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CHAOTIC ORBITS OF A PENDULUM WITH VARIABLE LENGTH

MASSIMO FURI, MARIO MARTELLI,
MIKE O'NEILL, & CAROLYN STAPLES

Abstract. The main purpose of this investigation is to show that a pendulum, whose pivot oscillates vertically in a periodic fashion, has uncountably many chaotic orbits. The attribute *chaotic* is given according to the criterion we now describe. First, we associate to any orbit a finite or infinite sequence as follows. We write 1 or -1 every time the pendulum crosses the position of unstable equilibrium with positive (counterclockwise) or negative (clockwise) velocity, respectively. We write 0 whenever we find a pair of consecutive zero's of the velocity separated only by a crossing of the stable equilibrium, and with the understanding that different pairs cannot share a common time of zero velocity. Finally, the symbol ω, that is used only as the ending symbol of a finite sequence, indicates that the orbit tends asymptotically to the position of unstable equilibrium. Every infinite sequence of the three symbols $\{1,-1,0\}$ represents a real number of the interval $[0,1]$ written in base 3 when -1 is replaced with 2. An orbit is considered chaotic whenever the associated sequence of the three symbols $\{1,2,0\}$ is an irrational number of $[0,1]$. Our main goal is to show that there are uncountably many orbits of this type.

1. Introduction

This introduction, although a bit technical in a couple of paragraphs, is largely descriptive. Its aim is to present the organization of the paper and to discuss its goal, strategies and results. It is written with the intent of generating enough interest and curiosity to entice our readers to follow us through the entire journey, including its most technical parts. First we present the main goal of the paper and the ideas needed to better understand its meaning and importance. Then we explain the organization of the paper and the strategies we use to prove the results. Finally, we talk about some results closely related to our effort and previously obtained by other authors.

The main purpose of our investigation is to show, with a rather simple argument, that a pendulum, whose pivot oscillates vertically in a periodic fashion, has uncountably many chaotic orbits that start, with zero velocity, from positions sufficiently close to the unstable equilibrium.

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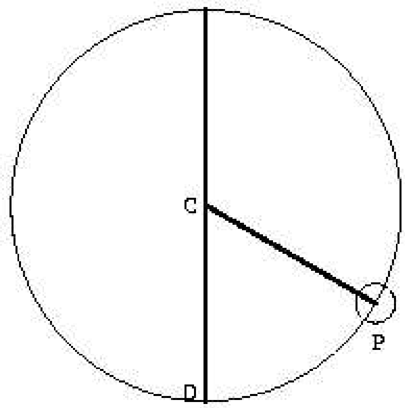

FIGURE 1. The pivot C of the pendulum moves vertically and periodically with period $\frac{2\pi}{\mu}$.

Our readers are certainly aware that there are many different definitions of chaos. Hence, we need to explain in what sense we say that an orbit is chaotic. To any orbit of the pendulum we associate a finite or infinite sequence as follows. We write 1 or -1 every time the position of unstable equilibrium is crossed with positive (counterclockwise) or negative (clockwise) velocity, respectively. We write 0 whenever we encounter a pair of consecutive zero's of the velocity separated only by a crossing of the stable equilibrium, and with the understanding that different pairs cannot share a common time of zero velocity. *Oscillation* is the name we shall use for each pair of this type. Finally, the symbol ω, that is used only as the ending symbol of a finite sequence, indicates that the orbit tends asymptotically to the position of unstable equilibrium. Every infinite sequence of the three symbols $\{1, -1, 0\}$ is a real number of the interval $[0, 1]$ written in base 3 when -1 is replaced with 2. An orbit of the pendulum will be considered chaotic whenever the associated sequence of the three symbols $\{1, 2, 0\}$ is an irrational number of $[0, 1]$.

In Section 4 we shall prove that given any infinite sequence S with entries taken exclusively from the three symbols $1, -1, 0$, or any finite sequence S ending with ω and with all remaining entries taken from the three symbols $1, -1, 0$, we can find infinitely many orbits of the pendulum to which the sequence S is associated according to the rules just described. For example, let us suppose that the sequence is $S = \{1, 1, -1, 0, 0, 1, -1, \dots\}$. Then, we can find infinitely many orbits that start with two counterclockwise crossings followed by one clockwise crossing, two oscillations, one counterclockwise and one clockwise crossing, etc. Since the irrational numbers of $[0, 1]$ are uncountable we obtain that the pendulum has uncountably many chaotic orbits.

The technical preparatory lemmas and theorems are presented in Section 3, and the main result is proved in Section 4. Although some parts of Section 3, and in particular Theorems 3.3 and 3.4, may look intimidating, they are based on a very simple idea that is explained below. A reader who feels comfortable with the idea can glance through Section 2, where notations and definitions are introduced, skip Section 3, and go directly to Section 4, where the main result is presented and some relevant consequences are derived.

To describe the idea let us model the motion of a pendulum, when the pivot oscillates vertically in a periodic manner, with the initial value problem

$$\begin{gathered} \ddot{x}(t) + (1 + r\sin\mu t)\sin x(t) = 0 \\ x(0) = \theta_0, \quad \dot{x}(0) = 0, \end{gathered} \tag{1.1}$$

with $(r,\mu) \in (0,1) \times (0,1]$. For simplicity, we assume that $\theta_0 \in (-\pi, -\frac{\pi}{2})$ is given and $\mu = 1$. Denote by $x(\theta_0, t)$ the corresponding solution of (1.1). Then, given $n \in N$, $x(\theta_0, t)$ will go over the top if the downward vertical position is reached for the first time when $t = (2n+1)\pi$, and will not go over the top if $t = 2n\pi$. The statement *going over the top* means that $\dot{x}(\theta_0, t) > 0$ as long as $x(\theta_0, t) \le \pi$, while *not going over the top* means that there exists $t_0 > 2n\pi$ such that $\dot{x}(\theta_0, t) > 0$ for $t \in (0, t_0)$, $\dot{x}(\theta_0, t_0) = 0$ and $x(\theta_0, t_0) < \pi$. Continuity with respect to initial conditions shows that the odd-even crossings just mentioned are sufficient but not necessary for going or not going over the top.

The reason why the odd-even crossings of the downward vertical position make such a big difference is the same for both cases. To better understand it, we need to remember that the function $u(t) = 2\arcsin(\tanh t)$ is a solution of the differential equation

$$\ddot{u}(t) + \sin u(t) = 0$$

and has the property

$$\lim_{t\to\mp\infty} u(t) = \mp\pi.$$

The function $u(t)$ is called *separatrix.* For $t \neq 0$ we have $\ddot{u}(t)u(t) < 0$. In other words the position and the acceleration of the separatrix have opposite sign.The total energy (kinetic and potential) of the separatrix is

$$E(t) = \frac{1}{2}\dot{u}^2(t) + 1 - \cos u(t).$$

Since $\dot{E}(t) = 0$, we have $E(t) = 2$ for every $t \in \mathbb{R}$. We now observe that the gain or loss of energy which is crucial for determining whether the solution $x(\theta_0, t)$ goes over or does not go over the top is taking place π *units of time before and after reaching the bottom position.* For example, let us consider the case when the first crossing is taking place at $t = (2n+1)\pi$ with $n \ge 1$. At $t = 2n\pi$ the solution is more negative than $2\arcsin(\tanh -\pi) < -2.9688$. In the time interval $[2n\pi, (2n+1)\pi]$ it gains energy over the separatrix since $(1 + r\sin(t)) > 1$. Then, in the time interval $[(2n+1)\pi, (2n+2)\pi]$ the negative acceleration is not as strong as the one acting on the separatrix, since $1 + r\sin t < 1$. Hence, at $t = (2n+2)\pi$ we have $x(\theta_0, (2n+2)\pi) > 2\arcsin(\tanh\pi) > 2.9688$ and the solution has enough energy to go over the top. The situation when $x(\theta_0, 2n\pi) = 0$ is just the opposite. We provide a technical proof of this very simple idea in Section 3. In Section 4 we show how this idea, in combination with continuity with respect to initial conditions, produces the infinitely many orbits to which a given sequence can be associated.

Several authors have worked on related problems during the last 20 years (see, for example,[2],[3],[5]). Many of them have been interested in proving the existence of chaotic orbits for the pendulum ([3],[8]). In some cases numerical evidence has been proposed as the main argument [2], while in others [8] chaos in the sense of Smale [7] has been proved using the Melnikov [4] method. Finally, some authors have proved the existence of chaotic orbits for planar systems ([5],[6]), while others have obtained the presence of specific type of chaotic orbits ([2],[3]) for systems similar to ours.

To the best of our knowledge the odd-even crossing idea is new. However, our investigation has been largely motivated by the paper of Hastings and McLeod [3]. We end this introduction by mentioning that the presence of a small friction term in (1.1) can easily be incorporated into the proofs of the results established in Sections 3 and 4.

2. Notation and Definitions

We mentioned in the Introduction that the motion of a pendulum, with the pivot oscillating vertically in a periodic manner, can be modeled by the second order ordinary differential equation

$$\ddot{x}(t) + (1 + r \sin \mu t) \sin x(t) = 0. \tag{2.1}$$

The function $x(t)$ is an angle and it measures the displacement of the pendulum's arm from the downward vertical position. It is taken positive when measured counterclockwise.

All results will be stated for the case $\mu = 1$. At the end of Section 4 we will make some observations regarding the cases $\mu < 1$ and $\mu > 1$. We shall indicate why the results proved in Sections 3 and 4 continue to be valid in the case $\mu < 1$. We can prove that the same conclusion holds when $\mu \in (1, \mu_0]$, where

$$\mu_0 = \frac{\pi}{\log(\sqrt{2}+1) - \log(\sqrt{2}-1)}.$$

The result is false for large values of μ (see [1]) although we are unable to specify what the values of μ might be. In Section 3 and 4 we usually deal with the Initial Value Problem

$$\begin{gathered} \ddot{x}(t) + (1 + r \sin t) \sin x(t) = 0 \\ x(0) = \theta_0, \quad \dot{x}(0) = 0, \end{gathered} \tag{2.2}$$

with $r \in (0, 1)$ and $\theta_0 \in [-\pi, 0)$. In the case when $r = 0$, (2.2) reduces to the well-known mathematical model of the motion of a simple pendulum

$$\begin{gathered} \ddot{x}(t) + \sin x(t) = 0 \\ x(0) = \theta_0, \quad \dot{x}(0) = 0. \end{gathered} \tag{2.3}$$

The solution of (2.3) reaches the downward vertical position at a time T given by the elliptic integral

$$T = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}\,, \tag{2.4}$$

where $k = \sin \frac{\theta_0}{2}$. The time $T = \pi$ is of special interest to us. In this case, we shall follow a standard notation used by other authors, and replace θ_0 by α. Hence the initial value problem will be

$$\begin{gathered} \ddot{q}(t) + \sin q(t) = 0 \\ q(0) = \alpha, \quad \dot{q}(0) = 0. \end{gathered} \tag{2.5}$$

An easy numerical estimate shows that the angle α such that $q(\alpha, \pi) = 0$ for the first time, satisfies the inequality $-2.78824 < \alpha < -2.78823$. We mentioned in the previous section that the separatrix of the equation of a simple pendulum is the function

$$x(t) = 2\arcsin(\tanh t) \tag{2.6}$$

The derivative is $\dot{x}(t) = 2\operatorname{sech} t$ and its value for $t = 0$ is 2.

A solution of (2.2) will be denoted by $x(\theta_0, t)$. When the initial velocity is $a \neq 0$, it will be incorporated in the notation and the solution will be denoted by $x(\theta_0, a, t)$. The energy of $x(\theta_0, t)$ is the function

$$E(t) = \frac{(\dot{x}(\theta_0, t))^2}{2} + 1 - \cos x(\theta_0, t) \tag{2.7}$$

and its derivative is

$$\dot{E}(t) = -r\dot{x}(\theta_0, t)\sin t \sin x(\theta_0, t). \tag{2.8}$$

The first term of (2.7) is the *kinetic energy*. The term $1-\cos x(\theta_0, t)$ is the potential energy, that sometimes will be called *height of the solution*, since it represents how far up is $x(\theta_0, t)$ with respect to the downward vertical position.

Observe that the differential equation

$$\ddot{x}(t) + (1 + r\sin t)\sin x(t) = 0 \tag{2.9}$$

has two equilibrium solution: one stable and one unstable. The stable one is obtained with the choice of 0 initial position and 0 initial velocity. The unstable one has the same initial velocity, but the initial position is changed to $\pm\pi$. Sometimes we shall call *bottom* and *top* the stable and unstable positions, respectively.

We explained in the previous section in what sense the orbits of a pendulum are considered chaotic. We add here a comment on how the symbol 0 is associated to a pair of 0's of the velocity separated only by a crossing of the stable equilibrium. Two different situations may arise. The symbols immediately before and after a string of k consecutive $0's$, $k = 1, 2, \dots$, may have the same or opposite sign. The velocity is $2k$ times equal to 0 in the first case, and $2k + 1$ times in the second case. The number of oscillations will be equal to k in both cases.

We are now ready for the technical details.

3. Over the top or not

This section is divided into two parts. After the preliminary Lemma 3.1 that is used in both parts, we prove, in Lemma 3.2 and Theorem 3.3, that a solution of (2.2) such that $x(\theta_0, (2n + 1)\pi) = 0, n \geq 1$, for the first time, will go over the top before its velocity changes sign. Then, in Theorem 3.4, we prove that when $x(\theta_0, 2n\pi) = 0, n \geq 2$, for the first time, the solution will not go over the top before its velocity changes sign.

Lemma 3.1. *Assume that $f : [a, b] \to \mathbb{R}$ is increasing and continuous. Then, for every $n \in \mathbb{N}$ such that $[0, 2n\pi] \subseteq [a, b]$ we have $\left|\int_0^{2n\pi} \cos t f(t)\,dt\right| \leq f(b) - f(a)$.*

Proof. We shall prove this result with the additional assumption that f is C^1. Integration by parts gives

$$\int_0^{2n\pi} \cos t f(t)\,dt = -\int_0^{2n\pi} \sin t \dot{f}(t)\,dt. \tag{3.1}$$

Since

$$\Big|\int_0^{2n\pi}\sin t\dot f(t)\,dt\Big|\le\int_0^{2n\pi}\dot f(t)\,dt=f(2n\pi)-f(0)\le f(b)-f(a),\tag{3.2}$$

the result follows. □

Note that Lemma 3.1 can be easily adjusted to the case when f is decreasing.

In what follows we shall denote by $u(\beta,\gamma,t)$ the solution of the initial value problem

$$\begin{gathered}\ddot u(t)+\sin u(t)=0\\ u(0)=\beta,\quad \dot u(0)=\gamma.\end{gathered}\tag{3.3}$$

We may simply write $u(\beta,t)$ when $\gamma=0$.

Let θ_1 be such that $u(\theta_2,a,\pi)=0$, where $a=\dot u(\theta_1,\pi)$ and $\theta_2=u(\theta_1,\pi)+a\pi$. Notice that θ_1 is selected so that in a time interval of 3π we reach the downward vertical position by first following $u(\theta_1,t)$ for $t\in[0,\pi]$, then advancing with constant speed $a=\dot u(\theta_1,\pi)$ for $t\in[\pi,2\pi]$ and, finally, following $u(\theta_2,a,t)$ for $t\in[2\pi,3\pi]$.

Lemma 3.2. *Let $r\in(0,1)$ be given. Denote by $x(\theta_0,t)$ the solution of the initial value problem (2.2) with $\theta_0\in(-\pi,0)$. Assume that $n\ge 1$ is such that $x(\theta_0,(2n+1)\pi)=0$ for the first time. Let $\beta=x(\theta_0,2n\pi)$. Then $\beta\le\theta_2$, where θ_2 was defined above.*

Proof. The proof is divided into three parts. First we establish that $a<\dot x(\theta_0,2n\pi)$ implies that $x(\theta_0,2n\pi)\le\theta_2$. Then we show that $\theta_1<\theta_0$ is not an acceptable alternative since it would require $a<\dot x(\theta_0,2n\pi)$ and $\theta_2<x(\theta_0,2n\pi)$. Finally we show that the only other option $\theta_0\le\theta_1$ always implies $x(\theta_0,2n\pi)\le\theta_2$.

For the first part assume that $a<\dot x(\theta_0,2n\pi)$, where a was defined above. We want to show that $x(\theta_0,2n\pi)\le\theta_2$. To see why this is true let $b=\dot x(\theta_0,2n\pi)$ and $\theta_3=x(\theta_0,2n\pi)$. The inequality $\theta_2<\theta_3$ would imply that $u(\theta_3,b,t)$ would reach the downward vertical position in a time $T_1<\pi$. In fact, the integral giving the time π for $u(\theta_2,a,t)$ to reach the downward vertical position is strictly larger than the integral that provides the time T_1 needed by $u(\theta_3,b,t)$ to reach the same position. At this point the conclusion for the first part is derived from the inequality $u(\theta_3,b,t)\le x(\theta_3,b,t)=x(\theta_0,2n\pi+t)$ for every $t\in(0,\pi)$, that can be easily established using an energy argument. In fact, since $\dddot u(\theta_3,b,0)<\dddot x(\theta_0,2n\pi)$ and the energy of $x(\theta_0,2n\pi+t)$ is larger than the energy of $u(\theta_3,b,t)$ for every $t\in(0,\pi]$ we obtain that $\dot u(\theta_3,b,t)<\dot x(\theta_0,2n\pi+t)$ for every $t\in(0,\pi]$.

For the second part let us assume that $\theta_1<\theta_0$. It is not hard to show that $x(\theta_0,2n\pi)<-\frac{\pi}{2}$. As a consequence of this we obtain that $\ddot u(\theta_1,t)<\ddot x(\theta_0,(2n-2)\pi+t)$ for every $t\in(0,\pi)$. It follows that $a<\dot x(\theta_0,(2n-1)\pi)$ and $u(\theta_1,\pi)<x(\theta_0,(2n-1)\pi)$. Consequently, $\theta_2<x(\theta_0,2n\pi)$ and $a<\dot x(\theta_0,2n\pi)$, in contradiction to the first part of the proof.

In the third part it remains to show that the only alternative left, namely $\theta_0\le\theta_1$, implies $x(\theta_0,2n\pi)\le\theta_2$. Using an energy argument we can show that $u(\theta_1,\pi)\le x(\theta_0,(2n-1)\pi)$ would imply $a=\dot u(\theta_1,\pi)<\dot x(\theta_0,(2n-1)\pi)$. Consequently, we would obtain the same unacceptable conclusion already seen in the second part of the proof. Hence, we must have $x(\theta_0,(2n-1)\pi)<u(\theta_1,\pi)$. Now let us look at $\dot x(\theta_0,2n\pi)$. On the one hand, we know that if this velocity is strictly larger than a then we must have $x(\theta_0,2n\pi)\le\theta_2$. On the other hand, if $\dot x(\theta_0,2n\pi)\le a$, then from the inequality $x(\theta_0,(2n-1)\pi)<u(\theta_1,\pi)$, we easily derive $x(\theta_0,2n\pi)\le\theta_2$. □

Theorem 3.3 can be labeled as the *over the top* theorem. Notice that, as a consequence of continuity with respect to initial conditions, the result stated in it continues to be valid for all solutions with initial conditions sufficiently close to the ones included in the theorem.

Theorem 3.3. *Let $r > 0$ be given and let $\phi \in (-\pi, 0)$ be such that $1+\cos\phi < 0.1r$. The following two statements hold.*

i. *There exists a positive integer $N \geq 1$ such that for every $n \geq N$ there is at least one initial position $\theta_0 \in (-\pi, \phi)$ such that the unique solution of the initial value problem (2.2) reaches the downward vertical position for the first time when $t = (2n+1)\pi$.*
ii. *There exists $t_2 > (2n+1)\pi$ such that $x(\theta_0, t_2) = \pi$ and $\dot{x}(\theta_0, t) > 0$ for every $t \in (0, t_2]$.*

Proof. The first part of Theorem 3.3 is an easy consequence of the continuity with respect to initial conditions, since, as we approach $-\pi$, the time needed to reach the downward vertical position goes to infinity.

To prove the second part we first show that when the solution arrives at the bottom position its energy is at least $2 + 0.8r$. Lastly, we prove that with this energy the solution will go over the top.

An easy computation shows that

$$\begin{aligned}\frac{\dot{x}^2(\theta_0,(2n+1)\pi)}{2} &= 2-\delta+2r\int_0^{2n\pi}\cos t\sin^2\frac{x(\theta_0,t)}{2}\,dt\\ &\quad+2r\int_{2n\pi}^{(2n+1)\pi}\cos t\sin^2\frac{x(\theta_0,t)}{2}\,dt,\end{aligned}\tag{3.4}$$

where $0 < \delta = 1+\cos\theta_0 < 0.1r$. Using the estimate provided by Lemma 3.2 we find that

$$x(\theta_0, 2n\pi) \leq -2.9688.$$

Hence, from Lemma 3.1, we derive

$$2r\Big|\int_0^{2n\pi}\cos t\sin^2\frac{x(t)}{2}\,dt\Big| \leq 0.014884r. \tag{3.5}$$

We now need to estimate the last integral of (3.4). To accomplish this task we split the integral into two parts: the first from $2n\pi$ to $2n\pi + \frac{\pi}{2}$ and the second from $2n\pi + \frac{\pi}{2}$ to $(2n+1)\pi$.

The first part of the integral is positive. We obtain a lower estimate of its value using the function $u(\theta_3, a_3, t)$ where $\theta_3 = 2\arcsin(\tanh(-\pi))$ and $a_3 = \frac{2}{\cosh(-\pi)}$. The given position and velocity are selected so that $u(\theta_3, a_3, \pi) = 0$ and

$$0 \leq \cos t\sin^2\frac{u(\theta_3,a_3,t)}{2} \leq \cos t\sin^2\frac{x(\theta_0,2n\pi+t)}{2},$$

for $t \in [0, \frac{\pi}{2}]$. The second part of the integral is negative and we provide a lower estimate of its value using the solution of the initial value problem

$$\begin{gathered}\ddot{v}(t)+2\sin v(t)=0\\ v(0)=\theta_4,\quad \dot{v}(0)=a_4,\end{gathered}\tag{3.6}$$

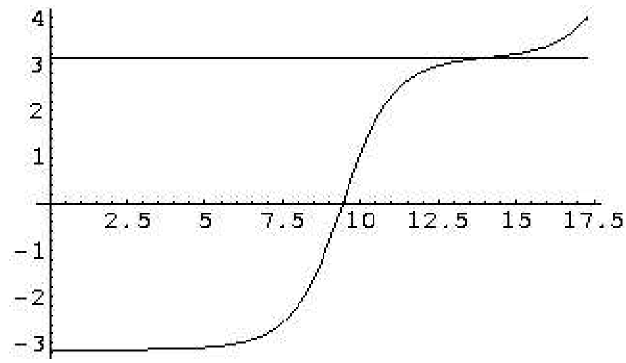

FIGURE 2. The solution with $r = 0.001$ that reaches the position $\theta = 0$ when $t = 5\pi$ crosses over the unstable equilibrium before 8π

where $\theta_4 = 2\arcsin(\tanh(-\sqrt{2}\,\pi))$ and $a_4 = \frac{2\sqrt{2}}{\cosh(-\sqrt{2}\,\pi)}$. The initial position and velocity are selected so that $v(\theta_4, a_4, \pi) = 0$ and

$$\cos t \sin^2 \frac{v(\theta_4, a_4, t)}{2} \le \cos t \sin^2 \frac{x(\theta_0, 2n\pi + t)}{2} \le 0\,,$$

for $t \in [\frac{\pi}{2}, \pi]$.

The first estimate provides a positive value exceeding 1.938527 and the second estimate provides a negative value not smaller than -0.829164. Putting together all estimates and assuming the worst possible situation we have

$$0.8r \le (-0.1 - 0.014884 + 1.938527 - 0.829164)r.$$

As $x(\theta_0, t)$ moves past the downward vertical position, it travels faster than the separatrix and at $t = (2n+2)\pi$ we have $x(\theta_0, (2n+2)\pi) > 2\arcsin\tanh\pi > 2.9688$. Hence, the energy needed to go over the top does not exceed $r(1 + \cos 2.9688) < 0.014892r$. Recall that at the bottom position the energy surplus was at least $0.8r$ and observe that in the interval $[(2n+1)\pi, (2n+2)\pi]$ the solution is losing less kinetic energy than the separatrix. Hence, the solution will make it over the top. □

Theorem 3.4 addresses the case when the solution reaches the downward vertical position for $t = 2n\pi$ with $n \ge 2$. The technical details are similar to the ones introduced in the proofs of Lemma 3.2 and Theorem 3.3. The estimates are obtained using different functions, but the basic ideas and strategy are the same. Therefore, we will simply mention the results without including the technical details.

Theorem 3.4. *Let $r > 0$ be given and let $\phi \in (-\pi, 0)$ be such that $1 + cos\phi \le 0.1r$. The following two statements hold.*

i. *There exists a positive integer $N \ge 1$ such that for every $n \ge N$ there is at least one initial position $\theta_0 \in (-\pi, \phi)$ such that the solution of the initial value problem (2.2) reaches the downward vertical position for the first time when $t = 2n\pi$.*
ii. *There exists $t_3 > 2n\pi$ such that $\dot{x}(\theta_0, t) > 0$ for $t \in (0, t_3)$, $\dot{x}(\theta_0, t_3) = 0$ and $x(\theta_0, t_3) < \pi$.*

Proof. The first part of Theorem 3.4 is an easy consequence of continuity with respect to initial conditions combined with the fact that as we approach $-\pi$ the time needed to reach the downward vertical position goes to infinity.

To prove the second part we first show that when the solution arrives at the downward vertical position its energy does not exceed $2 - 0.8r$. After, we prove that the solution will not go over the top.

An easy computation shows that

$$\begin{aligned}\frac{\dot{x}^2(2n\pi)}{2} &= 2 - \delta + 2r\int_0^{(2n-1)\pi} \cos t \sin^2 \frac{x(t)}{2}\,dt \\ &\quad + 2r\int_{(2n-1)\pi}^{2n\pi} \cos t \sin^2 \frac{x(t)}{2}\,dt,\end{aligned} \tag{3.7}$$

where $0 < \delta = 1 + \cos\theta_0 \leq 0.1r$. With a strategy similar to the one used in Lemma 3.2 we obtain that $x(\theta_0, (2n-1)\pi) \leq -2.65314$. Hence, by Lemma 3.1 we have

$$2r\Big|\int_0^{(2n-1)\pi} \cos t \sin^2 \frac{x(t)}{2}\,dt\Big| \leq 0.11694r. \tag{3.8}$$

We now estimate the last integral of (3.7). In the interval $[(2n-1)\pi, (2n-1)\pi + \frac{\pi}{2}]$ the integral is more negative than $-1.6308916r$ and in the interval $[(2n-1)\pi + \frac{\pi}{2}, 2n\pi]$ is bounded above by $0.5680262r$. Putting all estimates together we obtain that the energy of the solution at the downward vertical position does not exceed $2 - 0.8r$.

With this loss of energy the solution will not make it over the top. The proof of this last step is divided into two parts. In the first we consider those solutions such that

$$\dot{x}(\theta_0, 2n\pi) \leq \sqrt{2(1-\cos\alpha)},$$

where α was defined and numerically estimated in Section 2. In the second we consider those solutions whose velocity at the bottom is larger than $\sqrt{2(1-\cos\alpha)}$.

All solutions of the first group will come to a rest point at a time $t < \pi$ and before reaching the top position. Here is why. The solution

$$u(0, \sqrt{2(1-\cos\alpha)}, t)$$

reaches zero velocity at $t = \pi$, $u(0, \sqrt{2(1-\cos\alpha)}, \pi) < \pi$, and in the interval $(2n\pi, (2n+1)\pi)$ we have

$$\ddot{x}(\theta_0, t) < \ddot{u}(0, \sqrt{2(1-\cos\alpha)}, t) < 0.$$

For the solutions of the second group we use the estimate on the energy at the bottom to derive that $r \leq 0.077$. We now use the solution of the initial value problem

$$\begin{gathered}\ddot{w}(t) + (1.077)\sin w(t) = 0 \\ w(0) = 0, \quad \dot{w}(0) = \sqrt{2(1-\cos\alpha)}.\end{gathered} \tag{3.9}$$

It can be easily verified that

$$w(0, \sqrt{2(1-\cos\alpha)}, \pi) \leq x(\theta_0, (2n+1)\pi) \leq 2\arcsin\tanh\pi.$$

Hence, for any choice of $r \in (0, 0.077)$, a solution of (2.2) that reaches the bottom position at a time $t = 2n\pi$ will be at least as high as the solution of (3.9) at $t = (2n+1)\pi$. An easy numerical estimate shows that $1 - \cos w(0, \sqrt{2(1-\cos\alpha)}, \pi) \geq 2 - 0.2$. Consequently, the largest excess of energy $x(\theta_0, t)$ can gain to reach the

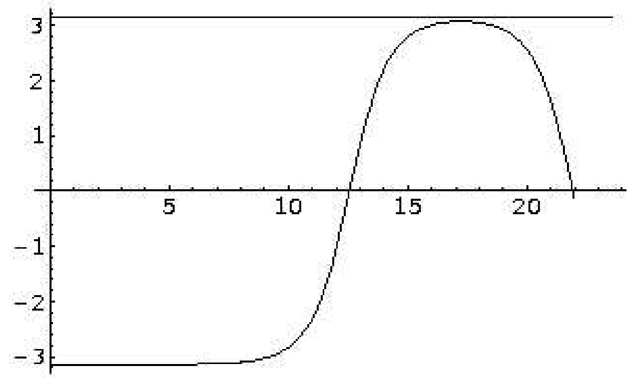

FIGURE 3. The solution with $r = 0.001$ that reaches the downward vertical position when $t = 4\pi$ does not cross over the unstable equilibrium before $t = 5\pi$

top after $t = (2n+1)\pi$ is at most $0.2r$. Since at the bottom there was already a loss of energy of at least $0.8r$ and in the following time interval $[2n\pi, (2n+1)\pi]$, the solution $x(\theta_0, t)$ is losing more energy than the separatrix, it will not have enough energy to reach the top before its velocity changes sign. □

We now add an important remark to the results established previously.

Remark 3.5. Systems of the form

$$\begin{gathered} \ddot{z}(t) + (1 + r\sin t)\sin z(t) = 0 \\ z(t_0) = \alpha_0, \quad \dot{z}(t_0) = 0. \end{gathered} \tag{3.10}$$

are sometimes considered and will be needed in the proof of our main result. Given $r > 0$ and t_0, we can determine $\alpha_0 \in (-\pi, 0)$ sufficiently close to $-\pi$ so that all conclusions reached before are still valid. In a similar manner we can handle cases in which $\alpha_0 = -\pi$ and small velocities are assumed in either direction. Both situations will arise in the proof of the next theorem.

4. Chaotic Orbits

We are now ready to state and prove the main result of this paper concerning orbits of a pendulum with an oscillating pivot. We introduce a preliminary definition.

Definition 4.1. The symbols $1, -1$ denote a crossing of the unstable equilibrium in a counterclockwise or clockwise direction, respectively. The symbol 0 denotes two times of zero velocity separated only by a crossing of the position of stable equilibrium. The symbol ω indicates that an orbit tends asymptotically to the position of unstable equilibrium.

Since an orbit may tend asymptotically to the top position either in a counterclockwise or clockwise manner, it would be more precise to use $+\omega$ in one case, and $-\omega$ in the other case. However, this distinction does not really add an important information on the orbit. Hence, we have decided not to use it.

The statement *solution starting from* followed by the indication of a position angle will be used to denote the solution of initial value problems like (4.1) below. We may also say that the solution *corresponds to* followed by the indication of the position angle. When the initial velocity is not mentioned it will be assumed equal to 0. The statement *a sequence corresponds to a solution* means that a sequence of symbols is associated to the solution according to the rules stated in Definition 4.1.

Theorem 4.2. *Let $r \in (0,1)$ be given. Select any infinite sequence of entries from the symbols $1, -1, 0$ or any finite sequence of entries from the same symbols and ending with ω. Then there are infinitely many initial conditions $(\theta_0, 0)$ such that the given sequence of symbols corresponds to the solution of the initial value problem*

$$\begin{gathered} \ddot{x}(t) + (1 + r\sin t)\sin x(t) = 0 \\ x(0) = \theta_0, \quad \dot{x}(0) = 0. \end{gathered} \tag{4.1}$$

Proof. The procedure to follow in the case of a finite sequence will be evident from the proof we present when the sequence is infinite. To obtain the desired result we will produce a family of nested intervals $I_n = [a_n, b_n]$, such that all orbits of (4.1) with initial position $\theta_0 \in I_n$ will complete the first n steps of the sequence, except when θ_0 is equal to either one of the border points. These two initial positions will produce solutions satisfying only the first $n-1$ steps of the sequence and terminating with ω. Since $\cap I_n = I_\infty \neq \emptyset$ we obtain the desired orbit by selecting $\theta_0 \in I_\infty$.

To better understand how the sequence of intervals can be constructed let us keep in mind that the set of initial conditions with corresponding orbits satisfying the first k entries of the infinite sequence is an open set. This is a direct consequence of the continuity with respect to initial conditions. Let us also keep in mind that to any solution we can associate a sequence, although different solutions need not have different sequences. For example, the sequence $\{0, 0, \dots, 0, \dots\}$ corresponds to all solutions that will indefinitely oscillate around the position of stable equilibrium.

Let us assume that the sequence starts with 1. The cases when the sequence starts with -1 or 0 are handled similarly.

Given $r \in (0,1)$ we select N large enough so that for all $n \geq N$ we can determine θ_0 so that the solution of the initial value problem

$$\begin{gathered} \ddot{x}(t) + (1 + r\sin t)\sin x(t) = 0 \\ x(0) = \theta_0, \quad \dot{x}(0) = 0 \end{gathered} \tag{4.2}$$

reaches the downward vertical position at time $t = (2n+1)\pi$. Hence, the solution will go over the top. Take the largest interval of the form $[a_1, b_1]$ where $a_1 < \theta_0 < b_1$ and $[a_1, b_1]$ is such that for all $\theta \in (a_1, b_1)$ the corresponding solution will go over the top at least once, while for $\theta = a_1$ or $\theta = b_1$ the corresponding solution will not go over the top but will tend asymptotically to it as $t \to +\infty$. Consequently, the sequence corresponding to these two orbits will be simply denoted by $S = \{\omega\}$. Set $I_1 = [a_1, b_1]$. Observe that this first interval can be selected in infinitely many different ways. In fact, for every $n \geq N$, we can find an open interval (a_1, b_1) such that every solution with initial position in (a_1, b_1) will go over the top at least once before its velocity changes sign.

We now indicate how to construct I_2. We shall assume that the second entry of the sequence is 0. The cases with second entry equal to ± 1 are handled similarly. I_2

will be constructed so that it is contained in I_1 and all its points, except the border points, provide solutions having $\{1, 0\}$ in the first two entries of the corresponding sequence. The solutions corresponding to the border points will have 1 in the first entry, and ω in the second entry. First we select in I_1 an initial condition θ_1 so that the velocity of the corresponding solution over the unstable equilibrium will be very small and the downward vertical position will be reached at time $t = 2k\pi$, with $2k\pi >> t_0$ and t_0 the time when the solution is over the top. This can obviously be accomplished, since as the initial condition in I_1 approaches b_1 (or a_1) the corresponding solution will arrive at the top with progressively smaller velocity. Hence, we can also consider an initial condition smaller than θ_1 and larger than a_1 so that the corresponding solution will reach the downward vertical position at a time that is an odd multiple of π. The two initial conditions will be separated by one generating a solution that after going over the top will tend asymptotically to the unstable equilibrium. From this discussion the interested reader can understand how the choice of θ_1 can be made so that the border points of the interval I_2 as selected below are contained in (a_1, b_1). Moreover, we can also satisfy the requirement imposed by the magnitude of r and mentioned in the statements of Theorems 3.3 and 3.4 of starting close enough to the unstable equilibrium to insure the validity of all inequalities previously established.

The solution starting from θ_1 will come to a rest on the right-hand-side before reaching the top a second time. Since the set of solutions with this property is open, we consider the largest interval in I_1 of the form (c_2, d_2) with $c_2 < \theta_1 < d_2$ and such that for all initial conditions of this interval the corresponding solution will come to a rest before reaching the top a second time. The solutions corresponding to $\theta = c_2$ or $\theta = d_2$ will go over the top once and then will tend asymptotically to the unstable equilibrium. Hence, the sequence corresponding to both will be $S = \{1, \omega\}$. Among the initial conditions of (c_2, d_2) there will be some with corresponding solution coming down to the downward vertical position at a time that is even multiple of π and others with corresponding solutions coming down at a time that is odd multiple of π. These will be separated by initial conditions with corresponding solutions that after going over the top and coming to a rest point on the right-hand-side, will tend asymptotically to the unstable equilibrium as they move up on the left-hand-side. Pick an initial condition in (c_2, d_2) so that the corresponding solution comes down again at a time that is an even multiple of π and consider the largest open interval in (c_2, d_2) containing this initial condition and such that for all θ in this open interval the corresponding solution will have a rest point on the left-hand-side separated from the one on the right-hand-side by a single crossing of the position of stable equilibrium. The closure of this interval is $I_2 = [a_2, b_2]$. Clearly, for all $\theta \in (a_2, b_2)$ the corresponding solution will be represented by a sequence having $\{1, 0\}$ in the first and second position. For $\theta = b_2$ or $\theta = a_2$ the corresponding solution will be represented by the sequence $\{1, \omega\}$. An induction argument can now be used to conclude the proof. □

Remark 4.3. We have not included the presence of a friction term $k\dot{x}(t)$ in our analysis. However, it is not difficult to see that the principles on which the proofs are based will continue to be valid if a small friction term is added. While it is hard to establish the magnitude of the constant k, we can say that given $r > 0$ there exists k_0 such that for all $k < k_0$ the inclusion of a friction term with constant $k > 0$ will not affect the validity of the results we have established.

Remark 4.4. We now examine the case when $\mu \neq 1$ and still $\mu > 0$. The separatrix of the problem

$$\ddot{u}(t) + c \sin u(t) = 0 \tag{4.3}$$

is given by

$$u(t) = 2 \arcsin(\tanh(\sqrt{c}t)).$$

At the downward vertical position its velocity is $2\sqrt{c}$.

The results proved before remain unchanged if $\mu < 1$. In fact, with a suitable change of variable we can rewrite equation (2.1) in the form

$$\ddot{\theta}(t) + c(1 + r \sin t) \sin \theta(t) = 0 \tag{4.4}$$

where $c = \frac{1}{\mu^2}$, and we see that all inequalities remain true due to the fact that the system must start from a more negative position to reach the downward vertical position at the required time. Hence, for every $(r, \mu) \in (0, 1) \times (0, 1]$ Theorem 4.2 holds true.

The case $\mu > 1$ is more complicated. The approach we used before is still valid for $\mu \in [1, \mu_0)$, where

$$\mu_0 = \frac{\pi}{\log(\sqrt{2}+1) - \log(\sqrt{2}-1)} \,.$$

For all these values of μ one can show, using exactly the same approach outlined in Lemmas 3.1 and 3.2, that the estimates of energy gain or loss are the same as previously determined. We simply have to multiply them by $\frac{1}{\mu^2}$. More precisely, we can prove that the kinetic energy of a solution that reaches the bottom position at an odd multiple of π is at least $\frac{2+0.8r}{\mu^2}$. For example, for $\mu = \mu_0$ and with $1 + \cos\theta_0 < 0.1r$ the energy surplus at the downward vertical position is at least $\frac{.8r}{\mu_0^2}$ and the same reasoning used in the proof of Theorem 3.3 shows that the solution will go over the top. Similarly, when the bottom position is reached at a time that is an even multiple of π and $\mu \leq \mu_0$, the kinetic energy cannot exceed $\frac{2-0.8r}{\mu^2}$ and the solution will not make it over the top. The only caveat is that the multiples of π may need to have n very large, but this is obviously not a problem.

For $\mu_0 < \mu$, the situation is more complex, particularly when $2\mu_0 < \mu$. Numerical experiments suggest that the result is still true when μ is not too large. One has to be careful in selecting the time needed to come down to the position of stable equilibrium. The reader would certainly remember that an appropriate choice was also included in Theorems 3.3 and 3.4. Hence, this is nothing new. The results on the stability of the inverted pendulum (see [1], Chapter 5) show the existence of large μ values for which it is hard to establish what the behavior of the system might be at least for certain choices of the initial position and velocity. Hence, from this point of view, some additional work needs to be done.

There are also some interesting questions we have not been able to answer. One of the most puzzling is the amount of energy an orbit can accumulate, given r and μ. We have done some experiments with $\mu = 1$ and we have observed that the energy fluctuates between specific values. In each case we have started with 0 initial velocity. The energy never grows too large or becomes too small. Although this behavior makes sense, we have not been able to prove it, let alone establish what an upper and lower bound for the energy must be.

We sincerely hope that some of these questions, and others that are not mentioned here, will rouse the curiosity of some interested readers, who will further explore the intricacies of these simple, yet fascinating systems.

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Massimo Furi
Dipartimento di Matematica Applicata "Giovanni Sansone", Università degli Studi di Firenze, Via S. Marta 3, 50139 Firenze, Italy
E-mail address: `furi@dma.unifi.it`

Mario Martelli
Department of Mathematics, Claremont McKenna College, Claremont, CA, 91711, USA
E-mail address: `mmartelli@mckenna.edu`

Mike O'Neill
Department of Mathematics, Claremont McKenna College, Claremont, CA, 91711, USA
E-mail address: `moneill@mckenna.edu`

Carolyn Staples
Department of Mathematics, Claremont McKenna College, Claremont, CA, 91711, USA
E-mail address: `cstaples@mckenna.edu`