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A NONLINEAR WAVE EQUATION WITH A NONLINEAR INTEGRAL EQUATION INVOLVING THE BOUNDARY VALUE

ABSTRACT. We consider the initial-boundary value problem for the nonlinear wave equation

\[ u_{tt} - u_{xx} + f(u, u_t) = 0, \quad x \in \Omega = (0, 1), \; 0 < t < T, \]

\[ u_x(0, t) = P(t), \quad u(1, t) = 0, \]

\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \]

where \( u_0, u_1, f \) are given functions, the unknown function \( u(x, t) \) and the unknown boundary value \( P(t) \) satisfy the nonlinear integral equation

\[ P(t) = g(t) + H(u(0, t)) - \int_0^t K(t - s, u(0, s)) ds, \]

where \( g, K, H \) are given functions. We prove the existence and uniqueness of weak solutions to this problem, and discuss the stability of the solution with respect to the functions \( g, H \) and \( K \). For the proof, we use the Galerkin method.

1. INTRODUCTION

In this paper we consider the problem of finding a pair of functions \((u, P)\) that satisfy

\[ u_{tt} - u_{xx} + f(u, u_t) = 0, \quad x \in \Omega = (0, 1), \; 0 < t < T, \]

\[ u_x(0, t) = P(t) \]  \hspace{1cm} (1.1)

\[ u(1, t) = 0, \]  \hspace{1cm} (1.2)

\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \] \hspace{1cm} (1.3)

where \( u_0, u_1, f \) are given functions satisfying conditions to be specified later and the unknown function \( u(x, t) \) and the unknown boundary value \( P(t) \) satisfy the nonlinear integral equation

\[ P(t) = g(t) + H(u(0, t)) - \int_0^t K(t - s, u(0, s)) ds, \]  \hspace{1cm} (1.5)

where \( g, H, K \) are given functions. Ang and Dinh \cite{2} established the existence of a unique global solution for the initial and boundary value problem \((1.1)-(1.4)\)

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with $u_0$, $u_1$, $P$ given functions and $f(u, u_t) = |u_t|^\alpha \text{sign}(u_t)$, $(0 < \alpha < 1)$. As a generalization of the results in [2], Long and Dinh [7, 9, 10] have considered problem (1.1)-(1.4) associated with the following nonhomogeneous boundary condition at $x = 0,$

$$u_x(0, t) = g(t) + H(u(0, t)) - \int_0^t K(t - s, u(0, s))ds. \quad (1.6)$$

We have considered it with $K \equiv 0$, $H(s) = hs$, where $h > 0$ [9]; $K \equiv 0$ [7], $H(s) = hs$, $K(t, u) = k(t)u$, where $h > 0$, $k \in H^2(0, T)$, for all $T > 0$ [10]. In the case of $H(s) = hs$, $K(t, u) = h\omega(\sin \omega t)u$, where $h > 0$, $\omega > 0$ are given constants, the problem (1.1)-(1.5) is formed from the problem (1.1)-(1.4) wherein, the unknown function $u(x, t)$ and the unknown boundary value $P(t)$ satisfy the following Cauchy problem

$$P''(t) + \omega^2 P(t) = h\epsilon(t), \quad 0 < t < T, \quad (1.7)$$
$$P(0) = P_0, \quad P'(0) = P_1, \quad (1.8)$$

where $\omega > 0$, $h \geq 0$, $P_0$, $P_1$ are given constants [10]. An and Trieu [1], studied a special case of problem (1.1)-(1.4), (1.7), (1.8) with $u_0 = u_1 = P_0 = 0$ and with $f(u, u_t)$ linear, i.e., $f(u, u_t) = Ku + \lambda u_t$ where $K$, $\lambda$ are given constants. In the later case the problem (1.1)-(1.4), (1.7), and (1.8) is a mathematical model describing the shock between a solid body and a linear viscoelastic bar with nonlinear elastic constraints at the side, and constraints associated with a viscous frictional resistance. From (1.7), (1.8) we represent $P(t)$ in terms of $P_0$, $P_1$, $\omega$, $h$, $u_{tt}(0, t)$ and then by integrating by parts, we have

$$P(t) = g(t) + hu(0, t) - \int_0^t k(t - s)u(0, s)ds, \quad (1.9)$$

where

$$g(t) = (P_0 - hu_0(0))\cos \omega t + (P_1 - hu_1(0))\frac{\sin \omega t}{\omega}, \quad (1.10)$$
$$k(t) = h\omega(\sin \omega t). \quad (1.11)$$

By eliminating an unknown function $P(t)$, we replace the boundary condition (1.2) by

$$u_x(0, t) = g(t) + hu(0, t) - \int_0^t k(t - s)u(0, s)ds. \quad (1.12)$$

Then, we reduce problem (1.1)-(1.4), (1.7), (1.8) to (1.1)-(1.4), (1.9)-(1.11) or to (1.1)-(1.3), (1.4), (1.10)-(1.12).

In this paper, we consider two main parts. In Part 1, we prove a theorem of global existence and uniqueness of a weak solution of problem (1.1)-(1.5). The proof is based on a Galerkin method associated to a priori estimates, weak-convergence and compactness techniques. We remark that the linearization method in [6, 11, 13] cannot be used for the problems in [2, 4, 5, 7, 9, 10]. In Part 2 we prove that the solution $(u, P)$ of this problem is stable with respect to the functions $g, H$ and $K$. The results obtained here generalize the ones in [1, 2, 4, 7, 9, 10].
2. THE EXISTENCE AND UNIQUENESS THEOREM

We first set notations $\Omega = (0, 1)$, $Q_T = \Omega \times (0, T)$, $T > 0$, $L^p = L^p(\Omega)$, $H^1 = H^1(\Omega)$, $H^2 = H^2(\Omega)$, where $H^1$, $H^2$ are the usual Sobolev spaces on $\Omega$.

The norm in $L^2$ is denoted by $\| \cdot \|$. We also denote by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2$ or pair of dual scalar product of continuous linear functional with an element of a function space. We denote by $\| \cdot \|_X$ the norm of a Banach space $X$ and by $X'$ the dual space of $X$. We denote by $L^p(0, T; X)$, $1 \leq p \leq \infty$ for the Banach space of the real functions $u : (0, T) \to X$ measurable, such that

$$
\| u \|_{L^p(0,T;X)} = \left( \int_0^T \| u(t) \|^p_X dt \right)^{1/p}
$$

for $1 \leq p < \infty$, and

$$
\| u \|_{L^\infty(0,T;X)} = \operatorname{esssup}_{0 < t < T} \| u(t) \|_X \text{ for } p = \infty.
$$

We put

$$
V = \{ v \in H^1 : v(1) = 0 \}, \quad a(u,v) = \left\langle \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \right\rangle = \int_0^1 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} dx.
$$

Here $V$ is a closed subspace of $H^1$ and on $V$, $\| v \|_{H^1}$ and $\| v \|_{V} = \sqrt{a(v,v)}$ are two equivalent norms.

**Lemma 2.1.** The imbedding $V \hookrightarrow C^0(\overline{\Omega})$ is compact and

$$
\| v \|_{C^0(\overline{\Omega})} \leq \| v \|_{V}
$$

for all $v \in V$.

The proof is straightforward and we omit it. We make the following assumptions:

(A) $u_0 \in H^1$ and $u_1 \in L^2$

(G) $g \in H^1(0, T)$ for all $T > 0$

(H) $H \in C^1(\mathbb{R})$, $H(0) = 0$ and there exists a constant $h_0 > 0$ such that

$$
\widehat{H}(y) = \int_0^y H(s) ds \geq -h_0
$$

(K1) $K$ and $\frac{\partial K}{\partial r}$ are in $C^0(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$

(K2) There exist the nonnegative functions $k_1 \in L^2(0, T)$, $k_2 \in L^1(0, T)$, $k_3 \in L^2(0, T)$, and $k_4 \in L^1(0, T)$, such that

(i) $|K(t, u)| \leq k_1(t)|u| + k_2(t),$

(ii) $|\frac{\partial K}{\partial r}(t, u)| \leq k_3(t)|u| + k_4(t).$

The function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies $f(0, 0) = 0$ and the following conditions:

(F1) $(f(u, v) - f(u, \overline{v}))(v - \overline{v}) \geq 0$ for all $u, v, \overline{v} \in \mathbb{R}$

(F2) There is a constant $\alpha$ in $[0, 1]$ and a function $B_1 : \mathbb{R}_+ \to \mathbb{R}_+$ continuous and satisfying

$$
|f(u, v) - f(u, \overline{v})| \leq B_1(|u|)|v - \overline{v}|^{\alpha} \text{ for all } u, v, \overline{v} \in \mathbb{R}
$$

(F3) There is a constant $\beta$ in $[0, 1]$ and a function $B_2 : \mathbb{R}_+ \to \mathbb{R}_+$ continuous and satisfying

$$
|f(u, v) - f(\overline{u}, v)| \leq B_2(|v|)|u - \overline{u}|^{\beta} \text{ for all } u, \overline{u}, v \in \mathbb{R}
$$
We will use the notation $u' = u_t = \partial u/\partial t$, $u'' = u_{tt} = \partial^2 u/\partial t^2$. Then we have the following theorem.

**Theorem 2.2.** Let $(A), (G), (H), (K1), (K2), (F1), (F3)$ hold. Then, for every $T > 0$, there exists a weak solution $(u, P)$ to problem (1.1)-(1.5) such that

$$u \in L^\infty(0, T; V), \quad u_t \in L^\infty(0, T; L^2), \quad u(0, \cdot) \in H^1(0, T),$$

$$P \in H^1(0, T).$$

Furthermore, if $\beta = 1$ in (F3) and the functions $H, K, f$ satisfying, in addition

(H1) $H \in C^2(\mathbb{R})$, $H'(s) > -1$ for all $s \in \mathbb{R}$

(K3) For all $M$ positive and all $T$ positive, there exists $p_{M,T}, q_{M,T}$ in $L^2(0, T)$, $p_{M,T}(t) \geq 0$, $q_{M,T}(t) \geq 0$ such that

(i) $|K(t, u) - K(t, v)| \leq p_{M,T}(t)|u - v|$ for all $u, v \in \mathbb{R}$, $|u|, |v| \leq M$,

(ii) $|\partial_K(t, u)| - \partial_K(t, v)| \leq q_{M,T}(t)|u - v|$ for all $u, v \in \mathbb{R}$, $|u|, |v| \leq M$.

(F4) $B_2(v) \in L^2(Q_T)$ for all $v \in L^2(Q_T)$ for all $T > 0$.

Then the solution is unique.

**Remark 2.3.** This result is stronger than that in [9]. Indeed, corresponding to the same problem (1.1)-(1.5) with $K(t, u) \equiv 0$ and $H(s) = hs$, $h > 0$ the following assumptions made in [9] are not needed here: $0 < \alpha < 1$, $B_1(|u|) \in L^{2/(1-\alpha)}(Q_T)$ for all $u \in L^\infty(0, T; V)$ and all $T > 0$; $B_1, B_2$ are nondecreasing functions.

**Proof of Theorem 2.2.** It is done in several steps.

**Step 1. The Galerkin approximation.** Consider the orthonormal basis on $V$ consisting of eigenvectors of the Laplacian, $-\partial^2/\partial x^2$,

$$w_j(x) = \sqrt{2/(1 + \lambda_j^2)} \cos(\lambda_j x), \quad \lambda_j = (2j - 1)\pi/2, \quad j = 1, 2, \ldots.$$  

Put

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t)w_j,$$

where $c_{mj}(t)$ satisfy the system of nonlinear differential equations

$$\langle u_m''(t), w_j \rangle + a(u_m(t), w_j) + P_m(t)w_j(0) + \langle f(u_m(t), u_m'(t)), w_j \rangle = 0,$$

$$P_m(t) = g(t) + H(u_m(0, t)) - \int_{0}^{t} K(t - s, u_m(0, s))ds,$$  

with

$$u_m(0) = u_{0m} = \sum_{j=1}^{m} \alpha_{mj}w_j \to u_0 \quad \text{strongly in } H^1,$$

$$u'_m(0) = u_{1m} = \sum_{j=1}^{m} \beta_{mj}w_j \to u_1 \quad \text{strongly in } L^2,$$

This system of equations is rewritten in form

$$c_{mj}''(t) + \lambda_j^2 c_{mj}(t) = \frac{-1}{\|w_j\|^2}(P_m(t)w_j(0) + \langle f(u_m(t), u_m'(t)), w_j \rangle),$$

$$P_m(t) = g(t) + H(u_m(0, t)) - \int_{0}^{t} K(t - s, u_m(0, s))ds,$$

$$c_{mj}(0) = \alpha_{mj}, \quad c_{mj}'(0) = \beta_{mj}, \quad 1 \leq j \leq m.$$
This system is equivalent to the system of integrodifferential equations
\[ c_{mj}(t) = G_{mj}(t) = \frac{1}{\|w_j\|^2} \int_0^t N_j(t - \tau) (H(u_m(0, \tau)) w_j(0) + \langle f(u_m(\tau), u_m'(\tau)), w_j \rangle) d\tau \]
\[ + \frac{w_j(0)}{\|w_j\|^2} \int_0^t N_j(t - \tau) d\tau \int_0^\tau K(\tau - s, u_m(0, s)) ds, \quad 1 \leq j \leq m, \] (2.7)

where \( N_j(t) = \sin(\lambda_j t) / \lambda_j \) and
\[ G_{mj}(t) = \alpha_{mj} N_j'(t) + \beta_{mj} N_j(t) - \frac{w_j(0)}{\|w_j\|^2} \int_0^t N_j(t - \tau) g(\tau) d\tau. \] (2.8)

We then have the following lemma.

**Lemma 2.4.** Let \((A), (G), (H), (K1), (K2), (F1), (F3)\) hold. For fixed \(T > 0\), the system \((1.10)-(1.11)\) has solution \(c_m = (c_{m1}, c_{m2}, \ldots, c_{mm})\) on an interval \([0, T_m] \subset [0, T)\).

**Proof.** Omitting the index \(m\), system (2.7), (2.8) is rewritten in the form
\[ c = Uc, \]
where \(c = (c_1, c_2, \ldots, c_m), Uc = ((Uc)_1, (Uc)_2, \ldots, (Uc)_m),\)
\[ (Uc)_j(t) = G_j(t) + \int_0^t N_j(t - \tau) (Vc)_j(\tau) d\tau, \] (2.9)
\[ (Vc)_j(t) = f_{1j}(c(t), c'(t)) + \int_0^t f_{2j}(t - s, c(s)) ds, \] (2.10)
\[ G_j(t) = \alpha_{mj} N_j'(t) + \beta_{mj} N_j(t) - \frac{w_j(0)}{\|w_j\|^2} \int_0^t N_j(t - \tau) g(\tau) d\tau, \] (2.11)

the functions \(f_{1j} : \mathbb{R}^{2m} \to \mathbb{R} \) \(f_{2j} : [0, T_m] \times \mathbb{R}^m \to \mathbb{R}\) satisfy
\[ f_{1j}(c, d) = -\frac{1}{\|w_j\|^2} \left[ H\left(\sum_{i=1}^m c_i w_i(0)\right) w_j(0) + \langle f\left(\sum_{i=1}^m c_i w_i, \sum_{i=1}^m d_i w_i\right), w_j \rangle\right], \] (2.12)
\[ f_{2j}(t, c) = \frac{w_j(0)}{\|w_j\|^2} K(t, \sum_{i=1}^m c_i w_i(0)), \quad 1 \leq j \leq m. \] (2.13)

For every \(T_m > 0, M > 0\) we put
\[ S = \{ c \in C^1([0, T_m]; \mathbb{R}^m) : \|c\|_1 \leq M \}, \quad \|c\|_1 = \|c\|_0 + \|c'\|_0, \]
\[ \|c\|_0 = \sup_{0 \leq t \leq T_m} |c(t)|_1, \quad |c(t)|_1 = \sum_{i=1}^m |c_i(t)|. \]

Clearly \(S\) is a closed convex and bounded subset of \(Y = C^1([0, T_m]; \mathbb{R}^m)\). Using the Schauder fixed point theorem we shall show that the operator \(U : S \to Y\) defined by (2.11)-(2.13) has a fixed point. This fixed point is the solution of (2.7).

(a) First we show that \(U\) maps \(S\) into itself. Note that \((Vc)_j \in C^0([0, T_m]; \mathbb{R})\) for all \(c \in C^1([0, T_m]; \mathbb{R}^m)\), hence it follows from (2.9), and the equality
\[ (Uc)_j(t) = G_j(t) + \int_0^t N_j(t - \tau) (Vc)_j(\tau) d\tau, \] (2.14)
that $U : Y \to Y$. Let $c \in S$, we deduce from (2.8), (2.13) that

$$
|(Uc)(t)|_1 \leq |G(t)|_1 + \frac{1}{\lambda_1}T_m\|Vc\|_0,
$$
(2.15)

$$
|(Uc)'(t)|_1 \leq |G'(t)|_1 + T_m\|Vc\|_0.
$$
(2.16)

On the other hand, it follows from (H), (K1), (K2), (F2), (F3), (2.10), (2.12), (2.13) that

$$
\|Vc\|_0 \leq \sum_{j=1}^m [N_1(f_{1j}, M) + TN_2(f_{2j}, M, T)] = \beta(M, T) \quad \text{for all } c \in S,
$$
(2.17)

where

$$
N_1(f_{1j}, M) = \sup\{|f_{1j}(y, z)| : \|y\|_{\mathbb{R}^m} \leq M, \|z\|_{\mathbb{R}^m} \leq M\},
$$

$$
N_2(f_{2j}, M, T) = \sup\{|f_{2j}(t, y) : 0 \leq t \leq T, \|y\|_{\mathbb{R}^m} \leq M\}.
$$
(2.18)

Hence, from (2.15)-(2.18) we obtain

$$
\|Uc\|_1 \leq \|G\|_{1T} + (1 + \frac{1}{\lambda_1})T_m\beta(M, T),
$$

where

$$
\|G\|_{1T} = \|G\|_{0T} + \|G'\|_{0T} = \sup_{0 \leq t \leq T} |G(t)|_1 + \sup_{0 \leq t \leq T} |G'(t)|_1.
$$

Choosing $M$ and $T_m > 0$ such that

$$
M > 2\|G\|_{1T} \quad \text{and} \quad (1 + \frac{1}{\lambda_1})T_m\beta(M, T) \leq M/2.
$$

Hence, $\|Uc\|_1 \leq M$ for all $c \in S$, that is, the operator $U$ maps $S$ the set into itself. (b) Now we show that the operator $U$ is continuous on $S$. Let $c, d \in S$, we have

$$
(Uc)_j(t) - ( Ud)_j(t) = \int_0^t N_j(t - \tau)((Vc)_j(\tau) - (Vd)_j(\tau))d\tau.
$$

Hence

$$
\|Uc - Ud\|_0 \leq \frac{1}{\lambda_1}T_m\|Vc - Vd\|_0.
$$
(2.19)

Similarly, we obtain from the equality

$$
(Uc)'_j(t) - ( Ud)'_j(t) = \int_0^t N_j'(t - \tau)((Vc)_j(\tau) - (Vd)_j(\tau))d\tau,
$$

which implies

$$
\|(Uc)' - ( Ud)'\|_0 \leq T_m\|Vc - Vd\|_0.
$$
(2.20)

By estimates (2.19), (2.20), we only have to prove that the operator $V : Y \to C^0([0, T_m]; \mathbb{R}^m)$ is continuous on $S$. We have

$$
(Vc)_j(t) - (Vd)_j(t) = f_{1j}(c(t), c'(t)) - f_{1j}(d(t), d'(t))
$$

$$
+ \int_0^t (f_{2j}(t - s, c(s)) - f_{2j}(t - s, d(s)))ds.
$$
(2.21)

From the assumptions (H), (F2) and (F3), it follows that there exists a constant $K_M > 0$ such that

$$
\sup_{0 \leq t \leq T_m} \sum_{j=1}^m |f_{1j}(c(t), c'(t)) - f_{1j}(d(t), d'(t))| \leq K_M(\|c - d\|_0 + \|c - d\|_0^\beta + \|c' - d'\|_0^\alpha),
$$
(2.22)
for all \(c, d \in S\). Then we have the following lemma.

**Lemma 2.5.** Let \(f_{2j} : [0, T_m] \times \mathbb{R}^m \to \mathbb{R}\) be continuous, and let

\[
(W_j c)(t) = \int_0^t f_{2j}(t-s, c(s))ds, c \in C^0([0, T_m]; \mathbb{R}^m). \tag{2.23}
\]

Then, the operator \(W_j : C^0([0, T_m]; \mathbb{R}^m) \to C^0([0, T_m]; \mathbb{R})\) is continuous on \(S\).

The proof of this lemma follows easily from \(f_{2j}\) being uniformly continuous on \([0, T_m] \times [-M, M]^m\). We omit the proof.

From (2.21), (2.22), (2.23), we deduce that

\[
\|Vc - Vd\|_0 = \sup_{0 \leq \tau \leq T_m} \left| \sum_{j=1}^m [(Vc)_j(\tau) - (Vd)_j(\tau)] \right|
\leq K_M \left( \|c - d\|_0 + \|c - d\|^2_0 + \|c' - d'\|^2_0 \right) + \sup_{0 \leq \tau \leq T_m} \left| \sum_{j=1}^m [(W_j c)(t) - (W_j d)(t)] \right|, \quad \forall c, d \in S.
\tag{2.24}
\]

Thus, Lemma 2.5 and inequality (2.24) show that \(V : S \to C^0([0, T_m]; \mathbb{R}^m)\) is continuous.

(c) Now, we shall show that the set \(\overline{US}\) is a compact subset of \(Y\). Let \(c \in S, t, t' \in [0, T_m]\). From (2.9), we rewrite

\[
(Uc)_j(t) - (Uc)_j(t') = G_j(t) - G_j(t') + \int_0^t N_j(t-\tau)(Vc)_j(\tau)d\tau - \int_0^{t'} N_j(t'-\tau)(Vc)_j(\tau)d\tau
= G_j(t) - G_j(t') + \int_0^t (N_j(t-\tau) - N_j(t'-\tau))(Vc)_j(\tau)d\tau
\tag{2.25}
- \int_t^{t'} N_j(t'-\tau)(Vc)_j(\tau)d\tau.
\]

From the inequality \(|N_j(t) - N_j(s)| \leq |t - s|\) for all \(t, s \in [0, T_m]\) and (2.17), we obtain

\[
|(Uc)(t) - (Uc)(t')|_1 = \sum_{j=1}^m |(Uc)_j(t) - (Uc)_j(t')| \leq |G(t) - G(t')|_1 + (T_m + \frac{1}{\lambda_1})|t - t'|\|Vc\|_0 \tag{2.26}
\leq |G(t) - G(t')|_1 + \beta(M, T)(T_m + \frac{1}{\lambda_1})|t - t'|.
\]

Similarly, from (2.14) and (2.17), we also obtain

\[
|(Uc)'(t) - (Uc)'(t')|_1 \leq |G'(t) - G'(t')|_1 + \beta(M, T)(\lambda_m T_m + 1)|t - t'|. \tag{2.27}
\]

Since \(US \subset S\), from estimates (2.26), (2.27) we deduce that the family of functions \(US = \{Uc, c \in S\}\), are bounded and equicontinuous with respect to the norm \(\| \cdot \|_1\) of the space \(Y\). Applying Arzela-Ascoli’s theorem to the space \(Y\), we deduce that \(\overline{US}\) is compact in \(Y\). By the Schauder fixed-point theorem, \(U\) has a fixed point \(c \in S\), which satisfies (2.7). The proof of Lemma 2.4 is complete. \(\square\)
Using Lemma 2.1, for $T > 0$, fixed, system (2.4) - (2.6) has solution $(u_m(t), P_m(t))$ on an interval $[0, T_m]$. The following estimates allow one to take $T_m = T$ for all $m$.

Step 2. A priori estimates. Substituting (2.5) into (2.4), then multiplying the equation of (2.4) by $c_j(t)$ and summing up with respect to $j$, integrating by parts with respect to the time variable from 0 to $t$, by (G) and (F1), we have

$$S_m(t) \leq -2\mathcal{H}(u_m(0, t)) + 2\mathcal{H}(u_{om}(0)) + S_m(0) + 2g(0)u_{om}(0)$$

$$- 2g(t)u_m(0, t) + 2 \int_0^t g'(s)u_m(0, s)ds - 2 \int_0^t \langle f(u_m(s), 0), u'_m(s) \rangle ds$$

$$+ 2 \int_0^t u'_m(0, s)ds \int_0^s K(s - \tau, u_m(0, \tau))d\tau,$$

(2.28)

where

$$S_m(t) = \|u'_m(t)\|^2 + \|u_m(t)\|^2_V. \quad (2.29)$$

Then, using (2.6), (2.29), (H), and Lemma 2.1 we have

$$- 2\mathcal{H}(u_m(0, t)) + 2\mathcal{H}(u_{om}(0)) + S_m(0) + 2g(0)u_{om}(0)$$

$$\leq 2h_0 + 2\mathcal{H}(u_{om}(0)) + S_m(0) + 2g(0)u_{om}(0)$$

(2.30)

$$\leq \frac{1}{4}C_1, \quad \text{for all } m \text{ and all } t,$$

where $C_1$ is a constant depending only on $u_0$, $u_1$, $h_0$, $H$, and $g$.

Again using Lemma 2.1 and the inequality $2ab \leq 4a^2 + \frac{1}{4}b^2$, we obtain

$$| - 2g(t)u_m(0, t) + 2 \int_0^t g'(s)u_m(0, s)ds|$$

$$\leq 4g^2(t) + 4 \int_0^t |g'(s)|^2 ds + \frac{1}{4}S_m(t) + \frac{1}{4} \int_0^t S_m(s)ds. \quad (2.31)$$

Using Lemma 2.1 from (F3) it follows that

$$| - 2 \int_0^t \langle f(u_m(s), 0), u'_m(s) \rangle ds| \leq 2B_2(0) \int_0^t S_m(s)^{(1 + \beta)/2} ds$$

$$\leq (1 + \beta)B_2(0) \int_0^t S_m(s)ds + (1 - \beta)B_2(0)t.$$

Note that the last integral in (2.28), after integrating by parts, gives

$$I = 2 \int_0^t u'_m(0, s)ds \int_0^s K(s - \tau, u_m(0, \tau))d\tau$$

$$= 2u_m(0, t) \int_0^t K(t - \tau, u_m(0, \tau))d\tau$$

$$- 2 \int_0^t u_m(0, s)ds \left[ K(0, u_m(0, 0)) + \int_0^s \frac{\partial K}{\partial s}(s - \tau, u_m(0, \tau))d\tau \right].$$
Hence
\[ |I| \leq 2\sqrt{S_m(t)} \int_0^t (k_1(t - \tau) \sqrt{S_m(\tau)} + k_2(t - \tau)) d\tau \]
\[ + 2 \int_0^t \sqrt{S_m(s)} ds [k_1(0) \sqrt{S_m(s)} + k_2(0)] \]
\[ + \int_0^t (k_3(s - \tau) \sqrt{S_m(\tau)} + k_4(s - \tau)) d\tau \]
\[ = 2\sqrt{S_m(t)} \int_0^t k_1(t - \tau) \sqrt{S_m(\tau)} d\tau + 2\sqrt{S_m(t)} \int_0^t k_2(\tau) d\tau \]
\[ + 2k_1(0) \int_0^t S_m(s) ds + 2k_2(0) \int_0^t \sqrt{S_m(s)} ds \]
\[ + 2 \int_0^t \sqrt{S_m(s)} ds \int_0^s k_3(s - \tau) \sqrt{S_m(\tau)} d\tau + 2 \int_0^t \sqrt{S_m(s)} ds \int_0^s k_4(\tau) d\tau \]
\[ \equiv I_1 + I_2 + 2k_1(0) \int_0^t S_m(s) ds + I_4 + I_5 + I_6. \]  

By the inequality \(2ab \leq 4a^2 + \frac{1}{4}b^2\) and the Cauchy-Schwarz inequality we estimate without difficulty the following integrals in the right-hand side of the above expression as follows

\[ I_1 = 2\sqrt{S_m(t)} \int_0^t k_1(t - \tau) \sqrt{S_m(\tau)} d\tau \leq \frac{1}{4}S_m(t) + 4 \int_0^t k_1^2(\tau) d\tau, \]
\[ I_2 = 2\sqrt{S_m(t)} \int_0^t k_2(\tau) \leq \frac{1}{4}S_m(t) + 4 \left( \int_0^t k_2^2(\tau) d\tau \right)^2, \]
\[ I_4 = 2k_4(0) \int_0^t S_m(s) ds \leq 4k_4^2(0) + \frac{1}{4} \int_0^t S_m(s) ds, \]
\[ I_5 = 2 \int_0^t \sqrt{S_m(s)} ds \int_0^s k_3(s - \tau) \sqrt{S_m(\tau)} d\tau \leq 2\sqrt{t} \left( \int_0^t k_3^2(\tau) d\tau \right)^{1/2} \int_0^t S_m(s) ds, \]
\[ I_6 = 2 \int_0^t \sqrt{S_m(s)} ds \int_0^s k_4(\tau) d\tau \leq \frac{1}{4} \int_0^t S_m(s) ds + 4t \left( \int_0^t k_4^2(\tau) d\tau \right)^2. \]

It follows from the estimates for \(I_1, I_2, I_4, I_5, I_6\) that

\[ |I| \leq 4 \left( \int_0^t k_2(\tau) d\tau \right)^2 + 4k_2^2(0) + 4t \left( \int_0^t k_4(\tau) d\tau \right)^2 + \frac{1}{2}S_m(t) \]
\[ + \frac{1}{4} \left[ 1 + t + 16 \int_0^t k_1^2(\tau) d\tau + 8k_1(0) + 8\sqrt{t} \left( \int_0^t k_3^2(\tau) d\tau \right)^{1/2} \right] \int_0^t S_m(s) ds. \]  

It follows from (2.28)-(2.30), (2.31)-(2.32), and (2.33) that

\[ S_m(t) \leq D_1(t) + D_2(t) \int_0^t S_m(\tau) d\tau, \]  

where

\[ D_1(t) = C_1 + 16k_2^2(0) + 4(1 - \beta)B_2(0)t + 16g^2(t) \]
\[ + 16 \int_0^t |g'(s)|^2 ds + 16 \left( \int_0^t k_2(\tau) d\tau \right)^2 + 16t \left( \int_0^t k_4(\tau) d\tau \right)^2, \]  

(2.35)
\[ D_2(t) = 2 + 4(1 + \beta)B_2(0) + 8k_1(0) + t + \int_0^t k_1^2(\tau)d\tau + 8\sqrt{t}\left(\int_0^t k_3^2(\tau)d\tau\right)^{1/2} \]
\[ \leq 2 + 4(1 + \beta)B_2(0) + 8k_1(0) + T + \|k_1\|_{L^2(0,T)}^2 + 8\sqrt{T}\|k_3\|_{L^2(0,T)} \equiv C_T^{(2)}. \]

Since \( H^1(0,T) \hookrightarrow C^0([0,T]) \), from the assumptions (G), (K2), we deduce that
\[ |D_1(t)| \leq C_T^{(1)}, \text{ a.e. in } [0,T], \quad (2.36) \]
where \( C_T^{(1)} \), is a constant depending only on \( T \). By Gronwall’s lemma, from (2.34)-(2.36) we obtain that
\[ S_m(t) \leq C_T^{(1)} \exp(C_T^{(2)}) \leq C_T \quad \forall t \in [0,T], \forall T > 0. \quad (2.37) \]

Now we need an estimate on the integral \( \int_0^t |u_m'(0,s)|^2 ds \). Put
\[ K_m(t) = \sum_{j=1}^m \frac{\sin(\lambda_j t)}{\lambda_j}, \quad (2.38) \]
\[ \gamma_m(t) = \sum_{j=1}^m w_j(0)\left[ \alpha_m \cos(\lambda_j t) + \beta_m \frac{\sin(\lambda_j t)}{\lambda_j} \right] \]
\[ - \sqrt{2} \sum_{j=1}^m \int_0^t \sin(\lambda_j(t-\tau)) \left( f(u_m(\tau), u_m'(\tau)) \cdot \frac{w_i}{\|w_j\|} \right) d\tau. \]

Then \( u_m(0,t) \) can be rewritten as
\[ u_m(0,t) = \gamma_m(t) - 2 \int_0^t K_m(t-\tau)P_m(\tau)d\tau. \quad (2.39) \]

We shall require the following lemma which proof can be found in [2].

**Lemma 2.6.** There exist a constant \( C_2 > 0 \) and a positive continuous function \( D(t) \) independent of \( m \) such that
\[ \int_0^t |\gamma_m'(\tau)|^2 d\tau \leq C_2 + D(t) \int_0^t \|f(u_m(\tau), u_m'(\tau))\|^2 d\tau \quad \forall t \in [0,T], \forall T > 0. \]

**Lemma 2.7.** There exist two positive constants \( C_T^{(3)} \) and \( C_T^{(4)} \) depending only on \( T \) such that
\[ \int_0^t ds \int_0^s K_m(s-\tau)P_m(\tau)d\tau \leq C_T^{(3)} + C_T^{(4)} \int_0^t ds \int_0^s |u_m'(0,\tau)|^2 d\tau, \quad (2.40) \]
for all \( t \in [0,T] \) and all \( T > 0 \).

**Proof.** Integrating by parts, we have
\[ \int_0^s K_m'(s-\tau)P_m(\tau)d\tau = K_m(s)P_m(0) + \int_0^t K_m(s-\tau)P_m'(\tau)d\tau, \]
From (2.5), we have

\[ \int_0^t ds \int_0^s K'_m(s - \tau)P_m(\tau)d\tau \leq 2P^2_m(0) \int_0^t K^2_m(s)ds + 2 \int_0^t ds \int_0^s K^2_m(\tau)d\tau \int_0^s |P'_m(\tau)|^2d\tau \]

\[ \leq 2 \int_0^t K^2_m(s)ds[P^2_m(0) + \int_0^t ds \int_0^s |P'_m(\tau)|^2d\tau]. \quad (2.41) \]

From (2.5), we have

\[ P_m(0) = g(0) + H(u_{0m}(0)), \quad (2.42) \]

Then

\[ P'_m(\tau) = g'(\tau) + H'(u_m(0, \tau))u'_m(0, \tau) - K(0, u_m(0, \tau)) - \int_0^\tau \frac{\partial K}{\partial \tau}(\tau - s, u_m(0, s))ds. \quad (2.43) \]

Using the inequality \((a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2)\), for all \(a, b, c, d \in \mathbb{R}\), we deduce from (2.37), (2.43), and (G), (H), (K2) that

\[ \int_0^s |P'_m(\tau)|^2d\tau \]

\[ \leq 4 \int_0^s |g'(\tau)|^2d\tau + 4 \max_{|s| \leq \sqrt{C_T}} |H'(s)|^2 \int_0^s |u'_m(0, \tau)|^2d\tau \]

\[ + 4 \int_0^s |K(0, u_m(0, \tau))|^2d\tau + 4 \int_0^s d\tau |\int_0^\tau \frac{\partial K}{\partial \tau}(\tau - s, u_m(0, s))ds|^2 \]

\[ \leq 4 \int_0^s |g'(\tau)|^2d\tau + 4 \max_{|s| \leq \sqrt{C_T}} |H'(s)|^2 \int_0^s |u'_m(0, \tau)|^2d\tau \]

\[ + 8k^2_3(0) \int_0^s |u_m(0, \tau)|^2d\tau + 8k^2_2(0)s \]

\[ + 8 \int_0^s d\tau \int_0^\tau k^3_3(s)ds \int_0^s u^2_m(0, s)ds + 8 \int_0^s d\tau (\int_0^\tau k_4(s)ds)^2 \]

\[ \leq 4 \int_0^s |g'(\tau)|^2d\tau + 8[k^2_3(0)C_T + k^2_2(0)s + 4C_Ts^2 \int_0^s k^2_3(\tau)d\tau] \]

\[ + 8s(\int_0^s k_4(\tau)d\tau)^2 + 4 \max_{|s| \leq \sqrt{C_T}} |H'(s)|^2 \int_0^s |u'_m(0, \tau)|^2d\tau. \]

Hence

\[ \int_0^t ds \int_0^s |P'_m(\tau)|^2d\tau \leq 4t \int_0^t |g'(\tau)|^2d\tau + 4[k^2_3(0)C_T + k^2_2(0)]t^2 \]

\[ + \frac{4}{3}C_Tt^3 \int_0^t k^3_3(\tau)d\tau + 4t^2 \left( \int_0^t k_4(\tau)d\tau \right)^2 \]

\[ + 4 \max_{|s| \leq \sqrt{C_T}} |H'(s)|^2 \int_0^t ds \int_0^s |u'_m(0, \tau)|^2d\tau. \]
From this inequality, (2.41), and (2.42), it follows that
\[
\int_0^t ds \int_0^s K_m'(s - \tau) P_m(\tau) d\tau \leq 2 \int_0^t K_m^2(s) ds \left[ (g(0) + H(u_{0m}(0)))^2 + 4t \int_0^t |g'(\tau)|^2 d\tau + 4[k_2^2(0)C_T + k_3^2(0)]^2 \right.+ \\
+ \frac{4}{3} C_T^3 t^3 \int_0^t k_2^3(\tau) d\tau + 4t^2 \left( \int_0^t k_4(\tau) d\tau \right)^2 + 4 \max_{|s| \leq \sqrt{C_T}} \left| H'(s) \right|^2 \int_0^t ds \int_0^s |u_m'(0, \tau)|^2 d\tau],
\]
(2.45)

Note that for every $T > 0$, $K_m \to \bar{K}$, strongly in $L^2(0, T)$ as $m \to +\infty$. Using the assumptions (G), (H), (K2) and the results (2.6) and (2.45), we obtain (2.40). The proof of Lemma 2.7 is complete. □

**Lemma 2.8.** There exist two positive constants $C_T^{(5)}$ and $C_T^{(6)}$ depending only on $T$ such that
\[
\int_0^t |u_m'(0, \tau)|^2 d\tau \leq C_T^{(5)} \quad \forall t \in [0, T], \forall T > 0.
\]
(2.46)
\[
\int_0^t |P_m'(\tau)|^2 d\tau \leq C_T^{(6)} \quad \forall t \in [0, T], \forall T > 0.
\]
(2.47)

**Proof.** Since (2.47) is a consequence of (2.44) and (2.46), we only have to prove (2.46). From (2.39), using Lemmas 2.6 and 2.7, we obtain
\[
\int_0^t |u_m'(0, \tau)|^2 d\tau \leq 2 \int_0^t |\gamma_m'(s)|^2 ds + 8 \int_0^t ds \int_0^s K_m'(s - \tau) P_m(\tau) d\tau \leq 2C_2 + 2D(t) \int_0^t \|f(u_m(\tau), u_m'(\tau))\| d\tau
\]
(2.48)
\[
+ 8C_T^{(3)} + 8C_T^{(4)} \int_0^t ds \int_0^s |u_m'(0, \tau)|^2 d\tau.
\]

On the other hand, from the assumptions (F2), (F3), we obtain
\[
\|f(u_m(t), u_m'(t))\|^2 \leq 2\left( \max_{|s| \leq \sqrt{C_T}} B_1^2(s) \right) \|u_m'(t)\|^2 + 2B^2_2(0) \|u_m(t)\|^2 \|u_m(t)\|_V,
\]
(2.49)
since $0 < \alpha \leq 1$ we have $\| \cdot \| \leq \| \cdot \|_{L^2 \alpha}$. Hence, using (2.37) and (2.49) we have
\[
\|f(u_m(t), u_m'(t))\| \leq C_T^{(7)}.
\]
(2.50)

At last from this inequality and (2.48) we obtain the inequality
\[
\int_0^t |u_m'(0, s)|^2 ds \leq C_T^{(8)} + 8C_T^{(4)} \int_0^t ds \int_0^s |u_m'(0, \tau)|^2 d\tau,
\]
which implies (2.46), by Grönwall’s lemma. Therefore, Lemma 2.8 is proved. □

**Step 2. Passing to limit.** From (2.5), (2.29), (2.37), (2.46), (2.47), and (2.50), we deduce that, there exists a subsequence of sequence $\{ (u_m, P_m) \}$, still denoted by
\{ (u_m, P_m) \}, such that
\begin{align}
  u_m &\to u \quad \text{in } L^\infty(0,T;V) \text{ weak*}, \\
  u'_m &\to u' \quad \text{in } L^\infty(0,T;L^2) \text{ weak*}, \\
  u'_m(0,t) &\to u(0,t) \quad \text{in } L^\infty(0,T) \text{ weak*}, \\
  u'_m(0,t) &\to u'(0,t) \quad \text{in } L^2(0,T) \text{ weak}, \\
  f(u_m, u'_m) &\to \chi \quad \text{in } L^\infty(0,T;L^2) \text{ weak*}, \\
  P_m &\to \hat{P} \quad \text{in } H^1(0,T) \text{ weak},
\end{align}
(2.51)

By the compactness lemma of Lions (see \[9\]), we can deduce from (2.51)-(2.54) that there exists a subsequence still denoted by \{u_m\} such that
\begin{align}
  u_m(0,t) &\to u(0,t) \quad \text{strongly in } C^0([0,T]), \\
  u_m &\to u \quad \text{strongly in } L^\infty(0,T;V) \text{ a.e. } (x,t) \in Q_T.
\end{align}
(2.57)

By (H),(K) and using (2.5), (2.57) we obtain
\begin{align}
P_m(t) &\to g(t) + H(u(0,t)) - \int_0^t K(t-s, u(0,s))ds \equiv P(t) \quad \text{strongly in } C^0([0,T]).
\end{align}
(2.59)

From (2.56) and (2.59) we have
\begin{align}
P &\equiv \hat{P} \quad \text{a.e. in } Q_T.
\end{align}
(2.60)

Passing to the limit in (2.4) by (2.51), (2.52), (2.59), and (2.60) we have
\begin{equation}
\frac{d}{dt} \langle u'(t), v \rangle + a(u(t), v) + P(t)v(0) + \langle \chi, v \rangle = 0 \quad \forall v \in V.
\end{equation}

As in \[9\], we can prove that
\begin{align}
u(0) &= u_0, \quad u'(0) = u_1.
\end{align}

To prove the existence of solution \( u \), we have to show that \( \chi = f(u, u') \). We need the following lemma which proof can be found in \[2\].

**Lemma 2.9.** Let \( u \) be the solution of the problem
\begin{align}
  u_{tt} - u_{xx} + \chi &= 0, \quad 0 < x < 1, \quad 0 < t < T, \\
  u_t(0,t) &= P(t), \quad u(1,t) = 0, \\
  u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x), \\
  u &\in L^\infty(0,T;V), \quad u' \in L^\infty(0,T;L^2) \\
  u(0,\cdot) &\in H^1(0,T).
\end{align}

Then
\begin{equation}
\frac{1}{2} \|u'(t)\|^2 + \frac{1}{2} \|u(t)\|_V^2 + \int_0^t P(s)u'(0,s)ds + \int_0^t \langle \chi(s), u'(s) \rangle ds \geq \frac{1}{2} \|u_1\|^2 + \frac{1}{2} \|u_0\|_V^2,
\end{equation}
a.e. \( t \in [0,T] \). Furthermore, if \( u_0 = u_1 = 0 \) there is equality in the above expression.
Now, from (2.4) - (2.6) we have
\[ \int_0^t \langle f(u_m(s), u'_m(s)), u'_m(s) \rangle ds = \frac{1}{2} \| u_{1m} \|^2 + \frac{1}{2} \| u_{0m} \|^2 - \frac{1}{2} \| u'_m(t) \|^2 - \frac{1}{2} \| u_m(t) \|^2 - \int_0^t P_m(s) u'_m(0, s) ds. \] (2.61)

By Lemma 2.9, it follows from (2.6), (2.51), (2.52), (2.54), (2.59) and (2.61), that
\[ \limsup_{m \to + \infty} \int_0^t \langle f(u_m(s), u'_m(s)), u'_m(s) \rangle ds \leq \frac{1}{2} \| u_1 \|^2 + \frac{1}{2} \| u_0 \|^2 - \frac{1}{2} \| u'(t) \|^2 - \frac{1}{2} \| u(t) \|^2 - \int_0^t P(s) u'(0, s) ds \leq \int_0^t \langle \chi(s), u'(s) \rangle ds, \quad \text{a.e. } t \in [0, T]. \]

Using the same arguments as in [9], we can show that \( \chi = f(u, u') \) a.e. in \( Q_T \). The existence of the solution is proved.

**Step 4. Uniqueness of the solution.** Assume now that \( \beta = 1 \) in (F3), and that \( H, K, f \) satisfy (H1), (K3), and (F4). Let \( (u_1, P_1), (u_2, P_2) \) be two weak solutions of the problem (1.1) - (1.5). Then \( u = u_1 - u_2, \ P = P_1 - P_2 \) satisfy the problem
\[
\begin{align*}
&u'' - u_{xx} + \chi = 0, \quad 0 < x < 1, \quad 0 < t < T, \\
&u_x(0, t) = P(t), \quad u(1, t) = 0, \\
&u(x, 0) = u'(x, 0) = 0, \\
&\chi = f(u_1, u'_1) - f(u_2, u'_2), \\
&P(t) = P_1(t) - P_2(t) \\
&= H(u_1(0, t)) - H(u_2(0, t)) \\
&\quad - \int_0^t (K(t - s, u_1(0, s)) - K(t - s, u_2(0, s))) ds, \\
&u_i \in L^\infty(0, T; V), \quad u'_i \in L^\infty(0, T; L^2), \quad u_i(0, \cdot) \in H^1(0, T), \\
&P_i \in H^1(0, T), \quad i = 1, 2.
\end{align*}
\]

Using Lemma 2.9 with \( u_0 = u_1 = 0 \), we obtain
\[ \frac{1}{2} \| u'(t) \|^2 + \frac{1}{2} \| u(t) \|^2 \leq \frac{1}{2} \| u'(t) \|^2 + \frac{1}{2} \| u(t) \|^2 + \int_0^t P(s) u'(0, s) ds + \int_0^t \langle \chi(s), u'(s) \rangle ds = 0, \] (2.62)
a.e. \( t \in [0, T] \). Put
\[
\begin{align*}
\sigma(t) &= \| u'(t) \|^2 + \frac{1}{2} \| u(t) \|^2, \\
\tilde{H}_1(t) &= H(u_1(0, t)) - H(u_2(0, t)), \\
\tilde{K}_1(t, s) &= K(t - s, u_1(0, s)) - K(t - s, u_2(0, s)).
\end{align*}
\]
Substituting $P(t)$, $\chi$ into (2.62) and using that $f$ is nondecreasing with respect to the second variable, we have
\[
\sigma(t) + 2 \int_0^t \bar{H}_1(s) u'(0, s) ds \\
\leq 2 \int_0^t ||f(u_1(s), u_2'(s)) - f(u_2(s), u_2'(s))|| ||u'(s)|| ds + 2 \int_0^t u'(0, s) ds \int_0^s \bar{K}_1(s, r) dr.
\]
(2.63)

Using assumption (F3),
\[
||f(u_1(s), u_2'(s)) - f(u_2(s), u_2'(s))|| \leq \|B_2([u_2'(s)])\| ||u(s)||_V.
\]

Using integration by parts in the last integral of (2.63), we get
\[
J = 2 \int_0^t u'(0, s) ds \int_0^s \bar{K}_1(s, r) dr \\
= 2u(0, t) \int_0^t \bar{K}_1(t, r) dr - 2 \int_0^t u(0, s) ds [\bar{K}_1(s, s) + \int_0^s \frac{\partial \bar{K}_1}{\partial s}(s, r) dr].
\]
(2.64)

From assumption (K3), we have
\[
|\bar{K}_1(s, r)| \leq p_{M, T}(t - r)|u(0, r)| \leq p_{M, T}(t - r)\sqrt{\sigma(r)}, \\
|\bar{K}_1(s, s)| \leq p_{M, T}(0)|u(0, s)| \leq p_{M, T}(0)\sqrt{\sigma(s)},
\]
(2.65)

where $M = \max_{i=1,2} \|u_i\|_{L^\infty(0, T, V)}$. It follows from (2.64) and (2.65) that
\[
|J| \leq 2\sqrt{\sigma(t)} \int_0^t p_{M, T}(t - r)\sqrt{\sigma(r)} dr + 2p_{M, T}(0) \int_0^t \sqrt{\sigma(s)} ds \\
+ 2 \int_0^t \sqrt{\sigma(s)} ds \int_0^s p_{M, T}(s - r)\sqrt{\sigma(r)} dr \\
\leq \beta_1 \sigma(t) + \frac{1}{\beta_1} \int_0^t p_{M, T}^2(r) dr \int_0^t \sigma(r) dr \\
+ 2p_{M, T}(0) \int_0^t \sigma(s) ds 2\sqrt{\bar{t}} \left( \int_0^t q_{M, T}^2(r) dr \right)^{1/2} \int_0^t \sigma(s) ds \\
= \beta_1 \sigma(t) + \left[ 2p_{M, T}(0) + \frac{1}{\beta_1} \int_0^t p_{M, T}^2(r) dr \right] \int_0^t \sigma(s) ds \\
+ 2\sqrt{\bar{t}} \left( \int_0^t q_{M, T}^2(r) dr \right)^{1/2} \int_0^t \sigma(s) ds,
\]
(2.66)

for all $\beta_1 > 0$. Put
\[
m_1 = \min_{|s| \leq M} H'(s), \quad m_2 = \max_{|s| \leq M} \max |H''(s)|.
\]
(2.67)

From assumption (H1) we have
\[
m_1 > -1.
\]
(2.68)
On the other hand, using integration by parts and (2.67) it follows that

\[ 2 \int_0^t \tilde{H}_1(s)u'(0,s)ds = 2 \int_0^t \left[ \int_0^1 \frac{d}{d\theta} H(u_2(0,s) + \theta u(0,s))d\theta \right] u'(0,s)ds \]

\[ = u^2(0,t) \int_0^1 H'(u_2(0,s) + \theta u(0,s))d\theta \]

\[ - \int_0^t u^2(0,s)ds \int_0^1 H''(u_2(0,s) + \theta u(0,s))(u'_2(0,s) + \theta u'(0,s))d\theta \]

\[ \geq m_1 u^2(0,t) - m_2 \int_0^t u^2(0,s)(|u'_1(0,s)| + |u'_2(0,s)|)ds \]

\[ \geq m_1 u^2(0,t) - m_2 \int_0^t \sigma(s)(|u'_1(0,s)| + |u'_2(0,s)|)ds. \]

From the above inequality, (2.63)-(2.64) and (2.66), we obtain

\[ \sigma(t) + m_1 u^2(0,t) \leq m_2 \int_0^t \sigma(s)(|u'_1(0,s)| + |u'_2(0,s)|)ds \]

\[ + \int_0^t \|B_2(|u'_2(s)|)\|\sigma(s)ds + |J| \equiv \eta(t). \] (2.69)

From (2.1), (2.68), and (2.69), we have

\[ (1 + m_1)u^2(0,t) \leq \sigma(t) + m_1 u^2(0,t) \leq \eta(t). \] (2.70)

It follows from (2.69) and (2.70) that

\[ \sigma(t) + [m_1 + \beta_2(1 + m_1)]u^2(0,t) \]

\[ \leq (1 + \beta_2)\eta(t) \]

\[ \leq (1 + \beta_2) \int_0^t \left[ m_2(|u'_1(0,s)| + |u'_2(0,s)|) + B_2(|u'_2(0,s)|)\right] \sigma(s)ds \]

\[ + (1 + \beta_2)\beta_1 \sigma(t) + (1 + \beta_2) \left[ 2p_{M,T}(0) + \frac{1}{\beta_1} \int_0^t p^2_{M,T}(r)dr \right. \]

\[ + 2\sqrt{T} \left( \int_0^t q^2_{M,T}(r)dr \right)^{1/2} \right] \int_0^t \sigma(s)ds, \] (2.71)

for all \( \beta_1 > 0, \beta_2 > 0 \). Choose \( \beta_1 > 0, \beta_2 > 0 \) such that \( m_1 + \beta_2(1 + m_1) \geq 1/2 \), \( (1 + \beta_2)\beta_1 \leq 1/2 \) and denote

\[ R_1(t) = 2(1 + \beta_2)[m_2(|u'_1(0,s)| + |u'_2(0,s)|) + B_2(|u'_2(s)|)\]

\[ + \frac{1}{\beta_1} \|p_{M,T}\|^2_{L^2(0,T)} + 2p_{M,T}(0) + 2\sqrt{T}\|q_{M,T}\|_{L^2(0,T)}]. \] (2.72)

Then from (2.71) and (2.72) we have

\[ \sigma(t) + u^2(0,t) \leq \int_0^t R_1(s)[\sigma(s) + u^2(0,s)]ds; \] (2.73)

i.e. \( \sigma(t) + u^2(0,t) = 0 \) by Gronwall’s lemma. Then Theorem 2.2 is proved. \( \square \)
In the special cases
\[
H(s) = hs, \quad h > 0;
\]
\[
K(t, u) = k(t)u, \quad k \in H^1(0, T), \quad \forall T > 0, k(0) = 0,
\]
the following theorem is a consequence of Theorem 2.2.

**Theorem 2.10.** Let (A), (G) and \((F_1)-(F_3)\) hold. Then, for every \(T > 0\), problem
\[
(1.1)- (1.4) \quad \text{and} \quad (1.9) \quad \text{has at least a weak solution } (u, P) \quad \text{satisfying } (2.2), (2.3).
\]
Furthermore, if \(\beta = 1\) in \((F_3)\) and \(B_2\) satisfies \((F_4)\), then this solution is unique.

We remark that Theorem 2.10 gives the same result as in [4], but we do not need the assumption “\(B_1\) is nondecreasing” used there.

In the special case with \(K(t, u) \equiv 0\), the following result is the consequence of Theorem 2.2.

**Theorem 2.11.** Let (A), (G), (H), \((F_1)-(F_3)\) hold. Then, for every \(T > 0\), the problem \((1.1) - (1.4)\) corresponding to \(P = g\) has at least a weak solution \(u\) satisfying \((2.2)\).

Furthermore, if \(\beta = 1\) in \((F_3)\) and the functions \(H, B_2\) satisfy the assumptions \((H_1), (F_4)\), then this solution is unique.

We remark that Theorem gives same result in [4] but without using the assumption “\(B_1\) is nondecreasing” used there.

3. Stability of the solutions

In this section, we assume that \(\beta = 1\) in \((F_3)\) and that the functions \(H, B_2\) satisfying \((H), (H_1), (F_4)\), respectively. By Theorem 2.2, problem \((1.1) - (1.3)\) admits a unique solution \((u, P)\) depending on \(g, H, K\):
\[
u = u(g, H, K), \quad P = P(g, H, K),
\]
where \(g, H, K\) satisfy the assumptions \((G), (H), (H_1), (K_1)-(K_3)\), and \(u_0, u_1, f\) are fixed functions satisfying \((A), (F_1)-(F_4)\).

Let \(h_0 > 0\) be a given constant and \(H_0 : \mathbb{R}^+ \to \mathbb{R}^+\) be a given function. We put
\[
\mathbb{R}(h_0, H_0) = \{ H \in C^2(\mathbb{R}) : H(0) = 0, \int_0^\infty H(s)ds \geq -h_0, \forall x \in \mathbb{R},
\]
\[
H'(s) > -1, \forall s \in \mathbb{R}, \sup_{|s| \leq M} \left( |H(s)| + |H'(s)| \right) \leq H_0(M), \forall M > 0 \}.
\]
Given \(t \geq 0, M > 0,\) and \(K \in C^0(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})\), we put
\[
N_h(M, K, t) = \sup\left[\sup_{u, v \leq M, u \neq v} |K(t, u) - K(t, v)| / (u - v)\right].
\]
Given the family \(\{p_{M,T}\}, M > 0, T > 0\) which consists of nonnegative functions \(p_{M,T}(t) = p(M, T, t), M > 0, T > 0\) such that \(p_{M,T} \in L^2(0, T)\), for all \(M, T > 0\).

Let \(k_1 \in L^2(0, T), k_2 \in L^1(0, T)\), for all \(T > 0\). We put
\[
\Gamma(k_1, k_2, \{p_{M,T}\})
\]
\[
= \{ K \in C^0(\mathbb{R}_+ \times \mathbb{R}) : \partial K / \partial t \in C^0(\mathbb{R}_+ \times \mathbb{R}),
\]
\[
N_h(M, K, t) + N_h(M, \partial K / \partial t, t) \leq p_{M,T}(t), \forall t \in [0, T], \forall M, T > 0,
\]
\[
|K(t, u)| + |\partial K / \partial t(t, u)| \leq k_1(t)|u| + k_2(t), \forall u \in \mathbb{R}, \forall t \in [0, T], \forall T > 0 \}.
\]
Then we have the following theorem.
Theorem 3.1. Let $\beta = 1$ and (A), (F1)–(F4) hold. Then, for every $T > 0$, the solutions of (1.1)-(1.5) are stable with respect to the data $g, H, K$; i.e., if $(g, H, K)$, $(g_j, H_j, K_j) \in H^1(0, T) \times \Re(h_0, H_0) \times \Gamma(k_1, k_2, \{p_{M,T}\})$, are such that

$$(g_j, H_j) \to (g, H) \quad \text{in} \quad H^1(0, T) \times C^1([-M, M]) \quad (3.1)$$

strongly, and

$$(K_j, \partial K_j/\partial t) \to (K, \partial K/\partial t) \quad \text{in} \quad [C_0^0([0, T] \times [-M, M])]^2 \quad (3.2)$$

strongly, as $j \to +\infty$, for all $M, T > 0$. Then

$$(u_j, u'_j, u_j(0, t), P_j) \to (u, u', u(0, t), P)$$

in $L^\infty(0, T; V) \times L^\infty(0, T; L^2) \times C^0([0, T]) \times C^0([0, T])$ strongly, as $j \to +\infty$, for all $M, T > 0$, where $u_j = u(g_j, H_j, K_j)$, $P_j = P(g_j, H_j, K_j)$.

Proof. First, we note that if the data $(g, H, K)$ satisfy

$$
\|g\|_{H^1(0, T)} \leq G_0, \quad H \in \Re(h_0, H_0), \quad K \in \Gamma(k_1, k_2, \{p_{M,T}\}),
$$

then, the a priori estimates of the sequences $\{u_m\}$ and $\{P_m\}$ in the proof of the Theorem 2.2 satisfy

$$
\|u'_m(t)\|^2 + \|u_m(t)\|_V^2 \leq C_T^2 \quad \forall t \in [0, T], \forall T > 0, \quad (3.4)
$$

$$
\int_0^t |u'_m(s)|^2 ds \leq C_T^2 \quad \forall t \in [0, T], \forall T > 0, \quad (3.5)
$$

$$
\int_0^t |P'_m(s)|^2 ds \leq C_T^2 \quad \forall t \in [0, T], \forall T > 0, \quad (3.6)
$$

where $C_T$ is a constant depending only on $T, u_0, u_1, f, G_0, h_0, H_0, k_1, k_2, \{p_{M,T}\}$ (independent of $g, H, K$). Hence, the limit $(u, P)$ in suitable function spaces of the sequence $\{(u_m, P_m)\}$ is defined by (2.4)–(2.6), which is a solution of (1.1)–(1.5) satisfying the a priori estimates (3.4)–(3.6).

Now, by (3.1)–(3.2) we can assume that there exists constant $G_0 > 0$ such that the data $(g_j, H_j, K_j)$ satisfy (3.3) with $(g, H, K) = (g_j, H_j, K_j)$. Then, by the above remark, we have that the solutions $(u_j, P_j)$ of problem (1.1)–(1.5) corresponding to $(g, H, K) = (g_j, H_j, K_j)$ satisfy

$$
\|u'_j(t)\|^2 + \|u_j(t)\|_V^2 \leq C_T^2 \quad \forall t \in [0, T], \forall T > 0, \quad (3.7)
$$

$$
\int_0^t |u'_j(s)|^2 ds \leq C_T^2 \quad \forall t \in [0, T], \forall T > 0, \quad (3.8)
$$

$$
\int_0^t |P'_j(s)|^2 ds \leq C_T^2 \quad \forall t \in [0, T], \forall T > 0, \quad (3.9)
$$

Put $\tilde{g}_j = g_j - g$, $\tilde{H}_j = H_j - H$, $\tilde{K}_j = K_j - K$. Then, $v_j = u_j - u$ and $Q_j = P_j - P$ satisfy the problem

$$
v''_j - v'_{jxx} + \chi_j = 0, \quad 0 < x < 1, \quad 0 < t < T,
$$

$$
v_j(x, 0) = Q_j(t), \quad v_j(1, t) = 0,
$$

$$
v_j(x, 0) = v'_j(x, 0) = 0,$$
where
\[ \chi_j = f(u_j, u_j') - f(u, u'), \]
\[ Q_j(t) = \tilde{g}_j(t) + H(u_j(0, t)) - H(u(0, t)) \]
\[ - \int_0^t [K(t - s, u_j(0, s)) - K(t - s, u(0, s))] ds, \]
\[ \tilde{g}_j(t) = \tilde{g}_j(t) + \tilde{H}_j(u_j(0, t)) - \int_0^t \tilde{K}_j(t - s, u_j(0, s)) ds. \] (3.11)

Applying Lemma 2.9 with \( u_0 = u_1 = 0, \chi = \chi_j, P = Q_j, \) we have
\[ \|v_j'(t)\|^2 + \|v_j(t)\|_V^2 + 2 \int_0^t Q_j(s)v_j'(0, s)ds + 2 \int_0^t \langle \chi_j(s), v_j'(s) \rangle ds = 0. \]

Let
\[ S_j(t) = \|v_j'(t)\|^2 + \|v_j(t)\|_V^2 + v_j^2(0, t), \]
\[ M = C_T, \quad m_1 = \min_{|s| \leq M} H'(s) > -1, \quad m_2 = \max_{|s| \leq M} |H''(s)|. \]

Then, we can prove the following inequality in a similar manner
\[ \|v_j'(t)\|^2 + \|v_j(t)\|_V^2 + m_1 v_j^2(0, t) \]
\[ \leq \int_0^t \|B_2(|u'(s)|)\| S_j(t) ds + m_2 \int_0^t (|u'(0, s)| + |u_j'(0, s)|) S_j(s) ds \]
\[ + 2\varepsilon S_j(t) + \varepsilon \int_0^t S_j(s) ds + \frac{1}{\varepsilon} \tilde{g}_j^2(t) + \int_0^t |\tilde{g}_j'(s)|^2 ds \]
\[ + \left( \frac{1}{\varepsilon} \|p_{M,T} \|_{L^2(0,T)}^2 + 2\sqrt{T} \|p_{M,T} \|_{L^2(0,T)} \right) \int_0^t S_j(s) ds \] (3.12)
\[ = 2\varepsilon S_j(t) + \frac{1}{\varepsilon} \tilde{g}_j^2(t) + \int_0^t |\tilde{g}_j'(s)|^2 ds \]
\[ + \int_0^t \|B_2(|u'(s)|)\| + m_2 (|u'(0, s)| + |u_j'(0, s)|) S_j(s) ds \]
\[ + \left( \varepsilon + \frac{1}{\varepsilon} \|p_{M,T} \|_{L^2(0,T)}^2 + 2\sqrt{T} \|p_{M,T} \|_{L^2(0,T)} \right) \int_0^t S_j(s) ds = y_j(t), \]
for all \( \varepsilon > 0 \) and \( t \in [0, T]. \)

We remark that \( v_j^2(0, t) \leq \|v_j(t)\|_V^2, \) consequently
\[ (1 + m_1)v_j^2(0, t) \leq \|v_j'(t)\|^2 + \|v_j(t)\|_V^2 + m_1 v_j^2(0, t) \leq y_j(t). \] (3.13)

Multiplying the two members of \( 3.13 \) by a number \( \beta_1 > 0 \) and adding to \( 3.12, \) we have
\[ \|v_j'(t)\|^2 + \|v_j(t)\|_V^2 + [(1 + m_1)\beta_1 + m_1]v_j^2(0, t) \]
\[ \leq (1 + \beta_1) y_j(t) \]
\[ \leq (1 + \beta_1)[2\varepsilon S_j(t) + \frac{1}{\varepsilon} \tilde{g}_j^2(t) + \int_0^t |\tilde{g}_j'(s)|^2 ds] \]
\[ + \int_0^t \tilde{R}_j(\varepsilon, T, s) S_j(s) ds, \quad \forall \varepsilon > 0, \beta_1 > 0, t \in [0, T]. \] (3.14)
where
\[
\tilde{R}_j(\varepsilon, T, s) = (1 + \beta_1) \left[ \varepsilon + \frac{1}{\varepsilon} \| p_{M,T} \|_{L^2(0,T)}^2 + 2\sqrt{T} \| p_{M,T} \|_{L^2(0,T)} + \| B_2(u'(t)) \| + m_2(\| u'(0, s) \| + \| u'(0, s) \|) \right].
\] (3.15)

Choose \( \beta_1 > 0 \) and \( \varepsilon > 0 \) such that \( (1 + m_1)\beta_1 + m_1 \geq 1 \), \( 2\varepsilon (1 + \beta_1) \leq 1/2 \). From \( H^1(0, T) \hookrightarrow C^0([0, T]) \), and (3.14) we have
\[
S_j(t) \leq 2(1 + \beta_1) \frac{1}{\varepsilon} C_T^{(9)} \| \tilde{g}_j \|_{H^1(0,T)}^2 + 2 \int_0^t \tilde{R}_j(\varepsilon, T, s) S_j(s) ds,
\] (3.16)
where \( C_T^{(9)} \) is a constant depending only on \( T \). By Gronwall’s lemma, we obtain from (3.16) that
\[
S_j(t) \leq 2(1 + \beta_1) \frac{1}{\varepsilon} C_T^{(9)} \| \tilde{g}_j \|_{H^1(0,T)}^2 \exp \left( 2 \int_0^T \tilde{R}_j(\varepsilon, T, s) S_j(s) ds \right),
\] (3.17)
for all \( t \in [0, T] \). On the other hand, we from (3.4), (3.10), (3.11), (3.15), and (3.17) obtain
\[
S_j(t) \leq C_T^{(10)} \| \tilde{g}_j \|_{H^1(0,T)}^2 \quad \forall t \in [0, T],
\] (3.18)
\[
|Q_j(t)| \leq |\tilde{g}_j(t)| + \max_{|s| \leq M} |H'(s)| \sqrt{S_j(t)} + \| p_{M,T} \|_{L^2(0,T)} \left( \int_0^t S_j(s) ds \right)^{1/2}.
\] (3.19)

We again use the embedding \( H^1(0, T) \hookrightarrow C^0([0, T]) \). Then, it follows from (3.18) and (3.19) that
\[
\| Q_j \|_{C^0([0, T])} \leq C_T^{(11)} \| \tilde{g}_j \|_{H^1(0,T)}^2.
\]
As a final step, we prove
\[
\lim_{j \to +\infty} \| \tilde{g}_j \|_{H^1(0,T)}^2 = 0.
\]
Indeed, from (3.11) combined with (3.8), we deduce the following inequality
\[
\| \tilde{g}_j \|_{H^1(0,T)} \leq \| \tilde{g}_j \|_{H^1(0,T)} + \sqrt{T + M^2} \| \tilde{H}_j \|_{C^1([-M,M])} + \sqrt{2T (1 + T^2)} \| \tilde{K}_j \|_{C^0([-0, T] \times [-M, M])} + \| \partial \tilde{K}_j / \partial t \|_{C^0([0, T] \times [-M, M])}.
\]
Then the proof is complete.

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