Existence of global solutions to reaction-diffusion systems with nonhomogeneous boundary conditions via a Lyapunov functional *

Said Kouachi

Abstract

Most publications on reaction-diffusion systems of $m$ components ($m \geq 2$) impose $m$ inequalities to the reaction terms, to prove existence of global solutions (see Martin and Pierre [10] and Hollis [4]). The purpose of this paper is to prove existence of a global solution using only one inequality in the case of 3 component systems. Our technique is based on the construction of polynomial functionals (according to solutions of the reaction-diffusion equations) which give, using the well known regularizing effect, the global existence. This result generalizes those obtained recently by Kouachi [6] and independently by Malham and Xin [9].

1 Introduction

We consider the reaction-diffusion system

$$\begin{align*}
\partial_t u - a\Delta u &= f(u,v,w) \quad \text{in } \mathbb{R}^+ \times \Omega, \\
\partial_t v - b\Delta v &= g(u,v,w) \quad \text{in } \mathbb{R}^+ \times \Omega, \\
\partial_t w - c\Delta w &= h(u,v,w) \quad \text{in } \mathbb{R}^+ \times \Omega,
\end{align*}$$

with the boundary conditions

$$\begin{align*}
\lambda_1 u + (1 - \lambda_1)\partial_n u &= \beta_1, \\
\lambda_2 v + (1 - \lambda_2)\partial_n v &= \beta_2, \\
\lambda_3 w + (1 - \lambda_3)\partial_n w &= \beta_3,
\end{align*}$$

on $\mathbb{R}^+ \times \partial\Omega$ (1.4)

and the initial data

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad w(0,x) = w_0(x) \quad \text{in } \Omega. \quad (1.5)$$

The boundary conditions are specified as follows:

*Mathematics Subject Classifications: 35K45, 35K57.*

*Key words: Reaction diffusion systems, Lyapunov functionals, global existence.*

©2002 Southwest Texas State University.

Existence of global solutions

(i) For nonhomogeneous Robin boundary conditions, we use $0 < \lambda_1, \lambda_2, \lambda_3 < 1, \beta_1 \geq 0, \beta_2$ and $\beta_3 \geq 0$.

(ii) For homogeneous Neumann boundary conditions, we use $\lambda_i = \beta_i = 0$, $i = 1, 2, 3$.

(iii) For homogeneous Dirichlet boundary conditions, we use $1 - \lambda_i = \beta_i = 0$, $i = 1, 2, 3$.

(iv) For a mixture of homogeneous Dirichlet with nonhomogeneous Robin boundary conditions, we use $1 - \lambda_i = \beta_i = 0$, $i = 1, 2$ or 3 and $0 < \lambda_j < 1, \beta_j \geq 0$, $j = 1, 2$, with $i \neq j$.

Here $\Omega$ is an open bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, $\partial_n$ denotes the outward normal derivative on $\partial \Omega$, $a$, $b$ and $c$ are positive constants, $0 \leq \lambda_1, \lambda_2, \lambda_3 \leq 1$ and $\beta_1, \beta_2$ and $\beta_3 \geq 0$ are in $C^1(\partial \Omega; \mathbb{R})$.

The initial data are assumed to be nonnegative. The functions $f$, $g$ and $h$ are continuously differentiable on $\mathbb{R}^3_+$ satisfying $f(0, v, w) \geq 0$, $g(u, 0, w) \geq 0$ and $h(u, v, 0) \geq 0$ for all $u, v, w \geq 0$ which imply, via the maximum principle (see Smoller [14]), the positivity of the solution on its interval of existence. We suppose that the functions $f$, $g$ and $h$ are of polynomial growth and satisfy

$$Df(u, v, w) + Eg(u, v, w) + h(u, v, w) \leq C_1(u + v + w + 1),$$

for all $u, v, w \geq 0$ where $D \geq \bar{D}$ and $E \geq \bar{E}$, where $\bar{D}$ and $\bar{E}$ are positive constants.

Morgan [11] generalized the results of Hollis, Martin and Pierre [4] to establish global existence for solutions of $m$-components systems $(m \geq 2)$ with the boundary conditions (1.4), where

$$0 < \lambda_1, \lambda_2, \lambda_3 < 1 \quad \text{or} \quad \lambda_1 = \lambda_2 = \lambda_3 = 1, \quad \beta_1, \beta_2, \beta_3 \geq 0,$$

or

$$\lambda_1 = \lambda_2 = \lambda_3 = \beta_1 = \beta_2 = \beta_3 = 0,$$

and where again the reaction terms are polynomially bounded and satisfy, in the case of our system, the conditions

$$a_{11}f(u, v, w) \leq c_{11}u + c_{12}v + c_{13}w + d_1,$$

$$(a_{21}f + a_{22}g)(u, v, w) \leq c_{21}u + c_{22}v + c_{23}w + d_2,$$

$$(a_{31}f + a_{32}g + a_{33}h)(u, v, w) \leq c_{31}u + c_{32}v + c_{33}w + d_3,$$

for all $u, v, w \geq 0$ where $a_{ij}, c_{ij}$ and $d_i$, $1 \leq i, j \leq 3$ are positive reals. Martin and Pierre [4] and Hollis [3] extended the results, under the same conditions, to the boundary conditions (1.4) where in (1.7), they took

$$0 \leq \lambda_1, \lambda_2, \lambda_3 \leq 1, \quad \beta_1, \beta_2, \beta_3 \geq 0,$$
but they imposed conditions of the form (1.9), at the same time, to the reaction terms whose corresponding components of the solution satisfy Neumann boundary conditions and to the others which satisfy Dirichlet boundary conditions. In other terms they imposed to the reaction terms to satisfy \( m \) inequalities. For example if \( 1 - \lambda_1 = \beta_1 = 0 \), they took in (1.9) \( a_{31} = a_{32} = 0 \). In this paper we show the global existence of a unique solution to problem (1.1)-(1.5) without using the first and the second conditions in (1.9), but only under the last one which is (1.6) and without distinguishing between the components of the solution which satisfy the one or the other type of boundary conditions; for this purpose, we use Lyapunov technique (see Kirane and Kouachi[5], Kouachi[6] and Kouachi and Youkana[7]).

2 Preliminary observations

The usual norms in spaces \( L^p(\Omega), L^\infty(\Omega) \) and \( C(\overline{\Omega}) \) are denoted respectively by

\[
\|u\|_p^p = \frac{1}{|\Omega|} \int_\Omega |u(x)|^p dx, \tag{2.1}
\]

\[
\|u\|_\infty = \max_{x \in \Omega} |u(x)|. \tag{2.2}
\]

It is well known that to prove global existence of solutions to (1.1)-(1.5) (see Henry [2], pp. 35-62), it suffices to derive a uniform estimate of \( \|f(u, v, w)\|_p \), \( \|g(u, v, w)\|_p \) and \( \|h(u, v, w)\|_p \) on \([0,T_{\text{max}}]\) in the space \( L^p(\Omega) \) for some \( p > N/2 \).

Our aim is to construct polynomial Lyapunov functionals allowing us to obtain \( L^p \) bounds on \( u, v \) and \( w \) that lead to global existence.

Since the functions \( f, g \) and \( h \) are continuously differentiable on \( \mathbb{R}^3_+ \), then for any initial data in \( C(\overline{\Omega}) \), it is easy to check directly their Lipschitz continuity on bounded subsets of the domain of a fractional power of the operator

\[
\begin{pmatrix}
-a\Delta & 0 & 0 \\
0 & -b\Delta & 0 \\
0 & 0 & -c\Delta
\end{pmatrix}
\]

(2.3)

Under these assumptions, the following local existence result is well known (see Friedman [1] and Pazy [12]).

**Proposition 2.1** The system (1.1)-(1.5) admits a unique, classical solution \((u, v, w)\) on \((0, T_{\text{max}}[\times \Omega]. If \( T_{\text{max}} < \infty \) then

\[
\lim_{t \to T_{\text{max}}} (\|u(t, .)\|_\infty + \|v(t, .)\|_\infty + \|w(t, .)\|_\infty) = \infty, \tag{2.4}
\]

where \( T_{\text{max}}(\|u_0\|_\infty, \|v_0\|_\infty, \|w_0\|_\infty) \) denotes the eventual blow-up time.
3 Results

Put \( A = \sqrt{\frac{(b+c)^2}{ab}} \), \( B = \sqrt{\frac{(a+c)^2}{ac}} \) and \( C = \sqrt{\frac{(a+b)^2}{ab}} \). Let \( \theta \) and \( \sigma \) be two positive constants such that

\[
(\sigma^2 - A^2)(\theta^2 - B^2) - (C - AB)^2 > 0 \quad \text{and} \quad \theta > C, \tag{3.1}
\]

and let

\[
\theta_q = \theta^q^2 \quad \text{and} \quad \sigma_p = \sigma^p^2, \quad \text{for} \quad q = 1, \ldots, p \quad \text{and} \quad p = 1, \ldots, n, \tag{3.2}
\]

where \( n \) is a positive integer. The main result of the paper reads as follows.

**Theorem 3.1** Suppose that the functions \( f, g \) and \( h \) are of polynomial growth and satisfy condition (1.6) for some positive constants \( D \) and \( E \) sufficiently large. Let \((u(t,\cdot), v(t,\cdot), w(t,\cdot))\) be a solution of (1.1)-(1.5) and let

\[
L(t) = \int_{\Omega} H_n(u(t,x), v(t,x), w(t,x)) \, dx, \tag{3.3}
\]

where

\[
H_n(u,v,w) = \sum_{p=0}^{n} \sum_{q=0}^{p} C_p^q C_n^p \theta_q \sigma_p u^q v^{p-q} w^{n-p}. \tag{3.4}
\]

Then the functional \( L \) is uniformly bounded on the interval \([0, T_{\text{max}}]\).

**Corollary 3.2** Under the hypotheses of theorem 3.1 all solutions of (1.1)-(1.5) with positive bounded initial data are global.

**Proposition 3.3** If \( \beta_1 = \beta_2 = \beta_3 = C_1 = 0 \), then under the hypotheses of theorem 3.1 all solutions of (1.1)-(1.5) with positive bounded initial data are global and uniformly bounded on \( \Omega \).

4 Proofs

For the proof of theorem 3.1, we need some preparatory Lemmata.

**Lemma 4.1** Let \( H_n \) be the homogeneous polynomial defined by (3.4). Then

\[
\partial_u H_n = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_{q+1} \sigma_{p+1} u^q v^{p-q} w^{(n-1)-p}, \tag{4.1}
\]

\[
\partial_v H_n = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_q \sigma_{p+1} u^q v^{p-q} w^{(n-1)-p}, \tag{4.2}
\]

\[
\partial_w H_n = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_q \sigma_q u^q v^{p-q} w^{(n-1)-p}. \tag{4.3}
\]
Proof. Differentiating $H_n$ with respect to $u$ yields

$$\partial_u H_n = \sum_{p = 0}^{n} p = 0^n \sum_{q = 0}^{n} qC_p^n C_q^n \theta_q \sigma_p u^{q-1} v^{p-q} w^{n-p}.$$ 

Using the fact that

$$qC_p^n = pC_{p-1}^q \quad \text{and} \quad pC_n^p = nC_{n-1}^{p-1},$$

for $q = 1, \ldots, p$ and $p = 1, \ldots, n$, we get

$$\partial_u H_n = n \sum_{p=1}^{n} \sum_{q=1}^{p} C_{n-1}^{p-1} C_{n-1}^{p} \theta_q \sigma_p u^{q-1} v^{p-q} w^{n-p},$$

while changing in the sums the indexes $q - 1$ by $q$ and $p - 1$ by $p$, we deduce (4.1). For the formula (4.2), differentiating $H_n$ with respect to $v$ gives

$$\partial_v H_n = \sum_{p=1}^{n} \sum_{q=0}^{p-1} (p - q) C_n^p C_p^n \theta_q \sigma_p u^{q-1} v^{p-q} w^{n-p}.$$ 

Taking account of

$$C_p^n = C_p^{p-q}, q = 1, \ldots, p \quad \text{and} \quad p = 1, \ldots, n,$$

using (4.4) and changing the index $p - 1$ by $p$, we get (4.2).

Finally, we have

$$\partial_w H_n = \sum_{p=0}^{n} \sum_{q=0}^{p} (n - p) C_n^p C_q^n \theta_q \sigma_p u^{q-1} v^{p-q} w^{n-p-1}.$$ 

Since $(n - p) C_p^n = (n - p) C_n^{n-p} = nC_{n-1}^{n-p-1} = nC_{n-1}^p$, then we get (4.3).

Lemma 4.2 The second partial derivatives of $H_n$ are given by

$$\partial_{uu} H_n = n(n - 1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^{p} C_{q}^{p} \theta_{q+2} \sigma_{p+2} u^{q} v^{p-q} w^{(n-2)-p},$$

$$\partial_{uv} H_n = n(n - 1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^{p} C_{q}^{p} \theta_{q+1} \sigma_{p+1} u^{q} v^{p-q} w^{(n-2)-p},$$

$$\partial_{uw} H_n = n(n - 1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^{p} C_{q}^{p} \theta_{q+1} \sigma_{p+1} u^{q} v^{p-q} w^{(n-2)-p},$$

$$\partial_{vw} H_n = n(n - 1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^{p} C_{q}^{p} \theta_{q} \sigma_{p+2} u^{q} v^{p-q} w^{(n-2)-p},$$

$$\partial_{ww} H_n = n(n - 1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^{p} C_{q}^{p} \theta_{q} \sigma_{p+2} u^{q} v^{p-q} w^{(n-2)-p}.$$
Existence of global solutions EJDE-2002/88

\[ \partial_v w H_n = n(n-1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_n^{-2} C_p^q \theta q \sigma p+1 u^q v^{p-q} w^{(n-2)-p}, \] (4.10)

\[ \partial_w w H_n = n(n-1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_n^{-2} C_p^q \theta q \sigma p+1 u^q v^{p-q} w^{(n-2)-p}. \] (4.11)

**Proof.** Differentiating \( \partial_u H_n \), given by the formula (4.1), with respect to \( u \) yields

\[ \partial_u u H_n = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_n^{-1} C_p^q \theta q+1 \sigma p+1 u^{q-1} v^{p-q} w^{(n-1)-p}. \]

Using (4.4) we get (4.6)

\[ \partial_u v H_n = \partial_v (\partial_u H_n) = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} q C_{n-1}^p C_p^q \theta q+1 \sigma p+1 u^q v^{p-q-1} w^{(n-1)-p}. \]

Applying (4.5) and then (4.4) we get (4.7).

\[ \partial_u w H_n = \partial_w (\partial_u H_n) = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} (p-q) C_{n-1}^p C_p^q \theta q+1 \sigma p+1 u^q v^{p-q-1} w^{n-2}. \]

Applying successively (4.5), (4.4) and (4.5) a second time we deduce (4.8).

\[ \partial_v w H_n = \partial_v (\partial_u H_n) = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} (p-q) C_{n-1}^p C_p^q \theta q \sigma (p+1) u^q v^{p-q-1} w^{(n-1)-p}. \]

An application of (4.5) and then (4.4) yields (4.9).

\[ \partial w w H_n = \partial_w (\partial_u H_n) = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} (p-q) C_{n-1}^p C_p^q \theta q \sigma p u^q v^{p-q-1} w^{(n-1)-p}. \]

One applies (4.5) and then (4.4), (4.10) yields. Finally we get (4.11), by differentiating \( \partial_w H_n \) with respect to \( w \) and applying successively (4.5), (4.4) and (4.5) a second time.

**Proof of Theorem 3.1.** Differentiating \( L \) with respect to \( t \) yields

\[ L'(t) = \int_{\Omega} \left[ \partial_u H_n \frac{\partial u}{\partial t} + \partial_v H_n \frac{\partial v}{\partial t} + \partial_w H_n \frac{\partial w}{\partial t} \right] dx, \]

\[ = \int_{\Omega} \left( a \partial_u H_n \Delta u + b \partial_v H_n \Delta v + c \partial_w H_n \Delta w \right) dx \]

\[ + \int_{\Omega} \left( f \partial_u H_n + g \partial_v H_n + h \partial_w H_n \right) dx \]

\[ = I + J. \]
Using Green’s formula and applying Lemma 4.1 we get \( I = I_1 + I_2 \), where

\[
I_1 = \int_{\partial \Omega} \left( a \partial_u H_n \partial_q u + b \partial_v H_n \partial_q v + c \partial_w H_n \partial_q w \right) dx,
\]

\[
I_2 = -n(n-1) \int_{\Omega} \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_p^q \left| (A_{pq} z) \right| dx,
\]

(4.12)

where

\[
A_{pq} = \begin{pmatrix}
  a \theta_{q+2} \sigma_{p+2} & \left( \frac{\alpha + \beta}{2} \right) \theta_{q+1} \sigma_{p+2} & \left( \frac{\alpha + \beta}{2} \right) \theta_{q+1} \sigma_{p+1} \\
  \left( \frac{\alpha + \beta}{2} \right) \theta_{q+1} \sigma_{p+2} & b \theta_{q} \sigma_{p+2} & \left( \frac{\beta}{2} \right) \theta_{q} \sigma_{p+1} \\
  \left( \frac{\alpha + \beta}{2} \right) \theta_{q+1} \sigma_{p+1} & \left( \frac{\beta}{2} \right) \theta_{q} \sigma_{p+1} & c \theta_{q} \sigma_{p}
\end{pmatrix}
\]

(4.13)

for \( q = 1, \ldots, p, p = 1, \ldots, n-2 \), and \( z = (\nabla u, \nabla v, \nabla w)^t \).

We prove that there exists a positive constant \( C_2 \) independent of \( t \in [0,T_{\text{max}}] \) such that

\[
I_1 \leq C_2 \text{ for all } t \in [0,T_{\text{max}}]
\]

(4.14)

and that

\[
I_2 \leq 0
\]

(4.15)

for several boundary conditions.

(i) If \( 0 < \lambda_1, \lambda_2, \lambda_3 < 1 \), using the boundary conditions (1.4) we get

\[
I_1 = \int_{\partial \Omega} \left( a \partial_u H_n (\gamma_1 - \alpha_1 u) + b \partial_v H_n (\gamma_2 - \alpha_2 v) + c \partial_w H_n (\gamma_3 - \alpha_3 w) \right) dx,
\]

where \( \alpha_i = \lambda_i/(1 - \lambda_i) \) and \( \gamma_i = \beta_i/(1 - \lambda_i), i = 1, 2, 3 \). Since \( H(u,v,w) = a \partial_u H_n (\gamma_1 - \alpha_1 u) + b \partial_v H_n (\gamma_2 - \alpha_2 v) + c \partial_w H_n (\gamma_3 - \alpha_3 w) = P_{n-1} (u,v,w) - Q_n (u,v,w) \), where \( P_{n-1} \) and \( Q_n \) are polynomials with positive coefficients and respective degrees \( n \) and \( n-1 \) and since the solution is positive, then

\[
\limsup_{(|u|+|v|+|w|) \to +\infty} H(u,v,w) = -\infty,
\]

(4.16)

which prove that \( H \) is uniformly bounded on \( \mathbb{R}^3_+ \) and consequently (4.12).

(ii) If \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \), then \( I_1 = 0 \) on \( [0,T_{\text{max}}] \).

(iii) The case of homogeneous Dirichlet conditions is trivial, since in this case the positivity of the solution on \( [0,T_{\text{max}}] \times \Omega \) implies \( \partial_q u \leq 0, \partial_q v \leq 0 \) and \( \partial_q w \leq 0 \) on \( [0,T_{\text{max}}] \times \partial \Omega \). Consequently one gets again (4.12) with \( C_2 = 0 \).

(iv) If one or two of the components of the solution satisfy homogeneous Dirichlet boundary conditions and the other (others) satisfies the nonhomogeneous Robin conditions; for example \( u = 0, \lambda_2 v + (1 - \lambda_2) \partial_q v = \beta_2 \) and \( \lambda_3 w + (1 - \lambda_3) \partial_q w = \beta_3 \) on \( [0,T_{\text{max}}] \times \partial \Omega \) with \( 0 < \lambda_2, \lambda_3 < 1 \) and \( \beta_2, \beta_3 \geq 0 \). Then, following the same reasoning as above we get

\[
\limsup_{(|v|+|w|) \to +\infty} H(0,v,w) = -\infty,
\]

(4.17)
and then (4.12).

Now we prove (4.15). The quadratic forms (with respect to $\nabla u$, $\nabla v$ and $\nabla w$) associated with the matrices $A_{pq}$, $q = 1, \ldots, p$ and $p = 1, \ldots, n - 2$ are positive since their main determinants $\Delta_1$, $\Delta_2$ and $\Delta_3$ are too according to Sylvester criterion. To see this, we have

1. $\Delta_1 = a \theta_{q+2} \sigma_{p+2} > 0$, for $q = 1, \ldots, p$ and $p = 1, \ldots, n - 2$.

2. 

$$\Delta_2 = \left| \begin{array}{c} \frac{a + b}{2} \theta_{q+1} \sigma_{p+2} \\ \frac{a + c}{2} \theta_{q+1} \sigma_{p+2} \\ b \theta_{q} \sigma_{p+2} \end{array} \right|$$

$$= ab((\theta_{q+2} \sigma_{p+2}) - (C \theta_{q+1} \sigma_{p+2})^2)$$

$$= ab \sigma_{p+2}^2 (\frac{\theta_{q+2} \sigma_{p+2}}{\theta_{q+1}^2} - C^2) \sigma_{p+1}^2 = ab \sigma_{p+2}^2 (\theta^2 - C^2) \sigma_{q+1}^2,$$

for $q = 1, \ldots, p$ and $p = 1, \ldots, n - 2$. Using (3.1), we get $\Delta_2 > 0$.

3. 

$$c \theta_{q} \sigma_{p} \Delta_3 = c \theta_{q} \sigma_{p} \left| \begin{array}{c} \frac{a + b}{2} \theta_{q+1} \sigma_{p+2} \\ \frac{a + c}{2} \theta_{q+1} \sigma_{p+2} \\ b \theta_{q} \sigma_{p+2} \end{array} \right|$$

$$= - (c(a + \frac{b}{2}) - (a + \frac{c}{2})b + \frac{c}{2})(\theta_{q} \sigma_{p+2})^2(\theta_{q+1} \sigma_{p+1})^2$$

$$+ abc^2(\sigma_{p+2} - A^2 \sigma_{p+1}^2)(\theta_{q+2} - B^2 \theta_{q+1}^2) \sigma_{p+2}^2$$

$$= [\frac{a \sigma_{p+2} - A^2}{\sigma_{p+1}^2} - \frac{A^2}{\sigma_{q+1}^2}](\frac{\theta_{q+2} \sigma_{p+2}}{\theta_{q+1}^2} - B^2) - (C - AB)^2 \sigma_{p+1}^2 \sigma_{q+1}^2$$

$$= [\sigma^2 - A^2]/(\theta^2 - B^2) - (C - AB)^2] \sigma_{p+1}^2 \sigma_{q+1}^2 > 0,$$

for $q = 1, \ldots, p$ and $p = 1, \ldots, n - 2$. Using again (3.1), we get $\Delta_3 > 0$. Consequently we have (4.15). Substituting the expressions of the partial derivatives given by lemma 4.1 in the second integral, yields

$$J = \int \Omega \left[ \sum_{p=0}^{n-1} \sum_{q=0}^{p} f_{\sigma_{p+} \sigma_{q}} \right] \left( \theta_{q+1} \sigma_{p+1} \theta_{q} \sigma_{p} \right) dx$$

$$= \int \Omega \left[ \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_{p}^{-1} C_{p}^{-1} \left( \theta_{q+1} \sigma_{p+1} \theta_{q} \sigma_{p} \right) \right] \left( \frac{\theta_{q+1} \sigma_{p+1} \theta_{q} \sigma_{p}}{C_{p}^{-1} C_{p}^{-1}} \right) dx$$

Using condition (1.6), we deduce

$$J \leq C_3 \int \Omega \left[ \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_{p}^{-1} C_{p}^{-1} \left( \frac{\theta_{q+1} \sigma_{p+1} \theta_{q} \sigma_{p}}{C_{p}^{-1} C_{p}^{-1}} \right) \right] \left( \frac{\theta_{q+1} \sigma_{p+1} \theta_{q} \sigma_{p}}{C_{p}^{-1} C_{p}^{-1}} \right) dx$$

Applying Holder’s inequality to the integrals

$$\int \Omega \left( \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_{p}^{-1} C_{p}^{-1} \left( \frac{\theta_{q+1} \sigma_{p+1} \theta_{q} \sigma_{p}}{C_{p}^{-1} C_{p}^{-1}} \right) \right) dx, \quad q = 1, \ldots, p \text{ and } p = 1, \ldots, n - 1,$$
and following the same reasoning as in [7], one gets that there exist positive con-
stants $C_4$ and $C_5$ such that the functional $L$ satisfies the differential inequality

$$L'(t) \leq C_4 L(t) + C_5 L^{(n-1)/n}(t),$$

which can be written

$$nZ' \leq C_4 Z + C_5,$$

if $Z = L^{1/n}$. A simple integration of the later inequality gives the uniform
bound of the functional $L$ on the interval $[0, T^*]$; this ends the proof of the
theorem.

**Proof of corollary 3.2.** The proof is an immediate consequence of theorem
3.1, the preliminary observations and the inequality

$$\int_{\Omega} (u(t, x) + v(t, x) + w(t, x)) \, dx \leq C_6 L(t) \quad \text{on} \quad [0, T^*],$$

(4.18)

for some $n > N/2$.

**Proof of proposition 3.3.** In this case the functional $L$ is of Lyapunov and
then gives

$$L(t) \leq L(0) \quad \text{on} \quad [0, T^*].$$

Using (4.18), we get the uniform boundedness of the solution on $[0, T^*] \times \Omega$.

5 Applications

In this section we apply corollary 3.2 and proposition 3.3 to some particular
biochemical and chemical reaction models. Throughout this section we assume
that all reactions take place in a bounded domain $\Omega$ with smooth boundary $\partial \Omega$.

Let us begin with the general three-component reaction

$$lU + qV \underset{k}{\rightleftharpoons} rW,$$

(5.1)

which leads to the reaction diffusion system

$$\frac{\partial u}{\partial t} - a \Delta u = -hu^lv^q + kw^r \quad \text{in} \quad \mathbb{R}^+ \times \Omega,$$

(5.2)

$$\frac{\partial v}{\partial t} - b \Delta v = -hu^lv^q + kw^r \quad \text{in} \quad \mathbb{R}^+ \times \Omega,$$

(5.3)

$$\frac{\partial w}{\partial t} - c \Delta w = hu^lv^q - kw^r \quad \text{in} \quad \mathbb{R}^+ \times \Omega,$$

(5.4)

with boundary conditions (1.4) and positive initial data in $L^\infty(\Omega)$, where $h, k, l, q,$ and $r$ are positive constants such that $r \leq 1$ or $l + q \leq 1$. The special case
$l = q = r = 1$ has been studied by Rothe [13] under homogeneous Neumann
boundary conditions where he showed that $T_{\text{max}} = \infty$ if $N \leq 5$. Morgan [11] generalized the results of Rothe for every integer $N \geq 1$ and when all the components satisfy the same boundary conditions (Neumann or Dirichlet). Hollis [3] completed the work of Morgan and established global existence if $w$ satisfies the same type of boundary conditions as either $u$ or $v$. But if boundary conditions of different types are imposed on $u$ and $v$, global existence follows regardless of the type of boundary condition that is imposed on $w$. Recently we have proved, in [6], global existence of solutions to system (5.2)-(5.4), under homogeneous Neumann boundary conditions, by studying the two coupled systems (5.2)-(5.4) and (5.3)-(5.4) when $r \leq 1$ or $l + q \leq 1$. However we have

**Proposition 5.1** Solutions of (5.2)-(5.4) with nonnegative uniformly bounded initial data and nonhomogeneous boundary conditions (1.4) are positive and exist globally for every positive constants $l, q$ and $r$ such that $r \leq 1$ or $l + q \leq 1$.

**Proof.** Conditions (1.6) is trivial when $r \leq 1$ by choosing $D + E \gg 1$. In the case $l + q \leq 1$, it is also satisfied by studying the system in the order (5.4)-(5.3)-(5.2), choosing $E + 1 \gg D$ and by applying the Young inequality to the term $u^l v^q$ (see [7] for more of details). Then corollary 3.2 implies that all components of the solution are in $L^\infty(0, T^*; L^n(\Omega))$ for all $n \geq 1$. Since the reaction terms are of polynomial growth, then $T_{\text{max}} = +\infty$. Another example, is as follows:

$$
\frac{\partial u_1}{\partial t} - d_1 \Delta u_1 = -k_1 u_1 u_2 + k_2 u_3 - k_3 u_1 u_4 + k_4 u_5, \quad (5.5)
$$

$$
\frac{\partial u_2}{\partial t} - d_2 \Delta u_2 = -k_1 u_1 u_2 + k_2 u_3, \quad (5.6)
$$

$$
\frac{\partial u_3}{\partial t} - d_3 \Delta u_3 = k_1 u_1 u_2 B - k_2 u_3, \quad (5.7)
$$

$$
\frac{\partial u_4}{\partial t} - d_4 \Delta u_4 = -k_3 u_1 u_4 + k_4 u_5, \quad (5.8)
$$

$$
\frac{\partial u_5}{\partial t} - d_5 \Delta u_5 = k_3 u_1 u_4 - k_4 u_5. \quad (5.9)
$$

Hollis [3] established global existence provided that: (1) $u_3$ satisfies the same type of boundary conditions as either $u_1$ or $u_2$, and (2) $u_5$ satisfies the same type of boundary condition as either $u_1$ or $u_4$. Our generalization is summarized as

**Proposition 5.2** Solutions of (5.5)-(5.9) with nonnegative uniformly bounded initial data and boundary conditions (1.4) exist globally.

**Proof.** Condition (1.6) is satisfied for the three component system (5.5)-(5.7)-(5.9) while choosing $u_3 \gg u_4 \gg 1$. Then corollary 3.2 implies that $u_1$, $u_3$ and $u_5$ are in $L^\infty(0, T^*; L^n(\Omega))$ for all $n \geq 1$. So, global existence of $u_3$ and $u_4$ is a trivial consequence of the maximum principle (see J. Smoller [14]) applied successively to equations (5.6) and (5.8).
Finally we illustrate our results with the system
\[
\frac{\partial u}{\partial t} - a \Delta u = -u^\alpha v^\beta - u^\gamma w^\rho + \mu_1 v + \mu_2 w, \tag{5.10}
\]
\[
\frac{\partial v}{\partial t} - b \Delta v = -u^\alpha v^\beta + u^\gamma w^\rho, \tag{5.11}
\]
\[
\frac{\partial w}{\partial t} - c \Delta w = u^\alpha v^\beta + u^\gamma w^\rho, \tag{5.12}
\]
where \(\alpha, \beta, \gamma, \rho \geq 1\) and \(\mu_1, \mu_2 \geq 0\). S. L. Hollis [3] established global existence provided that either: (1) \(\mu_1 = 0\) and \(u\) satisfies the same type of boundary conditions as \(w\), or else (2) \(\mu_2 = 0, \rho = 1,\) and \(u\) satisfies the same type of boundary conditions as \(v\).

It is trivial to verify that condition (1.6) is satisfied for system (5.10)-(5.12) by choosing \(D + 1 \gg E \gg 1\). The result of corollary 3.2 applied to this system is summarized in the following proposition.

**Proposition 5.3** Solutions of (5.5)-(5.9) with nonnegative uniformly bounded initial data and boundary conditions (1.4) exist for all \(t > 0\) and all constants \(\alpha, \beta, \gamma, \rho \geq 1, \mu_1, \mu_2 \geq 0\).

If \(\beta_1 = \beta_2 = \beta_3 = \mu_1 = \mu_2 = 0\), then proposition 3.3 applied to (5.5)-(5.9) permits us to give the following result.

**Corollary 5.4** All solutions of (5.5)-(5.9) with positive initial data in \(L^\infty(\Omega)\) are global and uniformly bounded on \([0, +\infty] \times \Omega\) provided that \(\beta_1 = \beta_2 = \beta_3 = \mu_1 = \mu_2 = 0\).

Another example, is

\[
\frac{\partial u}{\partial t} - a \Delta u = -a_{11} u^{r_1} + a_{12} v^{r_2} + a_{13} w^{r_3} - c_{11} u + c_{12} v + c_{13} w + d_1 \text{ in } \mathbb{R}^+ \times \Omega, \tag{5.13}
\]
\[
\frac{\partial v}{\partial t} - b \Delta v = a_{21} u^{r_1} - a_{22} v^{r_2} + a_{23} w^{r_3} + c_{21} u - c_{22} v + c_{23} w + d_2 \text{ in } \mathbb{R}^+ \times \Omega, \tag{5.14}
\]
\[
\frac{\partial w}{\partial t} - c \Delta w = a_{31} u^{r_1} + a_{32} v^{r_2} - a_{33} w^{r_3} + c_{31} u + c_{32} v - c_{33} w + d_3 \text{ in } \mathbb{R}^+ \times \Omega, \tag{5.15}
\]
where \(r_i > 1, a_{ij}, c_{ij}, d_i\) are positive for \(1 \leq i \leq j \leq 3\). It is clear that conditions of the form (1.9) are not satisfied and then techniques used by Hollis [3] are not applicable here, nevertheless the method of invariant regions (see Smoller [14]) can give the global existence of positive solutions under some complicated conditions on the constants \(a_{ij}, c_{ij}\) and \(d_i, 1 \leq i \leq j \leq 3\). However our technique is applicable and if

\[
A = \begin{pmatrix}
-a_{11} & a_{12} & a_{13} \\
a_{21} & -a_{22} & a_{23} \\
a_{31} & a_{32} & -a_{33}
\end{pmatrix}, \tag{5.16}
\]
Existence of global solutions

\textbf{Proposition 5.5} Suppose that for some \( k = \{1, 2, 3\} \), the cofactor of \(-a_{kk}\) is strictly positive, then solutions of (5.13)-(5.15) with nonnegative uniformly bounded initial data and boundary conditions (1.4) are positive and exist for all \( t > 0 \) for positive reals \( a_{ij}, c_{ij} \) and \( d_i, 1 \leq i \leq j \leq 3 \), provided that (1) \( r_k \leq 1 \) or (2) \( r_i > 1, 1 \leq i \leq 3 \) and \( \det A < 0 \) or (3) \( r_i > 1, 1 \leq i \leq 3 \) and \( a_{kk} \) is sufficiently large.

\textbf{Proof.} Take for example \( k = 3 \)

(1) Suppose that \( a_{11}a_{22} - a_{12}a_{21} > 0 \) and \( r_3 \leq 1 \), then (1.6) is satisfied if there exists \( D \) and \( E \) sufficiently large such that

\[-a_{11}D + a_{21}E + a_{31} \leq 0 \quad \text{and} \quad a_{12}D - a_{22}E + a_{32} \leq 0. \quad (5.17)\]

The study of these two inequalities implies (1.6) if we choose \( D \) and \( E \) satisfying

\[D \geq \frac{a_{21}a_{32} + a_{41}a_{22}}{a_{11}a_{22} - a_{12}a_{21}} \quad \text{and} \quad E \geq \frac{a_{11}a_{32} + a_{12}a_{31}}{a_{11}a_{22} - a_{12}a_{21}}. \quad (5.18)\]

(2) Now, suppose that \( r_i > 1, 1 \leq i \leq 3 \) and \( a_{33} \) is sufficiently large, then (1.6) is satisfied if there exists \( D \) and \( E \) sufficiently large such that (5.17) is too and

\[a_{13}D + a_{23}E - a_{33} \leq 0. \quad (5.19)\]

But if we develop the determinant of \( A \) with regard to the third column, then \( \det A < 0 \) is equivalent to

\[-a_{33}(a_{11}a_{22} - a_{12}a_{21}) + a_{13}(a_{21}a_{32} + a_{31}a_{22}) - a_{23}(-a_{11}a_{32} - a_{12}a_{31}) < 0. \quad (5.20)\]

and this inequality is equivalent to

\[a_{13}\left(\frac{a_{21}a_{32} + a_{41}a_{22}}{a_{11}a_{22} - a_{12}a_{21}}\right) + a_{23}\left(\frac{a_{11}a_{32} + a_{12}a_{31}}{a_{11}a_{22} - a_{12}a_{21}}\right) < a_{33}. \quad (5.21)\]

Then we can choose \( D \) and \( E \) such that (5.18) and (5.19) are satisfied. Consequently we get (1.6).

(3) This case is trivial by using (5.18) and (5.21).

\textbf{References}


Said Kouachi
Centre Univ. Tébessa, Dépt. Mathématiques, 12002, Tébessa, Algérie
and
Lab. Mathématiques, Université d’Annaba, B. P. 12, Annaba, 23200, Algérie
e-mail: kouachi.said@caramail.com