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ON THE GOURSAT PROBLEM FOR A SECOND ORDER EQUATION

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Abstract. We consider the Goursat problem for second order operators and show existence and uniqueness of smooth solutions. We prove one of the results of Hasegawa (J. Math. Soc. Japan 50 (1998), no. 3, 639–662) by the energy method. The same method is applied when one of the surfaces where the Goursat data are given is a non-characteristic.

1. Introduction

Hasegawa [1] studied the C^∞ wellposedness of the Goursat problem

$$\begin{aligned} \partial_x\partial_t u + A(t,x,y)\partial_y^2 u &= f(t,x,y) \quad (t,x,y) \in \mathbb{R}^3 \\ u(0,x,y) &= g_1(x,y) \quad (x,y) \in \mathbb{R}^2 \\ u(t,0,y) &= g_2(t,y) \quad (t,y) \in \mathbb{R}^2 \end{aligned} \tag{1.1}$$

and obtained very interesting results: when $A(t,x,y) = At^k x^l$ with A a non-zero real constant and k,l non-negative integers, the Goursat problem (1.1) is C^∞ well-posed if and only if

$$k \text{ and } l \text{ are odd and } A < 0. \tag{1.2}$$

This condition is equivalent to the following condition on the signature of the coefficient $A(t,x,y) = At^k x^l$ on each quadrant

$$\Pi_{p,q} = \big\{(t,x,y) \in \mathbb{R}^3 \mid (-1)^p t < 0 \text{ and } (-1)^q x < 0\big\}$$

with $p,q = 1,2$:

$$(-1)^{p+q} A(t,x,y) < 0 \quad \text{on } \Pi_{p,q}.$$

In other words, the polynomial of (τ,ξ,η), $\tau\xi + A(t,x,y)\eta^2$ is hyperbolic in the direction $(1,\delta,0)$ with some constant δ satisfying $(-1)^{p+q}\delta > 0$ on each quadrant $\Pi_{p,q}$.

Nishitani [4] gave necessary and sufficient conditions for the Goursat problem to be C^∞-wellposed in higher order differential operators with constant coefficients. According to his result, when $A(t,x,y)$ is constant, (1.1) is C^∞-wellposed if and only if there exists $\delta_0 > 0$ such that for $0 < |\delta| < \delta_0$, the polynomial $\tau\xi + A\eta^2$

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is hyperbolic in the direction $(1, \delta, 0)$. We note that $\tau\xi + A\eta^2$ is hyperbolic in the direction $(1, \delta, 0)$ with $\delta > 0$ [respectively with $\delta < 0$], if and only if

$$A \le 0 \quad \text{[respectively } A \ge 0].$$

Therefore, one may say that under condition (1.2), the polynomial $\tau\xi + A(t, x, y)\eta^2$ satisfies Nishitani's condition only in the "out-going" direction on each quadrant $\Pi_{p,q}$. From this point of view, in this paper, we draw the result of Hasegawa [1], that is to say, that the conditions (1.2) implies the C^∞-wellposedness of (1.1), by using the energy method, while Hasegawa used the fundamental solution.

Now consider the Goursat problem

$$\begin{gathered} \partial_x\partial_t u - t^{2k+1}x^{2l+1}A_2(t, x, y)\partial_y^2 u \\ +t^k x^l A_1(t, x, y)\partial_y u + a_0(t, x, y)u = f(t, x, y) \quad (t, x, y) \in \mathbb{R}^3 \\ u(0, x, y) = g_1(x, y) \quad (x, y) \in \mathbb{R}^2 \\ u(t, 0, y) = g_2(t, y) \quad (t, y) \in \mathbb{R}^2 \end{gathered} \tag{1.3}$$

where k and l are non-negative integers.

We assume that for $j = 1, 2$, $A_j(t, x, y)$ and $a_0(t, x, y)$ are C^∞-functions on $\mathbb{R}^3$ and bounded on $\mathbb{R}^3$ with derivatives of any order; that is to say, $A_j(t, x, y)$, $a_0(t, x, y)$ are in $B^\infty(\mathbb{R}^3)$. Also we assume that for some positive constant $\delta_0 > 0$,

$$A_2(t, x, y) \ge \delta_0 \quad \text{on } \mathbb{R}^3 . \tag{1.4}$$

Then we have the following statement.

Theorem 1.1. *For any $f(t, x, y) \in C^\infty(\mathbb{R}^3)$ and $g_1(x, y)$, $g_2(t, y) \in C^\infty(\mathbb{R}^2)$ satisfying the compatibility condition*

$$g_1(0, y) = g_2(0, y),$$

the Goursat problem (1.3) *has one and only one solution $u(t, x, y) \in C^\infty(\mathbb{R}^3)$.*

The plan of the proof is the following. First we reduce to the case where $g_1(x, y) = g_2(t, y) = 0$ and $f(t, x, y)$ is flat on both planes $\{(0, x, y) \mid (x, y) \in \mathbb{R}^2\}$ and $\{(t, 0, y) \mid (t, y) \in \mathbb{R}^2\}$. Then we consider (1.3) on each quadrant $\Pi_{(p,q)}$ $(p, q = 1, 2)$. For example, when we are on the first quadrant $\Pi_{(1,1)}$, we extend $f(t, x, y)$ as C^∞-function out of $\Pi_{(1,1)}$ by putting $f(t, x, y) = 0$ for $(t, x, y) \notin \Pi_{(1,1)}$. After approximating the operator $L_1 = \partial_x\partial_t - t^{2k+1}x^{2l+1}A_2(t, x, y)\partial_y^2 + t^k x^l A_0(t, x, y)\partial_y + a_0(t, x, y)$ by the strictly hyperbolic operator $L_{1,\varepsilon}$, we solve the Cauchy problem $L_{1,\varepsilon}u_\varepsilon = f(t, x, y)$ with zero data on the plane given by $t + x = 0$. We see that this solution u_ε supported on the closure of $\Pi_{(1,1)}$. Hence by taking the limit, we obtain the desired solution on $\Pi_{(1,1)}$. The uniqueness follows from the duality argument. The detail is given in the next section.

By using the similar argument we can consider the case that the plane $x = 0$ is not characteristic, that is to say

$$\begin{gathered} \partial_x\partial_t u - B(t, x, y)\partial_x^2 u - t^{2k+1}x^{2l+1}A_2(t, x, y)\partial_y^2 u \\ +t^k x^{l+1}A_1\partial_y u + a_0(t, x, y)u = f(t, x, y) \quad (t, x, y) \in \mathbb{R}^3 \\ u(0, x, y) = g_1(x, y) \quad (x, y) \in \mathbb{R}^2 \\ u(t, 0, y) = g_2(t, y) \quad (t, y) \in \mathbb{R}^2 \end{gathered} \tag{1.6}$$

where k and l are non-negative integers. We assume that $A_j(t,x,y)$ $(j = 1,2)$, $a_0(t,x,y)$, $B(t,x,y) \in B^\infty(\mathbb{R}^3)$, (1.4) and that $B(t,x,y)$ is real-valued and satisfies

$$|B(t,x,y)| \geq \sigma_0 \quad \text{on } \mathbb{R}^3$$

with some positive constant σ_0. Then we have the following statement.

Theorem 1.2. *For any $f(t,x,y) \in C^\infty(\mathbb{R}^3)$ and $g_1(x,y)$, $g_2(t,y) \in C^\infty(\mathbb{R}^2)$ satisfying the compatibility condition*

$$g_1(0,y) = g_2(0,y),$$

the Goursat problem (1.6) *has one and only one solution $u(t,x,y) \in C^\infty(\mathbb{R}^3)$.*

In this case we reduce the problem to that with $g_1(x,y) = 0$, $g_2(t,y) = 0$ and $f(t,x,y)$ that is flat on the plane $t = 0$. Then we consider the problem on the upper half space $t \geq 0$ and on the lower half space $t \leq 0$ separately. Here we remark that we may assume $B(t,x,y) > 0$ by putting $x = -x$ if necessary. When we work on the upper half space $t \geq 0$, we extend $f(t,x,y)$ to the lower half space by putting $f(t,x,y) = 0$ for $t < 0$. First we solve the mixed problem on the space $x \geq 0$ with the zero initial data on the plane $t + \delta x = 0$ with some $\delta > 0$ and zero Dirichlet data on the boundary $x = 0$. Then the solution u given for $x \geq 0$ is supported in the first quadrant. Since the given operator is hyperbolic with respect to x-variable in the second quadrant. We extend u as the solution of Cauchy problem with the initial plane $x = 0$. Then we obatin a solution supported on the upper half space. The detail is given in the section 4.

We use the following notation. The inner product in $L^2(\mathbb{R}_y)$ denoted by

$$(f,g) = \int_{-\infty}^{\infty} f(y)\overline{g(y)}\, dy,$$

and the norm given by $\|\cdot\| = \sqrt{(\cdot,\cdot)}$. For an open set Ω, $H^\infty(\Omega)$ is the space consisting of all smooth functions which and their derivatives of any order belong to $L^2(\Omega)$. The space $C_0^\infty(\Omega)$ consists of compactly supported C^∞ functions on Ω. The space $H^\infty_{\text{loc}}(\Omega)$ consists of functions f satisfying $\chi f \in H^\infty(\Omega)$ for any $\chi \in C_0^\infty(\Omega)$. For any closed set $F \subset \mathbb{R}^3$, we denote by $C_0^\infty(F)$ the space consisting of all functions f on F such that f can be extended as a function in $C_0^\infty(\mathbb{R}^3)$. Furthermore in the following, we denote by C with or without a subscript an arbitary constant which may be different line by line. And in the section 3 and 5, constants C are independent of $\varepsilon \in (0,1)$ even if not mentioned explicitly.

2. Proof of Theorem 1.1

We denote by L_1 the differential operator in (1.3):

$$L_1 = \partial_t\partial_x - t^{2k+1}x^{2l+1}A_2(t,x,y){\partial_y}^2 + t^k x^l A_1(t,x,y)\partial_y + a_0(t,x,y). \tag{2.1}$$

Suppose that $u \in C^\infty(\mathbb{R}^3)$ satisfies (1.3). Let $u_l(x,y)$ denote $\partial_t^l u(0,x,y)$. Then

$$u_0(x,y) = g_1(x,y).$$

Since $u(t,0,y) = g_2(t,y)$, we obtain

$$u_l(0,y) = \partial_t^l g_2(0,y). \tag{2.2}$$

From the equation $L_1 u(t,x,y) = f(t,x,y)$, we obtain that for $l \geq 1$

$$\partial_x u_l(x,y) = \sum_{j=1}^{l} \alpha_j^{(i)}(x,y,\partial_y) u_{l-j} + \partial_t^{l-1} f(0,x,y) \tag{2.3}$$

with some second order differential operators $\alpha_j^{(i)}(x,y,\partial_y)$. We can determine successively $u_l(x,y)$ from (2.2) and (2.3).

We see that $L_1(\sum_{l=0}^{k} \frac{u_l(x,y)t^l}{l!}) - f(t,x,y) = O(t^k)$ and that $\sum_{l=0}^{k} \frac{u_l(0,y)t^l}{l!} - g_2(t,y) = O(t^{k+1})$. Then by picking a C^∞ function $v(t,x,y)$ such that $\partial_t^l v(0,x,y) = u_l(x,y)$ for any $l \geq 0$, we see that $\tilde{f}(t,x,y) = f(t,x,y) - L_1 v(t,x,y)$ and $\tilde{g}_2(t,y) = g_2(t,y) - v(t,0,y)$ are flat on the plane given by $t = 0$. Hence $w(t,x,y) = u(t,x,y) - v(t,x,y)$ satisfies

$$\begin{gathered} L_1 w = \tilde{f}(t,x,y) \quad (t,x,y) \in \mathbb{R}^3 \\ w(0,x,y) = 0 \quad (x,y) \in \mathbb{R}^2 \\ w(t,0,y) = \tilde{g}_2(t,y) \quad (t,y) \in \mathbb{R}^2 \end{gathered}$$

Similarly by putting $w_l(t,y) = \partial_x^l w(t,0,y)$ $(l \geq 0)$, we see that

$$w_0(t,y) = \tilde{g}_2(t,y).$$

Since $w(0,x,y) = 0$, we obtain

$$w_l(0,y) = 0. \tag{2.4}$$

From the equation $L_1 w = \tilde{f}(t,x,y)$, we obtain that for $l \geq 1$

$$\partial_t w_l(t,y) = \sum_{j=1}^{l} \beta_j^{(i)}(t,y,\partial_y) w_{l-j} + \partial_x^{l-1} \tilde{f}(t,0,y) \tag{2.5}$$

with some second order differential operators $\beta_j^{(i)}(x,y,\partial_y)$. Since $w_0(t,y)$ and $\partial_x^l \tilde{f}(t,0,y)$ $(l \geq 0)$ are flat on the plane $t = 0$, we obtain from (2.4) and (2.5) $w_l(t,y)$ that is flat on the plane $t = 0$. Therefore by picking a C^∞ function $w(t,x,y)$ such that $\partial_x^l w(t,0,y) = w_l(t,y)$ for any $l \geq 0$ and $w(t,x,y)$ is flat on the plane $t = 0$, we see that $\tilde{f}(t,x,y) - L_1 w(t,x,y)$ is flat on the plane given by $t = 0$ and also on the plane given by $x = 0$ and that $0 = \tilde{g}_2(t,y) - w(t,0,y)$.

Lemma 2.1. *The problem to find a C^∞ solution to* (1.3) *is reduced to the problem to find a C^∞ solution to the following*

$$\begin{gathered} L_1 u = h(t,x,y) \quad (t,x,y) \in \mathbb{R}^3 \\ u(0,x,y) = 0 \quad (x,y) \in \mathbb{R}^2 \\ u(t,0,y) = 0 \quad (t,y) \in \mathbb{R}^2 \end{gathered} \tag{2.6}$$

where $h(t,x,y)$ is flat on the plane given by $t = 0$ and on the plane given by $x = 0$.

We remark that a C^∞ solution $u(t,x,y)$ to (2.6) is flat on the plane given by $t = 0$ and on the plane given by $x = 0$.

Since $h(t,x,y)$ is flat on the plane given by $t=0$ and on the plane given by $x=0$, we can define smooth functions $h_{p,q}(t,x,y)$ $(p,q=1,2)$ by

$$h_{p,q}(t,x,y)=\begin{cases} h(t,x,y) & (t,x,y)\in\Pi_{p,q}\\ 0 & \text{otherwise.}\end{cases} \tag{2.7}$$

Now we consider the problem

$$\begin{gathered} L_1u=h_{p,q}(t,x,y) \quad (t,x,y)\in\mathbb{R}^3\\ \operatorname{supp}u\subset\overline{\Pi_{p,q}} \end{gathered} \tag{2.8}$$

If $u_{p,q}(t,x,y)$ is the solution to (2.8), then $\sum_{p=1,2,q=1,2}u_{p,q}(t,x,y)$ satisfies (2.6). On the other hand for a solution $u(t,x,y)$ to (2.6), then $u_{p,q}(t,x,y)$ defined by

$$u_{p,q}(t,x,y)=\begin{cases} u(t,x,y) & (t,x,y)\in\Pi_{p,q}\\ 0 & \text{otherwise.}\end{cases}$$

is a solution to (2.8). Therefore the uniqueness for (2.8) implies that of (2.6).

We note that the problem (2.8) with $(p,q)\neq(1,1)$ is reduced to that of $(1,1)$ by the change of coordinates $t=(-1)^{p-1}t$, $x=(-1)^{q-1}x$. Then we have only to prove the following proposition in order to prove Theorem 1.1.

Proposition 2.2. *For any $h(t,x,y)\in C^\infty(R^3)$ satisfying $\operatorname{supp}h(t,x,y)\subset\overline{\Pi_{1,1}}$, there exists one and only one solution $u(t,x,y)\in C^\infty(R^3)$ to the equation*

$$\begin{gathered} L_1u=h(t,x,y) \quad (t,x,y)\in\mathbb{R}^3\\ \operatorname{supp}u\subset\overline{\Pi_{1,1}} \end{gathered} \tag{2.10}$$

Proof. To show the proposition above, we define the operator $L_{1,\varepsilon}$ with $1>\varepsilon>0$ by

$$L_{1,\varepsilon}=\partial_t\partial_x-t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}A_2(t,x,y)\partial_y^2+t_\varepsilon^k x_\varepsilon^l A_1(t,x,y)\partial_y+a_0(t,x,y).$$

where t_ε and x_ε are given by

$$t_\varepsilon=\varepsilon\chi(\frac{t}{\varepsilon}),\quad x_\varepsilon=\varepsilon\chi(\frac{x}{\varepsilon}) \tag{2.12}$$

by using a function $\chi(s)\in C^\infty(\mathbb{R})$ satisfying the following;

$$\begin{gathered} \chi(s)\geq\max\{\frac{1}{2},s+1\}\quad(s\in\mathbb{R})\\ \chi(s)=s+1\quad(s\geq0),\quad \chi(s)=\frac{1}{2}\quad(s\leq-1) \end{gathered}$$

We note that $t_\varepsilon,\ x_\varepsilon\in C^\infty(\mathbb{R})$ and that

$$t_\varepsilon,\ x_\varepsilon\geq\frac{\varepsilon}{2},\quad t_\varepsilon=t+\varepsilon\quad(t\geq0),\quad x_\varepsilon=x+\varepsilon\quad(x\geq0). \tag{2.13}$$

Since $t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}A_2(t,x,y)>0$ on $\mathbb{R}^3$, we see that the operator $L_{1,\varepsilon}$ is strictly hyperbolic in the direction $(\delta,1-\delta,0)$ with $0<\delta<1$. For any $T>0$, the coefficients of $L_{1,\varepsilon}$ are bounded on a closed domain Dom_T given by

$$\operatorname{Dom}_T=\big\{(t,x,y)\in\mathbb{R}^3\mid 2T\geq x+t\geq\frac{1}{2}(\sqrt{1+(t-x)^2}-1)\big\}.$$

Since $\overline{\Pi_{1,1}} \subset \{(t,x,y) \in \mathbb{R}^3 \mid x+t \geq \frac{1}{2}(\sqrt{1+(t-x)^2}-1)\}$, for a $h(t,x,y) \in C^\infty(R^3)$ satisfying $\operatorname{supp} h(t,x,y) \subset \overline{\Pi_{1,1}}$, there exists one and only one solution $u_\varepsilon(t,x,y) \in C^\infty(R^3)$ of the Cauchy problem

$$\begin{gathered} L_{1,\varepsilon} u_\varepsilon = h(t,x,y) \quad (t,x,y) \in \mathbb{R}^3 \\ \operatorname{supp} u_\varepsilon \subset \big\{(t,x,y) \in \mathbb{R}^3 \mid x+t \geq \frac{1}{2}(\sqrt{1+(t-x)^2}-1)\big\}. \end{gathered} \tag{2.14}$$

Since the plane $\{(t,x,y) \in \mathbb{R}^3 \mid \delta t + (1-\delta)x = 0\}$ with $0 < \delta < 1$ is space-like, we see that

$$\operatorname{supp} u_\varepsilon(t,x,y) \subset \overline{\Pi_{1,1}}.$$

Furthermore we have the finite propagation speed. That is to say, for any $T > 0$, there exists a positive constant λ independent of ε such that, for any $y_0 \in \mathbb{R}$ and $r > 0$,

$$\operatorname{supp} h(t,x,y) \cap \operatorname{Dom}_T \subset \{(t,x,y) \in \mathbb{R}^3 \mid \ |y-y_0| \leq r\}$$

implies

$$\operatorname{supp} u_\varepsilon(t,x,y) \cap \operatorname{Dom}_T \subset \{(t,x,y) \in \mathbb{R}^3 \mid \ |y-y_0| \leq r+\lambda\}.$$

If we can draw a sequence $u_{\varepsilon_j}(t,x,y)$ with $\varepsilon_j \to 0$ such that $u_{\varepsilon_j}(t,x,y)$ converges to a $u(t,x,y)$ in $C^\infty(\mathbb{R}^3)$, then we see that $u(t,x,y)$ satisfies (2.10). We see the existence of such a sequence from the following lemma whose proof is given in the section 3.

Lemma 2.3. *The family of solutions $\{u_\varepsilon(t,x,y)\}_{0<\varepsilon<1}$ to* (2.14) *is bounded in the space $H^\infty_{\rm loc}(\mathbb{R}^3)$.*

Concerning the uniqueness of solutions of the problem (2.10), we consider the adjoint problem; for any $T > 0$ and any $g(t,x,y) \in C_0^\infty(\mathbb{R}^3)$ whose support is contained in $\{(t,x,y) \in \mathbb{R}^3 \mid 0 \leq t \leq T \text{ and } 0 \leq x \leq T\}$, find a solution $w_\varepsilon(t,x,y)$ to

$$\begin{gathered} {}^tL_{1,\varepsilon} w_\varepsilon = g(t,x,y) \quad (t,x,y) \in \mathbb{R}^3 \\ \operatorname{supp} w_\varepsilon \subset \{(t,x,y) \in \mathbb{R}^3 \mid x+t \leq 2T\} \end{gathered} \tag{2.15}$$

where ${}^tL_{1,\varepsilon}$, the transpose of $L_{1,\varepsilon}$, is given by

$$\begin{aligned} {}^tL_{1,\varepsilon} w(t,x,y) &= \partial_t\partial_x w(t,x,y) - t_\varepsilon^{2k+1} x_\varepsilon^{2l+1} {\partial_y}^2 (A_2(t,x,y) w(t,x,y)) \\ &\quad - t_\varepsilon^k x_\varepsilon^l \partial_y (A_1(t,x,y) w(t,x,y)) + a_0(t,x,y) w(t,x,y). \end{aligned} \tag{2.16}$$

The coefficients of ${}^tL_{1,\varepsilon}$ are also bounded in Dom_T. Then solutions $w_\varepsilon(t,x,y)$ have a finite propagation speed independent of $0 < \varepsilon < 1$ in Dom_T. Then there exists a compact set $F \in \mathbb{R}^3$ such that

$$\operatorname{supp} w_\varepsilon(t,x,y) \cap \big\{(t,x,y) \in \mathbb{R}^3 \mid t \geq 0 \text{ and } x \geq 0\big\} \subset F.$$

Since the plane given by $(1-\delta)t + \delta x = C$ with $0 < \delta < 1$ is space like, we have

$$\operatorname{supp} w_\varepsilon(t,x,y) \subset (-\infty, T] \times (-\infty, T] \times \mathbb{R}.$$

Then we obtain the second assertion of the following lemma. The proof of the first assertion is given in the section 3.

Lemma 2.4. *The family of solutions* $\{w_\varepsilon(t,x,y)\}_{0<\varepsilon<1}$ *to* (2.15) *is bounded in the space* $H^\infty(\{(t,x,y)\in\mathbb{R}^3 \mid 0<x<T \text{ and } 0<x<T\})$. *Furtheremore there exists some constant* r_0 *such that*

$$\begin{aligned}&\operatorname{supp} w_\varepsilon(t,x,y)\cap\{(t,x,y)\in\mathbb{R}^3 \mid t\ge 0 \text{ and } x\ge 0\}\\ &\qquad\subset\{(t,x,y)\in\mathbb{R}^3 \mid 0\le t\le T,\ 0\le x\le T \text{ and } |y|\le r_0\}.\end{aligned}$$

Therefore we have a sequence $w_{\varepsilon_j}(t,x,y)$ with $\varepsilon_j\to 0$ such that $w_{\varepsilon_j}(t,x,y)$ converges to a $w(t,x,y)$ in $C^\infty(\{(t,x,y)\in\mathbb{R}^3 \mid 0\le t\le T \text{ and } 0\le x\le T\})$. Then we see by the integration by parts that for a solution $u(t,x,y)$ of the problem (2.10) with $h(t,x,y)=0$

$$\begin{aligned}0&=\int_{\mathbb{R}^3} L_1u(t,x,y)w(t,x,y)\,dt\,dx\,dy\\ &=\lim_{j\to\infty}\int_{\mathbb{R}^3} L_{1,\varepsilon_j}u(t,x,y)w_{\varepsilon_j}(t,x,y)\,dt\,dx\,dy\\ &=\int_{\mathbb{R}^3} u(t,x,y)g(t,x,y)\,dt\,dx\,dy\end{aligned}$$

from which we get $u(t,x,y)=0$. Hence we obtain the uniqueness of solutions of the problem (2.10). □

3. Proof of Lemmas 2.3 and 2.4

First we draw the estimates for $u(t,x,y)\in C^\infty(\mathbb{R}^3)$ carried by $\overline{\Pi_{1,1}}$ and vanishing for large $|y|$ by using the method of Oleinik [5]. Since

$$\begin{aligned}&2\Re(\partial_t\partial_x u - t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}A_2(t,x,\cdot){\partial_y}^2u,\partial_x u)\\ &\quad=\partial_t(\partial_x u,\partial_x u)+t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}\partial_x(A_2(t,x,\cdot)\partial_y u,\partial_y u)+R_1\end{aligned}\tag{3.1}$$

where

$$|R_1|\le Ct_\varepsilon^{2k+1}x_\varepsilon^{2l+1}(\|\partial_y u\|^2+\|\partial_x u\|^2)\tag{3.2}$$

and

$$\begin{aligned}&2\Re(\partial_t\partial_x u - t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}A_2(t,x,\cdot){\partial_y}^2u,\partial_t u)\\ &\quad=\partial_x(\partial_t u,\partial_t u)+t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}\partial_t(A_2(t,x,\cdot)\partial_y u,\partial_y u)+R_2\end{aligned}\tag{3.3}$$

where

$$|R_2|\le Ct_\varepsilon^{2k+1}x_\varepsilon^{2l+1}(\|\partial_y u\|^2+\|\partial_t u\|^2),$$

we have

$$\begin{aligned}&\partial_t(e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}\|\partial_x u\|^2)\\ &+\partial_x\big(e^{-\gamma(t+x)}t_\varepsilon^{-M+2k+1}x_\varepsilon^{-M+2l+1}(A_2(t,x,\cdot)\partial_y u,\partial_y u)\big)\\ &+e^{-\gamma(t+x)}(\gamma t_\varepsilon+Mt'_\varepsilon)t_\varepsilon^{-M-1}x_\varepsilon^{-M}\|\partial_x u\|^2\\ &+e^{-\gamma(t+x)}(\gamma x_\varepsilon+(M-2l-1)x'_\varepsilon)t_\varepsilon^{-M+2k+1}x_\varepsilon^{-M+2l}(A_2(t,x,\cdot)\partial_y u,\partial_y u)\\ &\le 2e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}\Re(\partial_t\partial_x u(t,x,y)-t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}A_2(t,x,\cdot){\partial_y}^2u,\partial_x u)\\ &\quad+Ce^{-\gamma(t+x)}t_\varepsilon^{-M+2k+1}x_\varepsilon^{-M+2l+1}(\|\partial_y u\|^2+\|\partial_x u\|^2)\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}&\partial_x(e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}\|\partial_t u\|^2)\\&+\partial_t\big(e^{-\gamma(t+x)}t_\varepsilon^{-M+2k+1}x_\varepsilon^{-M+2l+1}(A_2(t,x,\cdot)\partial_y u,\partial_y u)\big)\\&+e^{-\gamma(t+x)}(\gamma x_\varepsilon+Mx'_\varepsilon)t_\varepsilon^{-M}x_\varepsilon^{-M-1}\|\partial_t u\|^2\\&+e^{-\gamma(t+x)}(\gamma t_\varepsilon+(M-2k-1)t'_\varepsilon)t_\varepsilon^{-M+2k}x_\varepsilon^{-M+2l+1}(A_2(t,x,\cdot)\partial_y u,\partial_y u)\\&\le 2e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}\Re(\partial_t\partial_x u-t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}A_2(t,x,\cdot){\partial_y}^2u,\partial_t u)\\&\quad+Ce^{-\gamma(t+x)}t_\varepsilon^{-M+2k+1}x_\varepsilon^{-M+2l+1}(\|\partial_y u\|^2+\|\partial_t u\|^2).\end{aligned}\tag{3.6}$$

Since

$$\begin{aligned}&\partial_t\partial_x u(t,x,y)-t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}A_2(t,x,y){\partial_y}^2u(t,x,y)\\&\qquad=L_{1,\varepsilon}u(t,x,y)-t_\varepsilon^k x_\varepsilon^l A_1(t,x,y)\partial_y u(t,x,y)-a_0(t,x,y)u(t,x,y),\end{aligned}$$

we have

$$\begin{aligned}&|(\partial_t\partial_x u-t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}A_2(t,x,\cdot){\partial_y}^2u,\partial_x u)|\\&\qquad\le(\|L_{1,\varepsilon}u\|+C_0t_\varepsilon^k x_\varepsilon^l\|\partial_y u\|+C\|u\|)\|\partial_x u\|\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}&|(\partial_t\partial_x u-t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}A_2(t,x,\cdot){\partial_y}^2u,\partial_t u)|\\&\qquad\le(\|L_{1,\varepsilon}u\|+C_0t_\varepsilon^k x_\varepsilon^l\|\partial_y u\|+C\|u\|)\|\partial_t u\|\end{aligned}\tag{3.8}$$

where $C_0=\sup_{\mathbb{R}^3}|A_1(t,x,y)|$. Noting

$$\begin{aligned}C_0t_\varepsilon^k x_\varepsilon^l\|\partial_y u\|\|\partial_x u\|&\le\frac{C_0}{2}(t_\varepsilon^{2k+1}x_\varepsilon^{2l}\|\partial_y u\|^2+t_\varepsilon^{-1}\|\partial_x u\|^2),\\C_0t_\varepsilon^k x_\varepsilon^l\|\partial_y u\|\|\partial_t u\|&\le\frac{C_0}{2}(t_\varepsilon^{2k}x_\varepsilon^{2l+1}\|\partial_y u\|^2+x_\varepsilon^{-1}\|\partial_t u\|^2)\end{aligned}$$

and

$$t'_\varepsilon=1\quad(t\ge0),\ \ x'_\varepsilon=1\quad(x\ge0)$$

which follows from (2.13), we see that (3.5), (3.6), (3.7) and (3.8) imply the following; for $t\ge0$ and $x\ge0$

$$\begin{aligned}&\partial_t\big\{e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}\big(\|\partial_x u\|^2+t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}(A_2(t,x,\cdot)\partial_y u,\partial_y u)\big)\big\}\\&+\partial_x\big\{e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}\big(\|\partial_t u\|^2+t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}(A_2(t,x,\cdot)\partial_y u,\partial_y u)\big)\big\}\\&+e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}\big\{\gamma(\|\partial_t u\|^2+\|\partial_x u\|^2)\\&\qquad+(M-C_0)t_\varepsilon^{-1}\|\partial_x u\|^2+(M-C_0)x_\varepsilon^{-1}\|\partial_t u\|^2\big\}\\&+e^{-\gamma(t+x)}t_\varepsilon^{-M+2k+1}x_\varepsilon^{-M+2l+1}\big(2\gamma+(M-2k-1)t_\varepsilon^{-1}+(M-2l-1)x_\varepsilon^{-1}\big)\\&\qquad\times(A_2(t,x,\cdot)\partial_y u,\partial_y u)\\&\le e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}\big(\|L_{1,\varepsilon}u\|^2+\|\partial_x u\|^2+\|\partial_t u\|^2\\&\qquad+C_0t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}(t_\varepsilon^{-1}+x_\varepsilon^{-1})\|\partial_y u\|^2+C\|u\|^2\big).\end{aligned}\tag{3.9}$$

Since $A_2(t,x,y)\ge\delta_0>0$ and

$$\partial_t(e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}\|u\|^2)+e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}(\gamma+Mt'_\varepsilon t_\varepsilon^{-1})\|u\|^2$$

$$\le e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M} (\frac{\gamma}{2}\|u\|^2 + \frac{2}{\gamma}\|\partial_t u\|^2) \quad (3.10)$$

which follows from $\partial_t\|u\|^2 = 2\Re(\partial_t u, u)$ and the Schwarz inequality, we obtain, from (3.9), when

$$M \ge 2\max\{C_0, 2k+1+\frac{C_0}{\delta_0}, 2l+1+\frac{C_0}{\delta_0}\},$$

for $t_1 > 0$, $x_1 > 0$ and for $\gamma \ge \gamma_0$ with some constant $\gamma_0 > 0$ independent of ε,

$$\begin{aligned}
&\int_0^{t_1}\int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M} \big\{\frac{\gamma}{2}(\|\partial_t u\|^2 + \|\partial_x u\|^2 + \|u\|^2) \\
&\quad + \frac{M}{2} t_\varepsilon^{-1}(\|\partial_x u\|^2 + \|u\|^2) + \frac{M}{2} x_\varepsilon^{-1}\|\partial_t u\|^2\big\}\, dt\, dx \\
&+ \int_0^{t_1}\int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M+2k+1} x_\varepsilon^{-M+2l+1}\delta_0\big\{\gamma + \frac{M}{2}(t_\varepsilon^{-1} + x_\varepsilon^{-1})\|\partial_y u\|^2\big\}\, dt\, dx \\
&\le \int_0^{t_1}\int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M}\|L_{1,\varepsilon}u\|^2\, dt\, dx
\end{aligned}$$

from which we obtain

$$\begin{aligned}
&\frac{\gamma}{2}\int_0^{t_1}\int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M}\big(\|\partial_t u\|^2 + \|\partial_x u\|^2 + \|u\|^2\big)\, dt\, dx \\
&+ \delta_0\gamma\int_0^{t_1}\int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M+2k+1} x_\varepsilon^{-M+2l+1}\|\partial_y u\|^2\, dt\, dx \\
&\le \int_0^{t_1}\int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M}\|L_{1,\varepsilon}u\|^2\, dt\, dx.
\end{aligned} \tag{3.12}$$

Since

$$\partial_y L_{1,\varepsilon}u = \tilde{L}_{1,\varepsilon}\partial_y u + a_{0y}(t,x,y)u$$

where

$$\tilde{L}_{1,\varepsilon} = L_{1,\varepsilon} + t_\varepsilon^{2k+1} x_\varepsilon^{2l+1} A_{2y}(t,x,y)\partial_y + t_\varepsilon^k x_\varepsilon^l A_{1y}(t,x,y)$$

with $A_{2y}(t,x,y) = \partial_y A_2(t,x,y)$, $A_{1y}(t,x,y) = \partial_y A_1(t,x,y)$ and $a_{0y}(t,x,y) = \partial_y a_0(t,x,y)$. Noting that

$$\|t_\varepsilon^{2k+1} x_\varepsilon^{2l+1} A_{2y}(t,x,\cdot)\partial_y u\|^2 \le C t_\varepsilon^{4k+2} x_\varepsilon^{4l+2}\|\partial_y u\|^2,$$

we have the estimate similar to (3.12); with the same M as that of (3.12)

$$\begin{aligned}
&\frac{\gamma}{2}\int_0^{t_1}\int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M}\big(\|\partial_t u\|^2 + \|\partial_x u\|^2 + \|u\|^2\big)\, dt\, dx \\
&+ \frac{\delta_0\gamma}{2}\int_0^{t_1}\int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M+2k+1} x_\varepsilon^{-M+2l+1}\|\partial_y u\|^2\, dt\, dx \\
&\le \int_0^{t_1}\int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M}\|\tilde{L}_{1,\varepsilon}u\|^2\, dt\, dx
\end{aligned} \tag{3.15}$$

for $\gamma \geq \gamma_1$ with some constant $\gamma_1 > 0$ which is independent of ε but may depend on x_1 and t_1. Repeating the same argument, we have for any integer $l \geq 0$

$$\begin{aligned} &\sum_{j=0}^{l} \gamma \int_0^{t_1} \int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M} \big(\|\partial_t \partial_y^j u\|^2 \\ &\quad + \|\partial_x \partial_y^j u\|^2 + \|\partial_y^j u\|^2 \big)\, dt\, dx \\ &+ \sum_{j=0}^{l} \gamma \int_0^{t_1} \int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M+2k+1} x_\varepsilon^{-M+2l+1} \|\partial_y^{j+1} u\|^2 \, dt\, dx \\ &\leq \sum_{j=0}^{l} C \int_0^{t_1} \int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M} \|\partial_y^j L_{1,\varepsilon} u\|^2 \, dt\, dx \end{aligned} \tag{3.16}$$

for $\gamma \geq \gamma_l$ with some constant $\gamma_l > 0$ independent of ε and some constant $C > 0$ independent of ε and γ. Since

$$[\partial_t, L_{1,\varepsilon}] = R_t(t, x, y, \partial_y), \quad [\partial_x, L_{1,\varepsilon}] = R_x(t, x, y, \partial_y)$$

where $R_t(t, x, y, \partial_y)$ and $R_x(t, x, y, \partial_y)$ are second order differential operators, we obtain from (3.16)

$$\begin{aligned} &\sum\nolimits_{\alpha_1+\alpha_2 \leq 2} \sum_{j=0}^{l} \gamma \int_0^{t_1} \int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M} \|\partial_t^{\alpha_1} \partial_x^{\alpha_2} \partial_y^j u\|^2 \, dt\, dx \\ &\leq C \int_0^{t_1} \int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M} \Big(\sum_{\substack{\alpha_1+\alpha_2 \leq 1 \\ \alpha_1+\alpha_2+j \leq l+1}} \|\partial_t^{\alpha_1} \partial_x^{\alpha_2} \partial_y^j L_{1,\varepsilon} u\|^2 \\ &\quad + \|\partial_y^{l+2} L_{1,\varepsilon} u\|^2 \Big)\, dt\, dx \end{aligned}$$

for $\gamma \geq \gamma_{l,1}$ with some constant $\gamma_3 > 0$ independent of ε and some constant $C > 0$ independent of ε and γ. Repeating this argument we have the following lemma.

Lemma 3.1. *Let $u(t, x, y) \in C^\infty(\mathbb{R}^3)$ satisfy $\operatorname{supp} u(t, x, y) \subset \overline{\Pi_{1,1}}$ and vanish for large $|y|$. For any integer $k \geq 1$ and any integer $l \geq 0$*

$$\begin{aligned} &\sum_{\alpha_1+\alpha_2 \leq 2k-1} \sum_{j=0}^{l} \gamma \int_0^{t_1} \int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M} \|\partial_t^{\alpha_1} \partial_x^{\alpha_2} \partial_y^j u\|^2 \, dt\, dx \\ &\leq C \int_0^{t_1} \int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M} \big(\sum_{\substack{\alpha_1+\alpha_2 \leq 2k-2 \\ \alpha_1+\alpha_2+j \leq 2k-2+l}} \|\partial_t^{\alpha_1} \partial_x^{\alpha_2} \partial_y^j L_{1,\varepsilon} u\|^2 \big)\, dt\, dx \end{aligned} \tag{3.17}$$

$$\begin{aligned} &\sum_{\alpha_1+\alpha_2 \leq 2k} \sum_{j=0}^{l} \gamma \int_0^{t_1} \int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M} \|\partial_t^{\alpha_1} \partial_x^{\alpha_2} \partial_y^j u\|^2 \, dt\, dx \\ &\leq C \int_0^{t_1} \int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M} \big(\sum_{\substack{\alpha_1+\alpha_2 \leq 2k-1 \\ \alpha_1+\alpha_2+j \leq 2k+l-1}} \|\partial_t^{\alpha_1} \partial_x^{\alpha_2} \partial_y^j L_{1,\varepsilon} u\|^2. \\ &\quad + \|\partial_y^{2k+l} L_{1,\varepsilon} u\|^2 \big)\, dt\, dx \end{aligned} \tag{3.18}$$

for $\gamma \geq \gamma_{l,k}$ with some constant $\gamma_{l,k} > 0$ independent of ε and some constant $C > 0$ independent of ε and γ.

For the estimate of the right hand side of (3.17) or (3.18), we need the following lemma.

Lemma 3.2. *When a* $v(t,x,y) \in C^\infty(\mathbb{R}^3)$ *satisfies* $\operatorname{supp} v(t,x,y) \subset \overline{\Pi_{1,1}}$ *and vanishes for large* $|y|$*, we have, for* $t_1, x_1 \geq 0$ *and any integer* $K \geq 1$

$$\int_0^{t_1}\int_0^{x_1} t_\varepsilon^{-2K+1} x_\varepsilon^{-2K+1}\|v\|^2\,dt\,dx \leq C\int_0^{t_1}\int_0^{x_1}\|\partial_t^K\partial_x^K v\|^2\,dt\,dx$$

with some constant $C > 0$ *independent of* ε.

Proof. Since $v(t,x,y)$ is flat on the plane $t = 0$ and on the plane $x = 0$, we have for $t, x \geq 0$

$$\begin{aligned} v(t,x,y) &= \frac{1}{(K-1)!}\int_0^t (t-s)^{K-1}\partial_t^K v(s,x,y)ds \\ &= \frac{1}{(K-1)!^2}\int_0^t\int_0^x (t-s)^{K-1}(x-y)^{K-1}\partial_t^K\partial_x^K v(s,w,y)dsdw \end{aligned}$$

Then

$$|v(t,x,y)|^2 \leq Ct^{2K-1}x^{2K-1}\int_0^t\int_0^x |\partial_t^K\partial_x^K v(s,w,y)|^2 dsdw,$$

which implies

$$t_\varepsilon^{-2K+1}x_\varepsilon^{-2K+1}|v(t,x,y)|^2 \leq C\int_0^t\int_0^x |\partial_t^K\partial_x^K v(s,w,y)|^2 dsdw.$$

For $t/t_\varepsilon \leq 1$ $(t \geq 0)$ and $x/x_\varepsilon \leq 1$ $(x \geq 0)$. By integrating both sides of the inequality above, we obtain the desired estimate. □

Therefore, from Lemma 3.1 and Lemma 3.2 we see that any $u(t,x,y)$ that enjoys the assumption of Lemma 3.1 satisfies the following; for any integer $M_1 \geq 0$ there exists an integer $M_2 \geq 0$ such that for $t_1, x_1 \geq 0$

$$\begin{aligned} &\sum_{\alpha_1+\alpha_2+j\leq M_1}\int_0^{t_1}\int_0^{x_1}\|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j u\|^2\,dt\,dx \\ &\leq C\int_0^{t_1}\int_0^{x_1}\Big(\sum_{\alpha_1+\alpha_2+j\leq M_2}\|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j L_{1,\varepsilon}u\|^2\Big)\,dt\,dx \end{aligned} \tag{3.20}$$

Proof of Lemma 2.3. As we remarked in the previous section, solutions $u_\varepsilon(t,x,y)$ enjoy the finite speed of propagation. Hence for any compact set K in $\mathbb{R}^3$, there exists a compact set K_1 such that for any $h_1(t,x,y) \in C_0^\infty(\mathbb{R}^3)$ satisfying $\operatorname{supp} h_1(t,x,y) \subset \overline{\Pi_{1,1}}$ and $h_1(t,x,y) = h(t,x,y)$ on K_1 where $h(t,x,y)$ is the right hand side of (2.14), the solution of the problem (2.14) with $h_1(t,x,y)$ in the place of $h(t,x,y)$ coincides with $u_\varepsilon(t,x,y)$ on K. By using $\chi_1(t,x,y) \in C_0^\infty(\mathbb{R}^3)$ satisfying $\chi_1(t,x,y) = 1$ on K_1, we obtain from (3.20) that

$$\begin{aligned} &\sum_{\alpha_1+\alpha_2+j\leq M_1}\iiint_K |\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j u_\varepsilon(t,x,y)|^2\,dt\,dx\,dy \\ &\qquad \leq C\iiint_{\mathbb{R}^3}\Big(\sum_{\alpha_1+\alpha_2+j\leq M_2}|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j\chi_1 h(t,x,y)|^2\Big)\,dt\,dx\,dy \end{aligned}$$

which implies the assertion of Lemma 2.3. □

Next we prove Lemma 2.4. From (3.1), (3.2) and (3.3) we have

$$
\begin{aligned}
&-\partial_t(e^{\gamma(t+x)}t_\varepsilon^M x_\varepsilon^M\|\partial_x u\|^2)\\
&-\partial_x\big(e^{\gamma(t+x)}t_\varepsilon^{M+2k+1}x_\varepsilon^{M+2l+1}(A_2(t,x,\cdot)\partial_y u,\partial_y u)\big)\\
&+e^{\gamma(t+x)}(\gamma t_\varepsilon+Mt'_\varepsilon)t_\varepsilon^{M-1}x_\varepsilon^M\|\partial_x u\|^2\\
&+e^{\gamma(t+x)}(\gamma x_\varepsilon+(M+2l+1)x'_\varepsilon)t_\varepsilon^{M+2k+1}x_\varepsilon^{M+2l}(A_2(t,x,\cdot)\partial_y u,\partial_y u)\\
&\le -2e^{\gamma(t+x)}t_\varepsilon^M x_\varepsilon^M\Re(\partial_t\partial_x u-t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}A_2(t,x,\cdot)\partial_y{}^2u,\partial_x u)\\
&\quad+Ce^{\gamma(t+x)}t_\varepsilon^{M+2k+1}x_\varepsilon^{M+2l+1}(\|\partial_y u\|^2+\|\partial_x u\|^2)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\partial_x(e^{\gamma(t+x)}t_\varepsilon^M x_\varepsilon^M\|\partial_t u\|^2)\\
&-\partial_t\big(e^{\gamma(t+x)}t_\varepsilon^{M+2k+1}x_\varepsilon^{M+2l+1}(A_2(t,x,\cdot)\partial_y u,\partial_y u)\big)\\
&+e^{\gamma(t+x)}(\gamma x_\varepsilon+Mx'_\varepsilon)t_\varepsilon^{M}x_\varepsilon^{M-1}\|\partial_t u\|^2\\
&+e^{\gamma(t+x)}(\gamma t_\varepsilon+(M+2k+1)t'_\varepsilon)t_\varepsilon^{M+2k}x_\varepsilon^{M+2l+1}(A_2(t,x,\cdot)\partial_y u,\partial_y u)\\
&\le -2e^{\gamma(t+x)}t_\varepsilon^M x_\varepsilon^M\Re(\partial_t\partial_x u-t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}A_2(t,x,\cdot)\partial_y{}^2u,\partial_t u)\\
&\quad+Ce^{\gamma(t+x)}t_\varepsilon^{M+2k+1}x_\varepsilon^{M+2l+1}(\|\partial_y u\|^2+\|\partial_t u\|^2).
\end{aligned}
$$

The definition of ${}^tL_{1,\varepsilon}$ (2.16) and the argument used for estimates (3.12) and (3.15) imply that for any $w(t,x,y)\in C^\infty(\mathbb{R}^3)$ satisfying $\operatorname{supp} w(t,x,y)\subset\{(t,x,y)\in\mathbb{R}^3 \mid t\le t_1 \text{ and } x\le x_1\}$ with some $t_1,\ x_1>0$ and vanishing for large $|y|$, we have

$$
\begin{aligned}
&\frac{\gamma}{2}\int_0^{t_1}\int_0^{x_1} e^{\gamma(t+x)}t_\varepsilon^M x_\varepsilon^M\big(\|\partial_t w\|^2+\|\partial_x w\|^2+\|w\|^2\big)\,dt\,dx\\
&+\frac{\gamma\delta_0}{2}\int_0^{t_1}\int_0^{x_1} e^{\gamma(t+x)}t_\varepsilon^{M+2k+1}x_\varepsilon^{M+2l+1}\|\partial_y w\|^2\,dt\,dx\\
&\le\int_0^{t_1}\int_0^{x_1} e^{\gamma(t+x)}t_\varepsilon^M x_\varepsilon^M\|{}^tL_{1,\varepsilon}w\|^2\,dt\,dx
\end{aligned}
\tag{3.21}
$$

for $\gamma\ge\gamma_0$ and $M\ge M_0$ with some positive constants γ_0 and M_0 which are independent of ε. Similarly we obtain the estimates for the derivatives of $w(t,x,y)$. Hence

Lemma 3.3. *Let $w(t,x,y)\in C^\infty(\mathbb{R}^3)$ satisfy $\operatorname{supp} u(t,x,y)\subset\{(t,x,y)\in\mathbb{R}^3 \mid t\le t_1 \text{ and } x\le x_1\}$ with some $t_1,\ x_1>0$ and vanish for large $|y|$.*

For any integer $k\ge 1$ and any integer $l\ge 0$

$$
\begin{aligned}
&\sum_{\alpha_1+\alpha_2\le 2k-1}\sum_{j=0}^{l}\gamma\int_0^{t_1}\int_0^{x_1} e^{\gamma(t+x)}t_\varepsilon^M x_\varepsilon^M\|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j w\|^2\,dt\,dx\\
&\le C\int_0^{t_1}\int_0^{x_1}\Big(\sum_{\substack{\alpha_1+\alpha_2\le 2k-2\\ \alpha_1+\alpha_2+j\le 2k-2+l}} e^{\gamma(t+x)}t_\varepsilon^M x_\varepsilon^M\|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j\,{}^tL_{1,\varepsilon}w\|^2\Big)\,dt\,dx
\end{aligned}
\tag{3.22}
$$

$$
\sum_{\alpha_1+\alpha_2\le 2k}\sum_{j=0}^{l}\gamma\int_0^{t_1}\int_0^{x_1} e^{\gamma(t+x)}t_\varepsilon^M x_\varepsilon^M\|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j w\|^2\,dt\,dx
$$

$$\le C\int_0^{t_1}\int_0^{x_1} e^{-\gamma(t+x)} t_\varepsilon^M x_\varepsilon^M \Big(\sum_{\substack{\alpha_1+\alpha_2\le 2k-1\\ \alpha_1+\alpha_2+j\le 2k+l-1}} \|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^{j}\,{}^tL_{1,\varepsilon}w\|^2 + \|\partial_y^{2k+l}\,{}^tL_{1,\varepsilon}w\|^2\Big)\,dt\,dx \tag{3.23}$$

for $\gamma \ge \gamma_{l,k}$ and $M \ge M_0$ with some constant $\gamma_{l,k} > 0$ independent of ε and some constant $C > 0$ independent of ε and γ where M_0 is the constant appearing in (3.21).

In order to estimate L^2-norm of $w(t,x,y)$ and its derivatives by the left hand side of (3.22) or (3.23), we use the following lemma.

Lemma 3.4. *When a $v(t,x,y) \in C^\infty(\mathbb{R}^3)$ satisfies $\operatorname{supp} v(t,x,y) \subset \{(t,x,y) \in \mathbb{R}^3 \mid t \le t_1$ and $x \le x_1\}$ with some $t_1,\ x_1 > 0$ and vanishes for large $|y|$, we have for any integer $K \ge 0$,*

$$\int_0^{t_1}\int_0^{x_1} \|v\|^2\,dt\,dx \le C\int_0^{t_1}\int_0^{x_1} t_\varepsilon^{2K} x_\varepsilon^{2K}\|\partial_t^K\partial_x^K v\|^2\,dt\,dx$$

with some constant $C > 0$ independent of ε.

Proof. When $f(s) \in C^\infty(\mathbb{R})$ vanishes for $s \ge s_0 > 0$, we have for any integer $k \ge 0$,

$$(2k+1)\int_0^{s_0} s^{2k}|f^{(k)}(s)|^2 d\,s + 2\Re\int_0^{s_0} s^{2k+1} f^{(k)}(s)\overline{f^{(k+1)}(s)} d\,s = 0$$

from which we obtain

$$\int_0^{s_0} s^{2k}|f^{(k)}(s)|^2 d\,s \le \frac{4}{(2k+1)^2}\int_0^{s_0} s^{2k+2}|f^{(k+1)}(s)|^2 d\,s.$$

Hence, by the induction, for any positive integer K

$$\int_0^{s_0} |f(s)|^2 d\,s \le 4^K\int_0^{s_0} s^{2K}|f^{(K)}(s)|^2 d\,s.$$

The estimate above implies the desired assertion of Lemma 3.3. □

Therefore, from Lemma 3.3 and Lemma 3.4 we see that for any $w(t,x,y)$ that satisfies the assumption of Lemma 3.3 we have the following; for any integer $M_1 \ge 0$ there exists an integer $M_2 \ge 0$ such that for $t_1, x_1 \ge 0$

$$\sum_{\alpha_1+\alpha_2+j\le M_1}\int_0^{t_1}\int_0^{x_1} \|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^{j} w\|^2\,dt\,dx \le C\int_0^{t_1}\int_0^{x_1}\Big(\sum_{\alpha_1+\alpha_2+j\le M_2} \|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^{j}\,{}^tL_{1,\varepsilon}w\|^2\Big)\,dt\,dx \tag{3.25}$$

Then the estimate (3.25) shows that a solution of the problem (2.15) $w_\varepsilon(t,x,y)$ $(0 < \varepsilon < 1)$ is bounded in $H^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid 0 < t < T \text{ and } 0 < x < T\})$. Hence we see that the first assertion of Lemma 2.4 is also valid. Then the proof of Theorem 1.1 is completed.

4. Proof of Theorem 1.2

We denote by L_2 the differential operator in (1.6);

$$L_2 = \partial_x\partial_t - B(t,x,y)\partial_x^2 - t^{2k+1}x^{2l+1}A_2(t,x,y)\partial_y^2 + t^k x^{l+1}A_1\partial_y(t,x,y) + a_0(t,x,y)$$

where, by the assumption, $|B(t,x,y)| \geq \sigma_0$ and $A_2(t,x,y) \geq \delta_0$ with some positive constants σ_0 and δ_0.

As the proof of Lemma 2.1, we can construct a function $v(t,x,y) \in C^\infty(\mathbb{R}^3)$ such that

$$v(0,x,y) = g_1(x,y),\ v(t,0,y) = g_2(t,y)$$
$$L_2 v(t,x,y) - f(t,x,y) \text{ is flat on the plane } t = 0.$$

Then the problem (1.6) can be reduced to the problem

$$\begin{aligned} L_2 u = h(t,x,y) \quad & (t,x,y) \in \mathbb{R}^3 \\ u(0,x,y) = 0 \quad & (x,y) \in \mathbb{R}^2 \\ u(t,0,y) = 0 \quad & (t,y) \in \mathbb{R}^2 \end{aligned} \tag{4.1}$$

where $h(t,x,y)$ is flat on the plane $t = 0$.

We remark that a $C^\infty-$ solution $u(t,x,y)$ to (4.1) is flat on $t = 0$.

By putting

$$u_+(t,x,y) = \begin{cases} u(t,x,y) & (t \geq 0) \\ 0 & (t < 0) \end{cases}$$
$$u_-(t,x,y) = u(t,x,y) - u_+(t,x,y),$$

we see that $u_+(t,x,y)$ [resp. $u_-(t,x,y)$] satisfies

$$\begin{aligned} L_2 u_+ = h_+(t,x,y) \quad & (t,x,y) \in \mathbb{R}^3 \\ u_+(0,x,y) = 0 \quad & (x,y) \in \mathbb{R}^2 \\ \operatorname{supp} u_+(t,x,y) \subset \{(t,x,y) \in R^3 & \mid t \geq 0\} \end{aligned} \tag{4.2}$$

[resp.

$$\begin{aligned} L_2 u_- = h_-(t,x,y) \quad & (t,x,y) \in \mathbb{R}^3 \\ u_-(0,x,y) = 0 \quad & (x,y) \in \mathbb{R}^2 \\ \operatorname{supp} u_-(t,x,y) \subset \{(t,x,y) \in R^3 & \mid t \leq 0\}. \end{aligned} \tag{4.3}$$

] where

$$h_+(t,x,y) = \begin{cases} h(t,x,y) & (t \geq 0) \\ 0 & (t < 0) \end{cases}$$
$$h_-(t,x,y) = h(t,x,y) - h_+(t,x,y).$$

On the other hand the sum of solutions $u_+(t,x,y)$ to (4.2) and $u_-(t,x,y)$ to (4.3) satisfies (4.1). While by the change of coordinate $t = -t$, the problem (4.3) is reduced to that of (4.2), hence for the proof of Theorem 2 it suffices to prove the following proposition.

Proposition 4.1. *For any $h(t,x,y) \in C^\infty(\mathbb{R}^3)$ whose support is contained in $\{(t,x,y) \in \mathbb{R}^3 \mid t \geq 0\}$, there exists one and only one solution $u(t,x,y) \in C^\infty(\mathbb{R}^3)$ of the problem*

$$\begin{gathered} L_2 u = h(t,x,y) \quad \text{in } \mathbb{R}^3 \\ u(0,x,y) = 0 \quad \text{on } \mathbb{R}^2 \\ \operatorname{supp} u(t,x,y) \subset \{(t,x,y) \in R^3 \mid t \geq 0\}. \end{gathered} \tag{4.4}$$

Proposition 4.1 follows from the following two propositions where ${}^tL_2$ is the transpose of L_2, that is to say,

$$\begin{aligned} {}^tL_2 = \partial_t\partial_x - B(t,x,y)\partial_x^2 - 2B_x(t,x,y)\partial_x - B_{xx}(t,x,y) \\ - t^{2k+1}x^{2l+1}\big(A_2(t,x,y)\partial_y^2 + 2A_{2y}(t,x,y)\partial_y + A_{2yy}(t,x,y)\big) \\ - t^k x^{l+1}(A_1(t,x,y)\partial_y + A_{1y}(t,x,y)) + a_0(t,x,y) \end{aligned}$$

where $B_x(t,x,y) = \partial_x B(t,x,y)$, $B_{xx}(t,x,y) = \partial_x^2 B(t,x,y)$ and the similar notations are used for $A_j(t,x,y)$ $(j = 1,2)$.

As remarked in the section 1, we may assume $B(t,x,y) \geq \sigma_0$. Then in the following we assume

$$\frac{1}{\sigma_{00}} \geq B(t,x,y) \geq \sigma_0. \tag{4.5}$$

Proposition 4.2. a) *For any $h(t,x,y) \in C^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0\})$ satisfying $h(t,x,y) = 0$ for $t \leq 0$, there exists a solution $u(t,x,y) \in C^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0\})$ of the mixed problem*

$$\begin{gathered} L_2 u = h(t,x,y) \quad \text{in } \{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0\} \\ u(t,0,y) = 0 \quad \text{on } \mathbb{R}^2 \\ u(t,x,y) = 0 \quad (t \leq 0). \end{gathered} \tag{4.6}$$

b) *For any $\tilde{h}(t,x,y) \in C_0^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0\})$ and $g(t,y) \in C_0^\infty(\mathbb{R}^2)$, there exists a solution $w(t,x,y) \in C_0^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0 \text{ and } t \geq 0\})$ of the mixed problem*

$$\begin{gathered} {}^tL_2 w = \tilde{h}(t,x,y) \quad \text{in } \{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0 \text{ and } t \geq 0\} \\ w(t,0,y) = g(t,y) \quad \text{on } \{(t,y) \in \mathbb{R}^2 \mid t \geq 0\} \end{gathered} \tag{4.7}$$

Proposition 4.3. a) *For any $h(t,x,y) \in C^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \leq 0\})$ satisfying $h(t,x,y) = 0$ for $t \leq 0$ and any $g_1(t,y)$, $g_2(t,y) \in C^\infty(\mathbb{R}^2)$ carried on $\{(t,y) \in \mathbb{R}^2 \mid t \geq 0\}$, there exists a solution $u(t,x,y) \in C^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \leq 0\})$ satisfying $u(t,x,y) = 0$ for $t \leq 0$ of the Cauchy problem*

$$\begin{gathered} L_2 u = h(t,x,y) \quad \text{in } \{(t,x,y) \in \mathbb{R}^3 \mid x \leq 0\} \\ u(t,0,y) = g_1(t,y) \quad \text{on } \mathbb{R}^2 \\ \partial_x u(t,0,y) = g_2(t,y) \quad \text{on } \mathbb{R}^2 \end{gathered} \tag{4.8}$$

b) *For any $\tilde{h}(t,x,y) \in C_0^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \leq 0\})$, there exists a solution $w(t,x,y) \in C_0^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \leq 0 \text{ and } t \geq 0\})$ to*

$$ {}^tL_2 w = \tilde{h}(t,x,y) \quad \text{in } \{(t,x,y) \in \mathbb{R}^3 \mid x \leq 0 \text{ and } t \geq 0\} \tag{4.9}$$

We see that Proposition 4.2 and Proposition 4.3 imply Proposition 4.1. Indeed, for any $h(t,x,y) \in C^\infty(\mathbb{R}^3)$ whose support contained in $\{(t,x,y) \in \mathbb{R}^3 \mid t \geq 0\}$, we solve (4.6). Let $u_+(t,x,y) \in C^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0\})$ be its solution. Then by putting $g_1(t,y) = u_+(t,0,y)$ and $g_2(t,y) = \partial_x u_+(t,0,y)$, we solve the Cauchy problem (4.8) whose solution we denote by $u_-(t,x,y) \in C^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \leq 0\})$. Then $u(t,x,y) \in C^\infty(\mathbb{R}^3)$ defined by

$$u(t,x,y) = \begin{cases} u_+(t,x,y) & (x \geq 0) \\ u_-(t,x,y) & (x < 0) \end{cases}$$

satisfies (4.4).

On the other hand, for any $\tilde{h}(t,x,y) \in C_0^\infty(\mathbb{R}^3)$, we solve (4.9). Let $w_-(t,x,y) \in C_0^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \leq 0 \text{ and } t \geq 0\})$ be its solution. Then by putting $g(t,y) = w_-(t,0,y)$, we solve (4.7), whose solution is denoted by $w_+(t,x,y)$. Now we define $w(t,x,y)$ by

$$w(t,x,y) = \begin{cases} w_+(t,x,y) & (x \geq 0) \\ w_-(t,x,y) & (x < 0). \end{cases}$$

Since we have, if $u(t,0,y) = 0$,

$$\int_{-\infty}^{\infty} \partial_x^2 u(t,x,y) w(t,x,y) d\,x = \int_{-\infty}^{\infty} u(t,x,y) \partial_x^2 w(t,x,y) d\,x,$$

which implies for a solution $u(t,x,y)$ of (4.4)

$$\iiint_{\mathbb{R}^3} L_2 u(t,x,y) w(t,x.y)\, dt\, dx\, dy = \iiint_{\mathbb{R}^3} u(t,x,y)\, {}^t L_2 w(t,x.y)\, dt\, dx\, dy,$$

then for any solution $u(t,x,y)$ of (4.4) with $h(t,x,y) = 0$, we obtain

$$\iiint_{\mathbb{R}^3} u(t,x,y) \tilde{h}(t,x.y)\, dt\, dx\, dy = 0.$$

Then we see $u(t,x,y) = 0$, which implies the uniqueness of solution of (4.4).

For the proof of Proposition 4.2, using the functions t_ε and x_ε defined by (2.12), we introduce the opertors $L_{2,\varepsilon}$ and ${}^t L_{2,\varepsilon}$ by

$$\begin{aligned} L_{2,\varepsilon} &= \partial_t \partial_x - B(t,x,y) \partial_x^2 \\ &\qquad - t_\varepsilon^{2k+1} x_\varepsilon^{2l+1} A_2(t,x,y) \partial_y^2 + t_\varepsilon^{k+1} x_\varepsilon^{l+1} A_1(t,x,y) \partial_y + a_0(t,x,y) \end{aligned}$$

and

$$\begin{aligned} {}^t L_{2,\varepsilon} &= \partial_t \partial_x - B(t,x,y) \partial_x^2 - 2B_x(t,x,y) \partial_x - B_{xx}(t,x,y) \\ &\quad - t_\varepsilon^{2k+1} x_\varepsilon^{2l+1} \big(A_2(t,x,y) \partial_y^2 + 2A_{2y}(t,x,y) \partial_y + A_{2yy}(t,x,y) \big) \\ &\qquad - t_\varepsilon^{k+1} x_\varepsilon^{l+1} (A_1(t,x,y) \partial_y + A_{1y}(t,x,y)) + a_0(t,x,y). \end{aligned}$$

We see from the assumption (4.5) that both operators $L_{2,\varepsilon}$ and ${}^t L_{2,\varepsilon}$ are strictly hyperbolic in the direction $(1,\sigma,0)$ with $0 < \sigma < \sigma_{00}$ and the plane $x = 0$ is time like. Then both of the following two mixed problems are C^∞-wellposed (see for example [2] or [6]).

Find a solution $u_\varepsilon(t,x,y) \in C^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \ge 0\})$ satisfying

$$\begin{aligned} L_{2,\varepsilon} u_\varepsilon = h(t,x,y) \quad &\text{in } \{(t,x,y) \in \mathbb{R}^3 \mid x \ge 0\} \\ u_\varepsilon(t,0,y) = 0 \quad &\text{on } \mathbb{R}^2 \\ u_\varepsilon(t,x,y) = 0 \quad &(t + \frac{\sigma_{00}}{2}x \le 0). \end{aligned} \tag{4.12}$$

where $h(t,x,y) \in C^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \ge 0\})$ satisfies $h(t,x,y) = 0$ for $t \le 0$.

Find a solution $w_\varepsilon(t,x,y) \in C_0^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \ge 0 \text{ and } t + \frac{\sigma_{00}}{2}x \ge 0\})$ satisfying

$$\begin{aligned} {}^tL_{2,\varepsilon} w_\varepsilon = \tilde{h}(t,x,y) \quad &\text{in } \{(t,x,y) \in \mathbb{R}^3 \mid x \ge 0 \text{ and } t + \frac{\sigma_{00}}{2}x \ge 0\} \\ w_\varepsilon(t,0,y) = g(t,y) \quad &\text{on } \quad \{(t,y) \in \mathbb{R}^2 \mid t \ge 0\} \end{aligned} \tag{4.13}$$

where $\tilde{h}(t,x,y) \in C_0^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \ge 0\})$ and $g(t,y) \in C_0^\infty(\mathbb{R}^2)$.

First we remark that the surface given by

$$t + \frac{\sigma_{00}}{2}x + \frac{1}{4}(\sqrt{1 + (t - \frac{\sigma_{00}}{2}x)^2} - 1) = C$$

is space like for $L_{2,\varepsilon}$ and that on the closed domain V_T given by

$$\begin{aligned} V_T = \big\{(t,x,y) \in \mathbb{R}^3 \mid x \ge 0,\ t + \frac{\sigma_{00}}{2}x \ge 0 \text{ and} \\ t + \frac{\sigma_{00}}{2}x + \frac{1}{4}(\sqrt{1 + (t - \frac{\sigma_{00}}{2}x)^2} - 1) \le T\big\} \end{aligned}$$

the coefficients of $L_{2,\varepsilon}$ are bounded.

Concerning solutions $u_\varepsilon(t,x,y)$ to (4.12), since the plane $t + \sigma x = C$ with $0 < \sigma < \sigma_{00}$ is space like, we see that $u_\varepsilon(t,x,y) = 0$ for $t \le 0$. Then the following lemma implies the part a) of Proposition 4.2.

Lemma 4.4. *The family of solutions* $\{u_\varepsilon(t,x,y)\}_{0<\varepsilon<1}$ *to* (4.12) *is bounded in* $H^\infty_{\rm loc}(\{(t,x,y) \in \mathbb{R}^3 \mid x > 0\})$.

Indeed, thanks to the lemma above we can find a subsequence $\{u_{\varepsilon_j}(t,x,y)\}$ $(j = 1, 2, \cdots)$ which converges to a C^∞-function $u(t,x,y)$ that satisfies (4.6).

On the other hand, we note that the surface given by

$$t + \frac{\sigma_{00}}{2}x - \frac{1}{4}(\sqrt{1 + (t - \frac{\sigma_{00}}{2}x)^2} - 1) = 0$$

is space like for ${}^tL_{2,\varepsilon}$. Since on the closed domain W_T given by

$$\begin{aligned} W_T = \{(t,x,y) \in \mathbb{R}^3 \mid x \ge 0,\ t + \frac{\sigma_{00}}{2}x \le T \text{ and} \\ t + \frac{\sigma_{00}}{2}x - \frac{1}{4}(\sqrt{1 + (t - \frac{\sigma_{00}}{2}x)^2} - 1) \ge 0\} \end{aligned}$$

the coefficients of ${}^tL_{2,\varepsilon}$ are bounded, ${}^tL_{2,\varepsilon}$ has the finite propagation speed independent of $0 < \varepsilon < 1$ on W_T. Furthermore we see that

$$t + \frac{\sigma_{00}}{2}x - \frac{1}{4}(\sqrt{1 + (t - \frac{\sigma_{00}}{2}x)^2} - 1) \ge 0$$

for $t \geq 0$ and $x \geq 0$. Let $T_0 \geq 0$ satisfy

$$T_0 \geq \sup_{(t,x,y)\in \operatorname{supp} \tilde{h}(t,x,y)} t + \frac{\sigma_{00}}{2}x$$

and

$$T_0 \geq \sup_{(t,y)\in \operatorname{supp}\, \mathrm{g(t,y)}} t.$$

Then we see that $w_\varepsilon(t,x,y) = 0$ if $t + \frac{\sigma_{00}}{2}x \geq T_0$. Since ${}^tL_{2,\varepsilon}$ has the finite propagation speed independent of $0 < \varepsilon < 1$ on W_{T_0}, we see that there exists a compact set F such that

$$\operatorname{supp} w_\varepsilon(t,x,y) \cap \{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0 \text{ and } t \geq 0\} \subset F.$$

Hence, similarly to the case of the part a) of Proposition 4.2, the following lemma implies the part b) of Proposition 4.2.

Lemma 4.5. *The family of solutions* $\{w_\varepsilon(t,x,y)\}_{0<\varepsilon<1}$ *to* (4.13) *is bounded in* $H^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x > 0 \text{ and } t > 0\})$.

The proof of Lemma 4.4 and Lemma 4.5 is given in the next section.

For the proof of Proposition 4.3, we first change the problem in the half space $\{(t,x,y) \in \mathbb{R}^3 \mid x \leq 0\}$ to that in $\{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0\}$ by the change of coordinate $x = -x$. Let $\tilde{L}_2$ be

$$\tilde{L}_2 = \partial_x\partial_t + B(t,x,y)\partial_x^2 - t^{2k+1}x^{2l+1}A_2(t,x,y)\partial_y^2 + t^kx^{l+1}A_1\partial_y + a_0(t,x,y)$$

where $B(t,x,y) \geq \sigma_0$ and $A_2(t,x,y) \geq \delta_0$ with some positive constants σ_0 and δ_0. Then Proposition 4.3 is equivalent to the following proposition.

Proposition 4.6. a) *For any* $h(t,x,y) \in C^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0\})$ *satisfying* $h(t,x,y) = 0$ *for* $t \leq 0$ *and any* $g_1(t,y)$, $g_2(t,y) \in C^\infty(\mathbb{R}^2)$ *whose support contained in* $\{(t,y) \in \mathbb{R}^2 \mid t \geq 0\}$, *there exists a solution* $u(t,x,y) \in C^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0\})$ *satisfying* $u(t,x,y) = 0$ *for* $t \leq 0$ *of the Cauchy problem*

$$\begin{aligned} \tilde{L}_2u = h(t,x,y) &\quad \text{in } \{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0\} \\ u(t,0,y) = g_1(t,y) &\quad \text{on } \mathbb{R}^2 \\ \partial_xu(t,0,y) = g_2(t,y) &\quad \text{on } \mathbb{R}^2 \end{aligned} \tag{4.15}$$

b) *For any* $\tilde{h}(t,x,y) \in C_0^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0\})$, *there exists a solution* $w(t,x,y) \in C_0^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0 \text{ and } t \geq 0\})$ *to the equation*

$${}^t\tilde{L}_2w = \tilde{h}(t,x,y) \quad \text{in } \{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0 \text{ and } t \geq 0\}$$

where ${}^t\tilde{L}_2$ *is the transpose of* $\tilde{L}_2$.

First of all, we remark that the argument similar to Lemma 2.1 implies that there exists a function $v(t,x,y) \in C^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0\})$ supported in $\{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0 \text{ and } t \geq 0\}$ and satisfying the followings; $v(t,0,y) = g_1(t,y)$, $\partial_xv(t,0,y) = g_2(t,y)$ and $\tilde{L}_2v(t,x,y) - h(t,x,y)$ is flat on $x = 0$. Then by taking $u(t,x,y) - v(t,x,y)$, the problem (4.15) is reduced the case where $g_1(t,y) = 0$, $g_2(t,y) = 0$ and $h(t,x,y)$ is flat on $x = 0$.

For the proof of Proposition 4.6, using the functions t_ε and x_ε defined by (2.12), we introduce the operators $\tilde{L}_{2,\varepsilon}$ by

$$\begin{aligned} \tilde{L}_{2,\varepsilon} = {}& \partial_t\partial_x + B(t,x,y)\partial_x^2 \\ & - t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}A_2(t,x,y)\partial_y^2 + t_\varepsilon^kx_\varepsilon^{l+1}A_1(t,x,y)\partial_y + a_0(t,x,y). \end{aligned}$$

Let ${}^t\tilde{L}_{2,\varepsilon}$ be the transpose of $\tilde{L}_{2,\varepsilon}$. Since $\tilde{L}_{2,\varepsilon}$ and ${}^t\tilde{L}_{2,\varepsilon}$ are strictly hyperbolic in the direction $(\mu, 1, 0)$ with $\mu > -\sigma_0$, both of the following Cauchy problems are C^∞-wellposed (see for example [3]).

Find a solution $u_\varepsilon(t,x,y) \in C^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0\})$ that satisfies

$$\begin{aligned} \tilde{L}_{2,\varepsilon} u_\varepsilon = h(t,x,y) \quad &\text{in} \quad \{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0\} \\ u_\varepsilon(t,0,y) = 0 \quad &\text{on } \mathbb{R}^2 \\ \partial_x u_\varepsilon(t,0,y) = 0 \quad &\text{on } \mathbb{R}^2 \end{aligned} \tag{4.17}$$

where $h(t,x,y) \in C^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0\})$ which is supported in $\{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0 \text{ and } t \geq 0\}$ and flat on $x = 0$.

Find a solution $w_\varepsilon(t,x,y) \in C_0^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0\})$ satisfying

$${}^t\tilde{L}_{2,\varepsilon} w_\varepsilon = \tilde{h}(t,x,y) \quad \text{in } \{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0\} \tag{4.18}$$

where $\tilde{h}(t,x,y) \in C_0^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0\})$.

Since the plane $\mu t + x = 0$ $(\mu > -\sigma_0)$ is space like for $\tilde{L}_{2,\varepsilon}$, we see that a solution of (4.17) $u_\varepsilon(t,x,y)$ vanishes for $t \leq 0$. Hence, similarly to the case of the part a) of Proposition 4.2, the following lemma implies the part a) of Proposition 4.6.

Lemma 4.7. *The family of solutions* $\{u_\varepsilon(t,x,y)\}_{0<\varepsilon<1}$ *to* (4.17) *is bounded in* $H^\infty_{\text{loc}}(\{(t,x,y) \in \mathbb{R}^3 \mid x > 0\})$.

Similarly to the case (4.13), let $X \geq 0$ satisfy

$$X \geq \sup_{(t,x,y) \in \operatorname{supp} \tilde{h}(t,x,y)} x.$$

For any X_1 and X_2 satisfying $X_1 < X_2$ and any t_0, if $0 < \nu < \frac{\sigma_0^2}{4(X_2 - X_1)}$, the surface $x - \nu(t - t_0)^2 = X_1$ in $\{(t,x,y) \in \mathbb{R}^3 \mid x \leq X_2\}$ is space like and on the closed domain

$$\{(t,x,y) \in \mathbb{R}^3 \mid x \leq X_2,\ x - \nu(t-t_0)^2 \geq X_1\}$$

the coefficients of ${}^tL_{2,\varepsilon}$ are bounded. Then ${}^tL_{2,\varepsilon}$ has the finite propagation speed independent of $0 < \varepsilon < 1$ there. Hence we see that a solution of (4.18) $w_\varepsilon(t,x,y)$ vanishes if $x \geq X$ and that there exists a compact set F such that the solution $w_\varepsilon(t,x,y)$ of (4.18) satisfies

$$\operatorname{supp} w_\varepsilon(t,x,y) \cap \{(t,x,y) \in \mathbb{R}^3 \mid x \geq 0 \text{ and } t \geq 0\} \subset F.$$

Hence, similarly to the case of the part b) of Proposition 4.2, the following lemma implies the part b) of Proposition 4.6.

Lemma 4.8. *The family of solutions* $\{w_\varepsilon(t,x,y)\}_{0<\varepsilon<1}$ *to* (4.18) *is bounded in* $H^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x > 0 \textit{ and } t > 0\})$.

5. Proof of Lemmas 4.4, 4.5, 4.7, and 4.8

In this section also, we use the method of Oleinik [5] in order to draw the a priori estimates for $L_{1,\varepsilon}$, ${}^tL_{1,\varepsilon}$, $\tilde{L}_{2,\varepsilon}$ and ${}^t\tilde{L}_{2,\varepsilon}$ that are uniformly valid for $0 < \varepsilon < 1$. Let

$$L^0_{1,\varepsilon} = \partial_t\partial_x - B(t,x,y)\partial_x^2 - t_\varepsilon^{2k+1} x_\varepsilon^{2l+1} A_2(t,x,y)\partial_y^2.$$

Since

$$\begin{aligned} 2\Re(L^0_{1,\varepsilon}u, \partial_t u) =& \partial_x(\partial_t u, \partial_t u) - 2\partial_x \Re(B(t,x,\cdot)\partial_x u, \partial_t u) \\ &+ \partial_t(B(t,x,\cdot)\partial_x u, \partial_x u) + t_\varepsilon^{2k+1} x_\varepsilon^{2l+1} \partial_t(A_2(t,x,\cdot)\partial_y u, \partial_y u) + R \end{aligned} \tag{5.2}$$

where

$$|R| \le C\big(t_\varepsilon^{2k+1} x_\varepsilon^{2l+1}(\|\partial_y u\|^2 + \|\partial_t u\|^2) + \|\partial_x u\|\|\partial_t u\| + \|\partial_x u\|^2\big), \tag{5.3}$$

we have

$$\begin{aligned}
&\partial_x\big(e^{-\gamma(t+\sigma x)} t_\varepsilon^{-M}(\|\partial_t u\|^2 - 2\Re(B(t,x,\cdot)\partial_x u, \partial_t u))\big)\\
&+ \partial_t\big(e^{-\gamma(t+\sigma x)} t_\varepsilon^{-M}((B(t,x,\cdot)\partial_x u, \partial_x u) + t_\varepsilon^{2k+1} x_\varepsilon^{2l+1}(A_2(t,x,\cdot)\partial_y u, \partial_y u))\big)\\
&+ e^{-\gamma(t+\sigma x)}\gamma\sigma t_\varepsilon^{-M}\|\partial_t u\|^2 - 2e^{-\gamma(t+\sigma x)}\gamma\sigma t_\varepsilon^{-M}\Re(B(t,x,\cdot)\partial_x u, \partial_t u)\\
&+ e^{-\gamma(t+\sigma x)}(\gamma t_\varepsilon + M t'_\varepsilon) t_\varepsilon^{-M-1}(B(t,x,\cdot)\partial_x u, \partial_x u)\\
&+ e^{-\gamma(t+\sigma x)}(\gamma t_\varepsilon + (M-2k-1)t'_\varepsilon) t_\varepsilon^{-M+2k} x_\varepsilon^{2l+1}(A_2(t,x,\cdot)\partial_y u, \partial_y u)\\
&\le 2e^{-\gamma(t+\sigma x)} t_\varepsilon^{-M}\Re(L^0_{1,\varepsilon}u, \partial_t u) + C\big(e^{-\gamma(t+\sigma x)} t_\varepsilon^{-M+2k+1} x_\varepsilon^{2l+1}(\|\partial_y u\|^2 + \|\partial_t u\|^2)\\
&\quad + e^{-\gamma(t+\sigma x)} t_\varepsilon^{-M}(\|\partial_x u\|\|\partial_t u\| + \|\partial_x u\|^2)\big).
\end{aligned} \tag{5.4}$$

Since $1/\sigma_{00} \ge B(t,x,y) \ge \sigma_0$, we see that the plane $t + \sigma_1 x = T$ where $\sigma_1 = \sigma_{00}/2$, is space like. Then by integrating (5.4) on $\Delta_T = \{(t,x) \in \mathbb{R}^2 \mid t \ge 0,\ x \ge 0 \text{ and } t + \sigma_1 x \le T\}$ with $T > 0$, we obtain the following (see for example §24.1 of [2]). When $\gamma > \gamma_0$ with some $\gamma_0 > 0$ and $M > 2k+1$, for any smooth $u(t,x,y)$ satisfying $u(t,0,y) = 0$ and vanishing if $t \le 0$ or $|y|$ is large, we have

$$\begin{aligned}
&\int_{\Delta_T} e^{-\gamma(t+\sigma_1 x)}\gamma t_\varepsilon^{-M}\big(\|\partial_t u\|^2 + \|\partial_x u\|^2\big)\,dt\,dx\\
&+ \int_{\Delta_T} e^{-\gamma(t+\sigma_1 x)} M t_\varepsilon^{-M-1}\|\partial_x u\|^2\,dt\,dx\\
&+ \int_{\Delta_T} e^{-\gamma(t+\sigma_1 x)}(\gamma t_\varepsilon + (M-2k-1)) t_\varepsilon^{-M+2k} x_\varepsilon^{2l+1}\|\partial_y u\|^2\,dt\,dx\\
&\le C\int_{\Delta_T} e^{-\gamma(t+\sigma_1 x)} t_\varepsilon^{-M}\|L^0_{1,\varepsilon}u\|\|\partial_t u\|\,dt\,dx.
\end{aligned}$$

Similarly to (3.10), we have

$$\begin{aligned}
&\partial_t(e^{-\gamma(t+\sigma_1 x)} t_\varepsilon^{-M}\|u\|^2) + e^{-\gamma(t+\sigma_1 x)} t_\varepsilon^{-M}(\gamma + M t'_\varepsilon t_\varepsilon^{-1})\|u\|^2\\
&\le e^{-\gamma(t+\sigma_1 x)} t_\varepsilon^{-M}(\frac{\gamma}{2}\|u\|^2 + \frac{2}{\gamma}\|\partial_t u\|^2).
\end{aligned}$$

Hence, we obtain for $\gamma \ge \max\{\gamma_0, 1\}$ and $M \ge 2k+2$

$$\begin{aligned}
&\int_{\Delta_T} e^{-\gamma(t+\sigma_1 x)} t_\varepsilon^{-M}\big(\gamma(\|\partial_t u\|^2 + \|\partial_x u\|^2 + \|u\|^2) + t_\varepsilon^{-1}(\|\partial_x u\|^2 + \|u\|^2)\big)\,dt\,dx\\
&+ \int_{\Delta_T} e^{-\gamma(t+\sigma_1 x)} t_\varepsilon^{-M+2k+1} x_\varepsilon^{2l+1}(\gamma + t_\varepsilon^{-1})\|\partial_y u\|^2\,dt\,dx\\
&\le C\int_{\Delta_T} e^{-\gamma(t+\sigma_1 x)} t_\varepsilon^{-M}\big(\|L^0_{1,\varepsilon}u\|\|\partial_t u\| + \|\partial_t u\|^2\big)\,dt\,dx.
\end{aligned}$$

Since $\|L^0_{1,\varepsilon}u - L_{1,\varepsilon}u\| \le C\big(t_\varepsilon^k x_\varepsilon^{l+1}\|\partial_y u\| + \|u\|\big)$ and

$$t_\varepsilon^k x_\varepsilon^{l+1}\|\partial_y u\|\|\partial_t u\| \le (\frac{1}{2\sqrt{\gamma}} t_\varepsilon^{2k} x_\varepsilon^{2l+2}\|\partial_y u\|^2 + \frac{\sqrt{\gamma}}{2}\|\partial_t u\|^2), \tag{5.5}$$

there exists a $\gamma_1 > 0$ such that if $\gamma \geq \gamma_1$ and $M \geq 2k+2$

$$\begin{aligned}
&\int_{\Delta_T} e^{-\gamma(t+\sigma_1 x)} t_\varepsilon^{-M}\big(\gamma(\|\partial_t u\|^2 + \|\partial_x u\|^2 + \|u\|^2) + t_\varepsilon^{-1}(\|\partial_x u\|^2 + \|u\|^2)\big)\,dt\,dx \\
&+ \int_{\Delta_T} e^{-\gamma(t+\sigma_1 x)} t_\varepsilon^{-M+2k+1} x_\varepsilon^{2l+1}(\gamma + t_\varepsilon^{-1})\|\partial_y u\|^2\,dt\,dx \\
&\leq C\int_{\Delta_T} e^{-\gamma(t+\sigma_1 x)} t_\varepsilon^{-M}\|L_{1,\varepsilon}u\|\|\partial_t u\|\,dt\,dx
\end{aligned} \tag{5.6}$$

which implies

$$\int_{\Delta_T} t_\varepsilon^{-M}(\|\partial_t u\|^2 + \|\partial_x u\|^2 + \|u\|^2)\,dt\,dx \leq C\int_{\Delta_T} t_\varepsilon^{-M}\|L_{1,\varepsilon}u\|^2\,dt\,dx.$$

Since $[\partial_t, B(t,x,y)^{-1}L_{1,\varepsilon}]$ is equal to

$$\begin{aligned}
b(t,x,y)\partial_x\partial_t &- (2k+1)t'_\varepsilon t_\varepsilon^{2k} x_\varepsilon^{2l+1} a_2(t,x,y)\partial_y^2 \\
&- t_\varepsilon^{2k+1} x_\varepsilon^{2l+1} a_{2t}(t,x,y)\partial_y^2 + c_1(t,x,y)\partial_y + c_0(t,x,y)
\end{aligned}$$

and $[\partial_y, B(t,x,y)^{-1}L_{1,\varepsilon}]$ is equal to

$$\tilde{b}(t,x,y)\partial_x\partial_t - t_\varepsilon^{2k+1} x_\varepsilon^{2l+1} a_{2y}(t,x,y)\partial_y^2 + \tilde{c}_1(t,x,y)\partial_y + \tilde{c}_0(t,x,y)$$

where $a_2(t,x,y) = B(t,x,y)^{-1}A_2(t,x,y)$ and $b(t,x,y)$, $\tilde{b}(t,x,y)$, $c_j(t,x,y)$ and $\tilde{c}_j(t,x,y)$ $(j=1,2)$ are bounded smooth function on $\Delta_T \times \mathbb{R}$, then it follows from (5.6) where $u(t,x,y)$ is replaced by $\partial_t u(t,x,y)$ or $\partial_y u(t,x,y)$ that there exists a $\gamma_2 > 0$ such that for $\gamma \geq \gamma_2$ and $M \geq 2k+2$,

$$\begin{aligned}
&\int_{\Delta_T} e^{-\gamma(t+\sigma_1 x)} t_\varepsilon^{-M}\Big(\gamma\big(\|\partial_t^2 u\|^2 + \|\partial_x\partial_t u\|^2 + \|\partial_t u\|^2 + \|\partial_y\partial_t u\|^2 + \|\partial_y\partial_x u\|^2 \\
&+ \|\partial_y u\|^2 + \|u\|^2\big) + t_\varepsilon^{2k+1} x_\varepsilon^{2l+1}\gamma\|\partial_y^2 u\|^2\Big)\,dt\,dx \\
&\leq C\int_{\Delta_T} e^{-\gamma(t+\sigma_1 x)} t_\varepsilon^{-M}\big(\|\partial_t L_{1,\varepsilon}u\|^2 + \|\partial_y L_{1,\varepsilon}u\|^2 + \|L_{1,\varepsilon}u\|^2\big)\,dt\,dx,
\end{aligned}$$

where we used

$$\|\partial_y^2 u\|\|\partial_t^2 u\| \leq \frac{1}{2\sqrt{\gamma}}\|\partial_y^2 u\|^2 + \frac{\sqrt{\gamma}}{2}\|\partial_t^2 u\|^2.$$

Similarly for any positive integer N, there exists a $\gamma_N > 0$ such that if $\gamma \geq \gamma_N$ and $M \geq 2k+2$

$$\begin{aligned}
&\int_{\Delta_T} e^{-\gamma(t+\sigma_1 x)} t_\varepsilon^{-M}\gamma\Big(\sum_{\alpha_1+j\leq N \text{ and } j\leq N-1}\|\partial_t^{\alpha_1}\partial_y^j u\|^2 + \sum_{\alpha_1+j\leq N-1}\|\partial_t^{\alpha_1}\partial_y^j\partial_x u\|^2\Big)\,dt\,dx \\
&+ \int_{\Delta_T} e^{-\gamma(t+\sigma_1 x)} t_\varepsilon^{-M+2k+1} x_\varepsilon^{2l+1}\gamma\|\partial_y^N u\|^2\,dt\,dx \\
&\leq C\int_{\Delta_T} e^{-\gamma(t+\sigma_1 x)} t_\varepsilon^{-M}\Big(\sum_{\alpha_1+j\leq N-1}\|\partial_t^{\alpha_1}\partial_y^j(L_{1,\varepsilon}u)\|^2\Big)\,dt\,dx.
\end{aligned} \tag{5.7}$$

Since

$$\partial_x^2 = B(t,x,y)^{-1}(L_{1,\varepsilon} - \partial_x\partial_t$$

$$+ t_\varepsilon^{2k+1} x_\varepsilon^{2l+1} A_2(t,x,y)\partial_y^2 - t_\varepsilon^k x_\varepsilon^{l+1} A_1(t,x,y)\partial_y - a_0(t,x,y)),$$

we obtain from (5.7)

$$\begin{aligned}\int_{\Delta_T} t_\varepsilon^{-M}\Big(\sum_{\substack{\alpha_1+\alpha_2+j\le N\\ j\le N-1}} \|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j u\|^2\Big)\,dt\,dx \\ \le C\int_{\Delta_T} t_\varepsilon^{-M}\Big(\sum_{\substack{\alpha_1+\alpha_2+j\le N-1\\ \alpha_2\le \max\{N-2,0\}}} \|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j L_{1,\varepsilon}u\|^2\Big)\,dt\,dx. \quad (5.8)\end{aligned}$$

Hence, taking into account the proof of Lemma 3.2, we see from (5.8) that for any integer $M_1 \ge 0$ there exists an integer $M_2 \ge 0$ such that for any $u(t,x,y) \in C^\infty\{(t,x,y) \in \mathbb{R}^3 \mid x \ge 0\}$ satisfying $u(t,0,y) = 0$ and that $u(t,x,y) = 0$ if $|y|$ is large or $t \le 0$,

$$\begin{aligned}\int_{\Delta_T} \Big(\sum_{\alpha_1+\alpha_2+j\le M_1} \|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j u\|^2\Big)\,dt\,dx \\ \le C\int_{\Delta_T} \sum_{\alpha_1+\alpha_2+j\le M_2} \|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j L_{1,\varepsilon}u\|^2\,dt\,dx. \quad (5.9)\end{aligned}$$

Since solutions $\{u_\varepsilon(t,x,y)\}$ of (4.12) have a finite propagation speed that is independent of ε, as the proof of Lemma 2.3 we see from (5.9) that Lemma 4.4 is valid.

For the proof of Lemma 4.5, we first remark that by substracting some compactly supported function in $C^\infty\{(t,x,y) \in \mathbb{R}^3 \mid x \ge 0\}$ that is independent of ε, the problem (4.13) is reduced to that with $g(t,y) = 0$. In the following we assume it. Similarly to (5.4) it follows from (5.2) that, with $\sigma_1 = \sigma_{00}/2$,

$$\begin{aligned}&- \partial_x\big(e^{\gamma(t+\sigma_1 x)} t_\varepsilon(\|\partial_t u\|^2 - 2\Re(B(t,x,\cdot)\partial_x u, \partial_t u))\big) \\ &- \partial_t\big(e^{\gamma(t+\sigma_1 x)} t_\varepsilon(B(t,x,\cdot)\partial_x u, \partial_x u)\big) - \partial_t\big(e^{\gamma(t+\sigma_1 x)} t_\varepsilon^{2k+2} x_\varepsilon^{2l+1}(A_2(t,x,\cdot)\partial_y u, \partial_y u)\big) \\ &+ e^{\gamma(t+\sigma_1 x)}\gamma\sigma_1 t_\varepsilon\big(\|\partial_t u\|^2 - 2\Re(B(t,x,\cdot)\partial_x u, \partial_t u)\big) \\ &+ e^{\gamma(t+\sigma_1 x)}(\gamma t_\varepsilon + t'_\varepsilon)(B(t,x,\cdot)\partial_x u, \partial_x u) \\ &+ e^{\gamma(t+\sigma_1 x)}(\gamma t_\varepsilon + (2k+2)t'_\varepsilon) t_\varepsilon^{2k+1} x_\varepsilon^{2l+1}(A_2(t,x,\cdot)\partial_y u, \partial_y u) \\ &\le -2e^{-\gamma(t+\sigma_1 x)} t_\varepsilon \Re(L_{1,\varepsilon}^0 u, \partial_t u) \\ &+ C\big(e^{\gamma(t+\sigma_1 x)} t_\varepsilon^{2k+2} x_\varepsilon^{2l+1}(\|\partial_y u\|^2 + \|\partial_t u\|^2) + e^{\gamma(t+\sigma_1 x)} t_\varepsilon(\|\partial_x u\|\|\partial_t u\| + \|\partial_x u\|^2)\big),\end{aligned}$$

from which and from

$$-\partial_t(e^{\gamma(t+\sigma_1 x)} t_\varepsilon\|u\|^2) + e^{\gamma(t+\sigma_1 x)} t_\varepsilon(\gamma + t'_\varepsilon t_\varepsilon^{-1})\|u\|^2 \le e^{\gamma(t+\sigma_1 x)} t_\varepsilon\big(\frac{\gamma}{2}\|u\|^2 + \frac{2}{\gamma}\|\partial_t u\|^2\big)$$

we obtain the following estimate valid for any $u(t,x,y)$ in $C_0^\infty(\{(t,x,y) \in \mathbb{R}^3 \mid x \ge 0 \text{ and } t \ge 0\})$, vanishing when $t \ge T$ or $x \ge T$ with some $T > 0$ and satisfying

$u(t,0,y)=0$; there exists a $\gamma_0>0$ such that for $\gamma\ge\gamma_0$

$$\begin{aligned}&\int_{D_T} e^{\gamma(t+\sigma_1 x)}t_\varepsilon\Big(\gamma\big(\|\partial_t u\|^2+\|\partial_x u\|^2+\|u\|^2\big)+t_\varepsilon^{-1}(\|\partial_x u\|^2+\|u\|^2)\Big)\,dt\,dx\\&+\int_{D_T} e^{\gamma(t+\sigma_1 x)}(\gamma t_\varepsilon+(2k+2))t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}\|\partial_y u\|^2\,dt\,dx\\&\le C\int_{D_T} e^{\gamma(t+\sigma_1 x)}t_\varepsilon\|L^0_{1,\varepsilon}u\|\|\partial_t u\|\,dt\,dx\end{aligned}$$

where $D_T=[0,T]\times[0,T]$. Hence we see from the estimate on D_T

$$\|{}^tL_{1,\varepsilon}u-L^0_{1,\varepsilon}u\|\le C_0t_\varepsilon^k x_\varepsilon^{l+1}\|\partial_y u\|+C_1(\|\partial_x u\|+\|u\|)$$

and (5.5) that there exists $\gamma_1>0$ such that for $\gamma\ge\gamma_1$

$$\begin{aligned}&\int_{D_T} e^{\gamma(t+\sigma_1 x)}t_\varepsilon\Big(\gamma\big(\|\partial_t u\|^2+\|\partial_x u\|^2+\|u\|^2\big)+t_\varepsilon^{-1}(\|\partial_x u\|^2+\|u\|^2)\Big)\,dt\,dx\\&+\int_{D_T} e^{\gamma(t+\sigma_1 x)}(\gamma t_\varepsilon+1)t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}\|\partial_y u\|^2\,dt\,dx\\&\le C\int_{D_T} e^{\gamma(t+\sigma_1 x)}t_\varepsilon\|{}^tL_{1,\varepsilon}u\|\|\partial_t u\|\,dt\,dx,\end{aligned}\tag{5.10}$$

which implies

$$\begin{aligned}\int_{D_T} t_\varepsilon\big(\|\partial_t u\|^2+\|\partial_x u\|^2+\|u\|^2\big)\,dt\,dx+\int_{D_T} t_\varepsilon^{2k+2}x_\varepsilon^{2l+1}\|\partial_y u\|^2\,dt\,dx\\\le C\int_{D_T} t_\varepsilon\|{}^tL_{1,\varepsilon}u\|^2\,dt\,dx.\end{aligned}$$

From the expression of $[\partial_t,B(t,x,y)^{-1}\,{}^tL_{1,\varepsilon}]$ and $[\partial_y,B(t,x,y)^{-1}\,{}^tL_{1,\varepsilon}]$ and (5.10), we get

$$\begin{aligned}\int_{D_T} e^{\gamma(t+\sigma_1 x)}t_\varepsilon\sum_{\alpha_1+j\le 1}\big(\|\partial_t^{\alpha_1+1}\partial_y^j u\|^2+\|\partial_t^{\alpha_1}\partial_y^j\partial_x u\|^2+\|\partial_t^{\alpha_1}\partial_y^j u\|^2\big)\,dt\,dx\\+\int_{D_T} e^{\gamma(t+\sigma_1 x)}t_\varepsilon^{2k+2}x_\varepsilon^{2l+1}\|\partial_y^2 u\|^2\,dt\,dx\\\le\frac{C}{\gamma}\int_{D_T} e^{\gamma(t+\sigma_1 x)}t_\varepsilon\sum_{\alpha_1+j\le 1}\|\partial_t^{\alpha_1}\partial_y^j\,{}^tL_{1,\varepsilon}u\|^2\,dt\,dx\end{aligned}$$

Here γ is large. In general for any integer $N\ge 0$ we obtain

$$\begin{aligned}\int_{D_T} e^{\gamma(t+\sigma_1 x)}t_\varepsilon\sum_{\alpha_1+j\le N}\big(\|\partial_t^{\alpha_1+1}\partial_y^j u\|^2+\|\partial_t^{\alpha_1}\partial_y^j\partial_x u\|^2+\|\partial_t^{\alpha_1}\partial_y^j u\|^2\big)\,dt\,dx\\+\int_{D_T} e^{\gamma(t+\sigma_1 x)}t_\varepsilon^{2k+2}x_\varepsilon^{2l+1}\|\partial_y^{N+1} u\|^2\,dt\,dx\\\le\frac{C}{\gamma}\int_{D_T} e^{\gamma(t+\sigma_1 x)}t_\varepsilon\sum_{\alpha_1+j\le N}\|\partial_t^{\alpha_1}\partial_y^j\,{}^tL_{1,\varepsilon}u\|^2\,dt\,dx\end{aligned}\tag{5.11}$$

Here also γ is large. Since

$$\begin{aligned}\partial_x^2=&-B(t,x,y)^{-1}\big({}^tL_{1,\varepsilon}+t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}A_2(t,x,y)\partial_y^2\\&+l_1(t,x,y,\partial_t,\partial_y)\partial_x+l_2(t,x,y,\partial_t,\partial_y)\partial_t+l_3(t,x,y,\partial_t,\partial_y)\big)\end{aligned}$$

where $l_j(t,x,y,\partial_t,\partial_y)$ $(j=1,2,3)$ is a first order differential operator, then from (5.11) follows the estimate

$$\begin{aligned}
&\int_{D_T} t_\varepsilon \sum_{\alpha_1+\alpha_2+j\le N} \big(\|\partial_t^{\alpha_1+1}\partial_x^{\alpha_2}\partial_y^j u\|^2 + \|\partial_t^{\alpha_1}\partial_x^{\alpha_2+1}\partial_y^j u\|^2 + \|\partial_t^{\alpha_1}\partial_y^j u\|^2\big)\,dt\,dx \\
&\quad + \int_{D_T} t_\varepsilon^{2k+2} x_\varepsilon^{2l+1} \|\partial_y^{N+1}u\|^2\,dt\,dx \\
&\quad \le C\int_{D_T} t_\varepsilon \sum_{\substack{\alpha_1+\alpha_2+j\le N\\ \alpha_2\le\max\{N-1,0\}}} \|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j\, {}^tL_{1,\varepsilon}u\|^2\,dt\,dx.
\end{aligned} \tag{5.12}$$

From Lemma 3.4 and (5.12) we see that for any integer $M_1 \ge 0$ there exists an integer $M_2 \ge 0$ such that

$$\begin{aligned}
&\int_{D_T} \sum_{\alpha_1+\alpha_2+j\le M_1} \|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j u\|^2\,dt\,dx \\
&\qquad \le C\int_{D_T} \sum_{\alpha_1+\alpha_2+j\le M_2} \|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j\, {}^tL_{1,\varepsilon}u\|^2\,dt\,dx.
\end{aligned} \tag{5.13}$$

Since the constant C of the estimate above (5.13) is independent of $0<\varepsilon<1$, (5.13) implies that Lemma 4.5 is valid.

Now we prove Lemmas 4.7 and 4.8. Let

$$\tilde L^0_{2,\varepsilon} = \partial_t\partial_x + B(t,x,y)\partial_x^2 - t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}A_2(t,x,y){\partial_y}^2.$$

Since

$$\begin{aligned}
2\Re(\tilde L^0_{2,\varepsilon}u,\partial_x u) =&\partial_t(\partial_x u,\partial_x u) + \partial_x(B(t,x,\cdot)\partial_x u,\partial_x u) \\
&+ t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}\partial_x(A_2(t,x,\cdot)\partial_y u,\partial_y u) + R
\end{aligned} \tag{5.15}$$

where

$$|R| \le C\big(t_\varepsilon^{2k+1}x_\varepsilon^{2l+1}(\|\partial_y u\|^2 + \|\partial_x u\|^2) + \|\partial_x u\|^2\big), \tag{5.16}$$

we have for any $u(t,x,y)$ vanishing for large $|y|$,

$$\begin{aligned}
&\partial_t(e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}\|\partial_x u\|^2) + \partial_x\big(e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}(B(t,x,\cdot)\partial_x u,\partial_x u)\big) \\
&+ \partial_x\big(e^{-\gamma(t+x)}t_\varepsilon^{-M+2k+1}x_\varepsilon^{-M+2l+1}(A_2(t,x,\cdot)\partial_y u,\partial_y u)\big) \\
&+ e^{-\gamma(t+x)}(\gamma t_\varepsilon + Mt'_\varepsilon)t_\varepsilon^{-M-1}x_\varepsilon^{-M}\|\partial_x u\|^2 \\
&+ e^{-\gamma(t+x)}(\gamma x_\varepsilon + Mx'_\varepsilon)t_\varepsilon^{-M}x_\varepsilon^{-M-1}(B(t,x,\cdot)\partial_x u,\partial_x u) \\
&+ e^{-\gamma(t+x)}(\gamma x_\varepsilon + (M-2l-1)x'_\varepsilon)t_\varepsilon^{-M+2k+1}x_\varepsilon^{-M+2l}(A_2(t,x,\cdot)\partial_y u,\partial_y u) \\
&\le 2e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}\Re(\tilde L^0_{2,\varepsilon}u,\partial_x u) \\
&\quad + Ce^{-\gamma(t+x)}\big(t_\varepsilon^{-M+2k+1}x_\varepsilon^{-M+2l+1}(\|\partial_y u\|^2 + \|\partial_x u\|^2) + t_\varepsilon^{-M}x_\varepsilon^{-M}\|\partial_x u\|^2\big).
\end{aligned}$$

Noting

$$\begin{aligned}
&\partial_x(e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}\|u\|^2) + e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}(\gamma + Mx'_\varepsilon x_\varepsilon^{-1})\|u\|^2 \\
&\qquad \le e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}(\frac{\gamma}{2}\|u\|^2 + \frac{2}{\gamma}\|\partial_x u\|^2),
\end{aligned} \tag{5.17}$$

we see that, if $\gamma \geq \gamma_0$ with some γ_0 and $M \geq 2l+2$, we have for any $u(t,x,y)$ which is flat on the plane $t=0$ and on the plane $x=0$ and vanishes for large $|y|$,

$$\begin{aligned}\int_{D_T} & e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M} \big((\gamma + t_\varepsilon^{-1})\|\partial_x u\|^2 + (\gamma + x_\varepsilon^{-1})(\|\partial_x u\|^2 + \|u\|^2)\big)\,dt\,dx \\ &+ \int_{D_T} e^{-\gamma(t+x)}(\gamma x_\varepsilon + 1) t_\varepsilon^{-M+2k+1} x_\varepsilon^{-M+2l}\|\partial_y u\|^2\,dt\,dx \\ &\leq C\int_{D_T} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M}\|\tilde{L}^0_{2,\varepsilon}u\|\|\partial_x u\|\,dt\,dx\end{aligned} \tag{5.18}$$

where $D_T = [0,T]\times[0,T]$. Since $\|\tilde{L}^0_{2,\varepsilon}u - \tilde{L}_{2,\varepsilon}u\| \leq C\big(t_\varepsilon^k x_\varepsilon^{l+1}\|\partial_y u\| + \|u\|\big)$ and

$$t_\varepsilon^k x_\varepsilon^{l+1}\|\partial_y u\|\|\partial_x u\| \leq (\frac{\sqrt{\gamma}}{2} t_\varepsilon^{2k+1} x_\varepsilon^{2l+2}\|\partial_y u\|^2 + \frac{1}{2\sqrt{\gamma}} t_\varepsilon^{-1}\|\partial_x u\|^2),$$

then from (5.18) there exists a $\gamma_1 > 0$ such that if $\gamma \geq \gamma_1$ and $M \geq 2l+2$, we have

$$\begin{aligned}&\int_{D_T} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M}\big(\gamma(\|\partial_x u\|^2 + \|u\|^2) + t_\varepsilon^{-1}\|\partial_x u\|^2 + x_\varepsilon^{-1}(\|\partial_x u\|^2 + \|u\|^2)\big)\,dt\,dx \\ &+ \int_{D_T} e^{-\gamma(t+x)}(\gamma x_\varepsilon + 1) t_\varepsilon^{-M+2k+1} x_\varepsilon^{-M+2l}\|\partial_y u\|^2\,dt\,dx \\ &\leq C\int_{D_T} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M}\|\tilde{L}_{2,\varepsilon}u\|\|\partial_x u\| dt\,dx\end{aligned} \tag{5.19}$$

which implies

$$\int_{D_T} t_\varepsilon^{-M} x_\varepsilon^{-M}(\|\partial_x u\|^2 + \|u\|^2)\,dt\,dx \leq C\int_{D_T} t_\varepsilon^{-M} x_\varepsilon^{-M}\|\tilde{L}_{2,\varepsilon}u\|^2\,dt\,dx.$$

Since $[\partial_x, \tilde{L}_{2,\varepsilon}]$ is equal to

$$\begin{aligned}b_1(t,x,y)\partial_x^2 &- (2l+1)x'_\varepsilon t_\varepsilon^{2k+1} x_\varepsilon^{2l} A_2(t,x,y)\partial_y^2 \\ &- t_\varepsilon^{2k+1} x_\varepsilon^{2l+1} A_{2x}(t,x,y)\partial_y^2 + c_{1,1}(t,x,y)\partial_y + c_{1,0}(t,x,y),\end{aligned}$$

$$\begin{aligned}[\partial_t, \tilde{L}_{2,\varepsilon}] =& b_2(t,x,y)\partial_x^2 - (2k+1)t'_\varepsilon t_\varepsilon^{2k} x_\varepsilon^{2l+1} A_2(t,x,y)\partial_y^2 \\ &- t_\varepsilon^{2k+1} x_\varepsilon^{2l+1} A_{2t}(t,x,y)\partial_y^2 + c_{2,1}(t,x,y)\partial_y + c_{2,0}(t,x,y),\end{aligned}$$

and $[\partial_y, \tilde{L}_{2,\varepsilon}]$ is equal to

$$b_3(t,x,y)\partial_x^2 - t_\varepsilon^{2k+1} x_\varepsilon^{2l+1} A_{2y}(t,x,y)\partial_y^2 + c_{3,1}(t,x,y)\partial_y + c_{3,0}(t,x,y)$$

where $b_j(t,x,y)$, $c_{j,k}(t,x,y)$ $(j=1,2,3,\ k=1,2)$ are bounded smooth function on $D_T\times\mathbb{R}$, then it follows from (5.19) with $\partial_x u(t,x,y)$, $\partial_t u(t,x,y)$ or $\partial_y u(t,x,y)$ in the place of $u(t,x,y)$ that there exists a $\gamma_2 > 0$ such that if $\gamma \geq \gamma_2$ and $M \geq 2l+2$

$$\begin{aligned}&\int_{D_T} e^{-\gamma(t+x)} t_\varepsilon^{-M} x_\varepsilon^{-M}\gamma\big(\|\partial_x^2 u\|^2 + \|\partial_x\partial_t u\|^2 + \|\partial_x\partial_y u\|^2 \\ &+ \|\partial_x u\|^2 + \|\partial_t u\|^2 + \|\partial_y u\|^2 + \|u\|^2\big)\,dt\,dx \\ &+ \int_{D_T} e^{-\gamma(t+x)} t_\varepsilon^{-M+2k+1} x_\varepsilon^{-M+2l+1}\gamma\|\partial_y^2 u\|^2\,dt\,dx \\ &\leq C\int_{D_T} e^{-\gamma(t+x)} t_\varepsilon^{-M}\big(\|\partial_x\tilde{L}_{2,\varepsilon}u\|^2 + \|\partial_t\tilde{L}_{2,\varepsilon}u\|^2 + \|\partial_y\tilde{L}_{2,\varepsilon}u\|^2 + \|\tilde{L}_{2,\varepsilon}u\|^2\big)\,dt\,dx,\end{aligned}$$

where we used

$$t_\varepsilon^{2k+1}x_\varepsilon^{2l}\|\partial_y^2u\|\|\partial_x^2u\| \le t_\varepsilon^{2k+1}x_\varepsilon^{2l}(\frac{\sqrt{\gamma}}{2}x_\varepsilon\|\partial_y^2u\|^2+\frac{1}{2\sqrt{\gamma}}x_\varepsilon^{-1}\|\partial_x^2u\|^2)$$

and

$$t_\varepsilon^{2k}x_\varepsilon^{2l+1}\|\partial_y^2u\|\|\partial_x\partial_tu\| \le t_\varepsilon^{2k}x_\varepsilon^{2l+1}(\frac{\sqrt{\gamma}}{2}t_\varepsilon\|\partial_y^2u\|^2+\frac{1}{2\sqrt{\gamma}}t_\varepsilon^{-1}\|\partial_x\partial_tu\|^2).$$

In general for any positive integer N, there exists a $\gamma_N > 0$ such that if $\gamma \ge \gamma_N$ and $M \ge 2l+2$, we have

$$\begin{aligned}
&\int_{D_T} e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}\Big(\sum_{\alpha_1+\alpha_2+j=N-1}\|\partial_t^{\alpha_1}\partial_x^{\alpha_2+1}\partial_y^ju(t,x,y)\|^2\\
&\quad+\sum_{\alpha_1+\alpha_2+j\le N-1}\|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^ju\|^2\Big)\,dt\,dx\\
&\quad+\int_{D_T} e^{-\gamma(t+x)}t_\varepsilon^{-M+2k+1}x_\varepsilon^{-M+2l+1}\|\partial_y^Nu(t,x,y)\|^2\,dt\,dx\\
&\quad\le \frac{C}{\gamma}\int_{D_T} e^{-\gamma(t+x)}t_\varepsilon^{-M}x_\varepsilon^{-M}\Big(\sum_{\alpha_1+\alpha_2+j\le N-1}\|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j\tilde{L}_{2,\varepsilon}u\|^2\Big)\,dt\,dx.
\end{aligned} \tag{5.20}$$

When $u(t,x,y)$ is flat on $x=0$ and on $t=0$, Lemma 3.2 and (5.20) imply that for any integer $M_1 \ge 0$ there exists an integer $M_2 \ge 0$ such that

$$\sum_{\alpha_1+\alpha_2+j\le M_1}\int_{D_T}\|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^ju\|^2\,dt\,dx \le C\sum_{\alpha_1+\alpha_2+j\le M_2}\int_{D_T}\|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j\tilde{L}_{2,\varepsilon}u\|^2\,dt\,dx. \tag{5.21}$$

Since the right hand side $h(t,x,y)$ of (4.17) is flat on $x=0$, we remark that the solution $u_\varepsilon(t,x,y)$ to (4.17) is also flat on $x=0$. Therefore we see through the similar argument of the proof of Lemma 2.3 that the estimate (5.21) and the finite propagation speed that is independent of $0<\varepsilon<1$ show that Lemma 4.7 is valid.

Finally we prove Lemma 4.8. We can draw from (5.15) and (5.16) the following estimate for any $u(t,x,y)$ vanishing for large $|y|$;

$$\begin{aligned}
&-\partial_t(e^{\gamma(t+x)}t_\varepsilon x_\varepsilon\|\partial_xu\|^2)-\partial_x\big(e^{\gamma(t+x)}t_\varepsilon x_\varepsilon(B(t,x,\cdot)\partial_xu,\partial_xu)\big)\\
&-\partial_x\big(e^{\gamma(t+x)}t_\varepsilon^{2k+2}x_\varepsilon^{2l+2}(A_2(t,x,\cdot)\partial_yu,\partial_yu\big)+e^{\gamma(t+x)}t_\varepsilon x_\varepsilon(\gamma+t'_\varepsilon t_\varepsilon^{-1})\|\partial_xu\|^2\\
&+e^{\gamma(t+x)}t_\varepsilon x_\varepsilon(\gamma+x'_\varepsilon x_\varepsilon^{-1})(B(t,x,\cdot)\partial_xu,\partial_xu)\\
&+e^{\gamma(t+x)}(\gamma x_\varepsilon+(2l+2)x'_\varepsilon)t_\varepsilon^{2k+2}x_\varepsilon^{2l+1}(A_2(t,x,\cdot)\partial_yu,\partial_yu)\\
&\le -2e^{\gamma(t+x)}t_\varepsilon x_\varepsilon\Re(\tilde{L}_{2,\varepsilon}^0u,\partial_xu)\\
&\quad+Ce^{\gamma(t+x)}\big(t_\varepsilon^{2k+2}x_\varepsilon^{2l+2}(\|\partial_yu\|^2+\|\partial_xu\|^2)+t_\varepsilon x_\varepsilon\|\partial_xu\|^2\big),
\end{aligned}$$

from which and from

$$\begin{aligned}
&-\partial_x(e^{\gamma(t+x)}t_\varepsilon x_\varepsilon\|u\|^2)+e^{\gamma(t+x)}t_\varepsilon x_\varepsilon(\gamma+x'_\varepsilon x_\varepsilon^{-1})\|u\|^2\\
&\qquad\le e^{\gamma(t+x)}t_\varepsilon x_\varepsilon(\frac{\gamma}{2}\|u\|^2+\frac{2}{\gamma}\|\partial_xu\|^2),
\end{aligned}$$

we obtain the following estimate for any $u(t,x,y)$ in $C_0^\infty(\{(t,x,y)\in\mathbb{R}^3 \mid x\ge 0 \text{ and } t\ge 0\})$ vanishing when $t\ge T$ or $x\ge T$ with some $T>0$ with some $T>0$;

there exists a $\gamma_0 > 0$ such that for $\gamma \geq \gamma_0$

$$\begin{aligned}&\int_{D_T} e^{\gamma(t+x)} t_\varepsilon x_\varepsilon \Big(\gamma\big(\|\partial_x u\|^2 + \|u\|^2\big) + x_\varepsilon^{-1}(\|\partial_x u\|^2 + \|u\|^2) + t_\varepsilon^{-1}\|\partial_x u\|^2\Big)\,dt\,dx \\ &+ \int_{D_T} e^{\gamma(t+x)}(\gamma x_\varepsilon + (2l+2)) t_\varepsilon^{2k+2} x_\varepsilon^{2l+1}\|\partial_y u\|^2\,dt\,dx \\ &\leq C\int_{D_T} e^{\gamma(t+x)} t_\varepsilon x_\varepsilon \|\tilde{L}^0_{2,\varepsilon} u\|\|\partial_x u\|\,dt\,dx.\end{aligned}$$

Hence, we see from the estimate on D_T

$$\|{}^t\tilde{L}_{2,\varepsilon} u - \tilde{L}^0_{2,\varepsilon} u\| \leq C_0 t_\varepsilon^k x_\varepsilon^{l+1}\|\partial_y u\| + C_1(\|\partial_x u\| + \|u\|)$$

and (5.17) that there exists $\gamma_1 > 0$ such that for $\gamma \geq \gamma_1$,

$$\begin{aligned}&\int_{D_T} e^{\gamma(t+x)} t_\varepsilon x_\varepsilon \Big(\gamma\big(\|\partial_x u\|^2 + \|u\|^2\big) + x_\varepsilon^{-1}(\|\partial_x u\|^2 + \|u\|^2) + t_\varepsilon^{-1}\|\partial_x u\|^2\Big)\,dt\,dx \\ &+ \int_{D_T} e^{\gamma(t+x)}(\gamma x_\varepsilon + 1) t_\varepsilon^{2k+2} x_\varepsilon^{2l+1}\|\partial_y u\|^2\,dt\,dx \\ \leq& C\int_{D_T} e^{\gamma(t+x)} t_\varepsilon x_\varepsilon \|{}^t\tilde{L}_{2,\varepsilon} u\|\|\partial_x u\|\,dt\,dx,\end{aligned} \tag{5.22}$$

which implies

$$\begin{aligned}&\int_{D_T} t_\varepsilon x_\varepsilon\big(\|\partial_x u\|^2 + \|u\|^2\big)\,dt\,dx + \int_{D_T} t_\varepsilon^{2k+2} x_\varepsilon^{2l+2}\|\partial_y u\|^2\,dt\,dx \\ &\qquad\leq C\int_{D_T} t_\varepsilon x_\varepsilon \|{}^t L_{1,\varepsilon} u\|^2\,dt\,dx.\end{aligned}$$

Similarly to the estimates for $\tilde{L}_{2,\varepsilon}$, we obtain from (5.22) the following; for any integer $N \geq 0$ we have

$$\begin{aligned}&\int_{D_T} t_\varepsilon x_\varepsilon \sum_{\alpha_1+\alpha_2+j\leq N}\big(\|\partial_t^{\alpha_1}\partial_x^{\alpha_2+1}\partial_y^j u\|^2 + \|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j u\|^2\big)\,dt\,dx \\ &\quad+ \int_{D_T} t_\varepsilon^{2k+2} x_\varepsilon^{2l+2}\|\partial_y^{N+1} u\|^2\,dt\,dx \\ &\quad\leq C\int_{D_T} t_\varepsilon x_\varepsilon \sum_{\alpha_1+\alpha_2+j\leq N}\|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j\,{}^t\tilde{L}_{2,\varepsilon} u\|^2\,dt\,dx.\end{aligned} \tag{5.23}$$

From Lemma 3.4 and (5.23) we see that for any integer $M_1 \geq 0$ there exists an integer $M_2 \geq 0$ such that

$$\begin{aligned}&\int_{D_T} \sum_{\alpha_1+\alpha_2+j\leq M_1}\|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j u\|^2\,dt\,dx \\ &\qquad\leq C\int_{D_T} \sum_{\alpha_1+\alpha_2+j\leq M_2}\|\partial_t^{\alpha_1}\partial_x^{\alpha_2}\partial_y^j\,{}^t\tilde{L}_{2,\varepsilon} u\|^2\,dt\,dx.\end{aligned} \tag{5.24}$$

Since the constant C of the estimate above (5.24) is independent of $0 < \varepsilon < 1$, (5.24) implies that Lemma 4.8 is valid. Then the proof of Theorem 1.2 is complete.

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