Omega_2 = \{ x \in C^1[0,1] : \| x \| < \frac{R_2}{a^*} \}. \]

Then
\[ A_n x \not\subset x, \quad \forall x \in \partial \Omega_2 \cap P. \]

In fact, if there exists \( x_0 \in \partial \Omega_2 \cap P \) with \( x_0 \geq A_n x_0 \), by the definition of the cone and Lemma 2.3 one has
\[ x_0(t) \geq \gamma \| x_0 \|_1 \geq \gamma x(0) \geq \gamma \| x_0 \|_2, \]
\[ x_0(t) \geq \frac{R_2}{a^*} > R_2 \] for all \( t \in [0,1] \). Then \( x_0(t) + \frac{1}{n} > R_2 \). From (3.11),
\[ \gamma x_0(t) \geq \gamma A_n x_0(t) \]
\[ = \gamma \left( - \int_0^t (t-s)a(s)f(s, x_0(s) + \frac{1}{n}, x_0'(s))ds \right) \]
\[ \frac{1}{+1-\sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1-s) a(s) f(s, x_0(s) + \frac{1}{n}, x_0'(s)) ds \right) \]
\[ \geq \gamma \left( -\int_0^t (t-s) a(s) f(s, x_0(s) + \frac{1}{n}, x_0'(s)) ds \right) \]
\[ \geq \gamma \left( -\int_0^t \left( t-s \right) a(s) f(s, x_0(s) + \frac{1}{n}, x_0'(s)) ds \right) \]
\[ \geq \gamma \left( -\int_0^t \left( t-s \right) a(s) f(s, x_0(s) + \frac{1}{n}, x_0'(s)) ds \right) \]
\[ = \gamma \left( -\sum_{i=1}^{m-2} \int_0^1 a(s) f(s, x_0(s) + \frac{1}{n}, x_0'(s)) ds \right) \]
\[ \geq \gamma \left( -\sum_{i=1}^{m-2} a_i \int_0^1 (1-s) a(s) f(s, x_0(s) + \frac{1}{n}, x_0'(s)) ds \right) \]
\[ \geq \gamma \left( -\sum_{i=1}^{m-2} a_i \int_0^1 (1-s) a(s) f(s, x_0(s) + \frac{1}{n}, x_0'(s)) ds \right) \]
\[ = \gamma \left( -\sum_{i=1}^{m-2} \int_0^1 a(s) f(s, x_0(s) + \frac{1}{n}, x_0'(s)) ds \right) \]
\[ \geq \gamma \left( -\sum_{i=1}^{m-2} a_i \int_0^1 (1-s) a(s) f(s, x_0(s) + \frac{1}{n}, x_0'(s)) ds \right) \]
\[ = \gamma \left( -\sum_{i=1}^{m-2} a_i \int_0^1 (1-s) a(s) f(s, x_0(s) + \frac{1}{n}, x_0'(s)) ds \right) \]
\[ = \gamma \left( -\sum_{i=1}^{m-2} a_i \int_0^1 (1-s) a(s) f(s, x_0(s) + \frac{1}{n}, x_0'(s)) ds \right) \]
\[ = \gamma \left( -\sum_{i=1}^{m-2} a_i \int_0^1 (1-s) a(s) f(s, x_0(s) + \frac{1}{n}, x_0'(s)) ds \right) \]

that is, \( \|x_0\| \geq \gamma \|x_0\|_1 > \frac{R_2}{a^*} \), which is a contradiction to \( x_0 \in \partial \Omega_2 \cap P \). Then (5.11) holds. From Lemma 2.2

\[ i(A_n, \Omega_2 \cap P, P) = 0. \] (5.12)

This equality and (5.10) guarantee

\[ i(A_n, (\Omega_2 - \tilde{\Omega}_1 \cap P, P) = -1. \] (5.13)

From this equality and (5.10), \( A_n \) has two fixed points with \( x_{n,1} \in \Omega_1 \cap P, x_{n,2} \in (\Omega_2 - \tilde{\Omega}_1) \cap P \).

For each \( n \in N_0 \), there exists \( x_{n,1} \in \Omega_1 \cap P \) such that \( x_{n,1} = A_n x_{n,1} \); that is, for \( t \in [0, 1] \),

\[ x_{n,1}(t) = -\int_0^t (t-s) a(s) f(s, x_{n,1}(s) + \frac{1}{n}, -|x_{n,1}'(s)|) ds \]
\[ + \frac{1}{1-\sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1-s) a(s) f(s, x_{n,1}(s) + \frac{1}{n}, -|x_{n,1}'(s)|) ds \right) \]
\[ - \sum_{i=1}^{m-2} a_i \int_0^1 (1-s) a(s) f(s, x_{n,1}(s) + \frac{1}{n}, -|x_{n,1}'(s)|) ds \].
As in the proof of (3.5), \( x''_{n,1}(t) \leq 0, \ t \in (0, 1) \) and
\[
x''_{n,1}(t) = - \int_0^t a(s)f(s, x_{n,1}(s) + \frac{1}{n}, x'_{n,1}(s))ds, \quad n \in N_0, \ t \in (0, 1).
\]
Now we consider \( \{x_{n,1}(t)\}_{n \in N_0} \) and \( \{x'_{n,1}(t)\}_{n \in N_0} \), since \( \|x_{n,1}\| \leq R_1 \),
\[
\{x_{n,1}(t)\} \text{ is uniformly bounded on } [0, 1], \quad (5.15)
\]
\[
\{x'_{n,1}(t)\} \text{ is uniformly bounded on } [0, 1]. \quad (5.16)
\]
Then
\[
\{x_{n,1}(t)\} \text{ is equicontinuous on } [0, 1]. \quad (5.17)
\]
As in the proof of (3.6),
\[
x''_{n,1}(t) + a(t)f(t, x_{n,1}(t) + \frac{1}{n}, x'_{n,1}(t)) = 0, 0 < t < 1,
\]
\[
x'_{n,1}(0) = 0, x_{n,1}(1) = \sum_{i=1}^{m-2} a_i x_{n,1}(\xi_i). \quad (5.18)
\]
Now we show that for all \( t_1, t_2 \in [0, 1] \),
\[
|I(x'_{n,1}(t_2)) - I(x'_{n,1}(t_1))| \leq \max_{t \in [0, 1]} a(t) \left[ h(R_1 + \epsilon)(|x_{n,1}(t_2) - x_{n,1}(t_1)| + |t_2 - t_1|) \right.
\]
\[
+ \left. \int_{x_{n,1}(t_1)}^{x_{n,1}(t_2) + \frac{1}{n} (1 - t_2)} \omega(t)dt \right]. \quad (5.19)
\]
From (5.18),
\[
-x''_{n,1}(t) = a(t)f(t, x_{n,1}(t) + \frac{1}{n}, x'_{n,1}(t)) \leq a(t)[h(x_{n,1}(t) + \frac{1}{n}) + \omega(x_{n,1}(t) + \frac{1}{n})]r(x'_{n,1}(t)), \quad \forall t \in (0, 1),
\]
\[
x''_{n,1}(t) = -a(t)f(t, x_{n,1}(t) + \frac{1}{n}, x'_{n,1}(t)) \geq -a(t)[h(x_{n,1}(t) + \frac{1}{n}) + \omega(x_{n,1}(t) + \frac{1}{n})]r(x'_{n,1}(t)), \quad \forall t \in (0, 1),
\]
\[
\frac{-x''_{n,1}(t)x'_{n,1}(t)}{r(x'_{n,1}(t))} \geq a(t)[h(x_{n,1}(t) + \frac{1}{n}) + \omega(x_{n,1}(t) + \frac{1}{n})]x'_{n,1}(t) \quad (5.20)
\]
\[
= a(t)[h(R_1 + \epsilon)x'_{n,1}(t) + \omega(x_{n,1}(t) + \frac{1}{n} (1 - t))x'_{n,1}(t)], \quad \forall t \in (0, 1),
\]
\[
\frac{x''_{n,1}(t)x'_{n,1}(t)}{r(x'_{n,1}(t))} \leq -a(t)[h(x_{n,1}(t) + \frac{1}{n}) + \omega(x_{n,1}(t) + \frac{1}{n})]x'_{n,1}(t) \quad (5.21)
\]
\[
= -a(t)[h(R_1 + \epsilon)x'_{n,1}(t) + \omega(x_{n,1}(t) + \frac{1}{n} (1 - t))x'_{n,1}(t)], \quad \forall t \in (0, 1).
\]
Since the right-hand sides of (5.20) and (5.21) are positive, for all \( t_1, t_2 \in [0, 1] \) and \( t_1 < t_2 \),
\[
I(x_{n,2}^\prime(t_2)) - I(x_{n,1}^\prime(t_1)) \\
\geq \max_{t \in [0,1]} a(t)[h(R_1 + \epsilon) \left( \int_{t_1}^{t_2} x_{n,1}^\prime(s)dt + \int_{t_1}^{t_2} \omega(x_{n,1}(s) + \frac{1}{n})x_{n,1}^\prime(s)ds \right)] \\
\geq \max_{t \in [0,1]} a(t)[h(R_1 + \epsilon)(|x_{n,1}(t_2) - x_{n,1}(t_1)|) + \int_{x_{n,1}(t_1)}^{x_{n,1}(t_2)} \omega(s)dt], \\
I(x_{n,1}^\prime(t_1)) - I(x_{n,1}^\prime(t_2)) \\
\leq \max_{t \in [0,1]} a(t)[h(R_1 + \epsilon)(|x_{n,1}(t_2) - x_{n,1}(t_1)|) + \int_{x_{n,1}(t_1)}^{x_{n,1}(t_2)} \omega(s)dt],
\]
(5.19) holds.

Since \( I^{-1} \) is uniformly continuous on \([0, I(-R_1 - \epsilon)]\), for all \( \epsilon > 0 \), there exists \( \epsilon' > 0 \) such that
\[
|I^{-1}(s_1) - I^{-1}(s_2)| < \epsilon', \quad \forall |s_1 - s_2| < \epsilon', \quad s_1, s_2 \in [0, I(-R_1 - \epsilon)]. \tag{5.22}
\]
Also (5.19) guarantees that for \( \epsilon' > 0 \), there exists \( \delta' > 0 \) such that
\[
|I(x_{n,1}^\prime(t_2)) - I(x_{n,1}^\prime(t_1))| < \epsilon', \quad \forall |t_1 - t_2| < \delta', \quad t_1, t_2 \in [0, 1]. \tag{5.23}
\]
From this inequality and (5.22),
\[
|x_{n,1}^\prime(t_2) - x_{n,1}^\prime(t_1)| = |I^{-1}(I(x_{n,1}^\prime(t_2))) - I^{-1}(I(x_{n,1}^\prime(t_1)))| < \epsilon, \\
\forall |t_1 - t_2| < \delta', \quad t_1, t_2 \in [0, 1]; \tag{5.24}
\]
that is,
\[
\{x_{n,1}^\prime(t)\} \text{ is equi-continuous on } [0, 1]. \tag{5.25}
\]
From (5.15)–(5.17), (5.25) and the Arzela-Ascoli Theorem, \( \{x_{n,1}(t)\} \) and \( \{x_{n,1}^\prime(t)\} \) are relatively compact on \( C^1[0, 1] \). This implies, there exists a subsequence \( \{x_{n_j,1}(t)\} \) of \( \{x_{n,1}(t)\} \) and function \( x_{0,1}(t) \in C^1[0, 1] \) such that
\[
\lim_{j \to +\infty} \max_{t \in [0,1]} |x_{n_j,1}(t) - x_{0,1}(t)| = 0, \quad \lim_{j \to +\infty} \max_{t \in [0,1]} |x_{n_j,1}^\prime(t) - x_{0,1}^\prime(t)| = 0.
\]
Since \( x_{n_j,1}(0) = 0, x_{n_j,1}(1) = \sum_{i=1}^{m-2} a_i x_{n_j,1}^\prime(\xi_i), x_{n_j,1}^\prime(t) < 0, x_{n_j,1}(t) > 0, t \in (0, 1), j \in \{1, 2, \ldots\}, \)
\[
x_{0,1}(0) = 0, x_{0,1}(1) = \sum_{i=1}^{m-2} a_i x_{0,1}(\xi_i), x_{0,1}^\prime(t) \leq 0, x_{0,1}(t) \geq 0, t \in (0, 1). \tag{5.26}
\]
For \( (t, x_{n_j,1}(t) + \frac{1}{n_j}, x_{n_j,1}^\prime(t)) \in [0, 1] \times [0, R_1 + \epsilon] \times (-\infty, 0) \), from (S5) there exists a function \( \Psi_{R_1} \in C([0, 1], \mathbb{R}_+) \) such that
\[
f(t, x_{n_j,1}(t) + \frac{1}{n_j}, x_{n_j,1}^\prime(t))ds \geq \Psi_{R_1}(t)(-x_{n_j,1}^\prime(t))^\delta, \quad 0 \leq \delta < 1.
\]
Then, for \( n \in N_0, \)
\[
x_{n_j,1}^\prime(t) = -\int_{0}^{t} a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x_{n_j,1}^\prime(s))ds \\
\leq -\int_{0}^{t} a(s)\Psi_{R_1+\epsilon}(s)(-x_{n_j,1}^\prime(s))^\delta ds, \quad t \in [0, 1], n \in N_0,
\]
which implies
\[ x'_{n_j,1}(t) \leq -(1 - \delta)(\int_0^t a(s)\Psi_{R_1 + \epsilon}(s)ds)^{\frac{1}{(1 - \delta)}}. \]

and
\[
x_{n_j,1}(t) = -\int_0^t (t - s)a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s))ds
+ \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \left( \int_0^1 (1 - s)a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s))ds \right)
- \sum_{i=1}^{m-2} a_i \int_0^1 (\xi_i - s)a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s))ds
\geq -\int_0^1 (1 - s)a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s))ds
+ \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 (1 - s)a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s))ds
- \sum_{i=1}^{m-2} a_i \int_0^1 (\xi_i - s)a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s))ds
\geq \sum_{i=1}^{m-2} a_i \int_0^1 a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s))ds
- \sum_{i=1}^{m-2} a_i \int_0^1 (\xi_i - s)a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s))ds
= \sum_{i=1}^{m-2} a_i (1 - \xi_i) \int_0^1 a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s))ds
\geq \sum_{i=1}^{m-2} a_i (1 - \xi_i) \int_0^1 a(s)\Psi_{R_1 + \epsilon}(s)(-x'_{n_j,1}(s))\delta ds
\geq \sum_{i=1}^{m-2} a_i (1 - \xi_i) \int_0^1 a(s)\Psi_{R_1 + \epsilon}(s)((1 - \delta)(\int_0^t a(s)\Psi_{R_1 + \epsilon}(s)ds)^{\frac{1}{1 - \delta}})\delta ds
=: F, \quad t \in [0, 1].
\]

Since
\[
f(t, x_{n_j,1}(t) + \frac{1}{n_j}, x'_{n_j,1}(t))
\leq \left[ h(x_{n_j,1}(t) + \frac{1}{n_j}) + \omega(x_{n_j,1}(t) + \frac{1}{n_j}) \right] r(x'_{n_j,1}(t))
\leq \left[ h\left( \frac{R_1}{\gamma} + \epsilon \right) + \omega(F) \right] r(\delta - 1)\left( \int_0^t a(s)\Psi_{R_1 + \epsilon}(s)ds \right)^{\frac{1}{1 - \delta}}),
\]
and
\[
x'_{n_j,1}(t) - x'_{n_j,1}(\frac{1}{2}) = -\int_{1/2}^t a(s)f(s, x_{n_j,1}(s) + \frac{1}{n_j}, x'_{n_j,1}(s))ds, \quad t \in (0, 1),
\]
Then (1.1) has at least two positive solutions $x_{0,1}(t) = x_{0,1}(1/2) = -\int_{1/2}^{t} a(s)f(s, x_{0,1}(s), x'_{0,1}(s))ds$, $t \in (0, 1)$.

Differentiating, we have

$$x''_{0,1}(t) + a(t)f(t, x_{0,1}(t), x'_{0,1}(t)) = 0, \quad 0 < t < 1.$$  

From this equality and from (5.26), $x_{0,1}(t)$ is a positive solution of (1.1) with $x_{0,1} \in C^1[0, 1] \cap C^2(0, 1)$.

For the set $\{x_{n,2}\}_{n \in N_0} \subseteq (\Omega_2 - \Omega_1) \cap P$, as in the proof for the set $\{x_{n,1}\}_{n \in N_0}$, we obtain a convergent subsequence $\{x_{n,2}\}_{n \in N_0}$ of $\{x_{n,2}\}_{n \in N_0}$ with $\lim_{n \to +\infty} x_{n,2} = x_{0,2} \in C^1[0, 1] \cap C^2(0, 1)$. Moreover, $x_{0,2}$ is a positive solution to (1.1).

**Example 5.1** In (1.1), let $f(t, u, z) = \mu[1 + (-2)^a][1 + u^b + u^{-d}]$ and $a(t) \equiv 1$ with $0 \leq a < 1, b > 1, 0 < d < 1$ and $\mu > 0$. If

$$\mu < \sup_{c \in \mathbb{R}^+} \frac{I(c^{1-\sum_{i=1}^{m-2} a_i})}{\max_{t \in [0, 1]}(c + c^{1-d} + \frac{c^{d-1}}{1+t})},$$

Then (1.1) has at least two positive solutions $x_{0,1}, x_{0,2} \in C^1[0, 1] \cap C^2(0, 1)$.

**References**


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