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# EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR A CLASS OF NONAUTONOMOUS DIFFERENCE EQUATIONS

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Abstract. In this article, we investigate the existence of positive periodic solutions for a class of non-autonomous difference equations. Using the Krasnoselskii fixed point theorem, we establish sufficient criteria that are easily verifiable and that generalize and improve related studies in the literature. Numerical simulations are presented which support our theoretical results for some concrete models.

## 1. Introduction

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}_+$ the set of nonnegative numbers. Let $\mathbb{R}^n_+ = \{(x_1, \dots, x_n)^T : x_i \geq 0,\ 1 \leq i \leq n\}$. Let $\mathbb{Z}$ denote the set of integers and $\mathbb{Z}_+$ the set of nonnegative integers.

In this paper, we apply a cone fixed point theorem due to Krasnoselskii to investigate the existence of positive periodic solutions for the non-autonomous difference equations

$$\Delta x(k) = a(k)x(k) - f(k, u(k)), \tag{1.1}$$

and

$$\Delta x(k) = -a(k)x(k) + f(k, u(k)), \tag{1.2}$$

where $\Delta x(k) = x(k+1) - x(k)$, and for $k, s \in \mathbb{Z}$

$$u(k) = \Big(x(g_1(k)), x(g_2(k)), \dots, x(g_{n-1}(k)), \sum_{s=-\infty}^{k} h(k-s)x(s)\Big). \tag{1.3}$$

It is well-known that (1.1) includes many mathematical ecological difference models. For example, (1.1) includes the generalized discrete single species model

$$\Delta x(k) = x(k)[a(k) - \sum_{i=1}^{n} b_i(k)x(k-\tau_i(k)) - c(k)\sum_{s=-\infty}^{k} h(k-s)x(k)]. \tag{1.4}$$

Equation (1.1) includes also the single species discrete periodic population models [5, 9, 10, 12, 13, 16, 19, 20, 24, 31]

$$\Delta x(k) = a(k)x(k)\big[1 - \frac{x(k-\tau(k))}{H(k)}\big] \tag{1.5}$$

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and

$$\Delta x(k) = x(k)[a(k) - \sum_{i=1}^{n} b_i(k)x(k-\tau_i(k))]\,. \tag{1.6}$$

Equation (1.1) includes also the multiplicative delay periodic Logistic difference equation [5, 6, 12, 13, 20, 23, 24]

$$\Delta x(k) = a(k)x(k)\Big[1 - \prod_{i=1}^{n} \frac{x(k-\tau_i(k))}{H(k)}\Big] \tag{1.7}$$

In addition, (1.1) includes the periodic Michaelis-Menton discrete model [12, 18, 20]

$$\Delta x(k) = a(k)x(k)\Big[1 - \sum_{i=1}^{n} \frac{a_i(k)x(k-\tau_i(k))}{1+c_i(k)x(k-\tau_i(k))}\Big] \tag{1.8}$$

Similarly, model (1.2) includes many ecological equations. See for example the discrete Hematopoiesis model [10, 11, 13, 26, 28]

$$\Delta x(k) = -a(k)x(k) + b(k)\exp\{-\beta(k)x(k-\tau(k))\} \tag{1.9}$$

and the more general discrete models of blood cell production [6, 10, 11, 13, 20, 26]

$$\Delta x(k) = -a(k)x(k) + \frac{b(k)}{1+x(k-\tau(k))^n}, \quad n \in Z_+\,, \tag{1.10}$$

$$\Delta x(k) = -a(k)x(k) + \frac{b(k)x(k-\tau(k))}{1+x(k-\tau(k))^n}, \quad n \in Z_+ \tag{1.11}$$

Model (1.2) includes also the discrete Nicholson's blowflies model [8, 11, 14, 26, 28]

$$\Delta x(k) = -a(k)x(k) + b(k)x(k-\tau(k))\exp\{-\beta(k)x(k-\tau(k))\} \tag{1.12}$$

Studying the population dynamics, especial the existence of positive periodic solutions, has attracted much attention from both mathematicians and mathematical biologists recently. Many authors have investigated the existence of positive periodic solutions for several population models; see for example [1, 3, 9, 10, 11, 13, 15, 20, 21, 25, 26, 27, 28, 29, 30, 31, 32] and the references therein. In [3, 12, 20], the existence of one positive periodic solution was proved by using Mawhin's continuation theorem. In [4, 10, 11, 13, 26, 27, 30], the existence of multiple positive periodic solutions was studied by employing Krasnoselskii fixed point theorem in cones. The author in [30] obtained sufficient criteria for the existence of multiple positive periodic solutions to (1.1) and (1.2), in the continuous case by applying Krasnoselskii fixed point theorem.

To the best of the author's knowledge, there are very few works on the existence of positive periodic solutions for (1.1) and (1.2). In [31] periodic solutions of a single species discrete population model with periodic harvest/stock was discussed. The authors in [9, 21, 27] studied the existence of positive periodic solutions of some discrete equations, however, they are special cases of (1.1) and (1.2).

Motivated by the work above, in the present paper, we aim to study systematically the existence of positive periodic solution of (1.1) and (1.2) under general conditions by employing the Krasnoselskii fixed point theorem. The conditions in our main theorem can easily be checked in practice.

For the sake of convenience and simplicity, we will apply the below notations throughout this paper. Let

$$f^M = \max_{k\in I_\omega} f(k), \quad f^m = \min_{k\in I_\omega} f(k),$$
$$|u| = \max_{1\le j\le n}\{u_i\}, \quad u\in \mathbb{R}^n_+, \quad I_\omega = \{0,1,\dots,\omega-1\},$$

where $f$ is an $\omega$-periodic function from $\mathbb{Z}$ to $\mathbb{R}$.

Assume the following limits exist and let

$$\max f_0 = \lim_{|u|\downarrow 0}\max_{k\in I_\omega}\frac{f(k,u)}{|u|}, \quad \max f_\infty = \lim_{|u|\uparrow+\infty}\max_{k\in I_\omega}\frac{f(k,u)}{|u|},$$
$$\min f_0 = \lim_{|u|\downarrow 0,\ u_j\ge\sigma|u|,\ 1\le j\le n}\ \min_{k\in I_\omega}\frac{f(k,u)}{|u|},$$
$$\min f_\infty = \lim_{|u|\uparrow+\infty,\ u_j\ge\sigma|u|,\ 1\le j\le n}\ \min_{k\in I_\omega}\frac{f(k,u)}{|u|}.$$

The general assumptions are stated as follows:

| | | | |
|---|---|---|---|
| (P1) | $\min f_0 = \infty$ | (P2) | $\min f_\infty = \infty$ |
| (P3) | $\max f_\infty = 0$ | (P4) | $\max f_0 = 0$ |
| (P5) | $\max f_0 = \alpha_1 \in [0, \frac{1}{B\omega})$ | (P6) | $\min f_\infty = \beta_1 \in (\frac{1}{A\sigma\omega}, \infty)$ |
| (P7) | $\min f_0 = \alpha_2 \in (\frac{1}{A\sigma\omega}, \infty)$ | (P8) | $\max f_\infty = \beta_2 \in [0, \frac{1}{B\omega})$, |

with $A, B, \sigma$ to be defined below. In addition, the parameters in this paper are assumed to be not identically equal to zero.

To conclude this section, we state a few concepts and results that will be needed in this paper.

**Definition.** Let $X$ be Banach space and $E$ be a closed, nonempty subset. $E$ is said to be a cone if

(i) $\alpha u + \beta v \in E$ for all $u, v \in E$ and all $\alpha, \beta > 0$
(ii) $u, -u \in E$ imply $u = 0$

**Lemma 1.1** (Krasnoselskii fixed point theorem)**.** *Let $X$ be a Banach space,and let $E$ be a cone in $X$. Suppose $\Omega_1$ and $\Omega_2$ are open subsets of $X$ such that $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$. Suppose that*

$$T : E \cap (\bar{\Omega}_2 \setminus \Omega_1) \to E$$

*is a completely continuous operator and satisfies either*

(i) *$\|Tx\| \ge \|x\|$ for any $x \in E \cap \partial\Omega_1$ and $\|Tx\| \le \|x\|$ for any $x \in E \cap \partial\Omega_2$; or*
(ii) *$\|Tx\| \le \|x\|$ for any $x \in E \cap \partial\Omega_1$ and $\|Tx\| \ge \|x\|$ for any $x \in E \cap \partial\Omega_2$.*

*Then $T$ has a fixed point in $E \cap (\bar{\Omega}_2 \setminus \Omega_1)$.*

## 2. Positive periodic solutions of (1.1)

In this section, we establish sufficient criteria for the existence of positive periodic solutions to (1.1). We assume the following hypotheses:

(H1) $a : \mathbb{Z} \to (0, +\infty)$ is continuous and $\omega$-periodic , i.e., $a(k) = a(k+\omega)$, such that $a(k) \not\equiv 0$, where $\omega$ is a positive constant denoting the common period of the system;
(H2) $f : \mathbb{Z} \times \mathbb{R}^n_+ \to \mathbb{R}_+$ is continuous and $\omega$-periodic with respect to the first variable, i.e.,$f(k+\omega, u_1, \dots, u_n) = f(k, u_1, \dots, u_n)$ such that $f(k,u) \not\equiv 0$;

(H3) $h : \mathbb{Z}_+ \to \mathbb{R}_+$ is continuous and satisfies $\sum_{r=0}^{\infty} h(r) = 1, g_i : \mathbb{Z} \to \mathbb{Z}$ is continuous $\omega$-periodic function and satisfies $g_i(k) < k$.

Let $X = \{x(k) : x(k+\omega) = x(k)\}$, $\|x\| = \max\{|x(k)| : x \in X\}$, and

$$\sigma = [\prod_{r=0}^{\omega-1}(1+a(r))]^{-1} .$$

Then $X$ is a Banach space endowed with the norm $\|\cdot\|$.

To prove the existence of positive solutions to (1.1), we first give the following lemmas.

**Lemma 2.1.** *If $x(k)$ is a positive $\omega$-periodic solution of* (1.1)*, then $x(k) \geq \sigma\|x\|$.*

*Proof.* It is clear that (1.1) is equivalent to

$$x(k+1) = x(k)(a(k)+1) - f(k, u(k)),$$

and that it can be written as

$$\Delta\Big(x(k)\prod_{s=0}^{k-1}\frac{1}{1+a(s)}\Big) = -\prod_{s=0}^{k}\frac{1}{1+a(s)}f(k,u(k)),$$

By summing the above equation from $k = n$ to $k = n+\omega-1$, since $x(k) = x(k+\omega)$, we obtain

$$x(k) = \sum_{s=0}^{\omega-1} G(k,s)f(s,u(s)), \quad k, s \in Z \tag{2.1}$$

where

$$G(k,s) = \frac{\prod_{r=s+1}^{k+\omega-1}(1+a(r))}{\prod_{r=0}^{\omega-1}(1+a(r))-1}, \quad k \leq s \leq k+\omega-1$$

Then, $x(k)$ is an $\omega$-periodic solution of (1.1) if and only if $x(k)$ is an $\omega$-periodic solution of difference equation (2.1). A direct calculation shows that

$$A := \frac{1}{\prod_{s=0}^{\omega-1}(1+a(s))-1} \leq G(k,s) \leq \frac{\prod_{s=0}^{\omega-1}(1+a(s))}{\prod_{s=0}^{\omega-1}(1+a(s))-1} =: B \tag{2.2}$$

Clearly

$$A = \frac{\sigma}{1-\sigma}, \quad B = \frac{1}{1-\sigma}, \quad \sigma = \frac{A}{B} < 1,$$

$$\|x\| \leq B\sum_{k=0}^{\omega-1} f(s,u(s)), \quad x(t) \geq A\sum_{k=0}^{\omega-1} f(s,u(s)).$$

Therefore,

$$x(k) \geq A\sum_{k=0}^{\omega-1} f(s,u(s)) \geq \frac{A}{B}\|x\| = \sigma\|x\|.$$

□

Define a mapping $T : X \to X$ by

$$(Tx)(k) = \sum_{s=0}^{\omega-1} G(k,s)f(s,u(s)), \tag{2.3}$$

for $x \in X, k \in Z$. Clearly, $T$ is a continuous and completely continuous operator on $X$. Notice that finding a periodic solution of (1.1) is equivalent to establishing a fixed point of operator $T$.

Define
$$E = \{x \in X : x(k) \geq 0, x(k) \geq \sigma\|x\|\}.$$
One may readily verify that $E$ is a cone.

**Lemma 2.2.** *With the definitions above, $TE \subset E$.*

*Proof.* In view of the arguments in the proof of Lemma 2.1, for each $x \in E$, we have
$$\|Tx\| \leq B\sum_{t=0}^{\omega-1} f(s, u(s)),$$
By (2.3), one can obtain
$$(Tx)(k) \geq A\sum_{k=0}^{\omega-1} f(s, u(s)) \geq \frac{A}{B}\|Tx\| = \sigma\|Tx\|.$$
Therefore, $Tx \in E$. This completes the proof. □

Now, we are in the position to state the main results in this section.

**Theorem 2.3.** *If* (P1) *and* (P3) *are satisfied,then* (1.1) *has at least one positive $\omega$-periodic solution.*

*Proof.* By (P1), for any $M_1 > 1/(A\sigma\omega)$, one can find a $r_0 > 0$ such that
$$f(k, u) \geq M_1|u|, \quad \text{for } u_j \geq \sigma|u|, 1 \leq j \leq n, |u| \leq r_0. \tag{2.4}$$
Let $\Omega_{r_0} = \{x \in X : \|x\| < r_0\}$. Note, if $x \in E \cap \partial\Omega_{r_0}$ with $\|x\| = r_0$, then $x(k) \geq \sigma\|x\| = \sigma r_0$. So, from Lemma 2.1 and $u(k)$ defined by (1.3), we obtain
$$u_j(k) = x(g_j(k)) \geq \sigma\|x\| \geq \sigma|u|, \quad j = 1, \dots, n-1,$$
$$u_n(k) = \sum_{s=-\infty}^{k} h(k-s)x(s) \geq \sigma\|x\| \sum_{s=-\infty}^{k} h(k-s) = \sigma\|x\| \geq \sigma|u|.$$
Then
$$|u| = \max_{1\leq j\leq n-1}\{x(g_j(k)), \sum_{s=-\infty}^{k} h(k-s)x(s)\} \geq \sigma\|x\| \geq \sigma|u|.$$
Therefore, by (2.3) and (2.4), we have
$$(Tx)(k) \geq A\sum_{s=0}^{\infty} f(s, u(s)) \geq AM_1\omega|u| \geq AM_1\sigma\omega r_0 \geq r_0 = \|x\|.$$
This implies that $\|Tx\| \geq \|x\|$ for any $x \in E \cap \partial\Omega_{r_0}$. Again, by (P3), for any $0 < \varepsilon \leq 1/(2B\omega)$, there exists an $N_1 > r_0$ such that
$$f(k, u) \leq \varepsilon|u|, \quad \text{for } |u| \geq N_1. \tag{2.5}$$
Choose
$$r_1 > N_1 + 1 + 2B\omega \max\{f(k, u) : k \in I_\omega,\ |u| \leq N_1,\ u \in \mathbb{R}^n_+\}.$$

Let $\Omega_{r_1} = \{x \in X : \|x\| < r_1\}$. If $x \in E \cap \partial\Omega_{r_1}$, then

$$\begin{aligned}(Tx)(k) &\le B \sum_{s=0}^{\omega-1} f(s, u(s)) \\ &\le B \sum_{s=0,\ |u(s)|\le N_1}^{\omega-1} f(s, u(s)) + B \sum_{s=0,\ |u(s)|>N_1}^{\omega-1} f(s, u(s)) \\ &\le \frac{r_1}{2} + B\omega\varepsilon\|x\| \\ &\le \frac{r_1}{2} + \frac{\|x\|}{2} = \|x\|.\end{aligned}$$

This implies that $\|Tx\| \le \|x\|$ for any $x \in \partial\Omega_{r_1}$.

In conclusion, under the assumptions (P1) and (P3), $T$ satisfies all the requirements in Lemma 1.1. Then $T$ has a fixed point $E \cap (\bar{\Omega}_{r_1} \setminus \Omega_{r_0})$. Clearly, we have $r_0 \le \|x\| \le r_1$ and $x(k) \ge \sigma\|x\| \ge \sigma r_0 > 0$, which shows that $x(k)$ is a positive $\omega$-periodic solution of (2.1). By Lemma 2.1, $x(k)$ is a positive $\omega$-periodic solution of (1.1). This completes the proof. □

**Theorem 2.4.** *If* (P2) *and* (P4) *are satisfied, then* (1.1) *has at least one positive $\omega$-periodic solution.*

*Proof.* By (P4), for any $0 < \varepsilon \le 1/(B\omega)$, there exists $r_2 > 0$, such that

$$f(k, u) \le \varepsilon|u|, \quad \text{for } |u| \le r_2. \tag{2.6}$$

Define $\Omega_{r_2} = \{x \in X : \|x\| < r_2\}$. If $x \in E \cap \partial\Omega_{r_2}$, then by (2.2), (2.3) and (2.6), we have

$$(Tx)(k) \le B \sum_{s=0}^{\omega-1} f(s, u(s)) \le B\varepsilon|u|\omega \le B\varepsilon\|x\|\omega \le \|x\|.$$

In particular, $\|Tx\| \le \|x\|$, for all $x \in E \cap \partial\Omega_{r_2}$. Next, by (P2), for any $M_2 \ge 1/(A\sigma\omega)$, there exists a $r_3 > \frac{r_2}{\sigma}$ such that

$$f(k, u) \ge M_2|u| \quad \text{for } u_j \ge \sigma|u|, \quad 1 \le j \le n, |u| \ge \sigma r_3. \tag{2.7}$$

Define $\Omega_{r_3} = \{x \in X : \|x\| < r_3\}$. If $x \in E \cap \partial\Omega_{r_3}$, then

$$u_j(k) = x(g_j(k)) \ge \sigma\|x\| = \sigma r_3 \ge \sigma|u| \quad j = 1, \dots, n-1,$$

$$u_n(k) = \sum_{s=-\infty}^{k} h(k-s)x(s) \ge \sigma\|x\| = \sigma r_3 \ge \sigma|u|,$$

$$|u(k)| = \max_{1\le j\le n-1}\{x(g_j(k)), \sum_{s=-\infty}^{k} h(k-s)x(s)d\} \ge \sigma r_3,$$

Therefore, by (2.2), (2.3), and (2.7), we get

$$(Tx)(k) \ge A \sum_{s=0}^{\omega-1} f(s, u(s)) \ge AM_2\sigma\|x\|\omega \ge \|x\|.$$

In particular, $\|Tx\| \ge \|x\|$ for all $x \in E \cap \partial\Omega_{r_3}$. By Lemma 1.1, there exists a fixed point $x \in E \bigcap(\bar{\Omega}_{r_3} \setminus \Omega_{r_2})$ satisfying $r_2 \le \|x\| \le r_3$. That is, $x(k)$ is a positive $\omega$-periodic solution of (1.1). □

Now, we introduce two extra assumptions to be used in the next theorems.

(P9) There exists $d_1 > 0$ such that $f(k,u) > d_1/(A\omega)$, for $|u| \in [\sigma d_1, d_1]$.
(P10) There exists $d_2 > 0$ such that $f(k,u) < d_2/(B\omega)$, for $|u| \le d_2$.

**Theorem 2.5.** *If* (P3), (P4), (P9) *are satisfied, then* (1.1) *has at least two positive $\omega$-periodic solutions $x_1$ and $x_2$ satisfying* $0 < \|x_1\| < d_1 < \|x_2\|$.

*Proof.* By assumption (P4), for any $0 < \varepsilon \le 1/(B\omega)$, there exists a $r_4 < d_1$ such that
$$f(k,u) \le \varepsilon |u|, \quad \text{for } |u| \le r_4 \tag{2.8}$$
Define $\Omega_{r_4} = \{x \in X : \|x\| < r_4\}$. Then for any $x \in E \cap \partial\Omega_{r_4}$, we have $\|x\| = r_4$. From (2.2), (2.3) and (2.8), we obtain
$$(Tx)(k) \le B \sum_{s=0}^{\omega-1} f(s,u(s)) \le B\varepsilon r_4 \omega \le r_4 = \|x\|,$$
which implies $\|Tx\| \le \|x\|$ for all $x \in E \cap \partial\Omega_{r_4}$. Likewise, from (P3), for any $0 < \varepsilon \le 1/(2B\omega)$, there exists an $N_2 > d_1$ such that
$$f(k,u) \le \varepsilon |u|, \quad \text{for } |u| \ge N_2. \tag{2.9}$$
Choose
$$r_5 > N_2 + 1 + 2B\omega \max\{f(k,u) : k \in I_\omega,\ |u| \le N_2,\ u \in \mathbb{R}^n_+\} \tag{2.10}$$
Let $\Omega_{r_5} = \{x \in X : \|x\| < r_5\}$. If $x \in E \cap \partial\Omega_{r_5}$, then by (2.3), (2.4), and (2.10), we have
$$\begin{aligned}
(Tx)(k) &\le B \sum_{s=0}^{\omega-1} f(s,(u(s)) \\
&= B \sum_{s=0,\ |u(s)| \le N_2}^{\omega-1} f(s,u(s)) + B \sum_{s=0,\ |u(s)| > N_2}^{\omega-1} f(s,u(s)) \\
&\le \frac{r_5}{2} + \frac{\|x\|}{2} = \|x\|.
\end{aligned}$$
Which shows that $\|Tx\| \le \|x\|$ for all $x \in E \cap \partial\Omega_{r_5}$.

Set $\Omega_{d_1} = \{x \in X : \|x\| < d_1\}$. Then, for any $x \in E \cap \partial\Omega_{d_1}$, we have $x(k) \ge \sigma\|x\| = \sigma d_1$. Consequently,
$$u_j(k) = x(g_j(k)) \ge \sigma\|x\| = \sigma d_1 \quad j = 1, \dots, n-1,$$
$$u_n(k) = \sum_{s=-\infty}^{k} h(k-s)x(s) \ge \sigma\|x\| = \sigma d_1,$$
That is
$$|u(k)| = \max_{1 \le j \le n-1} \{x(g_j(k)), \sum_{s=-\infty}^{k} h(k-s)x(s)\} \ge \sigma d_1.$$
Thus, by (1.3), (2.3), (P9), we have
$$(Tx)(k) \ge A \sum_{s=o}^{\omega-1} f(s,u(s)) > A \frac{d_1}{A\omega}\omega = d_1 = \|x\|.$$
This yields $\|Tx\| > \|x\|$ for all $x \in E \cap \partial\Omega_{d_1}$. By Lemma 1.1, there exist two positive $\omega$-periodic solutions $x_1$ and $x_2$ satisfying $0 < \|x_1\| < d_1 < \|x_2\|$. This completes the proof. $\square$

From the arguments in the above proof, we have the following result immediately.

**Corollary 2.6.** *If* (P1), (P2), (P10) *are satisfied, then* (1.1) *has at least two $\omega$-periodic solutions $x_1$ and $x_2$ satisfying* $0 < \|x_1\| < d_2 < \|x_2\|$.

To obtain better results in this section, we give a more general criterion in the following, which plays an important role later.

**Theorem 2.7.** *Suppose that* (P9) *and* (P10) *are satisfied, then* (1.1) *has at least one positive $\omega$-periodic solution $x$ with $\|x\|$ lying between $d_2$ and $d_1$, where $d_1$ and $d_2$ are defined in* (P9) *and* (P10)*, respectively.*

*Proof.* Without loss of generality, we assume that $d_2 < d_1$. Set $\Omega_{d_2} = \{x \in X : \|x\| < d_2\}$. If $x \in E \cap \partial\Omega_{d_2}$, then from (2.2), (2.3) and (P10), we get

$$(Tx)(k) \le B \sum_{s=0}^{\omega-1} f(s, u(s)) < B \frac{d_2}{B\omega}\omega = d_2 = \|x\|,$$

In particular, $\|Tx\| < \|x\|$ for all $x \in E \cap \partial\Omega_{d_2}$.

Choose $\Omega_{d_1} = \{x \in X : \|x\| < d_1\}$. For any $x \in E \cap \partial\Omega_{d_1}$, we have $x(k) \ge \sigma\|x\| = \sigma d_1$. Thus,

$$\sigma d_1 = \sigma\|x\| \le |u| = \max_{1 \le j \le n-1}\{x(g_j(k)), \sum_{s=-\infty}^{k} h(k-s)x(s)\} \le \|x\| = d_1,$$
$$u_j \ge \sigma\|x\| \ge \sigma|u| \quad (j = 1, 2, \dots, n).$$

From (2.3) and (P9), one has

$$(Tx)(k) \ge A \sum_{s=0}^{\omega-1} f(s, u(s)) > A \frac{d_1}{A\omega}\omega = d_1 = \|x\|.$$

This implies $\|Tx\| > \|x\|$ for all $x \in E \cap \partial\Omega_{d_1}$. Therefore, by Lemma 1.1, we can obtain the conclusion and this completes the proof. □

**Theorem 2.8.** *If* (P5) *and* (P6) *are satisfied, then* (1.1) *has at least one positive $\omega$-periodic solution.*

*Proof.* By assumption (P5), for any $\varepsilon = \frac{1}{B\omega} - \alpha_1 > 0$, there exists a sufficiently small $d_2 > 0$ such that

$$\max_{k \in I_\omega} \frac{f(k,u)}{|u|} < \alpha_1 + \varepsilon = \frac{1}{B\omega}, \quad \text{for } |u| \le d_2;$$

that is,

$$f(k,u) < \frac{1}{B\omega}|u| \le \frac{d_2}{B\omega} \quad \text{for } |u| \le d_2,\ k \in I_\omega$$

So, (P10) is satisfied. By the assumption (P6), for $\varepsilon = \beta_1 - \frac{1}{A\sigma\omega} > 0$, there exists a sufficiently large $d_1 > 0$ such that

$$\min_{k \in I_\omega} \frac{f(k,u)}{|u|} > \beta_1 - \varepsilon = \frac{1}{A\sigma\omega}, \quad \text{for } |u| \ge \sigma d_1,\ u_j \ge \sigma|u|,$$

Which leads to

$$f(k,u) > \frac{1}{A\sigma\omega}\sigma d_1 = \frac{d_1}{A\omega}, \quad \text{for } |u| \in [\sigma d_1, d_1], u_j \ge \sigma|u|$$

That is, (P9) holds. By Theorem 2.7 we complete the proof. □

**Theorem 2.9.** *If* (P7) *and* (P8) *are satisfied, then* (1.1) *has at least one positive $\omega$-periodic solution.*

*Proof.* By assumption (P7), for any $\varepsilon = \alpha_2 - \frac{1}{A\sigma\omega} > 0$, there exists a sufficiently small $d_1 > 0$ such that

$$\min_{k\in I_\omega} \frac{f(k,u)}{|u|} > \alpha_2 - \varepsilon = \frac{1}{A\sigma\omega} \quad \text{for } 0 \le |u| \le d_1,\ u_j \ge \sigma|u|$$

Therefore,

$$f(k,u) > \frac{1}{A\sigma\omega}\sigma d_1 = \frac{d_1}{A\omega} \quad \text{for } |u| \in [\sigma d_1, d_1], u_j \ge \sigma|u|$$

for $j = 1, 2, \dots, n$ and $k \in I_\omega$; that is, (P9) holds. By assumption (P8), for $\varepsilon = \frac{1}{B\omega} - \beta_2 > 0$, there exists a sufficiently large $d$ such that

$$\max_{k\in I_\omega} \frac{f(k,u)}{|u|} < \beta_2 + \varepsilon = \frac{1}{B\omega} \quad \text{for } |u| > d. \tag{2.11}$$

In the following, we consider two cases to prove (P10): $\max_{k\in I_\omega} f(k,u)$ bounded and unbounded. The bounded case is clear. If $\max_{k\in I_\omega} f(k,u)$ is unbounded, then there exists $u^* \in \mathbb{R}^n_+, |u^*| = d_2 > d$ and $k_0 \in I_\omega$ such that

$$f(k,u) \le f(k_0,u^*) \quad \text{for } 0 < |u| \le |u^*| = d_2. \tag{2.12}$$

Since $|u^*| = d_2 > d$, by (2.11) and (2.12), we have

$$f(k,u) \le f(k_0,u^*) < \frac{1}{B\omega}|u^*| = \frac{d_2}{B\omega} \quad \text{for } 0 < |u| \le d_2, k \in I_\omega$$

Which implies condition (P10) holds. Therefore, using Theorem 2.7 we complete the proof. □

**Theorem 2.10.** *Suppose that* (P6), (P7), (P10) *are satisfied, then* (1.1) *has at least two positive $\omega$-periodic solutions $x_1$ and $x_2$ satisfying* $0 < \|x_1\| < d_2 < \|x_2\|$, *where $d_2$ is defined in* (P10).

*Proof.* From (P6) and the proof of Theorem 2.8, we know that there exists a sufficiently large $d_1 > d_2$, such that $f(k,u) > d_1/(A\omega)$ for $|u| \in [\sigma d_1, d_1]$, $u_j \ge \sigma|u|$ ($j = 1, 2, \dots, n$). From (P7) and the proof of Theorem 2.9, we can find a sufficiently small $d_1^* \in (0, d_2)$ such that $f(k,u) > d_1^*/(A\omega)$ for $|u| \in [\sigma d_1^*, d_1^*]$, $u_j \ge \sigma|u|$ ($j = 1, 2, \dots, n$). Therefore, from the proof of Theorem 2.7, there exists two positive solutions $x_1$ and $x_2$ satisfying $d_1^* < \|x_1\| < d_2 < \|x_2\| < d_1$. □

From the arguments in the above proof, we have the following statement.

**Corollary 2.11.** *Suppose that* (P5), (P8) *and* (P9) *are satisfied, then* (1.1) *has at least two positive $\omega$-periodic solutions $x_1$ and $x_2$ satisfying* $0 < \|x_1\| < d_2 < \|x_2\|$, *where $d_1$ is defined in* (P9).

**Theorem 2.12.** *If* (P1) *and* (P8) *are satisfied, then* (1.1) *has at least one positive $\omega$-periodic solution.*

*Proof.* Let $\Omega_{r_0} = \{x \in X : \|x\| < r_0\}$. From assumption (P1) and the proof of Theorem 2.5, we know that $\|Tx\| \ge \|x\|$ for all $x \in E \cap \partial\Omega_{r_0}$. Choose $\Omega_{r_1} =$

$\{x \in X : \|x\| < r_1\}$. From (P8) and Theorem 2.9, as $|u| \le r_1$, we know that $f(k,u) < \frac{r_1}{B\omega}$ and

$$(Tx)(k) \le B \sum_{s=0}^{\omega-1} f(s,u(s)) < B \frac{r_1}{B\omega} \omega = r_1 = \|x\|.$$

Which implies $\|Tx\| < \|x\|$ for all $x \in E \cap \partial\Omega_{r_1}$. This completes the proof. $\square$

Similar to Theorem 2.12, one immediately has the following statements.

**Theorem 2.13.** *If* (P2) *and* (P5) *are satisfied, then* (1.1) *has at least one positive $\omega$-periodic solution.*

**Theorem 2.14.** *If* (P3) *and* (P7) *are satisfied, then* (1.1) *has at least one positive $\omega$-periodic solution.*

**Theorem 2.15.** *If* (P4) *and* (P6) *are satisfied, then* (1.1) *has at least one positive $\omega$-periodic solution.*

**Theorem 2.16.** *If* (P1), (P6) *and* (P10) *are satisfied, then* (1.1) *has at least two positive $\omega$-periodic solutions $x_1$ and $x_2$ satisfying* $0 < \|x_1\| < d_2 < \|x_2\|$, *where $d_2$ is defined in* (P10).

*Proof.* Let $\Omega_{r_*} = \{x \in X : \|x\| < r_*\}$, where $r_* < d_2$. By assumption (P1) and the proof of Theorem 2.3, we know $\|Tx\| \ge \|x\|$ for all $x \in E \cap \partial\Omega_{r_*}$. Let $\Omega_{d_1} = \{x \in X : \|x\| < d_1\}$. By the assumption (P6) and the proof of Theorem 2.7, we see that $f(k,u) > \frac{d_1}{A\omega}$ for $|u| \in [\sigma d_1, d_1]$. From (P10) and the proof of Theorem 2.7, we know that there exist two positive $\omega$-periodic solutions $x_1$ and $x_2$ satisfying $0 < \|x_1\| < d_2 < \|x_2\|$. $\square$

**Theorem 2.17.** *If* (P2), (P7), (P10) *are satisfied, then* (1.1) *has at least two positive $\omega$-periodic solutions $x_1$ and $x_2$ satisfying* $0 < \|x_1\| < d_2 < \|x_2\|$, *where $d_2$ is defined in* (P10).

**Theorem 2.18.** *If* (P3), (P5), (P9) *are satisfied, then* (1.1) *has at least two positive $\omega$-periodic solutions $x_1$ and $x_2$ satisfying* $0 < \|x_1\| < d_1 < \|x_2\|$, *where $d_1$ is defined in* (P9).

**Theorem 2.19.** *If* (P4), (P8), (P9) *are satisfied, then* (1.1) *has at least two positive $\omega$-periodic solutions $x_1$ and $x_2$ satisfying* $0 < \|x_1\| < d_1 < \|x_2\|$, *where $d_1$ is defined in* (P9).

## 3. Existence of periodic solution to (1.2)

In this section, we prove the existence of positive periodic solution to (1.2). By carrying out similar arguments as in Section 2, one can easily obtain sufficient criteria for the existence of positive periodic solutions of (1.2). We assume that (H2) and (H3) hold. Moreover we will use the assumption

- (H1') $a : \mathbb{Z} \to (0,1)$ is continuous and $\omega$-periodic function, i.e., $a(k) = a(k+\omega)$, such that $a(k) \not\equiv 0$, where $\omega$ is a positive constant denoting the common period of the system.

Define

$$\sigma = \prod_{s=0}^{\omega-1}(1-a(s)), \tag{3.1}$$

$$H(k,s) = \frac{\prod_{r=s+1}^{k+\omega-1}(1-a(r))}{1-\prod_{r=0}^{\omega-1}(1-a(r))},\ k,s \in Z, \tag{3.2}$$

From the definition of $H(k,s)$, for any $s \in [k, k+\omega-1]$, we have

$$A := \frac{\prod_{s=0}^{\omega-1}(1-a(s))}{1-\prod_{s=0}^{\omega-1}(1-a(s))} \le H(k,s) \le \frac{1}{1-\prod_{s=0}^{\omega-1}(1-a(s))} := B \tag{3.3}$$

Clearly,

$$A = \frac{\sigma}{1-\sigma}, \quad B = \frac{1}{1-\sigma}, \quad \sigma = \frac{A}{B} < 1.$$

**Lemma 3.1.** *$x(k)$ is an $\omega$-periodic solution of* (1.2) *if and only if it is an $\omega$-periodic solution of the difference equation*

$$x(k) = \sum_{s=k}^{k+\omega-1} H(k,s)f(s,u(s)). \tag{3.4}$$

Similarly, we can establish sufficient criteria for the existence of periodic solutions of (1.2). Now we list the corresponding criteria without proof.

**Theorem 3.2.** *Suppose that one of the following pairs of conditions holds:* (P1) *and* (P3)*, or* (P1) *and* (P8)*, or* (P2) *and* (P4)*, or* (P2) *and* (P5)*, or* (P3) *and* (P7)*, or* (P4) *and* (P6)*, or* (P5) *and* (P6)*, or* (P7) *and* (P8)*, or* (P9) *and* (P10)*. Then* (1.2) *has at least one positive $\omega$-periodic solution.*

**Theorem 3.3.** *Suppose that* (P9)*holds and one of the following pairs of conditions is satisfied:* (P3) *and* (P4)*, or* (P3) *and* (P5)*, or* (P4) *and* (P8)*, or* (P5) *and* (P8)*. Then* (1.2) *has at least two $\omega$-periodic solutions $x_1$ and $x_2$ satisfying $0 < \|x_1\| < d_1 < \|x_2\|$, where $d_1$ is defined in* (P9).

**Theorem 3.4.** *Suppose that* (P10) *and one of the following pairs of conditions is satisfied:* (P1) *and* (P2)*, or* (P1) *and* (P6)*, or* (P2) *and* (P7)*, or* (P6) *and* (P7)*. Then* (1.2) *has at least two $\omega$-periodic solutions $x_1$ and $x_2$ satisfying $0 < \|x_1\| < d_2 < \|x_2\|$, where $d_2$ is defined in* (P10)

**Remark 3.5.** In this section, the values of $A, B, \sigma$ in (P1)–(P8) should be replaced by corresponding values defined in (3.1) and (3.3).

## 4. Examples and numerical simulations

In this section, we apply our main results to investigate some classical biological models and test our theoretical results.

**Theorem 4.1.** *Assume that $a(k), c(k), b_i(k) \in C(Z,(0,+\infty))$, $\tau_i(k) \in C(Z,Z)$ $(i=1,\dots,n)$ are all $\omega$-periodic, then* (1.4) *has at least one positive $\omega$-periodic solution.*

*Proof.* Note that

$$f(k,u) = u_0(\sum_{i=1}^{n} b_i(k)u_i + c(k)u_{n+1}), \quad u = (u_0,\dots,u_{n+1}) \in \mathbb{R}_+^{n+2}.$$

Clearly, conditions (H1)–(H3) are satisfied. Moreover, when $|u| \to 0$, we have

$$\max_{k\in I_\omega} \frac{|f(k,u)|}{|u|} \le (\sum_{i=1}^{n} b_i^M + c^M)|u| \to 0,$$

Hence, $\max f_0 = 0$. In addition, if $u \in \mathbb{R}_+^{n+2}$ and $u_i > \sigma|u|$, then

$$\min_{k\in I_\omega} \frac{|f(k,u)|}{|u|} \ge \sigma^2(\sum_{i=1}^{n} b_i^m + c^m)|u| \to +\infty, \quad \text{as } |u| \to +\infty\,;$$

that is, $\min f_\infty = \infty$. Therefore (P2) and (P4) are satisfied. The claim follows from Theorem 2.4. $\square$

**Corollary 4.2.** *Assume that $a(k), b(k), H(k) \in C(Z,(0,+\infty))$, $\tau_i(k) \in C(Z,Z)$ ($i = 1,\dots,n$) are all $\omega$-periodic, then* (1.5) *has at least one positive $\omega$-periodic solution.*

**Corollary 4.3.** *Assume that $a(k), b(k), H(k) \in C(Z,(0,+\infty))$, $\tau_i(k) \in C(Z,Z)$ ($i = 1,\dots,n$) are all $\omega$-periodic, then* (1.6) *has at least one positive $\omega$-periodic solution.*

**Corollary 4.4.** *Assume that $a(k), H(k) \in C(Z,(0,+\infty))$, $\tau_i(k) \in C(Z,Z)$ ($i = 1,\dots,n$) are all $\omega$-periodic, then* (1.7) *has at least one positive $\omega$-periodic solution.*

**Theorem 4.5.** *Assume that $a(k), a_i(k), c_i(k) \in C(Z,(0,+\infty))$, $\tau_i(k) \in C(Z,Z)$ ($i = 1,\dots,n$) are all $\omega$-periodic. Also assume that*

$$a^m \sum_{i=1}^{n} \frac{a_i^m}{c_i^M} > \frac{1-\sigma}{\sigma^2\omega},$$

*where $\sigma = \prod_{k=0}^{\omega-1}(1+a(k))^{-1}$. Then* (1.8) *has at least one positive $\omega$-periodic solution.*

*Proof.* Note that

$$f(k,u) = a(k)u_0 \sum_{i=1}^{n} \frac{a_i(k)u_i}{1+c_i(k)u_i},$$

It is clear that (H1)–(H3) are satisfied. Moreover,

$$\max_{k\in I_\omega} \frac{|f(k,u)|}{|u|} \le a^M \sum_{i=1}^{n} \frac{a_i^M|u|}{1+c_i^m|u|} \to 0, \quad \text{as } |u| \to 0\,.$$

Thus $\max f_0 = 0$. In addition, when $|u| \to +\infty$, we have

$$\min_{k\in I_\omega} \frac{|f(k,u)|}{|u|} \ge a^m \sum_{i=1}^{n} \frac{a_i^m|u|}{1+c_i^M|u|} \to a^m \sum_{i=1}^{n} \frac{a_i^m}{c_i^M},$$

By the assumption, one has $\min f_\infty > \frac{1}{A\sigma\omega}$. Therefore, by Theorem 2.15,the proof is completed. $\square$

In [[31], the authors studied the periodic solution of a single species discrete population model with periodic harvest. Their model is

$$x(k+1) = \mu x(k)\big[1 - \frac{x(k)}{T}\big] + b(k), \quad k \in \mathbb{Z},$$

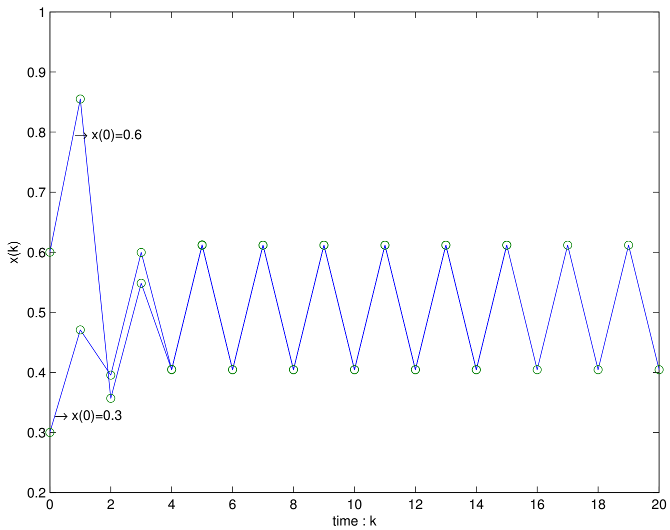


FIGURE 1. Existence of periodic solutions of (4.1)

where $\mu > 0, T > 0, b(t) \in C(Z, R), a(k) = a(k + \omega)$. Now we consider

$$\Delta x(k) = x(k)\big[a(k) - \frac{b(k)x(k)}{1 + cx(k)}\big],$$

whose growth law obeys Michaelis-Menton type growth equation. Moreover, we assume that the population subjects to harvesting. Under the catch-per-unit-effort hypothesis, the harvest population's growth equation can be written as

$$\Delta x(k) = x(k)\big[a(k) - \frac{b(k)x(k)}{1 + cx(k)}\big] - qEx(k), \tag{4.1}$$

where $a(k), b(k) \in C(Z, (0, +\infty))$ are $\omega$-periodic, $c$ is positive constant, $q$ and $E$ are positive constants denoting the catch-ability-coefficient and the harvesting effort,respectively.

**Theorem 4.6.** *If*

$$0 < qE < \frac{1 - \sigma}{\omega}, \quad \frac{b^m}{c} + qE > \frac{1 - \sigma}{\sigma^2 \omega},$$

*Then* (4.1) *has at least one positive $\omega$-periodic solution, where*

$$\sigma = \prod_{k=0}^{\omega - 1} (1 + a(k))^{-1}.$$

*Proof.* Note that

$$f(k, u) = \frac{b(k)u^2}{1 + cu} + qEu, \quad u \geq 0.$$

It is not difficult to show that

$$\max f_0 = qE, \quad \min f_\infty = \frac{b^m}{c} + qE.$$

The conditions in Theorem 4.6 guarantee that (P5) and (P6) hold. Then by Theorem 3.2, the proof is complete. $\square$

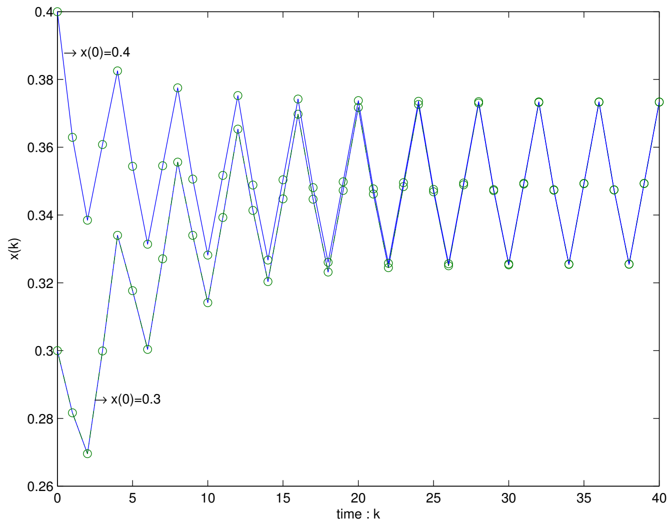


FIGURE 2. Existence of periodic solutions of (1.9)

**Theorem 4.7.** *Assume that* $a(k) \in C(Z,(0,1)), b(k), \beta(k) \in C(Z,(0,\infty))$, $\tau \in C(Z,Z)$ *are all* $\omega$*-periodic, then* (1.9) *has at least one positive* $\omega$*-periodic solution.*

*Proof.* It is clear that (H1')–(H3) are satisfied. Note that

$$f(k,u) = b(k)\exp\{-\beta(k)u\}, \quad u \in \mathbb{R}_+$$

Then

$$\min_{k\in I_\omega]} \frac{|f(k,u)|}{|u|} = \min_{k\in I_\omega} \frac{b(k)}{u\exp\{\beta(k)u\}} \geq \frac{b^m}{u\exp\{\beta^M u\}} \to +\infty, \quad u \to 0,$$

$$\max_{k\in I_\omega} \frac{|f(k,u)|}{|u|} \leq \frac{b^M}{u\exp\{\beta^m u\}} \to 0, \quad u \to +\infty.$$

Thus, $\min f_0 = \infty$ and $\max f_\infty = 0$. Thus (P1) and (P3) are satisfied. Theorem 2.3 proves the claim. □

**Example 4.8.** Consider the system (4.1) with $a(k) = 1 + \sin(k\pi)$, $b(k) = 2 + \cos(k\pi)$, $qE = 0.2$, $c = 1$. Then a sketch of the existence of periodic solutions is shown in figure 1.

**Example 4.9.** Consider again the system (1.9) with $a(k) = \frac{1}{8} + \frac{1}{16}\sin\frac{k\pi}{2}$, $b(k) = \frac{1}{16} + \frac{1}{32}\cos\frac{k\pi}{2}$, $\beta(k) \equiv 1$, $\tau(k) \equiv 1$. Periodic solutions of (1.9) is shown in figure 2.

**Theorem 4.10.** *Assume that* $a(k) \in C(Z,(0,1))$, $b(k) \in C(Z,(0,+\infty))$ *are all* $\omega$*-periodic. Moreover,* $b^m > \frac{1-\sigma}{\sigma^2\omega}$, *then* (1.11) *has at least one positive* $\omega$*-periodic solution. where* $\sigma = \prod_{k=0}^{\omega-1}(1+a(k))^{-1}$.

*Proof.* Note that $f(t,u) = b(k)u/(1+u^n)$. Then

$$\max_{k\in I_\omega} \frac{|f(k,u)|}{|u|} = \frac{b^M}{1+u^n}, \quad \min_{k\in I_\omega} \frac{|f(k,u)|}{|u|} = b^m, \quad u \geq 0.$$

In view of $b^m > \frac{1-\sigma}{\sigma^2\omega}$, then $\max f_\infty = 0$, $\min f_0 = b^m > \frac{1}{A\sigma\omega}$. Therefore, by Theorem 3.2, we conclude that (1.11) has at least one positive $\omega$-periodic solution. □

**Corollary 4.11.** *Assume that $a(k) \in C(Z,(0,1))$, $b(k) \in C(Z,(0,+\infty))$ are all $\omega$-periodic, then* (1.10) *has at least one positive $\omega$-periodic solution.*

**Theorem 4.12.** *Assume that $a(k) \in C(Z,(0,1))$, $b(k), \beta(k) \in C(Z,(0,\infty))$, $\tau(k) \in C(Z,Z)$ are all $\omega$-periodic. If $b^m > \frac{1-\sigma}{\sigma^2\omega}$, then* (1.12) *has at least one positive $\omega$-periodic solution.*

The proof is exactly the same as that of Theorem 4.10; so we omit it here.

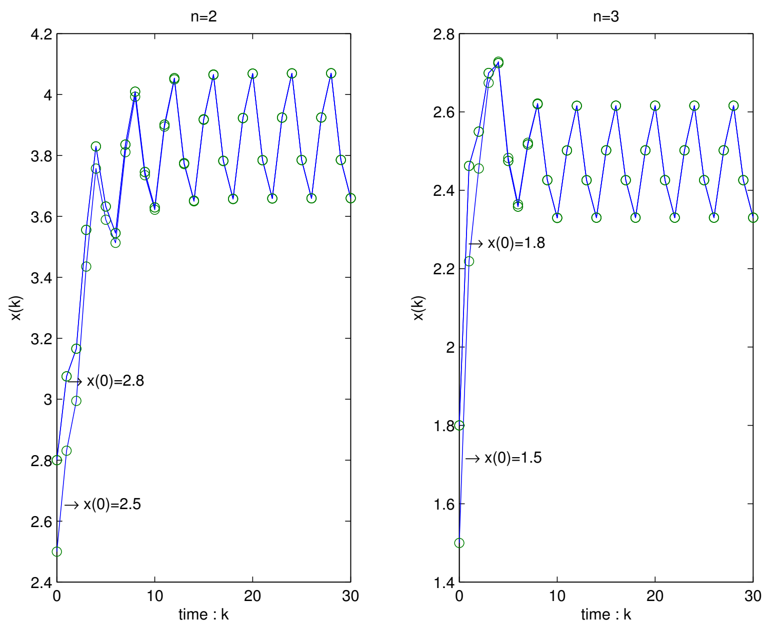


Figure 3. Periodic solutions of (1.11)

**Example 4.13.** Consider the difference equation

$$\Delta x(k) = -\frac{7}{8}\sin\frac{k\pi}{2}x(k) + \Big(\frac{1}{16} + \frac{1}{32}\cos\frac{k\pi}{2}\Big)(x^\alpha(k) + x^\beta(k)), \tag{4.2}$$

where $0 < \alpha < 1$ and $\beta > 1$ are constants. It is cleat that

$$f(k,u) = \Big(\frac{1}{16} + \frac{1}{32}\cos(\frac{k\pi}{2})\Big)(u^\alpha + u^\beta), \quad u \ge 0$$

$$\min_{k\in[0,3]} \frac{|f(k,u)|}{|u|} = \frac{1}{32}(u^{\alpha-1} + u^{\beta-1}).$$

Then,it follows that $\min f_0 = \min f_\infty = \infty$. That is (P1) and (P2) are valid. Let $r_2 = 1$, then for any $0 \le u \le r_2$, we have

$$f(k,u) \le \frac{3}{16} < \frac{r_2}{B\omega} = \frac{49}{256},$$

Hence, (P10) is satisfied. By Theorem 3.4, there exist two positive periodic solutions $x_1^*(t)$ and $x_2^*(t)$ satisfying $0 < \|x_1^*\| < 1 < \|x_2^*\|$.

**Example 4.14.** Consider the system (1.11) with $a(k) = \frac{1}{8} + \frac{1}{16}\sin\frac{k\pi}{2}$, $b(k) = 2 + \frac{1}{2}\cos\frac{k\pi}{2}$, $\tau(k) = 1$, when $n = 2, 3$. The sketches of positive periodic solutions are shown in figure 3.

**Example 4.15.** Consider the difference equation (4.2) with $\alpha = 1/4$, $\beta = 2$. The solution curves satisfying $x(0) = 0.01$, $x(0) = 0.03$ and $x(0) = 0.05$ are illustrated in extended phase space and the periodic solutions is shown in Figure 4.

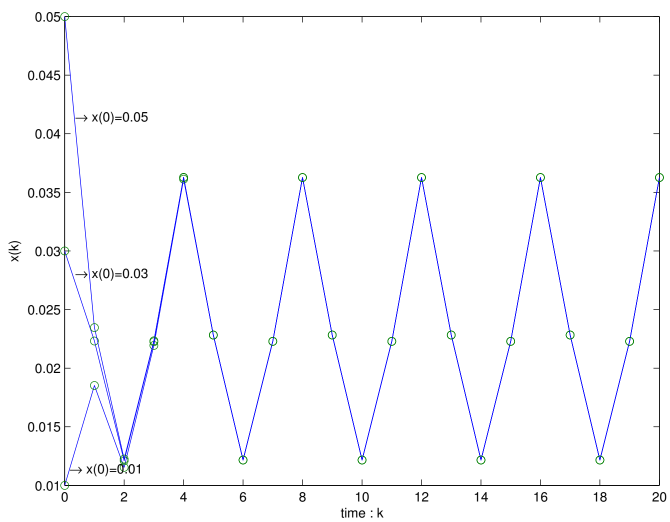


FIGURE 4. Periodic solutions of (4.2)

**Concluding remarks.** In this paper, we employed Krasnoselskii fixed point theorem to investigate systematically the existence and multiplicity of positive periodic solutions of difference equations (1.1) and (1.2). From our arguments, the famous theorem is effective in dealing with the difference equations. However, all of the considered difference equations are those equations with no delays or retarded types. It still remains open to test it with forward or neutral or mixed types.

Though we discuss the existence and multiplicity of positive periodic solutions of difference equations (1.1) and (1.2) in detail based on four key numbers, i.e., $\max f_0$, $\min f_0$, $\max f_\infty$, $\min f_\infty$, there are some cases not covered; for example, the cases of $\min f_\infty = 0$, $\max f_\infty = \infty$, $\min f_0 = 0$, $\max f_0 = \infty$. In fact, solving these cases is beyond our ability by using Krasonselskii fixed point theorem. In [27], the author established the non-existence criteria of periodic solutions, then whether or not can we establish corresponding non-existence criteria of periodic solutions for the rest cases by employing the same method applied in [27].

Our numerical simulations strongly support the analytical achievements. From the above figures, we find that the positive periodic solutions are stable, although we concern about the existence of periodic solutions only. Therefore, we leave an open question, whether or not our concise criteria guarantee the stability of positive periodic solutions.

## References


[1] R. P. Agarwal,Y. H. Pang. *On a generalized difference system*. Nonlinear Anal, 30 (1997), no. 1, 365-376.

[2] K. Deimling. *Nonlinear Functional Analysis*. New York, Springer-Verlag, 1985.

[3] M. Fan, K. Wang. *Periodic solutions of a discrete time nonautonomous ratio-dependent predator-prey system*. Math. Comput. Model, 35 (2002), 951-961.

[4] Y. Gao, G. Zhang, W. G. Ge, *Existence of periodic positive solutions for delay difference equations*. J. Sys. Sci. Math. Scis, 4, 23(2) (2003), 155-162.

[5] K. Gopalsamy. *Stability and Oscillations in Delay Differential Equations of Population Dynamics*. Boston, Kluwer Academic Publishers, 1992.

[6] K. Gopalsamy, B. S. Lalli. *Oscillatory and asymptotic behavior of a multiplicative delay logistic equation*, Dynam. Stability. Syst, 7 (1992), 35-42.

[7] K. Gopalsamy, P. Weng. *Global attactivity and level crossing in model of Hoematopoiesis*, Bull. Inst. Math. Acad. Sin. 22 (1994), no. 4, 341-360.

[8] W. S. C. Gurney, S. P. Blythe, R. M. Nisbet. *Nicholson's blowflies revisited*. Nature, 287 (1980), 17-20.

[9] D. Q. Jiang, D. O'Regan, R. P. Agarwal. *Optimal existence theory for single and multiple positive periodic solutions to functional difference equations*. Appl. Math. Comput. 161 (2005), no. 2, 441–462.

[10] D. Q. Jiang, J. J. Wei, B. Zhang. *Positive periodic solutions of functional differential equations and population models*. Electron. J. Diff. Equat, 2002 (2002), no. 71, 1-13.

[11] D. Q. Jiang, J. J. Wei. *Existence of positive periodic solutions for Volterra integro-differential equations*, Acta. Math. Sci, 21B (2002), no. 4, 553-560.

[12] D. Q. Jiang, R. P. Agarwal. *Existence of positive periodic solutions for a class of difference equations with several deviating arguments*, Advance in Difference Equation IV, J. Comput. Math. Appl, 45 (2003), 1303-1309.

[13] D. Q. Jiang, J. J. Wei. *Existence of positive periodic solution of nonautonomous delay differential equation*, Chin. Ann of Math., 20A (1999), no. 6, 715-720.

[14] W. Joseph, H. So., J. Yu. *Global attractivity and uniformly persistence in Nicholson's blowflies*, Differ. Equat. Dynam. Syst. 2 (1994), no. 1, 11-18.

[15] I. Katsunori. *Asympototic analysis for linear difference equations*, Tran. Amer. Math. Soc. 349 (1997), 4107-4142.

[16] W. G. Kelley, A. C. Peterson. *Difference Equations: An Introduction with Applications*, Acadamic Press, New York, 1991.

[17] M. A. Krasnoselskii. *Positive Solution of Operator Equation*. Gomingen: P. Noordhoff Ltd. Groningen 1964.

[18] Y. Kuang. *Global stability for a class of nonlinear non-autonomous delay logistic equations*. Nonlinear Anal, 17 (1991), 627-634.

[19] S. Lenhart, C. Travis, *Global stability of a biology model with time delay*, Proc. Amer. Math. Soc. 96 (1986), 75-78.

[20] Y. K. Li. *Existence and global attractivity of positive periodic solutions for a class of differential equations*. Chinese Science, 28A (1998), no. 2, 108-118.

[21] M. J. Ma, J. S. Yu. *Existence of multiple positive periodic solutions for nonlinear functional difference equations*. J. Math. Anal. Appl. 305 (2002), 483-490.

[22] M. C. Mackey, L. Glass. *Oscillations and chaos in physiological control systems*. Sciences, 197 (1977), no. 2, 287-289.

[23] J Mallet-Paret, R. Nussbaum, *Global continuation and asymptotic behavior for periodic solutions of a differ ential-delay equation*, Ann. Math. Pure. Appl. 145 (1986), 33-128.

[24] E. C. Pielou. *Mathematics Ecology*. New York, Wiley-Interscience, 1997.

[25] M. Ryszard, P. Jerzy. *On periodic solutions of a first order difference equation*. An Stiint Univ, "Al.I.Cuza" Iasi Sect I a Mat (N.S.), 34 (1998), no. 2, 25-133.

[26] A. Y. Wan, D. Q. Jiang. *Existence of positive periodic solutions for functional differential equations*. Kyushu J.Math, 56 (2002), No. 1, 193-202.

[27] H. Y. Wang. *Positive periodic solutions of functional differential equations*, J. Diff. Equat. 202 (2004), 354-366.

[28] P. Weng, M. Liang. *The existence and behavior of periodic solution of Hematopoiesis model*, Math. Appl. 8 (1995), no. 4, 434-439.

[29] P. Weng. *Existence and global attractivity of periodic solution of integro-differential equation in population dynamics*, Acta Appl. Math. 12 (1996), no. 4, 427-434.

[30] D. Ye. *Periodic solutions for scalar functional differential equations,Master thesis.* Northeast Normal University, Changchun, China, 2004.

[31] R. Y. Zhang, Z. C. Wang, Y. Cheng, J. Wu. *Periodic solutions of a single species discrete population model with periodic harvest/stock*, Comput. Math. Appl., 39 (2000), 77-90.

[32] G. Zhang, S. S. Cheng. *Periodic solutions of a discrete population model.* Functional Differential Equations, 7 (2000), nos. 3-4, 223-230.

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